On the advection-diffusion equation with rough coefficients: weak solutions and vanishing viscosity

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ON THE ADVECTION-DIFFUSION EQUATION WITH ROUGH COEFFICIENTS: WEAK SOLUTIONS AND VANISHING VISCOSITY

PAOLO BONICATTO, GENNARO CIAMPA, AND GIANLUCA CRIPPA

Abstract. In the first part of the paper, we study the Cauchy problem for the advection-diffusion equation $\partial_t v + \operatorname{div}(v\boldsymbol{b}) = \Delta v$ associated with a merely integrable, divergence-free vector field $\boldsymbol{b}$ defined on the torus. We first introduce two notions of solutions (distributional and parabolic), recalling the corresponding available results of existence and uniqueness. Then, we establish a regularity criterion, which in turn provides uniqueness for distributional solutions. This is motivated by the recent results in [31] where the authors showed non-uniqueness of distributional solutions to the advection-diffusion equation despite the parabolic one is unique. In the second part of the paper, we precisely describe the vanishing viscosity scheme for the transport/continuity equation drifted by $\boldsymbol{b}$, i.e. $\partial_t u + \operatorname{div}(u\boldsymbol{b}) = 0$. Under Sobolev assumptions on $\boldsymbol{b}$, we give two independent proofs of the convergence of such scheme to the Lagrangian solution of the transport equation. The first proof slightly generalizes the original one of [21]. The other one is quantitative and yields rates of convergence. This offers a completely general selection criterion for the transport equation (even beyond the distributional regime) which compensates the wild non-uniqueness phenomenon for solutions with low integrability arising from convex integration schemes, as shown in recent works [10, 31, 32, 33], and rules out the possibility of anomalous dissipation.

Keywords: transport/continuity equation, advection-diffusion equation, vanishing viscosity, regular Lagrangian flow, uniqueness, stochastic Lagrangian flow, anomalous dissipation.

2020 Mathematics Subject Classification: 35F10, 35K15, 35Q35

Contents

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1. Introduction

The goal of this paper is twofold. First of all, we are interested in the so-called *advection-diffusion equation*
$$\partial_t v + \operatorname{div}(v\boldsymbol{b}) = \Delta v$$
under general, low regularity assumptions on the (divergence-free) vector field $\boldsymbol{b}$. Furthermore, we want to exploit the advection-diffusion equation to set up the *vanishing viscosity scheme* for the linear transport/continuity equation
$$\partial_t u + \operatorname{div}(u\boldsymbol{b}) = 0$$
under Sobolev regularity assumptions on $\boldsymbol{b}$, in a framework of low integrability for the solution which does not guarantee uniqueness. More precisely, we construct a family $(v_\varepsilon)_\varepsilon$ of solutions to
$$\partial_t v_\varepsilon + \operatorname{div}(v_\varepsilon\boldsymbol{b}) = \varepsilon\Delta v_\varepsilon$$
and establish the rigourous convergence $v_\varepsilon \to u^{\mathsf{L}}$ as $\varepsilon \downarrow 0$, where u^{L} is the Lagrangian solution of the transport equation, that is the solution transported by the flow of $\boldsymbol{b}$. Such a result fits into a well understood physical mechanism (the zero diffusivity/viscosity limit) and has also its own, mathematical interest: similar schemes have been proposed over the years for different equations (e.g. for hyperbolic conservation laws [4, 24]). Since the Lagrangian solution preserves all the L^q-norms (if finite at the initial time), this also rules out the possibility of anomalous dissipation in the vanishing viscosity limit.

1.1. Part I. The advection-diffusion equation.

Given a vector field $\boldsymbol{b}\colon [0,T]\times\mathbb{T}^d \to \mathbb{R}^d$ on the d-dimensional torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, we study the initial value problem for the advection-diffusion equation associated with $\boldsymbol{b}$, i.e.
$$\begin{cases} \partial_t v + \operatorname{div}(v\boldsymbol{b}) = \Delta v \\ v|_{t=0} = v_0, \end{cases} \tag{ADE}$$
where $v_0\colon \mathbb{T}^d \to \mathbb{R}$ is a given initial datum. Due to the presence of the Laplacian, (ADE) is a second-order parabolic partial differential equation. If the vector field $\boldsymbol{b}$ is smooth, classical existence and uniqueness results are available and can be found in standard PDEs textbooks (see e.g. [23]). The problem (ADE) has been also studied outside the smooth framework in many classical references (besides quoting again [23], we mention the monograph [27]).
Nevertheless, our approach is closer to the one developed in the more recent book [29], which is intimately related to a fluid dynamics context. Typically, existence results are obtained by a simple approximation argument: under global bounds on the vector field, one easily establishes energy estimates for the solutions of suitable approximate problems. Such estimates allow to apply standard weak compactness results and the linearity of the equation ensures that the weak limit is a solution to (ADE).

At a closer look, however, an interesting feature of (ADE) arises: it is possible to give several, a priori different, notions of "weak" solutions. This opens a wide spectrum of possibilities and taming this complicated scenario, understanding the relationships among different notions of solutions, is one of the aims of the present work.

Distributional solutions. We first deal with divergence-free vector fields $\boldsymbol{b}$, satisfying a general $L^1_t L^p_x$ integrability condition in space-time, for some $1 \le p \le \infty$. Correspondingly, we assume that the initial datum $\overline{v} \in L^q(\mathbb{T}^d)$, for some $1 \le q \le \infty$, with $1/p + 1/q \le 1$. This allows to introduce distributional solutions to (ADE), i.e. functions $v \in L^\infty_t L^q_x$ solving the equation in the sense of distributions. Notice that a mild regularity in time of solutions is always granted for evolutionary PDEs, which allows to give a meaning to the initial condition in the Cauchy problem (ADE). It is then easily seen that distributional solutions always exist; yet, such a notion seems too vague and uniqueness is, in general, false.

Parabolic solutions. The general lack of uniqueness for distributional solutions motivates the introduction of another notion of solution. Hopefully, such alternative notion will share the same existence results as the distributional ones, offering at the same time some uniqueness results. It turns out that such a notion can be found for fields having enough integrability in the space variable, $\boldsymbol{b} \in L^1_t L^2_x$. If this is the case, exploiting the divergence-free constraint one can show the basic available energy estimate for smooth solutions

$$\frac{1}{2}\int_{\mathbb{T}^d} |v(t,x)|^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{T}^d} |\nabla v(\tau,x)|^2 \,\mathrm{d}\tau \,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{T}^d} |v_0(x)|^2 \,\mathrm{d}x,$$

for every $t \in [0,T]$. The energy estimate entices one to look for solutions possessing L^2 gradient, i.e. solutions that are H^1 in the space variable. We therefore say that a distributional solution $v \in L^\infty_t L^q_x$ to (ADE) is *parabolic* if it holds $v \in L^2_t H^1_x$. The energy estimate suggests that we could look for solutions possessing L^2 gradient, i.e. solutions that are H^1 in the space variable: we say that a distributional solution $v \in L^\infty_t L^q_x$ to (ADE) is *parabolic* if it holds $v \in L^2_t H^1_x$. Crucially, parabolic solutions carry the exact regularity needed to establish their uniqueness (under a suitable integrability assumption of the field $\boldsymbol{b}$ w.r.t. the time variable as well). This uniqueness result is proven via a well-known technique, i.e. resorting to commutators' estimates. The $L^2_t H^1_x$ regularity of the solution allows to obtain a better control on the error one commits when considering smooth approximations of the solution. Such error (which is commonly known as *commutator*) always goes to 0 in the sense of distributions; however, in order to prove uniqueness, a better control is needed. In particular, in [28] it is shown that the commutator for parabolic solutions converges strongly to 0 in $L^1_{t,x}$. This is made possible by the fact that, asymptotically, the commutator is related to the quantity $\boldsymbol{b} \cdot \nabla v$ and bounds for this product can be established (for parabolic solutions $v \in L^2_t H^1_x$) if $\boldsymbol{b} \in L^2_t L^2_x$.

A new regularity result for distributional solutions. Besides existence and uniqueness results for distributional and for parabolic solutions, a legitimate question concerns the mutual relationship between these two notions; according to our definitions, parabolic solutions cannot always be defined, but if they can, then they are always distributional. The converse implication is, in general, not true: in [31] it is shown that there exist infinitely many distributional solutions $v \in L^\infty_t L^2_x$ to (ADE) with a vector field $\boldsymbol{b} \in L^\infty_t L^2_x$, while the parabolic one is unique. This motivates our search for a condition that guarantees *parabolic regularity* of a distributional solution. Our first main result shows that, in the regime $1/p + 1/q \leq 1/2$ (and under a L^2 integrability assumption of $\boldsymbol{b}$ w.r.t. time), every distributional solution is parabolic (hence, a fortiori, unique). The precise statement is the following:

Regularity Theorem. Let $p, q \in [1,\infty)$ such that $1/p + 1/q \leq 1/2$. If $\boldsymbol{b} \in L^2_t L^p_x$ is a divergence-free vector field and $u \in L^\infty_t L^q_x$ is a distributional solution to (ADE), then $u \in L^2_t H^1_x$.

Such a regularity result is, to our knowledge, new and it is obtained using a refined commutator estimate: we show that, in the current regime, the convergence to zero of the commutators takes place in $L^2_t H^{-1}_x$ and this is enough to obtain our regularity result (see Lemma 3.8 for the precise commutator estimate). We remark, en passant, that the L^2 integrability seems critical in our argument. Recent, groundbreaking works (which will be discussed more thoroughly later on in this introduction) have shown that, at lower integrability, a severe phenomenon of non-uniqueness may arise. In particular, using convex integration techniques, in [32, 33, 31] the authors constructed divergence-free vector fields $\boldsymbol{b} \in C^0_t L^p_x$, with $1 \leq p < \gamma(d) < 2$, such that (ADE) admits infinitely many solutions in the class $C^0_t H^1_x$. Here $\gamma(d) = 2d/d+2$ denotes a dimensional constant, which is indeed strictly smaller than the critical exponent 2. The situation in the intermediate regime $\gamma(d) \le p < 2$ is still open and it is the object of one question we formulate. See also [7], where nonuniqueness of weak solutions (not necessarily in the Leray class) of the Navier-Stokes equations is shown via convex integration with a beautiful argument exploiting time-intermittency, and [9], in which it is shown that the integrability of weak solutions plays an essential role for weak-strong uniqueness results for the Navier-Stokes equations.

Finally, we observe that also the integrability in time could play a non trivial role (in a similar spirit to e.g. [10]): it seems conceivable that non-uniqueness of parabolic solutions arises when $\boldsymbol{b} \in L^2_t L^p_x$ (instead of $\boldsymbol{b} \in C^0_t L^p_x$) for a larger class of exponents p.

We refer the reader to Figure 1 and Figure 2 for a visual summary of the results concerning advection-diffusion equations.

A comparison with LeBris-Lions' theory of renormalized solutions. Yet another approach to (ADE) builds on the notion of *renormalized solution*. In a nutshell, such a concept allows one to define the transport term $v\boldsymbol{b}$ in a completely general framework (i.e. for any choice of exponents p, q) and this is achieved by prescribing that the equation in (ADE) holds not for u but for a (non-linear) function of u (together with some additional assumptions on the regularity of u). We have opted not to pursue this direction here and we refer the reader to the monograph [29] where one can find, besides the theory of bounded parabolic solutions, an extensive and comprehensive study of renormalized solutions (see in particular, [29, Chapter 2, Remark 16] for an interesting comparison between distributional and renormalized solutions).

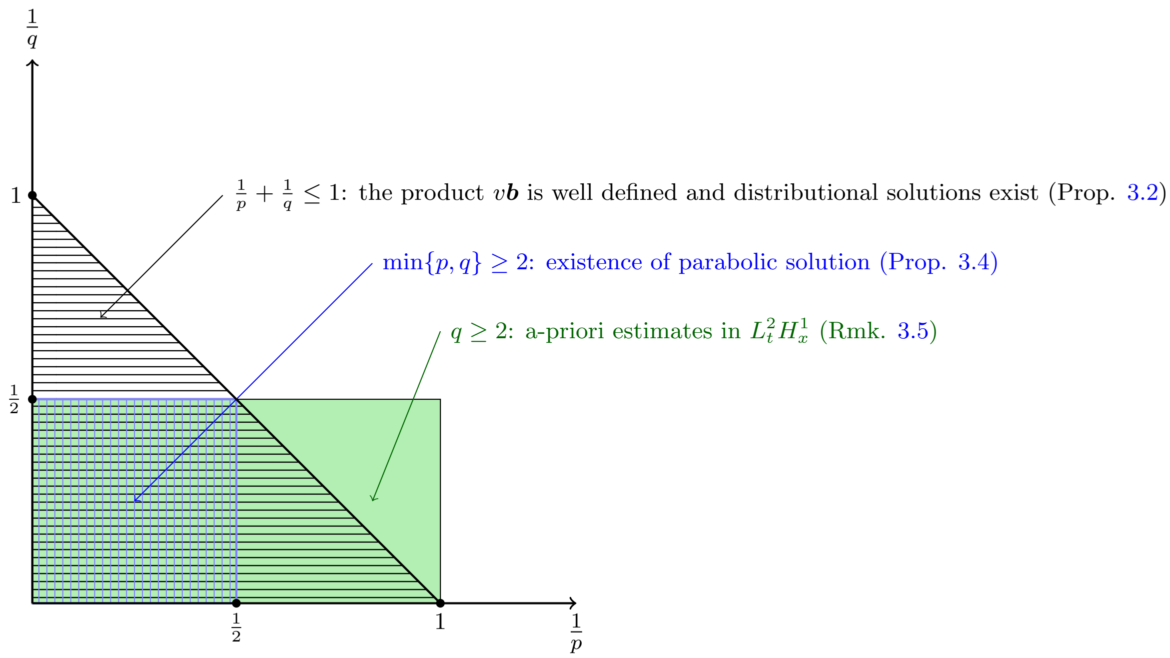

Figure 1. Visual depiction of the *existence* results for distributional and parabolic solutions for vector fields $\boldsymbol{b} \in L^1_t L^p_x$ and initial datum $v_0 \in L^q$.

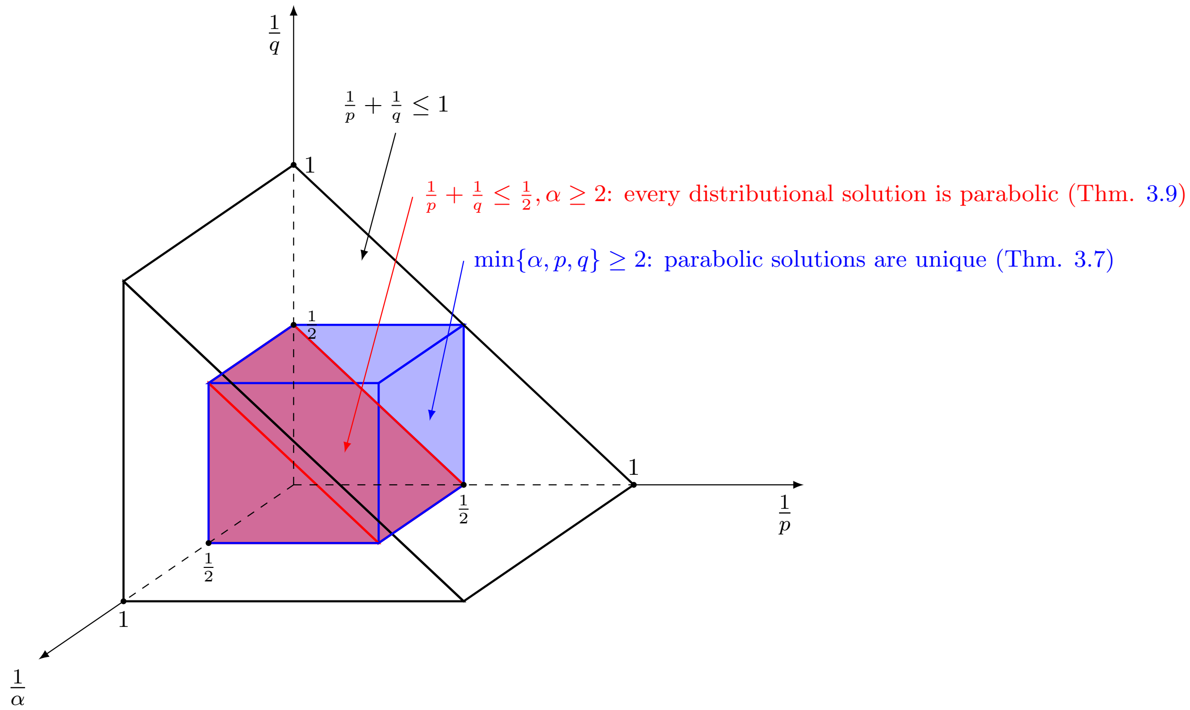

Figure 2. Visual depiction of the *uniqueness* and *regularity* results for distributional and parabolic solutions for fields $\boldsymbol{b} \in L^\alpha_t L^p_x$ and initial datum $v_0 \in L^q$. Distributional solutions $v \in L^\infty_t L^q_x$ are well defined in the black wedge. In the blue cube parabolic solutions are unique and in the red wedge every distributional solution is parabolic.

1.2. **Part II. Vanishing viscosity scheme and rates of convergence.** The second part of the paper deals instead with a different, though closely related, problem. The focus now becomes the linear transport equation associated with a Sobolev vector field $\boldsymbol{b}$, i.e.

$$\begin{cases} \partial_t u + \operatorname{div}(u\boldsymbol{b}) = 0 \\ u|_{t=0} = u_0, \end{cases} \tag{TE}$$

for a given initial datum $u_0 \colon \mathbb{R}^d \to \mathbb{R}$. It is by now well known that (TE) is deeply connected with the ordinary differential equation associated to $\boldsymbol{b}$ and more precisely with the *regular Lagrangian flow* of $\boldsymbol{b}$ (see Definition 4.3 below). This concept has proven to be the right generalization of the classical notion of flow in connection with such problems (see e.g. [3, 5]). On the torus, if $\boldsymbol{b} \in L^1_t W^{1,p}_x$, existence and uniqueness of the regular Lagrangian flow $\boldsymbol{X}$ of $\boldsymbol{b}$ hold [21]: in turn, a straightforward computation allows to check that the transport of the initial datum along the characteristics selected by the regular Lagrangian flow always defines a solution to (TE). More precisely, the function $u^{\mathsf{L}}(t,x) := u_0(\boldsymbol{X}^{-1}(t,\cdot)(x))$ is a distributional solution to (TE), whenever $u_0 \in L^q$ for some $q \ge 1$ with $1/p + 1/q \le 1$. We will refer to u^{L} as the *Lagrangian solution*. Remarkably, u^{L} turns out to be the *unique* distributional solution within the aforementioned range of integrability.

The need for a selection criterion for (TE). A few years ago, in a series of innovative contributions, Modena and Székelyhidi constructed a plethora of counterexamples, showing ill-posedness of the problem (TE). More precisely, in [32, 33] the authors have produced divergence-free, Sobolev vector fields $\boldsymbol{b} \in C_t W^{1,p}_x$ such that (TE) admits infinitely many distributional solutions $u \in C_t L^q_x$, with $1/p + 1/q > 1 + \sigma(d)$ and $u\boldsymbol{b} \in L^1$. Here $\sigma(d)$ is a dimensional constant, which has been refined in [31] to $\sigma(d) := 1/d$. Yet the situation in the intermediate regime

$1 < 1/p + 1/q \leq 1 + \sigma(d)$ is an open problem (see, however, [10]). Remarkably, in the works just mentioned, the authors build distributional solutions that do *not* enjoy typical properties of smooth solutions, such as the conservation of the L^p norms. It is therefore natural to ask whether such solutions can be obtained as limit of (physically or numerically) significant approximation procedures, as for instance the vanishing viscosity method.

The second part of the present paper is devoted to establishing the following theorem, together with some related results.

Selection Theorem (Vanishing viscosity for linear transport)**.** Let $\boldsymbol{b} \in L^1_t W^{1,1}_x$ be a divergence-free vector field on the torus $\mathbb{T}^d$ and let $u_0 \in L^1$ be a given initial datum. Let $(v_0^\varepsilon)_\varepsilon \subset L^\infty_x$ be *any* sequence of functions such that $v_0^\varepsilon \to u_0$ strongly in L^1_x. Consider the *vanishing viscosity solutions*, i.e. the sequence $(v_\varepsilon)_{\varepsilon>0} \subseteq L^\infty((0,T);L^\infty(\mathbb{T}^d)) \cap L^2((0,T);H^1(\mathbb{T}^d))$ of parabolic solutions to

$$\begin{cases} \partial_t v_\varepsilon + \boldsymbol{b} \cdot \nabla v_\varepsilon = \varepsilon \Delta v_\varepsilon & \text{in } (0,T) \times \mathbb{T}^d \\ v_\varepsilon|_{t=0} = v_0^\varepsilon & \text{in } \mathbb{T}^d. \end{cases} \tag{VV}$$

Then $(v_\varepsilon)_{\varepsilon>0}$ converges in $C([0,T];L^1(\mathbb{T}^d))$ to the Lagrangian solution u^{L} to (TE).

We highlight the applicability range of our result, which is completely general: dealing with the (extreme) case of a $W^{1,1}$ field and an L^1 solution (in space), we prove that the vanishing viscosity scheme *always* acts as a selection principle (even in an integrability regime where the product $u\boldsymbol{b}$ cannot be defined distributionally) and that the family $(v_\varepsilon)_\varepsilon$ always selects, in the limit, the Lagrangian solution u^{L}. Observe that, for the Lagrangian solution, all its L^q-norms are conserved (recall that we assume the vector field to be divergence-free). In particular, for $u_0 \in L^2$, our result rules out the possibility of anomalous dissipation in the vanishing viscosity limit, that is, it implies that

$$\varepsilon \int_0^T \|\nabla v_\varepsilon\|^2_{L^2} \, \mathrm{d}t \to 0 \quad \text{as } \varepsilon \to 0.$$

As in the case of the advection-diffusion equation (ADE), distributional solutions of (TE) exist even if we require only integrability assumptions on $\boldsymbol{b}$. However, in contrast to the case of (ADE), for vector fields outside the Sobolev class there is a wide literature of counterexamples to the uniqueness for (TE), see for example [1, 2, 21, 20]. There are many contexts in PDEs (conservation laws, fluid dynamics, etc...) in which the notion of distributional solution is too general to ensure uniqueness and therefore selection criteria are needed to characterize particularly meaningful solutions. The selection problem for the transport equation has already been posed in [11] (see also [19]) where the authors considered as a (non) selection criterion the smooth approximation of the vector field. In particular, it is shown that a smooth approximation may produce different (Lagrangian) solutions in the limit. Moreover, results of Lagrangianity for weak solutions of the 2D Euler equations obtained via vanishing viscosity have been established in [16, 17], see also [13] for the Lagrangianity of solutions obtained via vortex-blob approximations.

We present two proofs of the Selection Theorem. The first one is more Eulerian in spirit and is a slightly expanded version of the one contained in DiPerna-Lions' original contribution [21], which is based on a duality argument. We offer a comprehensive, detailed proof which ultimately reveals the complete generality of the vanishing viscosity scheme, which is able to bypass the distributional regime. In particular, we cover also the case $p = 1$ which was left somehow implicit in [21]. The second proof we provide, instead, has a more Lagrangian nature and is based on the use of stochastic flows ([8, 12]). At the price of a technical, additional integrability assumption (which is not needed in the Eulerian proof), this alternative proof of the Selection Theorem yields *quantitative* rates of convergence of v^ε to u^{L} and also quantitative (in the viscosity) stability estimates for solutions of the advection-diffusion equation. Such rates are compared with the ones obtained in the recent works [6] and [34].

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2. Preliminaries and notations

We begin by fixing the notation and recalling some basic facts we will need in the following.

2.1. **Notation.** Throughout the paper, $d \geq 1$ will be a fixed integer. We will denote by $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ the d-dimensional flat torus and by $\mathscr{L}^d$ the Lebesgue measure on it. We identify the d-dimensional flat torus with the cube $[0,1)^d$ and we denote with d the geodesic distance on $\mathbb{T}^d$, which is given by

$$\mathsf{d}(x,y) = \min\{|x-y-k| : k \in \mathbb{Z}^d \text{ such that } |k| \leq 2\}.$$

We will use the letters p, q to denote real numbers in $[1,+\infty]$ and p' will be the (Hölder) conjugate to p. We will adopt the customary notation for Lebesgue spaces $L^p(\mathbb{T}^d)$ and for Sobolev spaces $W^{k,p}(\mathbb{T}^d)$; in particular, $H^k(\mathbb{T}^d) := W^{k,2}(\mathbb{T}^d)$. We will denote with $\|\cdot\|_{L^p}$ (respectively $\|\cdot\|_{W^{k,p}}, \|\cdot\|_{H^k}$) the norms of the aforementioned functional spaces, omitting the domain dependence when not necessary. Every definition below can be adapted in a standard way to the case of spaces involving time, like e.g. $L^1([0,T]; L^p(\mathbb{T}^d))$.

Equi-integrability. We recall the definition of equi-integrability for a family of functions in L^1:

Definition 2.1 (Equi-integrability). A family $\{\varphi_i\}_{i\in I} \subset L^1(\mathbb{T}^d)$ is *equi-integrable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every Borel set $E \subset \mathbb{T}^d$ with $\mathscr{L}^d(E) \leq \delta$ it holds

$$\int_E |\varphi_i| \mathrm{d}x \leq \varepsilon \qquad \text{for every } i \in I.$$

The following well-known results offer useful criteria to check the equi-integrability of a family of functions in L^1:

Theorem 2.2 (Dunford-Pettis, de la Vallée-Poussin). *Let $\{\varphi_i\}_{i\in I} \subset L^1(\mathbb{T}^d)$ be a bounded family. Then the following are equivalent:*

(i) *$\{\varphi_i\}_{i\in I}$ is equi-integrable;*
(ii) *$\{\varphi_i\}_{i\in I}$ is weakly sequentially pre-compact in $L^1(\mathbb{T}^d)$;*
(iii) *there exists a non-negative, increasing, convex function $\Psi\colon [0,+\infty) \to [0,+\infty)$ such that*

$$\lim_{t\to+\infty} \frac{\Psi(t)}{t} = +\infty \qquad \textit{and} \qquad \sup_{i\in I} \int_{\mathbb{T}^d} \Psi(|\varphi_i|) \mathrm{d}x < +\infty.$$

We finish this subsection with the following useful lemma.

Lemma 2.3. *Let $\{\varphi_i\}_{i\in I} \subset L^1(\mathbb{T}^d)$ be a bounded family. Then, $\{\varphi_i\}_{i\in I}$ is equi-integrable if and only if for any $r \in [1,\infty]$ and $\varepsilon > 0$ there exist $\{g_i^1\}_{i\in I} \subset L^1(\mathbb{T}^d)$, $\{g_i^2\} \subset L^r(\mathbb{T}^d)$, and a constant $C_\varepsilon > 0$ such that*

$$f_i = g_i^1 + g_i^2, \qquad \sup_{i\in I} \|g_i^1\|_{L^1} \leq \varepsilon, \qquad \sup_{i\in I} \|g_i^2\|_{L^r} \leq C_\varepsilon.$$

Proof. Let $\{\varphi_i\}_{i\in I} \subset L^1(\mathbb{T}^d)$ be an equi-integrable sequence such that $\sup_{i\in I} \|\varphi_i\|_{L^1} \le C$. Let $\varepsilon > 0$ be fixed and let $\delta > 0$ as in Definition 2.1. Then, we define the set

$$A^i_\delta := \left\{ x \in \mathbb{T}^d : |\varphi_i(x)| > \frac{C}{\delta} \right\},$$

and by Chebishev inequality we know that

$$\sup_{i\in I} \mathscr{L}^d(A^i_\delta) \le \frac{\delta}{C} \|\varphi_i\|_{L^1} \le \delta.$$

So, by the equi-integrability

$$\sup_{i\in I} \int_{A^i_\delta} |\varphi_i| \mathrm{d}x \le \varepsilon,$$

and it is now clear that, by defining $g^1_i = \varphi_i \chi_{A^i_\delta}$ and $g^2_i = \varphi_i(1 - \chi_{A^i_\delta})$, we have that

$$\sup_{i\in I} \|g_i\|_{L^1} \le \varepsilon, \qquad \sup_{i\in I} \|g^2_i\|_{L^r} \le C_\varepsilon,$$

since $\mathbb{T}^d$ has finite measure.

We now prove the opposite implication. Let $r \in [1, \infty]$ and $\varepsilon > 0$ be fixed, we consider a decomposition such that

$$\|g^1_i\|_{L^1} \le \varepsilon/2, \qquad \|g^2_i\|_{L^r} \le C_\varepsilon.$$

Let us check that we can take $\delta = \left(\frac{\varepsilon}{2C_\varepsilon}\right)^{r/(r-1)}$ in the definition of equi-integrability. Indeed, if $A \subset \mathbb{T}^d$ is such that $\mathscr{L}^d(A) \le \delta$, we have that

$$\int_A |\varphi_i| \mathrm{d}x \le \int_A |g^1_i| \mathrm{d}x + \int_A |g^2_i| \mathrm{d}x \le \|g^1_i\|_{L^1} + \|g^2_i\|_{L^r} \mathscr{L}^d(A)^{(r-1)/r} \le \varepsilon/2 + \varepsilon/2 = \varepsilon. \qquad \square$$

Some Harmonic Analysis tools. We will need to work with weak Lebesgue spaces, denoted by $M^p(\mathbb{T}^d)$: for the sake of completeness, we recall here their definition and some useful lemmata.

Definition 2.4. Let $u : \mathbb{T}^d \to \mathbb{R}$ be a measurable function. For any $1 \le p < \infty$ we define

$$|||u|||^p_{M^p} = \sup_{\lambda>0} \left\{ \lambda^p \mathscr{L}^d \left(\{x \in \mathbb{T}^d : |u(x)| > \lambda\} \right) \right\},$$

and we define the weak Lebesgue space $M^p(\mathbb{T}^d)$ as the set of the functions $u : \mathbb{T}^d \to \mathbb{R}$ with $|||u|||_{M^p} < \infty$. By convention, for $p = \infty$ we set $M^\infty(\mathbb{T}^d) = L^\infty(\mathbb{T}^d)$.

Note that $||| \cdot |||_{M^p}$ is not subadditive, hence it is *not* a norm. As a consequence, $M^p(\mathbb{T}^d)$ is not a Banach space. Moreover, since for every $\lambda > 0$

$$\lambda^p \mathscr{L}^d \left(\{x \in \mathbb{T}^d : |u(x)| > \lambda\} \right) = \int_{|u|>\lambda} \lambda^p \mathrm{d}x \le \int_{|u|>\lambda} |u(x)|^p \mathrm{d}x \le \|u\|^p_{L^p},$$

we have the inclusion $L^p(\mathbb{T}^d) \subset M^p(\mathbb{T}^d)$ and in particular $|||u|||_{M^p} \le \|u\|_{L^p}$. The following lemma shows that we can interpolate the spaces M^1 and M^p, with $p > 1$, obtaining a bound on the L^1 norm.

Lemma 2.5. *Let $u : \mathbb{T}^d \to [0, \infty)$ be a non-negative measurable function. Then for every $p \in (1, \infty)$ we have the interpolation estimate*

$$\|u\|_{L^1} \le \frac{p}{p-1} |||u|||_{M^1} \left[1 + \log \left(\frac{|||u|||_{M^p}}{|||u|||_{M^1}} \right) \right],$$

while for $p = \infty$ we have

$$\|u\|_{L^1} \le |||u|||_{M^1} \left[1 + \log \left(\frac{\|u\|_{L^\infty}}{|||u|||_{M^1}} \right) \right].$$

We recall the definition of the Hardy-Littlewood maximal function.

Definition 2.6. Let $f \in L^1(\mathbb{T}^d)$, we define Mf the maximal function of f as

$$Mf(x) = \sup_{r>0} \frac{1}{\mathscr{L}^d(B_r)} \int_{B_r(x)} |f(y)| \mathrm{d}y \qquad \text{for every } x \in \mathbb{T}^d.$$

The following estimates hold.

Lemma 2.7. *For every $1 < p \leq \infty$ we have the strong estimate*

$$\|Mf\|_{L^p} \leq C_{d,p} \|f\|_{L^p},$$

while for $p = 1$ only the weak estimate

$$|||Mf|||_{M^1} \leq C_d \|f\|_{L^1}$$

holds.

Finally, we recall the following estimate on the different quotients of a $W^{1,1}$ function.

Lemma 2.8. *Let $f \in W^{1,1}(\mathbb{T}^d)$. Then there exists a negligible set $\mathcal{N} \subset \mathbb{T}^d$ such that*

$$|f(x) - f(y)| \leq C(d) \mathsf{d}(x,y) \left(MDf(x) + MDf(y) \right),$$

for every $x, y \in \mathbb{T}^d \setminus \mathcal{N}$, where Du is the distributional derivative of u.

3. On the advection-diffusion equation: L^p theory

In this section, we are interested in the Cauchy problem for the advection-diffusion equation, namely

$$\begin{cases} \partial_t u + \operatorname{div}(\boldsymbol{b} u) = \Delta u & \text{in } (0,T) \times \mathbb{T}^d \\ u|_{t=0} = u_0 & \text{in } \mathbb{T}^d \end{cases} \tag{3.1}$$

where the data of the problem are $T > 0$, the vector field $\boldsymbol{b}$ and the initial datum u_0. More precisely, we want first to present some different notions of solutions (distributional and parabolic) and then discuss existence, uniqueness and mutual relationship under general integrability assumptions on $\boldsymbol{b}$ and u_0. For the sake of completeness, we decided to include a self-contained proof of every result, citing the respective references whenever appropriate.

3.1. Distributional solutions.

We start by giving the following definition.

Definition 3.1 (Distributional solution)**.** Let $\boldsymbol{b} \in L^1((0,T); L^p(\mathbb{T}^d))$ be a divergence-free vector field and $u_0 \in L^q(\mathbb{T}^d)$ for p, q such that $1/p + 1/q \leq 1$. A function $u \in L^\infty((0,T); L^q(\mathbb{T}^d))$ is a *distributional solution* to (3.1) if for any $\varphi \in C_c^\infty([0,T) \times \mathbb{T}^d)$ the following equality holds:

$$\int_0^T \int_{\mathbb{T}^d} u(\partial_t \varphi + \boldsymbol{b} \cdot \nabla \varphi + \Delta \varphi) \mathrm{d}x \mathrm{d}t + \int_{\mathbb{T}^d} u_0 \varphi(0, \cdot) \mathrm{d}x = 0.$$

Notice that in the definition of distributional solutions the assumption that p, q satisfy $1/p + 1/q \leq 1$ is the minimum requirement we need in order to have $u\boldsymbol{b} \in L^1$ so that the definition makes sense. The proof of existence of distributional solutions is well-known and immediately follows from a classical a priori estimate.

Proposition 3.2. *Let $\boldsymbol{b} \in L^1((0,T); L^p(\mathbb{T}^d))$ be a divergence-free vector field and $u_0 \in L^q(\mathbb{T}^d)$ for p, q such that $1/p + 1/q \leq 1$. Then there exists a distributional solution $u \in L^\infty((0,T); L^q(\mathbb{T}^d))$ to (3.1).*

Proof. Let $(\rho^\delta)_\delta$ be a standard family of mollifiers and let us define $\boldsymbol{b}^\delta = \boldsymbol{b} * \rho^\delta$, $u_0^\delta = u_0 * \rho^\delta$. Then, we consider the approximating problem

$$\begin{cases} \partial_t u^\delta + \operatorname{div}(\boldsymbol{b}^\delta u^\delta) = \Delta u^\delta \\ u^\delta(0,\cdot) = u_0^\delta. \end{cases} \tag{3.2}$$

Being $\boldsymbol{b}^\delta$ and u_0^δ smooth, there exists a unique smooth solution u^δ to (3.2) (see [23]). It is readily checked that the sequence u^δ is equi-bounded in $L^\infty((0,T);L^q(\mathbb{T}^d))$. Indeed, we can multiply the equation in (3.2) by $\beta'(u^\delta)$, where $\beta\colon \mathbb{R} \to \mathbb{R}$ is a smooth, convex function: by an easy application of the chain rule and integrating in space, we get

$$\frac{d}{dt}\int_{\mathbb{T}^d} \beta(u^\delta(t,x))\,\mathrm{d}x = -\int_{\mathbb{T}^d} \beta''(u^\delta(t,x))|\nabla\beta(u^\delta(t,x))|^2\,\mathrm{d}x \le 0.$$

In particular, fixing $t > 0$ and integrating in time on $(0,t)$ we obtain

$$\int_{\mathbb{T}^d} \beta(u^\delta(t,x))\,\mathrm{d}x \le \int_{\mathbb{T}^d} \beta(u_0^\delta(x))\,\mathrm{d}x. \tag{3.3}$$

By considering a sequence of smooth, convex functions, uniformly convergent to $\beta(s) = |s|^q$, for $1 < q < \infty$, we obtain the following uniform bounds on the L^q-norm of the solutions u^δ:

$$\|u^\delta(t,\cdot)\|_{L^q(\mathbb{T}^d)} \le \|u_0^\delta\|_{L^q(\mathbb{T}^d)} \le \|u_0\|_{L^q(\mathbb{T}^d)}. \tag{3.4}$$

For $q > 1$ by standard compactness arguments, we can extract a sub-sequence which converges weakly-star to a function $u \in L^\infty((0,T);L^q(\mathbb{T}^d))$ and it is immediate to deduce that u is a distributional solution of (3.1) because of the linearity of the equation. For $q = \infty$, the estimate (3.4) still holds for every $\delta > 0$: we send $q \to \infty$ in (3.4) and then we can conclude as in the previous case. For the case $q = 1$, the compactness in $L^\infty((0,T);L^1(\mathbb{T}^d))$ can be obtained as a consequence of equi-integrability of the family $(u^\delta)_\delta$ which follows from (3.3) and (iii) in Theorem 2.2: we do not present here the full details of this equi-integrability argument and we refer the reader to the proof of Theorem 4.5 where the same idea is exploited and described in full details in a slightly more complicated case. □

3.2. **Parabolic solutions.** A special sub-class of distributional solutions is given by the so-called *parabolic solutions*, whose peculiar property is the Sobolev regularity in the space variable. As we are going to see, this notion of solution is natural for vector fields possessing enough integrability in the space variable.

Definition 3.3. Let $\boldsymbol{b} \in L^1((0,T);L^2(\mathbb{T}^d))$ a divergence-free vector field and $u_0 \in L^2(\mathbb{T}^d)$. A function $u \in L^\infty((0,T);L^2(\mathbb{T}^d))$ is a *parabolic solution* to (3.1) if it is a distributional solution to (3.1) and furthermore $u \in L^2((0,T);H^1(\mathbb{T}^d))$.

We will sometimes refer to the space $L^2((0,T);H^1(\mathbb{T}^d))$ as the *parabolic class*.

3.2.1. *Existence.* We now prove that, under the assumptions above, there exists at least one solution in the parabolic class:

Proposition 3.4. *Let $\boldsymbol{b} \in L^1((0,T);L^2(\mathbb{T}^d))$ be a divergence-free vector field and $u_0 \in L^2(\mathbb{T}^d)$. Then there exists at least one parabolic solution.*

Proof. The proof follows the same idea of the one of Proposition 3.2. We consider the approximating problems (3.2) and their unique smooth solutions u^δ. Choosing $\beta(s) = s^2/2$ and integrating in time on $(0,t)$, we get the following energy balance

$$\frac{1}{2}\int_{\mathbb{T}^d} |u^\delta(t,x)|^2\mathrm{d}x + \int_0^t\int_{\mathbb{T}^d} |\nabla u^\delta(s,x)|^2\mathrm{d}x\mathrm{d}s = \frac{1}{2}\int_{\mathbb{T}^d} |u_0^\delta(x)|^2\mathrm{d}x. \tag{3.5}$$

Standard weak compactness arguments yield the conclusion. □

Remark 3.5. *In the proofs of Proposition 3.2 and Proposition 3.4, we have constructed solutions as limit of solutions $(u^\delta)_\delta$ associated with a regularization $(\boldsymbol{b}^\delta)_\delta$ of the vector field and $(u_0^\delta)_\delta$ of the initial datum. This strategy will be used once more later in the paper and we explicitly remark here that the family $(u^\delta)_\delta$ satisfies two a-priori estimates:*

(E1) $\sup_{t\in(0,T)} \|u^\delta\|_{L^q} \le C\|u_0\|_{L^q}$ *if* $u_0 \in L^q(\mathbb{T}^d)$;

(E2) $\int_0^T \|\nabla u^\delta(t,\cdot)\|_{L^2}\,\mathrm{d}t \le C\|u_0\|_{L^2}$ *if* $u_0 \in L^2(\mathbb{T}^d)$.

These bounds follow integrating by parts and exploiting the divergence-free assumption on the vector field, in particular they are independent of the integrability of $\boldsymbol{b}$. However, we need the assumption $\boldsymbol{b} \in L^1((0,T); L^2(\mathbb{T}^d))$ in Proposition 3.4 to give a distributional meaning to the product $u\boldsymbol{b}$.

3.2.2. *Uniqueness of solutions in the parabolic class.* The uniqueness of solutions in the parabolic class is a consequence of the following lemma, which is a straightforward modification of [28, Lemma 5.1].

Lemma 3.6 (Commutator estimates I)**.** *Consider a vector field $\boldsymbol{b} \in L^2([0,T]; L^p(\mathbb{T}^d))$ and a function $w \in L^\infty([0,T]; L^q(\mathbb{T}^d))$, where p, q are positive real numbers with $1/p + 1/q \le 1$. Let $(\rho^\delta)_\delta$ be a family of smooth convoutions kernels. Define the* commutator of w and $\boldsymbol{b}$ *as follows:*

$$r^\delta := \boldsymbol{b}\cdot\nabla(w*\rho^\delta) - (\boldsymbol{b}\cdot\nabla w)*\rho^\delta. \tag{3.6}$$

If $\nabla w \in L^2([0,T]; L^q(\mathbb{T}^d))$, then r^δ converges to 0 in $L^1([0,T]\times\mathbb{T}^d)$.

Proof. Observe that, for a.e. $t \in [0,T]$ and a.e. $x \in \mathbb{T}^d$, we can explicitly write the commutator in the following form:

$$\begin{aligned} r^\delta(t,x) &= [\boldsymbol{b}\cdot\nabla(w*\rho^\delta)](t,x) - [(\boldsymbol{b}\cdot\nabla w)*\rho^\delta](t,x) \\ &= \boldsymbol{b}(t,x)\cdot\nabla\int_{\mathbb{T}^d} w(t,x-y)\rho^\delta(y)\mathrm{d}y - \int_{\mathbb{T}^d}\boldsymbol{b}(t,x-y)\cdot\nabla w(t,x-y)\rho^\delta(y)\mathrm{d}y \\ &= \int_{\mathbb{T}^d}\rho^\delta(y)\left(\boldsymbol{b}(t,x)-\boldsymbol{b}(t,x-y)\right)\cdot\nabla w(t,x-y)\mathrm{d}y \\ &= \int_{\mathbb{T}^d}\rho(z)\left(\boldsymbol{b}(t,x)-\boldsymbol{b}(t,x-\delta z)\right)\cdot\nabla w(t,x-\delta z)\mathrm{d}z. \end{aligned}$$

We thus have that

$$\begin{aligned} \iint_{[0,T]\times\mathbb{T}^d}|r^\delta(t,x)|\mathrm{d}t\mathrm{d}x &= \iint_{[0,T]\times\mathbb{T}^d}\left|\int_{\mathbb{T}^d}\rho(z)\left(\boldsymbol{b}(t,x)-\boldsymbol{b}(t,x-\delta z)\right)\cdot\nabla w(t,x-\delta z)\mathrm{d}z\right|\mathrm{d}t\mathrm{d}x \\ &\le \int_{\mathbb{T}^d}\rho(z)\int_0^T\int_{\mathbb{T}^d}|\boldsymbol{b}(t,x)-\boldsymbol{b}(t,x-\delta z)||\nabla w(t,x-\delta z)|\mathrm{d}x\mathrm{d}t\mathrm{d}z. \end{aligned} \tag{3.7}$$

Since $(t,x)\mapsto \boldsymbol{b}(t,x)-\boldsymbol{b}(t,x-\delta z)$ converges to 0 in measure (for every fixed z), the conclusion follows by the Dominated Convergence Theorem. □

Having at our disposal the previous lemma, we can now show the uniqueness of solutions in the parabolic class, arguing as in [28].

Theorem 3.7 (Uniqueness of parabolic solutions)**.** *Consider a divergence-free vector field $\boldsymbol{b} \in L^2([0,T]; L^2(\mathbb{T}^d))$. Then there exists at most one parabolic solution to (3.1).*

Proof. The uniqueness is a rather straightforward consequence of the strong convergence of commutators established in Lemma 3.6. More precisely, since the problem is linear, it suffices to show that, if u is a parabolic solution to (3.1) with $u_0 = 0$, then $u = 0$. Consider again a standard family of mollifiers $(\rho^\delta)_\delta$ and set $u^\delta := u * \rho^\delta$. Then, a direct computation shows that u^δ solves the following equation

$$\begin{cases} \partial_t u^\delta + \operatorname{div}(\boldsymbol{b} u^\delta) = \Delta u^\delta + r^\delta \\ u^\delta(0,\cdot) = 0 \end{cases} \tag{3.8}$$

where r^δ is the commutator between u and $\boldsymbol{b}$, defined as in (3.6). Consider now a smooth function $\beta \in C^2(\mathbb{R})$, with the following properties: $\beta(s) \geq 0$, $|\beta'(s)| \leq C$ for some $C > 0$ and $\beta''(s) \geq 0$ for any $s \in \mathbb{R}$ with $\beta(s) = 0$ if, and only if, $s = 0$ (e.g. one could easily verify that the function which satisfies $\beta'(s) = \arctan(s)$ with $\beta(0) = 0$ is an admissible choice). Multiplying the equation by $\beta'(u^\delta)$ and integrating on $[0,t] \times \mathbb{T}^d$ we obtain

$$\int_{\mathbb{T}^d} \beta(u^\delta) \mathrm{d}x + \int_0^t \int_{\mathbb{T}^d} \beta''(u^\delta) |\nabla u^\delta|^2 \mathrm{d}x \mathrm{d}s = \int_0^t \int_{\mathbb{T}^d} \beta'(u^\delta) r^\delta \mathrm{d}x \mathrm{d}s.$$

We now let $\delta \to 0$: using the uniform bound on β' and Lemma 3.6 we deduce that the right-hand side converges to 0 and thus

$$\int_{\mathbb{R}^d} \beta(u(t,x)) \mathrm{d}x = -\int_0^t \int_{\mathbb{R}^d} \beta''(u) |\nabla u|^2 \mathrm{d}x \mathrm{d}s \leq 0.$$

Since $t \in [0,T]$ is arbitrary, the conclusion $u = 0$ easily follows. □

If the vector field is less integrable than $L^2(\mathbb{T}^d)$, then a severe phenomenon of non-uniqueness may arise. In particular, in [31] counter-examples are constructed via convex integration techniques: it is shown that there exist infinitely many solutions to (3.1) in the class $C([0,T]; H^1(\mathbb{T}^d))$ with a divergence-free vector field $\boldsymbol{b} \in C([0,T]; L^p(\mathbb{T}^d))$ with $1 \leq p < {}^{2d}/_{d+2}$, for which it additionally holds $u\boldsymbol{b} \in L^1((0,T) \times \mathbb{T}^d)$. This, however, leaves open the following questions.

(Q1) *What happens in the case* ${}^{2d}/_{d+2} \leq p < 2$*?*
(Q2) *If uniqueness holds for p as in (Q1), is it possible to show non-uniqueness of solutions in the larger class* $L^2((0,T); H^1(\mathbb{T}^d))$ *for vector fields which are merely* L^2 *in time (instead than continuous)?*
(Q3) *For a vector field* $\boldsymbol{b} \in L^r((0,T); L^2(\mathbb{T}^d))$ *with* $1 \leq r < 2$*, are parabolic solutions unique?*

A strategy to tackle (Q2) could be to exploit "time-intermettency" as in [7, 10], which allows to increase the space integrability at the expense of the time integrability.

3.3. **The regularity result.** At this point, a natural question is under which conditions a distributional solution is a parabolic solution. In order to address this question, we will need the following version of the commutator lemma which, to the best of our knowledge, is not present in the literature:

Lemma 3.8 (Commutator estimates II). *Consider a vector field* $\boldsymbol{b} \in L^2([0,T]; L^p(\mathbb{T}^d))$ *and a function* $w \in L^\infty([0,T]; L^q(\mathbb{T}^d))$*, where* p, q *are positive real numbers with* ${}^1/_p + {}^1/_q \leq {}^1/_2$*. Let* $(\rho^\delta)_\delta$ *be a family of smooth convoutions kernels and define* r^δ *as in (3.6). Then* r^δ *converges to 0 in* $L^2([0,T]; H^{-1}(\mathbb{T}^d))$*.*

Proof. We write the commutator as

$$r^\delta = [\boldsymbol{b} \cdot \nabla(w * \rho^\delta)] - [(\boldsymbol{b} \cdot \nabla w) * \rho^\delta] = \operatorname{div}[\boldsymbol{b}(w * \rho^\delta)] - [\operatorname{div}(\boldsymbol{b}w) * \rho^\delta] = \operatorname{div}[\boldsymbol{b}(w * \rho^\delta) - (\boldsymbol{b}w) * \rho^\delta],$$

in the sense of distributions on $[0,T] \times \mathbb{T}^d$. We can thus write

$$r^\delta(t,x) = \operatorname{div}_x \left(\int_{\mathbb{T}^d} [\boldsymbol{b}(t,x) - \boldsymbol{b}(t,x-y)] w(t,x-y) \rho^\delta(y) \mathrm{d}y \right)$$

and we can estimate

$$\begin{aligned}
\|r^\delta\|_{L^2(H^{-1})} &= \sup_{\|\varphi\|_{L^2H^1}\leq 1}\left|\iint_{[0,T]\times\mathbb{T}^d} r^\delta(t,x)\varphi(t,x)\mathrm{d}t\mathrm{d}x\right| \\
&= \sup_{\|\varphi\|_{L^2H^1}\leq 1}\left|\iint_{[0,T]\times\mathbb{T}^d}\left(\int_{\mathbb{T}^d}[\boldsymbol{b}(t,x)-\boldsymbol{b}(t,x-y)]w(t,x-y)\rho^\delta(y)\mathrm{d}y\right)\nabla\varphi(t,x)\mathrm{d}t\mathrm{d}x\right| \\
&\leq \sup_{\|\varphi\|_{L^2H^1}\leq 1}\int_{\mathbb{T}^d}\rho(z)\int_0^T\int_{\mathbb{T}^d}|\boldsymbol{b}(t,x)-\boldsymbol{b}(t,x-\delta z)||w(t,x-\delta z)||\nabla\varphi(t,x)|\mathrm{d}x\mathrm{d}t\mathrm{d}z.
\end{aligned}$$

Notice now that, as in the proof of Lemma 3.6, the map $(t,x)\mapsto \boldsymbol{b}(t,x)-\boldsymbol{b}(t,x-\delta z)$ converges to 0 in measure (for every fixed z). Hölder inequality on the product space $[0,T]\times\mathbb{T}^d$ with exponents $(p,q,2)$ (in space) and $(2,\infty,2)$ (in time) allows to apply Lebesgue Dominated Convergence Theorem and we can therefore conclude that $r^\delta\to 0$ in $L^2(H^{-1})$. □

We can now present a regularity result which guarantees that a distributional solution in the class $L^\infty((0,T);L^q(\mathbb{T}^d))$ is actually in $L^2((0,T);H^1(\mathbb{T}^d))$ whenever $1/p+1/q\leq 1/2$.

Theorem 3.9. *Let $p,q\geq 1$ such that $1/p+1/q\leq 1/2$. If $\boldsymbol{b}\in L^2([0,T];L^p(\mathbb{T}^d))$ is a divergence-free vector field and $u\in L^\infty((0,T);L^q(\mathbb{T}^d))$ is a distributional solution to (3.1), then $u\in L^2((0,T);H^1(\mathbb{T}^d))$ and satisfies*

$$\frac{1}{2}\int_{\mathbb{T}^d}|u|^2\mathrm{d}x+\int_0^T\int_{\mathbb{T}^d}|\nabla u|^2\mathrm{d}x\mathrm{d}t=\frac{1}{2}\int_{\mathbb{T}^d}|u_0|^2\mathrm{d}x. \tag{3.9}$$

Proof. To commence, we observe that $1/p+1/q\leq 1/2$ clearly implies that both $p,q\geq 2$ and, since we are on the torus, any $u\in L^\infty((0,T);L^q(\mathbb{T}^d))$ lies also in $L^\infty((0,T);L^2(\mathbb{T}^d))$. We thus need to prove $\nabla u\in L^2((0,T);L^2(\mathbb{T}^d))$ and this will be achieved exhibiting an approximating sequence $(u^\delta)_\delta$ enjoying uniform bounds on ∇u^δ: in turn, this estimate will be obtained as a consequence of Lemma 3.8.

Let $(\rho^\delta)_\delta$ be a standard family of mollifiers. As in the proof of Theorem 3.7, the function $u^\delta := u*\rho^\delta$ solves (3.8). Let us now prove an estimate on the H^1-norm of u^δ which is independent of δ: multiply the equation (3.8) by u^δ and integrate by parts to obtain

$$\frac{1}{2}\int_{\mathbb{T}^d}|u^\delta|^2\mathrm{d}x+\int_0^T\int_{\mathbb{T}^d}|\nabla u^\delta|^2\mathrm{d}x\mathrm{d}t=\frac{1}{2}\int_{\mathbb{T}^d}|u_0^\delta|^2\mathrm{d}x+\int_0^T\int_{\mathbb{T}^d}r^\delta u^\delta\mathrm{d}x\mathrm{d}t. \tag{3.10}$$

On the one hand, by standard properties of convolutions, we can estimate the first term in the right-hand side of (3.10) as

$$\int_{\mathbb{T}^d}|u_0^\delta|^2\mathrm{d}x=\|u_0^\delta\|_{L^2}^2\leq C_d\|u_0^\delta\|_{L^q}^2\leq C_d\|u_0\|_{L^q}^2 \tag{3.11}$$

On the other hand, for the second term in the right-hand side of (3.10) we can apply Young's inequality to obtain

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d} r^\delta(t,x)u^\delta(t,x)\mathrm{d}x\mathrm{d}t &\le \|r^\delta\|_{L^2H^{-1}}\|u^\delta\|_{L^2H^1}\\ &\le C(T)\|r^\delta\|^2_{L^2H^{-1}} + \frac{1}{4(1+T)}\|u^\delta\|^2_{L^2H^1}\\ &= C(T)\|r^\delta\|^2_{L^2H^{-1}} + \frac{1}{4(1+T)}\left(\|u^\delta\|^2_{L^2L^2} + \|\nabla u^\delta\|^2_{L^2L^2}\right)\\ &\le C(T)\|r^\delta\|^2_{L^2H^{-1}} + \frac{1}{4(1+T)}\left(T\|u^\delta\|^2_{L^\infty L^2} + \|\nabla u^\delta\|^2_{L^2L^2}\right)\\ &\le C(T)\|r^\delta\|^2_{L^2H^{-1}} + \frac{1}{4}\left(\|u^\delta\|^2_{L^\infty L^2} + \|\nabla u^\delta\|^2_{L^2L^2}\right).\end{aligned} \tag{3.12}$$

Since r^δ goes to 0 in $L^2(H^{-1})$, the term $\|r^\delta\|_{L^2H^{-1}}$ is equi-bounded. Combining (3.11), (3.12) and plugging them into (3.10) we can conclude

$$\|u^\delta\|^2_{L^\infty L^2} + \|\nabla u^\delta\|^2_{L^2L^2} \le C(d, \|u_0\|_{L^q}),$$

for some constant C which does not depend on δ: this shows that the distributional solution u is parabolic and thus unique thanks to Theorem 3.7. Finally, (3.9) immediately follows from (3.10) sending $\delta \to 0$. □

The assumption on the time-integrability of the vector field in the above theorem suggests the following question.

(Q4) *Let $u \in L^\infty((0,T);L^q(\mathbb{T}^d))$ be a distributional solution associated to a divergence-free vector field $\boldsymbol{b} \in L^r((0,T);L^p(\mathbb{T}^d))$ with $1 \le r < 2$, and assume that $1/p + 1/q \le 1/2$. Is u a parabolic solution?*

Combining Theorem 3.9 and Theorem 3.7, we obtain the following corollary.

Corollary 3.10. *Let $p, q \ge 1$ such that $1/p + 1/q \le 1/2$. If $\boldsymbol{b} \in L^2([0,T];L^p(\mathbb{T}^d))$ is a divergence-free vector field, then there exists at most one distributional solution $u \in L^\infty((0,T);L^q(\mathbb{T}^d))$.*

At this point, it is natural to wonder whether in the regime $1/2 < 1/p + 1/q \le 1$ there exist distributional solutions that are not parabolic and, therefore, whether uniqueness of parabolic solutions holds but uniqueness of distributional solutions does not. A partial answer to this, in dimension $d > 2$, can be obtained using [31, Theorem 1.4], which gives non uniqueness of distributional solutions in the regime $1/p + 1/q = 1$ and $p < d$ (notice that in those examples the vector field and the solution are bounded in time). A particular case of interest (somewhat reminiscent of the case of the Navier-Stokes equations in [7]) is when $p = q = 2$: with such a choice, one obtains an example where there exist infinitely many distributional solutions, despite the parabolic one is unique in view of Theorem 3.7.

However, the convex integration schemes of [31] are not able to cover the case $d = 2$. We therefore formulate the following question.

(Q5) *Does it exist a divergence-free vector field $\boldsymbol{b} \in L^2((0,T);L^2(\mathbb{T}^2))$ and a distributional solution $u \in L^\infty((0,T);L^2(\mathbb{T}^2))$ which is not parabolic, i.e. $u \notin L^2((0,T);H^1(\mathbb{T}^2))$? What if the vector field $\boldsymbol{b} \in L^2(\mathbb{T}^2)$ is autonomous?*

As a last point, we observe that the situation in the intermediate regime $1/2 < 1/p + 1/q < 1$ is completely open:

(Q6) *Let $1/2 < 1/p + 1/q < 1$. Does it exist a divergence-free vector field $\boldsymbol{b} \in L^2((0,T);L^p(\mathbb{T}^d))$ and a distributional solution $u \in L^\infty((0,T);L^q(\mathbb{T}^d))$ which is not parabolic, i.e. $u \notin L^2((0,T);H^1(\mathbb{T}^d))$?*

It is worth noticing that a partial answer to (Q6) is given in [28, pag. 70]: when $1 \le q < 2$ we cannot expect $\nabla u \in L^2_t L^2_x$ since this does not hold for the heat equation.

4. The vanishing viscosity scheme I: setup and Eulerian proof

Starting from this section onwards, we slightly change the focus of our investigation and we move towards the transport equation side. Our overarching goal is indeed the study of the well-posedeness of the transport equation in the regime of Sobolev vector fields: we plan to tackle such question by setting up a vanishing viscosity scheme and establishing its convergence towards a *unique* limit, also ruling out the possibility of anomalous dissipation in this setting. In order to achieve such a result, we need first to collect and formulate in our notations some (mostly known) results on the advection-diffusion equation drifted by Sobolev vector fields. This is indeed the content of the next paragraph.

4.1. **On the advection-diffusion equation drifted by a Sobolev vector field.** We recall the following proposition.

Proposition 4.1 ([28, Proposition 5.3]). *Let $\boldsymbol{b} \in L^1((0,T);W^{1,1}(\mathbb{T}^d))$ be a divergence-free vector field and let $v_0 \in L^\infty(\mathbb{T}^d)$ be given. Then the problem*

$$\begin{cases} \partial_t v + \operatorname{div}(\boldsymbol{b} v) = \Delta v & \text{in } (0,T)\times\mathbb{T}^d \\ v|_{t=0} = v_0 & \text{in } \mathbb{T}^d \end{cases} \tag{4.1}$$

admits a unique parabolic solution $v \in L^\infty((0,T);L^2(\mathbb{T}^d)) \cap L^2((0,T),H^1(\mathbb{T}^d))$. Furthermore, it holds $v \in L^\infty((0,T);L^\infty(\mathbb{T}^d))$ and

$$\|v\|_{L^\infty((0,T);L^s(\mathbb{T}^d))} \le \|v_0\|_{L^s(\mathbb{T}^d)} \tag{4.2}$$

for any real number $s \in [1,+\infty]$.

Remark 4.2 (Equation with a forcing term). For future use, we explicitly observe that the same conclusions of Proposition 4.1 apply as well to the equation with a forcing term. More precisely, if $\chi \in C^\infty((0,T)\times\mathbb{T}^d)$ is a smooth function and $v_0 \in L^\infty(\mathbb{T}^d)$, then the problem

$$\begin{cases} \partial_t v + \boldsymbol{b}\cdot\nabla v = \Delta v + \chi & \text{in } (0,T)\times\mathbb{T}^d \\ v|_{t=0} = v_0 & \text{in } \mathbb{T}^d \end{cases}$$

has a unique parabolic solution. Observe also that via the transformation $v(t,x) \mapsto v(T-t,-x)$ we deduce well-posedness results also for the backward equation

$$\begin{cases} -\partial_t v - \boldsymbol{b}\cdot\nabla v = \Delta v + \chi & \text{in } (0,T)\times\mathbb{T}^d \\ v|_{t=T} = v_T & \text{in } \mathbb{T}^d. \end{cases} \tag{4.3}$$

For future use, notice that if $v_T = 0$ then the problem (4.3) admits a unique solution in $L^\infty((0,T);L^\infty(\mathbb{T}^d))$ and that it holds

$$\|v\|_{L^\infty((0,T);L^\infty(\mathbb{T}^d))} \le C(\|\chi\|_{C^0(\mathbb{T}^d)}) < +\infty.$$

4.2. **The transport equation.** In the following, we will consider the initial value problem for the tranport/continuity equation

$$\begin{cases} \partial_t u + \operatorname{div}(\boldsymbol{b} u) = 0 & \text{in } (0,T)\times\mathbb{T}^d \\ u|_{t=0} = u_0 & \text{in } \mathbb{T}^d \end{cases} \tag{4.4}$$

where $T>0$, $\boldsymbol{b}\colon [0,T]\times\mathbb{T}^d \to \mathbb{R}^d$ is a given divergence-free vector field and $u_0\colon \mathbb{T}^d \to \mathbb{R}$ is the initial datum. As already done in Section 4, we will work in Sobolev classes for the velocity field and the equation (4.4) will be understood in the sense of distributions. We explicitly observe that, since we are working on the torus, the integrability of $\boldsymbol{b}$ is sufficient to prevent the blow up of its trajectories and thus we can work with the *regular Lagrangian flow* of $\boldsymbol{b}$:

Definition 4.3 (Regular Lagrangian flow)**.** Let $\boldsymbol{b} \in L^1((0,T); L^1(\mathbb{T}^d))$ be a divergence-free vector field. A map $\boldsymbol{X}\colon (0,T) \times (0,T) \times \mathbb{T}^d \to \mathbb{T}^d$ is a *regular Lagrangian flow* of $\boldsymbol{b}$ if the following conditions hold:

- for a.e. $x \in \mathbb{T}^d$ and for any $t \in [0,T]$ the map $s \in [0,T] \mapsto \boldsymbol{X}(t,s,x) = \boldsymbol{X}_{t,s}(x) \in \mathbb{T}^d$ is an absolutely continuous solution of

$$\begin{cases} \partial_s \boldsymbol{X}_{t,s}(x) = \boldsymbol{b}(s, \boldsymbol{X}_{t,s}(x)) & s \in [0,T], \\ \boldsymbol{X}_{t,t}(x) = x. \end{cases}$$

- For any $t \in [0,T]$ and $s \in [0,T]$ the map $x \in \mathbb{T}^d \mapsto \boldsymbol{X}_{t,s}(x) \in \mathbb{T}^d$ is measure-preserving.

Existence and uniqueness of the regular Lagrangian flow of a Sobolev, divergence-free vector field $\boldsymbol{b}$ are ensured by [21] and therefore we can give the following definition:

Definition 4.4. Let $\boldsymbol{b} \in L^1((0,T); W^{1,1}(\mathbb{T}^d))$ be a divergence-free vector field and let $\boldsymbol{X}\colon (0,T)\times(0,T) \times \mathbb{T}^d \to \mathbb{T}^d$ be its regular Lagrangian flow. If $u_0 \in L^1(\mathbb{T}^d)$, then the map

$$u^{\mathsf{L}}(t,x) := u_0(\boldsymbol{X}_{t,0}(x))$$

is called *Lagrangian solution* to (4.4).

Observe that, under the assumption that $\boldsymbol{b}$ is divergence-free, if $u_0\boldsymbol{b} \in L^1(\mathbb{T}^d)$ then the Lagrangian solution is also a distributional solution to (4.4).

4.3. **Setup of the vanishing viscosity scheme.** We consider the problem

$$\begin{cases} \partial_t u + \boldsymbol{b} \cdot \nabla u = 0 & \text{in } (0,T) \times \mathbb{T}^d \\ u|_{t=0} = u_0 & \text{in } \mathbb{T}^d \end{cases} \tag{4.5}$$

with a Sobolev, divergence-free vector field $\boldsymbol{b}$ and an initial datum $u_0 \in L^1(\mathbb{T}^d)$. For each $\varepsilon > 0$ we introduce the parabolic problem

$$\begin{cases} \partial_t v_\varepsilon + \boldsymbol{b} \cdot \nabla v_\varepsilon = \varepsilon \Delta v_\varepsilon & \text{in } (0,T) \times \mathbb{T}^d \\ v_\varepsilon|_{t=0} = v_0^\varepsilon & \text{in } \mathbb{T}^d \end{cases} \tag{4.6}$$

being v_0^ε a suitable bounded approximation of the initial datum u_0. In view of Proposition 4.1, the problem (4.6) is well-posed within parabolic solutions, namely the family $(v_\varepsilon)_{\varepsilon>0}$ is well-defined. Our goal is to establish (weak) compactness bounds on the family $(v_\varepsilon)_{\varepsilon>0}$ and characterize its limit points: we show that a "selection principle" holds: the sequence $(v_\varepsilon)_\varepsilon$ always converges as $\varepsilon \to 0$ to the Lagrangian solution to (4.5).

The precise statement of the main result of this section is a refinement of [21, Theorem IV.1] and reads as follows:

Theorem 4.5. *Let $\boldsymbol{b} \in L^1((0,T); W^{1,1}(\mathbb{T}^d))$ be a divergence-free vector field and let $u_0 \in L^1(\mathbb{T}^d)$ be a given initial datum. Let $(v_0^\varepsilon)_\varepsilon \subset L^\infty(\mathbb{T}^d)$ be any sequence of functions such that $v_0^\varepsilon \to u_0$ strongly in $L^1(\mathbb{T}^d)$. Then the sequence $(v_\varepsilon)_{\varepsilon>0} \subseteq L^\infty((0,T); L^\infty(\mathbb{T}^d)) \cap L^2((0,T); H^1(\mathbb{T}^d))$ of solutions to (4.6) converges in $C([0,T]; L^1(\mathbb{T}^d))$ to the (unique) Lagrangian solution to (4.4).*

Proof. We split the proof in several steps.

Step 1. Parabolic well-posedness and compactness (equi-integrability). We begin with the study of the problem (4.6). From Proposition 4.1, we deduce that for every fixed $\varepsilon > 0$ there exists a unique function $v_\varepsilon \in L^\infty((0,T); L^\infty(\mathbb{T}^d)) \cap L^2((0,T); H^1(\mathbb{T}^d))$ solving (4.6), which moreover satisfies

$$\|v_\varepsilon\|_{L^\infty((0,T);L^s(\mathbb{T}^d))} \leq \|v_0^\varepsilon\|_{L^s(\mathbb{T}^d)},$$

for any $s \in [1,+\infty]$. Since $u_0 \in L^1$, the family $(v_\varepsilon)_\varepsilon$ is in general not equi-bounded neither in $L^\infty((0,T);L^\infty(\mathbb{T}^d))$ nor in $L^2((0,T);H^1(\mathbb{R}^d))$. However, since $v_0^\varepsilon \to u_0$ strongly in $L^1(\mathbb{R}^d)$, we get for $s=1$

$$\|v_\varepsilon\|_{L^\infty((0,T);L^1(\mathbb{T}^d))} \le \|v_0^\varepsilon\|_{L^1(\mathbb{T}^d)} \le C < +\infty \tag{4.7}$$

for some constant $C>0$ independent of ε. This is, however, still not sufficient to obtain weak compactness in L^1, as we need to show the equi-integrability of the family $(v_\varepsilon)_{\varepsilon>0}$. To do so, we argue in the following way: since $v_0^\varepsilon \to u_0$ strongly in $L^1(\mathbb{T}^d)$, by Theorem 2.2, there exists a convex, increasing function $\Psi\colon [0,+\infty] \to [0,+\infty]$ such that $\Psi(0)=0$ and

$$\lim_{s\to\infty} \frac{\Psi(s)}{s} = \infty \qquad \text{and} \qquad \sup_{\varepsilon>0} \int_{\mathbb{R}^d} \Psi(|v_0^\varepsilon(x)|)\,dx =: C < \infty. \tag{4.8}$$

Without loss of generality, we can assume that Ψ is smooth. By an easy approximation argument (as already done several times before), we can multiply the equation (4.6) by $\Psi'(|v_\varepsilon|)$ and we obtain

$$\frac{d}{dt}\int_{\mathbb{T}^d} \Psi(|v_\varepsilon(\tau,x)|)\,dx + \varepsilon \int_{\mathbb{T}^d} \Psi''(|v_\varepsilon(\tau,x)|)|\nabla(|v_\varepsilon|)|^2\,dx = 0.$$

The convexity of Ψ and an integration in time on $(0,t)$ give

$$\int_{\mathbb{T}^d} \Psi(|v_\varepsilon(t,x)|)\,dx \le C,$$

where C is the same constant as in (4.8). Since t is arbitrary,

$$\sup_{t\in(0,T)} \int_{\mathbb{T}^d} \Psi(|v_\varepsilon(t,x)|)\,dx \le C. \tag{4.9}$$

Since the constant C is independent of ε, we can resort to Point (iii) of Theorem 2.2 and we infer that the family $(v_\varepsilon)_{\varepsilon>0}$ is weakly-precompact in $L^\infty((0,T);L^1(\mathbb{T}^d))$. Therefore, there exists a function $u^\vee \in L^\infty((0,T);L^1(\mathbb{T}^d))$ such that $v_\varepsilon \rightharpoonup u^\vee$ in $L^\infty((0,T);L^1(\mathbb{T}^d))$ (up to a non-relabelled subsequence).

Step 2. Identification of the limit via duality I. We now want to exploit a duality argument. Let $\chi \in C^\infty((0,T)\times\mathbb{T}^d)$ be arbitrary. By Remark 4.2, for every $\varepsilon>0$, there exists a unique function $\vartheta_\varepsilon \in L^\infty((0,T);L^\infty(\mathbb{T}^d)) \cap L^2((0,T);H^1(\mathbb{T}^d))$ solving

$$\begin{cases} -\partial_t \vartheta_\varepsilon - \boldsymbol{b}\cdot\nabla\vartheta_\varepsilon = \varepsilon\Delta\vartheta_\varepsilon + \chi & \text{in } (0,T)\times\mathbb{T}^d \\ \vartheta_\varepsilon|_{t=T} = 0 & \text{in } \mathbb{T}^d. \end{cases} \tag{4.10}$$

The family $(\vartheta_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty((0,T);L^\infty(\mathbb{T}^d))$ so, up to a subsequence, the family $(\vartheta_\varepsilon)_{\varepsilon>0}$ converges in $C([0,T];w^*-L^\infty(\mathbb{T}^d))$ to a function $\vartheta \in C([0,T];w^*-L^\infty(\mathbb{T}^d))$ solving the backward, inhomogenous transport equation

$$\begin{cases} -\partial_t \vartheta - \boldsymbol{b}\cdot\nabla\vartheta = \chi & \text{in } (0,T)\times\mathbb{T}^d \\ \vartheta|_{t=T} = 0 & \text{in } \mathbb{T}^d. \end{cases} \tag{4.11}$$

By [21], the problem (4.11) is well-posed in $L^\infty((0,T);L^\infty(\mathbb{T}^d))$ and thus ϑ coincides with *the* unique solution to (4.11) which lies in $C([0,T];L^\infty(\mathbb{T}^d))$. In addition, this implies that the whole sequence $(\vartheta_\varepsilon)_{\varepsilon>0}$ converges to ϑ (in other words, the passage to a subsequence is not needed). For future use, observe that it also holds that

$$\vartheta_\varepsilon(0,\cdot) \rightharpoonup \vartheta(0,\cdot) \qquad \text{in } L^\infty(\mathbb{T}^d) \tag{4.12}$$

and via a straightforward computation one also obtains the Duhamel representation formula

$$\vartheta(t,\boldsymbol{X}_{0,t}(x)) = \int_t^T \chi(s,\boldsymbol{X}_{0,s}(x))\,\mathrm{d}s, \qquad \forall x\in\mathbb{T}^d,\ t\in[0,T]. \tag{4.13}$$

Step 3. Identification of the limit via duality II. We now consider the regularized versions of problem (4.6) and (4.10). Let ρ be a non-negative, radially symmetric convolution kernel and set for $\varepsilon, \delta > 0$

$$v_\varepsilon^\delta := v_\varepsilon * \rho^\delta, \qquad \vartheta_\varepsilon^\delta := \vartheta_\varepsilon * \rho^\delta.$$

The smooth functions $v^{\varepsilon,\delta}$ and $\vartheta^{\varepsilon,\delta}$ solve respectively the problems

$$\begin{cases} \partial_t v_\varepsilon^\delta + \boldsymbol{b} \cdot \nabla v_\varepsilon^\delta = r_v^{\varepsilon,\delta} + \varepsilon \Delta v_\varepsilon^\delta & \text{in } (0,T) \times \mathbb{T}^d \\ v_\varepsilon^\delta|_{t=0} = v_0^{\varepsilon,\delta} & \text{in } \mathbb{T}^d \end{cases} \tag{4.14}$$

and

$$\begin{cases} -\partial_t \vartheta_\varepsilon^\delta - \boldsymbol{b} \cdot \nabla \vartheta_\varepsilon^\delta = r_\vartheta^{\varepsilon,\delta} + \varepsilon \Delta \vartheta_\varepsilon^\delta + \chi^\delta & \text{in } (0,T) \times \mathbb{T}^d \\ \vartheta_\varepsilon^\delta|_{t=T} = 0, & \text{in } \mathbb{T}^d \end{cases} \tag{4.15}$$

where

$$\chi^\delta := \chi * \rho^\delta$$

and the commutators are defined as

$$r_v^{\varepsilon,\delta} := \boldsymbol{b} \cdot \nabla v_\varepsilon^\delta - (\boldsymbol{b} \cdot \nabla v_\varepsilon) * \rho^\delta \qquad \text{and} \qquad r_\vartheta^{\varepsilon,\delta} := \boldsymbol{b} \cdot \nabla \vartheta_\varepsilon^\delta - (\boldsymbol{b} \cdot \nabla \vartheta_\varepsilon) * \rho^\delta.$$

Multiplying (4.14) times $\vartheta_\varepsilon^\delta$, applying Fubini's Theorem and integrating by parts in time and space we obtain

$$\begin{aligned} 0 &= \iint_{(0,T)\times\mathbb{T}^d} \left[(\partial_t v_\varepsilon^\delta)\vartheta_\varepsilon^\delta + \boldsymbol{b} \cdot (\nabla v_\varepsilon^\delta)\vartheta_\varepsilon^\delta - r_v^{\varepsilon,\delta}\vartheta_\varepsilon^\delta - \varepsilon(\Delta v_\varepsilon)\vartheta_\varepsilon^\delta \right] \mathrm{d}t\mathrm{d}x \\ &= \iint_{(0,T)\times\mathbb{T}^d} v_\varepsilon^\delta \left[-\partial_t \vartheta_\varepsilon^\delta - \boldsymbol{b} \cdot \nabla \vartheta_\varepsilon^\delta - \varepsilon \Delta \vartheta_\varepsilon^\delta \right] \mathrm{d}t\,\mathrm{d}x - \int_{\mathbb{T}^d} v^{\varepsilon,\delta}(0,x)\vartheta_\varepsilon^\delta(0,x)\,\mathrm{d}x \\ &\qquad - \iint_{(0,T)\times\mathbb{T}^d} r_v^{\varepsilon,\delta}\vartheta_\varepsilon^\delta\,\mathrm{d}t\mathrm{d}x \\ &\overset{(4.15)}{=} \iint_{(0,T)\times\mathbb{T}^d} v_\varepsilon^\delta (r_\vartheta^{\varepsilon,\delta} + \chi^\delta)\,\mathrm{d}t\,\mathrm{d}x - \int_{\mathbb{T}^d} v^{\varepsilon,\delta}(0,x)\vartheta_\varepsilon^\delta(0,x)\,\mathrm{d}x - \iint_{(0,T)\times\mathbb{T}^d} r_v^{\varepsilon,\delta}\vartheta_\varepsilon^\delta\,\mathrm{d}t\mathrm{d}x \\ &= \iint_{(0,T)\times\mathbb{T}^d} v_\varepsilon^\delta \chi^\delta\,\mathrm{d}t\mathrm{d}x + \iint_{(0,T)\times\mathbb{T}^d} (v_\varepsilon^\delta r_\vartheta^{\varepsilon,\delta} - r_v^{\varepsilon,\delta}\vartheta_\varepsilon^\delta)\,\mathrm{d}t\mathrm{d}x - \int_{\mathbb{T}^d} v^{\varepsilon,\delta}(0,x)\vartheta_\varepsilon^\delta(0,x)\,\mathrm{d}x \\ &=: \text{(I)} + \text{(II)} + \text{(III)}. \end{aligned}$$

Keeping $\varepsilon > 0$ fixed, we now send $\delta \to 0$. The two commutators can be written in the form

$$r_v^{\varepsilon,\delta}(t,x) = \int_{\mathbb{T}^d} v_\varepsilon(t, x+\delta y) \left[\frac{\boldsymbol{b}(t,x+\delta y) - \boldsymbol{b}(t,x)}{\delta} \right] \cdot \nabla\rho(y)\,\mathrm{d}y$$

and

$$r_\vartheta^{\varepsilon,\delta}(t,x) = \int_{\mathbb{T}^d} \vartheta_\varepsilon(t, x+\delta y) \left[\frac{\boldsymbol{b}(t,x+\delta y) - \boldsymbol{b}(t,x)}{\delta} \right] \cdot \nabla\rho(y)\,\mathrm{d}y.$$

Since $v_\varepsilon, \vartheta_\varepsilon \in L^\infty((0,T); L^\infty(\mathbb{T}^d))$, arguing as in the proof of Proposition 4.1, we can conclude that both $r_v^{\varepsilon,\delta}$ and $r_\vartheta^{\varepsilon,\delta}$ converge to 0 strongly in $L^1((0,T)\times\mathbb{T}^d)$ as $\delta \to 0$. This observation, combined with the uniform L^∞ bounds on v_ε^δ and $\vartheta_\varepsilon^\delta$, shows that (II) $\to 0$ as $\delta \to 0$.

For the term (I), instead, we can use the strong convergence of $v_\varepsilon^\delta \to v_\varepsilon$ and the uniform convergence of $\chi^\delta \to \chi$. Finally, for (III), by standard results about convolutions

$$v^{\varepsilon,\delta}(0,\cdot) \to v_0^\varepsilon$$

strongly in $L^1(\mathbb{T}^d)$ as $\delta \to 0$; furthermore, we have

$$\vartheta_\varepsilon^\delta(0,\cdot) \to \vartheta_\varepsilon(0,\cdot)$$

weakly* in $L^\infty(\mathbb{T}^d)$ as $\delta \to 0$. Such convergence follows from a standard argument in the framework of evolutionary PDEs (see e.g. [18, Lemma 3.7]) which establishes the weak continuity in time of the solutions to advection-diffusion or transport equations.

Thus, for any $\varepsilon > 0$, it holds

$$\iint_{(0,T)\times\mathbb{T}^d} v_\varepsilon(t,x)\chi(t,x)\,\mathrm{d}t\mathrm{d}x = \int_{\mathbb{T}^d} v_0^\varepsilon(x)\vartheta_\varepsilon(0,x)\,\mathrm{d}x.$$

We now send $\varepsilon \to 0$, getting

$$\iint_{(0,T)\times\mathbb{T}^d} u^{\mathsf{V}}(t,x)\chi(t,x)\,\mathrm{d}t\mathrm{d}x = \int_{\mathbb{T}^d} u_0(x)\vartheta(0,x)\,\mathrm{d}x. \tag{4.16}$$

In the last passage, we have used:

- $v^\varepsilon \rightharpoonup u^{\mathsf{V}}$ weakly in $L^\infty((0,T);L^1(\mathbb{T}^d))$;
- $v_0^\varepsilon \to u_0$ strongly in $L^1(\mathbb{T}^d)$;
- $\vartheta_\varepsilon(0,\cdot) \rightharpoonup \vartheta(0,\cdot)$ weakly* in $L^\infty(\mathbb{T}^d)$ by (4.12).

Step 4. Duality of the Lagrangian solution. A direct computation shows that the Lagrangian solution u^{L} satisfies

$$\begin{aligned}\iint_{(0,T)\times\mathbb{T}^d} u^{\mathsf{L}}(t,x)\chi(t,x)\,\mathrm{d}t\mathrm{d}x &= \iint_{(0,T)\times\mathbb{T}^d} u_0(\boldsymbol{X}_{t,0}(x))\chi(t,x)\,\mathrm{d}t\mathrm{d}x \\ &= \int_0^T \int_{\mathbb{T}^d} u_0(y)\chi(t,\boldsymbol{X}_{0,t}(y))\,\mathrm{d}t\,\mathrm{d}y \\ &= \int_{\mathbb{T}^d} u_0(y)\int_0^T \chi(t,\boldsymbol{X}_{0,t}(y))\,\mathrm{d}t\,\mathrm{d}y \\ &\overset{(4.13)}{=} \int_{\mathbb{T}^d} u_0(x)\vartheta(0,x)\,\mathrm{d}x.\end{aligned} \tag{4.17}$$

Hence, comparing (4.16) and (4.17), we obtain

$$\iint_{(0,T)\times\mathbb{T}^d} (u^{\mathsf{V}}(t,x) - u^{\mathsf{L}}(t,x))\chi(t,x)\,\mathrm{d}t\mathrm{d}x = 0.$$

Being $\chi \in C^\infty((0,T)\times\mathbb{T}^d)$ arbitrary, we have thus obtained $u^{\mathsf{V}} = u^{\mathsf{L}}$ a.e. and this concludes the proof.

Step 5. Upgrade to strong convergence. The convergence of $(v^\varepsilon)_{\varepsilon>0}$ to the Lagrangian solution is strong in $C([0,T];L^1(\mathbb{T}^d))$. This follows from [12, Lemma 3.3]: indeed, the regularity assumption (H1') of [12, Lemma 3.3] includes the case $\boldsymbol{b} \in L^1((0,T);W^{1,1}(\mathbb{T}^d))$, and the growth assumption (H2) is trivially satisfied as already remarked in Section 2. □

Remark 4.6. If $u_0 \in L^2(\mathbb{T}^d)$, the same argument as in the proof of Theorem 4.5 gives that $(v_\varepsilon)_\varepsilon$ converges in $C([0,T];L^2(\mathbb{T}^d))$ to the Lagrangian solution of (4.4). In particular, from the identity

$$\frac{1}{2}\|v_\varepsilon(t,\cdot)\|_{L^2}^2 + \varepsilon\int_0^t \|\nabla v_\varepsilon(s,\cdot)\|_{L^2}^2\mathrm{d}s = \frac{1}{2}\|v_\varepsilon(0,\cdot)\|_{L^2}^2$$

valid for every $\varepsilon > 0$ we deduce that

$$\varepsilon\int_0^t \|\nabla v_\varepsilon(s,\cdot)\|_{L^2}^2\mathrm{d}s \to 0 \quad \text{as } \varepsilon \to 0$$

for every $t > 0$. This means that *no anomalous dissipation is possible* for the vanishing viscosity limit in the case $\boldsymbol{b} \in L^1((0,T);W^{1,1}(\mathbb{T}^d))$ and $u \in L^\infty((0,T);L^2(\mathbb{T}^d))$, even though the solution lacks the integrability required for the DiPerna-Lions' theory to apply.

In a similar spirit, if $u_0 \in L^q(\mathbb{T}^d)$, we obtain that

$$\|u(t,\cdot)\|_{L^q(\mathbb{T}^d)} = \|u_0\|_{L^q(\mathbb{T}^d)},$$

and more generally all Casimirs of the solution obtained as vanishing viscosity limit are conserved, that is for every f and for every $t > 0$ it holds

$$\int_{\mathbb{T}^d} f(u(t,x))\,\mathrm{d}x = \int_{\mathbb{T}^d} f(u_0(x))\,\mathrm{d}x.$$

On the other hand, vector fields in the class $L^1((0,T);C^\alpha(\mathbb{T}^d))$, with $d \geq 2$ and $\alpha \in [0,1)$, may exhibit anomalous dissipation as shown in [22].

5. The vanishing viscosity scheme II: Stochastic flows and Lagrangian proof

5.1. **Preliminaries on stochastic flows.** We introduce the *stochastic Lagrangian* formulation of the system (4.6). Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a given filtered probability space, and let $\boldsymbol{W}_t$ be a $\mathbb{T}^d$-valued Brownian motion adapted to the backward filtration, i.e. for any fixed $t \in [0,T]$ and any $s \in [0,t]$, the Brownian motion $\boldsymbol{W}_s$ is such that $\boldsymbol{W}_t = 0$. We have the following definition.

Definition 5.1 (Stochastic flows)**.** Let $\varepsilon > 0$ and let $\boldsymbol{b} \in L^1((0,T);L^1(\mathbb{T}^d))$ be a divergence-free vector field. A map $\boldsymbol{X}^\varepsilon\colon (0,T)\times(0,T)\times\mathbb{T}^d\times\Omega \to \mathbb{T}^d$ is a *stochastic flow* of $\boldsymbol{b}$ if

- for any $t \in [0,T]$, for any $x \in \mathbb{T}^d$ and for a.e. $\omega \in \Omega$, the map $s \in [0,t] \mapsto \boldsymbol{X}^\varepsilon(t,s,x,\omega) = \boldsymbol{X}^\varepsilon_{t,s}(x,\omega) \in \mathbb{T}^d$ is a continuous solution to

$$\begin{cases} \mathrm{d}\boldsymbol{X}^\varepsilon_{t,s}(x,\omega) = \boldsymbol{b}(s,\boldsymbol{X}^\varepsilon_{t,s}(x,\omega))\mathrm{d}s + \sqrt{2\varepsilon}\,\mathrm{d}\boldsymbol{W}_s(\omega), \qquad s \in [0,t), \\ \boldsymbol{X}^\varepsilon_{t,t} = x, \end{cases} \tag{5.1}$$

- for any $t \in [0,T]$ and $s \in [0,t]$ and a.e. $\omega \in \Omega$ the map $x \in \mathbb{T}^d \mapsto \boldsymbol{X}^\varepsilon_{t,s}(x,\omega) \in \mathbb{T}^d$ is measure preserving.

The celebrated Feynman-Kac formula, see [25], gives an explicit representation of the solution v_ε of (4.6) in terms of the stochastic flow of $\boldsymbol{b}$, that is

$$v_\varepsilon(t,x) = \mathbb{E}[v_0^\varepsilon(\boldsymbol{X}^\varepsilon_{t,0}(x))],$$

where we have used the standard notation $\mathbb{E}[f]$ to denote the average with respect to $\mathbb{P}$, that is

$$\mathbb{E}[f] := \int_\Omega f(\omega)\mathrm{d}\mathbb{P}(\omega).$$

We remark that by considering a divergence-free vector field $\boldsymbol{b} \in L^1((0,T);W^{1,1}(\mathbb{T}^d))$ we have *strong* existence and pathwise uniqueness for (5.1): this means that we can construct a solution $\boldsymbol{X}^\varepsilon$ to (5.1) on any given filtered probability space equipped with any given adapted Brownian motion, see [8]. We remark that, since we are working on the torus, the boundedness assumption in [8] can be dropped.

5.2. **The Lagrangian proof.** In this subsection, we aim at giving another proof (exploiting Lagrangian tecnhiques) of the convergence of the vanishing viscosity scheme. In order to do that, we first establish some stability estimates between the stochastic and the deterministic flows.

Lemma 5.2. *Let $\boldsymbol{b} \in L^1((0,T);W^{1,p}(\mathbb{T}^d))$ be a divergence-free vector field, where $p \geq 1$. Let $\boldsymbol{X}, \boldsymbol{X}^\varepsilon$ be, respectively, the regular Lagrangian flow and the stochastic flow of $\boldsymbol{b}$. Then,*

(i) *if $p = 1$ and $\boldsymbol{b} \in L^q((0,T)\times\mathbb{T}^d)$ for some $q > 1$, then for every $\gamma > 0$ there exists a constant C_γ such that for a.e. $t \in [0,T]$ and $s \in [0,t]$*

$$\int_{\mathbb{T}^d} \mathbb{E}[\mathsf{d}(\boldsymbol{X}^\varepsilon_{t,s}(x), \boldsymbol{X}_{t,s}(x))]\mathrm{d}x \leq C(T,p)\left(\sqrt[4]{\varepsilon} + \frac{C_\gamma}{|\ln\varepsilon|} + \frac{1}{|\ln\sqrt{\varepsilon}|}\gamma\left[1 + \ln^+\left(\frac{\|\boldsymbol{b}\|_{L^q}}{\sqrt{\varepsilon}\gamma}\right)\right]\right). \tag{5.2}$$

(ii) If $p>1$, there exists a constant $C(T,p)$ such that for a.e. $t\in[0,T]$ and $s\in[0,t]$

$$\int_{\mathbb{T}^d}\mathbb{E}[\mathsf{d}(\boldsymbol{X}^\varepsilon_{t,s}(x),\boldsymbol{X}_{t,s}(x))]\mathrm{d}x\le C(T,p)\left(\sqrt[4]{\varepsilon}+\frac{\|\nabla\boldsymbol{b}\|_{L^1L^p}}{|\ln\varepsilon|}\right). \tag{5.3}$$

Moreover, the estimates (5.2), (5.3) give the L^1-convengence of $\boldsymbol{X}^\varepsilon_{t,s}$ towards $\boldsymbol{X}_{t,s}$ as $\varepsilon\to0$.

Proof. We divide the proof in several steps.

Step 1. The case $p=1$. For any $t\in(0,T)$, a.e. $\omega\in\Omega$ and a.e. $x\in\mathbb{T}^d$, the difference of the flows $\boldsymbol{X}^\varepsilon-\boldsymbol{X}$ satisfies the following S.D.E. for $s\in[0,t]$

$$\begin{cases}\mathrm{d}(\boldsymbol{X}^\varepsilon_{t,s}(x,\omega)-\boldsymbol{X}_{t,s}(x))=(\boldsymbol{b}(s,\boldsymbol{X}^\varepsilon_{t,s}(x,\omega))-\boldsymbol{b}(s,\boldsymbol{X}_{t,s}(x)))\,\mathrm{d}s+\sqrt{2\varepsilon}\mathrm{d}\boldsymbol{W}_s(\omega),\\ \boldsymbol{X}^\varepsilon_{t,t}(x,\omega)-\boldsymbol{X}_{t,t}(x)=0.\end{cases} \tag{5.4}$$

We define the function the function $q_\delta(y)=\ln\left(1+\frac{|y|^2}{\delta^2}\right)$ and the related functional Q^δ_ε as

$$Q^\delta_\varepsilon(t,s,x,\omega):=q_\delta(\boldsymbol{X}^\varepsilon_{t,s}(x,\omega)-\boldsymbol{X}_{t,s}(x))=\ln\left(1+\frac{|\boldsymbol{X}^\varepsilon_{t,s}(x,\omega)-\boldsymbol{X}_{t,s}(x)|^2}{\delta^2}\right), \tag{5.5}$$

where $\delta>0$ is a fixed parameter that will be chosen later. An application of Itô's formula gives that

$$\begin{aligned}\int_{\mathbb{T}^d}\mathbb{E}\left[Q^\delta_\varepsilon(t,s,x)\right]\mathrm{d}x&=\int_s^t\int_{\mathbb{T}^d}\mathbb{E}\left[\nabla_y q_\delta(t,\tau,\boldsymbol{X}^\varepsilon_{t,\tau}(x)-\boldsymbol{X}_{t,\tau}(x))\cdot\left(\boldsymbol{b}(\tau,\boldsymbol{X}^\varepsilon_{t,\tau}(x))-\boldsymbol{b}(\tau,\boldsymbol{X}_{t,\tau}(x))\right)\right]\mathrm{d}x\mathrm{d}\tau\\&\quad+\varepsilon\int_s^t\int_{\mathbb{T}^d}\mathbb{E}\left[\nabla^2_y q_\delta(t,\tau,\boldsymbol{X}^\varepsilon_{t,\tau}(x)-\boldsymbol{X}_{t,\tau}(x))\right]\mathrm{d}x\mathrm{d}\tau,\end{aligned}$$

and from the inequalities

$$\left|\nabla\ln\left(1+\frac{|y|^2}{\delta^2}\right)\right|\le\frac{C}{\delta+|y|},\qquad\left|\nabla^2\ln\left(1+\frac{|y|^2}{\delta^2}\right)\right|\le\frac{C}{\delta^2+|y|^2},$$

we obtain the following bound

$$\int_{\mathbb{T}^d}\mathbb{E}\left[Q^\delta_\varepsilon(t,s,x)\right]\mathrm{d}x\le\frac{\varepsilon(t-s)}{\delta^2}+C\int_s^t\int_{\mathbb{T}^d}\mathbb{E}\left[\frac{\left|\boldsymbol{b}(\tau,\boldsymbol{X}^\varepsilon_{t,\tau}(x))-\boldsymbol{b}(\tau,\boldsymbol{X}_{t,\tau}(x))\right|}{\delta+\left|\boldsymbol{X}^\varepsilon_{t,\tau}(x)-\boldsymbol{X}_{t,\tau}(x)\right|}\right]\mathrm{d}x\mathrm{d}\tau. \tag{5.6}$$

We now use the characterization of the equi-integrability as in Lemma 2.3. We fix $r>1$ and let $\gamma>0$ a parameter that will be chosen later. Then, using Lemma 2.3 we decompose $\nabla\boldsymbol{b}$ as

$$|\nabla\boldsymbol{b}|=g_1^\gamma+g_2^\gamma,$$

with

$$\|g_1^\gamma\|_{L^1}\le\gamma,\qquad\|g_2^\gamma\|_{L^r}\le C_\gamma,$$

where the constant C_γ is increasing as $\gamma\to0$. Finally, we introduce the function

$$\varphi(t,s,x,\omega):=\min\left\{\frac{|\boldsymbol{b}(s,\boldsymbol{X}^\varepsilon_{t,s}(x,\omega))|+|\boldsymbol{b}(s,\boldsymbol{X}_{t,s}(x))|}{\delta};g_1^\gamma(s,\boldsymbol{X}^\varepsilon_{t,s}(x,\omega))+g_1^\gamma(s,\boldsymbol{X}_{t,s}(x))\right\}.$$

Going back to (5.6), using the definition of φ, we get that

$$\begin{aligned}&\int_s^t\int_{\mathbb{T}^d}\mathbb{E}\left[\frac{\left|\boldsymbol{b}(\tau,\boldsymbol{X}^\varepsilon_{t,\tau}(x))-\boldsymbol{b}(\tau,\boldsymbol{X}_{t,\tau}(x))\right|}{\delta+\left|\boldsymbol{X}^\varepsilon_{t,\tau}(x)-\boldsymbol{X}_{t,\tau}(x)\right|}\right]\mathrm{d}x\mathrm{d}\tau\\&\le\int_s^t\int_{\mathbb{T}^d}\mathbb{E}\left[\varphi(t,\tau,x)\right]\mathrm{d}x\mathrm{d}\tau\\&\quad+\int_s^t\int_{\mathbb{T}^d}\mathbb{E}\left[g_2^\gamma(\tau,\boldsymbol{X}^\varepsilon_{t,\tau}(x))+g_2^\gamma(\tau,\boldsymbol{X}_{t,\tau}(x))\right]\mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

Since $g_2^\gamma \in L^r((0,T)\times\mathbb{T}^d)$, by Holder inequality we have that

$$\int_s^t \int_{\mathbb{T}^d} \mathbb{E}\left[g_2^\gamma(\tau, \boldsymbol{X}^\varepsilon_{t,\tau}(x)) + g_2^\gamma(\tau, \boldsymbol{X}_{t,\tau}(x))\right] \mathrm{d}x\mathrm{d}\tau \le 2T^{(r-1)/r} C_\gamma. \tag{5.7}$$

We now want to apply the interpolation inequality of Lemma 2.5 to φ: first, by using the measure preserving property of $\boldsymbol{X}$ and $\boldsymbol{X}^\varepsilon$, we have that

$$\|\varphi\|_{L^q} \le \frac{C}{\delta}\|\boldsymbol{b}\|_{L^q}. \tag{5.8}$$

Secondly, by Chebishev inequality

$$|||\varphi|||_{M^1((0,T)\times(0,T)\times\mathbb{T}^d\times\Omega)} \le C|||g_1^\gamma|||_{M^1((0,T)\times\mathbb{T}^d)} \le C\|g_1^\gamma\|_{L^1((0,T)\times\mathbb{T}^d)}. \tag{5.9}$$

We apply Lemma 2.5 to φ. The fact that the function $z \in [0,\infty) \mapsto z\left[1+\ln^+\left(\frac{C}{z}\right)\right] \in [0,\infty)$ is non-decreasing (where $\ln^+(w) := \max\{0, \ln(w)\}$ for every $w \ge 0$) and the bounds (5.8) and (5.9), give

$$\|\varphi\|_{L^1((0,T)\times(0,T)\times\mathbb{T}^d\times\Omega)} \le C\frac{q}{q-1}\|g_1^\gamma\|_{L^1}\left[1+\ln^+\left(\frac{\|\boldsymbol{b}\|_{L^q}}{\|g_1^\gamma\|_{L^1}}\frac{T^{1-\frac{1}{q}}}{\delta}\right)\right]. \tag{5.10}$$

Substituting (5.7) and (5.10) in (5.6) we finally obtain

$$\int_{\mathbb{T}^d} \mathbb{E}\left[Q^\delta_\varepsilon(t,s,x)\right]\mathrm{d}x \le \frac{\varepsilon T}{\delta^2} + 2T^{(r-1)/r}C_\gamma + \frac{Cq}{q-1}\gamma\left[1+\ln^+\left(\frac{\|\boldsymbol{b}\|_{L^q}T^{1-\frac{1}{q}}}{\delta\gamma}\right)\right].$$

Next, by defining

$$A_\delta(t,s) := \left\{(x,\omega) \in \mathbb{T}^d\times\Omega : \mathsf{d}(\boldsymbol{X}^\varepsilon_{t,s}(x,\omega), \boldsymbol{X}_{t,s}(x)) > \sqrt{\delta}\right\}, \tag{5.11}$$

we obtain that

$$\begin{aligned}\sup_{t,s\in(0,T)}\left(\mathscr{L}^d\otimes\mathbb{P}\right)(A_\delta(t,s)) &\le \frac{C}{|\ln\delta|}\int_{\mathbb{T}^d}\mathbb{E}\left[\ln\left(1+\frac{(\mathsf{d}(\boldsymbol{X}^\varepsilon_{t,s}(x),\boldsymbol{X}_{t,s}(x)))^2}{\delta^2}\right)\right]\mathrm{d}x \\ &\le \frac{C}{|\ln\delta|}\int_{\mathbb{T}^d}\mathbb{E}\left[Q^\delta_\varepsilon(t,s)\right]\mathrm{d}x \\ &\le C(T,q,r)\left(\frac{\varepsilon}{\delta^2|\ln\delta|} + \frac{C_\gamma}{|\ln\delta|} + \frac{1}{|\ln\delta|}\gamma\left[1+\ln^+\left(\frac{\|\boldsymbol{b}\|_{L^q}}{\delta\gamma}\right)\right]\right),\end{aligned} \tag{5.12}$$

where we have used that $\mathsf{d}(x,y) \le |x-y|$ for any $x,y\in\mathbb{T}^d$. Therefore,

$$\begin{aligned}\int_{\mathbb{T}^d}\mathbb{E}[\mathsf{d}(\boldsymbol{X}^\varepsilon_{t,s}(x),\boldsymbol{X}_{t,s}(x))]\mathrm{d}x &= \int_{(\mathbb{T}^d\times\Omega)\setminus A_\delta(t,s)}\mathsf{d}(\boldsymbol{X}^\varepsilon_{t,s}(x,\omega),\boldsymbol{X}_{t,s}(x))\mathrm{d}\mathbb{P}(\omega)\mathrm{d}x \\ &\quad + \int_{A_\delta(t,s)}\mathsf{d}(\boldsymbol{X}^\varepsilon_{t,s}(x,\omega),\boldsymbol{X}_{t,s}(x))\mathrm{d}\mathbb{P}(\omega)\mathrm{d}x \\ &\le \sqrt{\delta} + \left(\mathscr{L}^d\otimes\mathbb{P}\right)(A_\delta(t,s))\end{aligned} \tag{5.13}$$

where we have used that $\mathscr{L}^d\otimes\mathbb{P}$ is a probability measure on $\mathbb{T}^d\times\Omega$ and the distance d on the torus is bounded. Finally, we choose $\delta = \sqrt{\varepsilon}$ and plugging (5.12) in (5.13), we get that

$$\int_{\mathbb{T}^d}\mathbb{E}[\mathsf{d}(\boldsymbol{X}^\varepsilon_{t,s}(x),\boldsymbol{X}_{t,s}(x))]\mathrm{d}x \le C(T,q,r)\left(\sqrt[4]{\varepsilon} + \frac{C_\gamma}{|\ln\varepsilon|} + \frac{1}{|\ln\sqrt{\varepsilon}|}\gamma\left[1+\ln^+\left(\frac{\|\boldsymbol{b}\|_{L^q}}{\sqrt{\varepsilon}\gamma}\right)\right]\right),$$

and this concludes the proof of the estimate (5.2).

Step 2. The case $p > 1$. The proof easily follows from the arguments of Step 1. Since $\nabla\boldsymbol{b}(t,\cdot) \in L^p(\mathbb{T}^d)$ for a.e. $t \in (0,T)$, we apply Lemma 2.3 pointwise in time choosing $r = p$,

$g_1^\gamma = 0$, $\gamma = 0$, $g_2^\gamma(t,\cdot) = |\nabla \boldsymbol{b}(t,\cdot)|$ and $C_\gamma(t) = \|\nabla \boldsymbol{b}(t,\cdot)\|_{L^p}$. In particular, note that the bound in (5.7) changes into

$$\int_s^t \int_{\mathbb{T}^d} \mathbb{E}\left[g_2^\gamma(\tau, \boldsymbol{X}^\varepsilon_{t,\tau}(x)) + g_2^\gamma(\tau, \boldsymbol{X}_{t,\tau}(x))\right] \mathrm{d}x\mathrm{d}\tau \le 2\|C_\gamma\|_{L^1} = 2\|\nabla \boldsymbol{b}\|_{L^1L^p},$$

and by substituting in (5.2) we get (5.3).

Step 3. Convergence of the flows. We now prove the convergence of $X^\varepsilon_{t,s}$ towards $X_{t,s}$ as $\varepsilon \to 0$. If $p > 1$ it follows directly by (5.3) by letting $\varepsilon \to 0$. Then we analyze the case (i): the strategy is to choose properly the parameter γ in (5.2) independently from ε. In this regards, note that the last term on the right hand side of (5.2) is uniformly bounded in γ for ε small and converges to 0 as $\gamma \to 0$. Hence, for any given $\eta > 0$ there exists γ_0 independent from ε such that for all $\gamma \le \gamma_0$

$$\frac{C(T,q,r)}{|\ln\sqrt{\varepsilon}|}\gamma\left[1 + \ln^+\left(\frac{\|\boldsymbol{b}\|_{L^q}}{\sqrt{\varepsilon}\gamma}\right)\right] < \frac{\eta}{3}.$$

Now that the constant γ is fixed, and so is C_γ, we can infer that there exists $\varepsilon_0(M) > 0$ such that for all $\varepsilon \le \varepsilon_0(\gamma)$

$$C(T,q)\left(\sqrt[4]{\varepsilon} + \frac{C_\gamma}{|\ln\varepsilon|}\right) < \frac{2}{3}\eta,$$

and this concludes the proof of the convergence of the flows. □

The convergence result for $\varepsilon \to 0$ to the Lagrangian solution reads as follows:

Theorem 5.3. *Let $\boldsymbol{b} \in L^1((0,T); W^{1,1}(\mathbb{T}^d)) \cap L^q((0,T)\times\mathbb{R}^d)$ be a divergence-free vector field for some $q > 1$ and let $u_0 \in L^1(\mathbb{T}^d)$ be a given initial datum. Let $(v_0^\varepsilon)_\varepsilon \subset L^\infty(\mathbb{T}^d)$ be any sequence of functions such that $v_0^\varepsilon \to u_0$ strongly in $L^1(\mathbb{T}^d)$. Then the sequence $(v_\varepsilon)_{\varepsilon>0} \subset L^\infty((0,T); L^\infty(\mathbb{T}^d)) \cap L^2((0,T); H^1(\mathbb{T}^d))$ of solutions to (4.6) converges in $C([0,T]; L^1(\mathbb{T}^d))$ to the (unique) Lagrangian solution to (4.5).*

Proof. First of all, from Proposition 4.1 we deduce that for every fixed $\varepsilon > 0$ there exists a unique function $v_\varepsilon \in L^\infty((0,T); L^\infty(\mathbb{T}^d)) \cap L^2((0,T); H^1(\mathbb{T}^d))$ solving (4.6). Moreover, by the Feynman-Kac formula we know that v_ε satisfies

$$v_\varepsilon(t,x) = \mathbb{E}[v_0^\varepsilon(\boldsymbol{X}^\varepsilon_{t,0}(x))].$$

On the other hand, the Lagrangian solution to (4.5) is given by

$$u^{\mathsf{L}}(t,x) = u_0(\boldsymbol{X}_{t,0}(x)).$$

Having both v_ε and u^{L} a representation formula in terms of the flow, we use the stability of the flows to prove the convergence in the inviscid limit. We consider a sequence u_0^n of Lipschitz approximations of u_0, then for any $t \in (0,T)$ we have that

$$\begin{aligned}
\|v_\varepsilon(t,\cdot) - u^{\mathsf{L}}(t,\cdot)\|_{L^1} &= \|\mathbb{E}[v_0^\varepsilon(\boldsymbol{X}^\varepsilon_{t,0})] - u_0(\boldsymbol{X}_{t,0})\|_{L^1} \\
&\le \int_{\mathbb{T}^d}\int_\Omega |v_0^\varepsilon(\boldsymbol{X}^\varepsilon_{t,0}(x,\omega)) - u_0(\boldsymbol{X}^\varepsilon_{t,0}(x,\omega))|\mathrm{d}\mathbb{P}(\omega)\mathrm{d}x \\
&+ \int_{\mathbb{T}^d}\int_\Omega |u_0^n(\boldsymbol{X}^\varepsilon_{t,0}(x,\omega)) - u_0(\boldsymbol{X}^\varepsilon_{t,0}(x,\omega))|\mathrm{d}\mathbb{P}(\omega)\mathrm{d}x \\
&+ \int_{\mathbb{T}^d} |u_0^n(\boldsymbol{X}_{t,0}(x)) - u_0(\boldsymbol{X}_{t,0}(x))|\mathrm{d}x \\
&+ \int_{\mathbb{T}^d}\int_\Omega |u_0^n(\boldsymbol{X}^\varepsilon_{t,0}(x,\omega)) - u_0^n(\boldsymbol{X}_{t,0}(x))|\mathrm{d}\mathbb{P}(\omega)\mathrm{d}x.
\end{aligned}$$

In particular, by using that u_0^n is Lipschitz and the measure preserving property of the flows, we get that

$$\|v_\varepsilon(t,\cdot) - u^{\mathsf{L}}(t,\cdot)\|_{L^1} \leq \|v_0^\varepsilon - u_0\|_{L^1} + 2\|u_0^n - u_0\|_{L^1} + C_n\|\mathbb{E}[\mathsf{d}(\boldsymbol{X}_{t,0}^\varepsilon, \boldsymbol{X}_{t,0})]\|_{L^1}. \tag{5.14}$$

We first fix n big enough, independently from t and ε, in order to make the second term in (5.14) as small as we want. Then the conclusion follows from Lemma 5.2. $\square$

6. Rates of convergence

The goal of this section is to show that Lagrangian techniques are particularly useful in order to obtain explicit rates of convergence for the vanishing viscosity limit. To find such rates, we need slightly stronger integrability/regularity assumptions on the data. The first result deals with bounded initial data.

Proposition 6.1. *Let $\boldsymbol{b} \in L^1((0,T);W^{1,p}(\mathbb{T}^d))$ be a divergence-free vector field with $p>1$ and $u_0 \in L^\infty(\mathbb{T}^d)$ be a given initial datum. Let $(v_0^\varepsilon)_\varepsilon \subset L^\infty(\mathbb{T}^d)$ be* any *sequence of functions such that $v_0^\varepsilon \to u_0$ strongly in $L^1(\mathbb{T}^d)$ and consider v^ε and u be, respectively, the unique solutions of (4.6) and (4.5) with initial datum v_0^ε and u_0. Then, there exist $\bar\varepsilon$ and a continuous function $\phi_{u_0}:\mathbb{R}^+\to\mathbb{R}^+$ with $\phi_{u_0}(0)=0$, such that for any $1 \leq q < \infty$*

$$\sup_{t\in(0,T)} \|v_\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^q} \leq C(T,p,q,\|u_0\|_{L^\infty}, \|\nabla \boldsymbol{b}\|_{L^1L^p}) \left(\delta(\varepsilon) + \frac{1}{|\ln\delta(\varepsilon)|} + \phi_{u_0}(\delta(\varepsilon))\right)^{1/q}, \tag{6.1}$$

for any $\varepsilon \leq \bar\varepsilon$, where

$$\delta(\varepsilon) := \max\{\sqrt{\varepsilon}, \|v_0^\varepsilon - u_0\|_{L^1}\}. \tag{6.2}$$

Proof. We show the estimate (6.1) in the case $q=1$, the general case will follow by a straightforward interpolation of the spaces $L^1(\mathbb{T}^d)$ and $L^\infty(\mathbb{T}^d)$. Since $u_0 \in L^\infty(\mathbb{T}^d) \subset L^1(\mathbb{T}^d)$, using the continuity of translations in L^1 we can infer that there exists a continuous function ϕ_{u_0} as in the statement of the theorem and $h_0>0$ such that

$$\|u_0(\cdot + h) - u_0\|_{L^1} \leq \phi_{u_0}(h), \qquad \text{for all } h \leq h_0.$$

Then, for any $\delta \leq h_0$, we can compute

$$\begin{aligned}
\|v_\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^1} &= \|\mathbb{E}[v_0^\varepsilon(\boldsymbol{X}_{t,0}^\varepsilon)] - u_0(\boldsymbol{X}_{t,0})\|_{L^1} \\
&\leq \|\mathbb{E}[v_0^\varepsilon(\boldsymbol{X}_{t,0}^\varepsilon)] - u_0(\boldsymbol{X}_{t,0}^\varepsilon)\|_{L^1} \\
&+ \iint_{A_\delta(t,0)} |u_0(\boldsymbol{X}_{t,0}^\varepsilon(x,\omega)) - u_0(\boldsymbol{X}_{t,0}(x))| \mathrm{d}\mathbb{P}(\omega)\mathrm{d}x \\
&+ \iint_{(\mathbb{T}^d\times\Omega)\setminus A_\delta(t,0)} |v_0^\varepsilon(\boldsymbol{X}_{t,0}^\varepsilon(x,\omega)) - u_0(\boldsymbol{X}_{t,0}(x))| \mathrm{d}\mathbb{P}(\omega)\mathrm{d}x \\
&\leq \|v_0^\varepsilon - u_0\|_{L^1} + 2\|u_0\|_{L^\infty}\mathscr{L}^d \otimes \mathbb{P}(A_\delta(t,0)) + \phi_{u_0}(\delta) \\
&\leq \|v_0^\varepsilon - u_0\|_{L^1} + 2C(T,p)\|u_0\|_{L^\infty}\left(\frac{\varepsilon}{\delta^2|\ln\delta|} + \frac{\|\nabla\boldsymbol{b}\|_{L^1L^p}}{|\ln\delta|}\right) + \phi_{u_0}(\delta),
\end{aligned}$$

where the set A_δ is defined as in (5.11) and in the last line we have used the estimate in Lemma 5.2. The proof follows by choosing $\delta(\varepsilon)$ as in (6.2) and $\delta(\bar\varepsilon) = h_0$. $\square$

It is clear that the rate provided by Proposition 6.1 is not completely explicit for two reasons: on the one hand, the convergence of the initial datum depends upon the choice of the approximation v_0^ε; on the other hand, the function ϕ_{u_0} is implicitly related to the regularity of the initial datum. For the former issue, since we deal with bounded initial datum, existence and uniqueness of solutions of (3.1) and (4.4) are guaranteed by Proposition 4.1 and [21], thus we do not need the approximating sequence v_0^ε. Concerning the latter issue, the function ϕ_{u_0} can be explicitly constructed once the regularity of u_0 is known. Motivated by the results in [6], we provide the following example.

Corollary 6.2. *Let $u_0 \in H^1(\mathbb{T}^d)$. Assume that the hypothesis of Theorem 6.1 hold with $v_0^\varepsilon = u_0$. Then,*

$$\sup_{t\in(0,T)} \|v_\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^2} \le \frac{C}{\sqrt{|\ln\varepsilon|}}, \tag{6.3}$$

where the constant $C > 0$ depends on $T, p, \|u_0\|_{L^\infty}, \|\nabla u_0\|_{L^2}, \|\nabla \boldsymbol{b}\|_{L^1L^p}$.

Proof. It is enough to compute the function ϕ_{u_0}. We have that

$$\|u_0(\cdot + h) - u_0\|_{L^2} \le h\|\nabla u_0\|_{L^2},$$

and then we conclude by applying Proposition 6.1 with $\delta = \sqrt{\varepsilon}$ and $\phi_{u_0}(\delta) = \delta\|\nabla u_0\|_{L^2}$. □

It is interesting to compare the rate given by Corollary 6.2 and the one in [6, Theorem 3.3]. Under the same assumption on the initial datum, Corollary 6.2 provides a rate of convergence for a more general class of vector fields, namely $\boldsymbol{b} \in L^1((0,T); W^{1,p}(\mathbb{T}^d))$ with $p > 1$ instead of $\boldsymbol{b} \in L^\infty((0,T); W^{1,p}(\mathbb{T}^d))$ with $p > 2$. On the other hand, we do not improve completely the rate in [6]: the rate in (6.3) is better if $2 \le p \le 3$, while it is worst for $p > 3$. We also observe that a key tool in [6] is a propagation-of-regularity result, which is not needed in our argument.

We finally show how with these techniques it is possible to give a quantitative stability estimate for advection-diffusion equations. We address this issue motivated by the recent results in [34]:

Lemma 6.3. *Let $\boldsymbol{b} \in L^1((0,T); W^{1,p}(\mathbb{T}^d))$ be a divergence-free vector field, where $p > 1$. Let $\boldsymbol{X}_{t,s}^{\varepsilon_1}, \boldsymbol{X}_{t,s}^{\varepsilon_2}$ be the stochastic flows of $\boldsymbol{b}$ associated respectively to $\varepsilon_1, \varepsilon_2 > 0$. Then,*

$$\int_{\mathbb{T}^d} \mathbb{E}[\mathsf{d}(\boldsymbol{X}_{t,s}^{\varepsilon_1}(x), \boldsymbol{X}_{t,s}^{\varepsilon_2}(x))]\mathrm{d}x \le C(T,p)\left(\sqrt[4]{|\varepsilon_1 - \varepsilon_2|} + \frac{\|\nabla \boldsymbol{b}\|_{L^1L^p}}{|\ln|\varepsilon_1 - \varepsilon_2||}\right). \tag{6.4}$$

Proof. We just sketch the proof since it follows the same computations of Step 2 in Lemma 5.2. Notice that the S.D.E. solved by the difference $\boldsymbol{X}_{t,s}^{\varepsilon_1} - \boldsymbol{X}_{t,s}^{\varepsilon_2}$ is

$$\begin{cases} \mathrm{d}(\boldsymbol{X}_{t,s}^{\varepsilon_1}(x,\omega) - \boldsymbol{X}_{t,s}^{\varepsilon_2}(x,\omega)) = (\boldsymbol{b}(s, \boldsymbol{X}_{t,s}^{\varepsilon_1}(x,\omega)) - \boldsymbol{b}(s, \boldsymbol{X}_{t,s}^{\varepsilon_2}(x,\omega)))\,\mathrm{d}s + (\sqrt{2\varepsilon_1} - \sqrt{2\varepsilon_2})\mathrm{d}\boldsymbol{W}_s(\omega), \\ \boldsymbol{X}_{t,t}^{\varepsilon_1}(x,\omega) - \boldsymbol{X}_{t,t}^{\varepsilon_2}(x,\omega) = 0. \end{cases}$$

Then, by defining the function $q_\delta(y) = \ln\left(1 + \frac{|y|^2}{\delta^2}\right)$ and the related $Q_{\varepsilon_1,\varepsilon_2}^\delta$ as

$$Q_{\varepsilon_1,\varepsilon_2}^\delta(t,s,x,\omega) := q_\delta(\boldsymbol{X}_{t,s}^{\varepsilon_1}(x,\omega) - \boldsymbol{X}_{t,s}^{\varepsilon_2}(x,\omega)) = \ln\left(1 + \frac{|\boldsymbol{X}_{t,s}^{\varepsilon_1}(x,\omega) - \boldsymbol{X}_{t,s}^{\varepsilon_2}(x,\omega)|^2}{\delta^2}\right),$$

when we apply Itô's formula the contribution of the stochastic part is different, namely

$$\int_{\mathbb{T}^d} \mathbb{E}\left[Q_{\varepsilon_1,\varepsilon_2}^\delta(t,s,x)\right]\mathrm{d}x \le \frac{|\varepsilon_1 - \varepsilon_2|(t-s)}{\delta^2} + C\int_s^t \int_{\mathbb{T}^d} \mathbb{E}\left[\frac{\left|\boldsymbol{b}(\tau, \boldsymbol{X}_{t,\tau}^{\varepsilon_1}(x)) - \boldsymbol{b}(\tau, \boldsymbol{X}_{t,\tau}^{\varepsilon_2}(x))\right|}{\delta + \left|\boldsymbol{X}_{t,\tau}^{\varepsilon_1}(x) - \boldsymbol{X}_{t,\tau}^{\varepsilon_2}(x)\right|}\right]\mathrm{d}x\mathrm{d}\tau.$$

The conclusion follows by defining the set A_δ as

$$A_\delta(t,s) := \left\{(x,\omega) \in \mathbb{T}^d \times \Omega : \mathsf{d}(\boldsymbol{X}_{t,s}^{\varepsilon_1}(x,\omega), \boldsymbol{X}_{t,s}^{\varepsilon_2}(x,\omega)) > \sqrt{\delta}\right\},$$

and doing the same computations as in Step 2 of Lemma 5.2. □

Then, the estimate on the flows yields the following rate of convergence for the solutions of (3.1).

Proposition 6.4. *Let $\boldsymbol{b} \in L^1((0,T); W^{1,p}(\mathbb{T}^d))$ be a divergence-free vector field with $p > 1$ and $u_0 \in L^\infty(\mathbb{T}^d)$. Let ε_n be a sequence such that $\varepsilon_n \to \varepsilon > 0$ and let $v_{\varepsilon_n}, v_\varepsilon$ the unique solutions of (4.6) with initial datum u_0 and viscosity $\varepsilon_n, \varepsilon$, respectively. Then, there exist $N(u_0, T)$ and a continuous function $\phi_{u_0} : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi_{u_0}(0) = 0$, such that*

$$\sup_{t\in(0,T)} \|v_{\varepsilon_n}(t,\cdot) - v_\varepsilon(t,\cdot)\|_{L^1} \leq C\left(\frac{1}{|\ln|\varepsilon_n - \varepsilon||} + \phi_{u_0}\left(\sqrt{|\varepsilon_n - \varepsilon|}\right)\right), \tag{6.5}$$

for any $n \geq N(u_0, T)$, where the constant C depends upon $T, p, \|u_0\|_{L^\infty}$, and $\|\nabla \boldsymbol{b}\|_{L^1L^p}$.

Proof. The proof follows by arguing exactly as in the one of Proposition 6.4 and using Lemma 6.3. □

One can compare the rate given by Proposition 6.4 with the ones in [30] and [34]. The rate in (6.5) depends upon ϕ_{u_0} and cannot be better than $O\left(\frac{1}{|\ln|\varepsilon_n-\varepsilon||}\right)$, but provides convergence in the *strong* norm $C([0,T]; L^1(\mathbb{T}^d))$. On the other hand, the rates of [30] and [34] are of order $\sqrt{|\varepsilon_n - \varepsilon|}$ and $|\varepsilon_n - \varepsilon|$, respectively, but they are given for a logarithmic distance which instead metrizes *weak* convergence.

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