Faculty Working Papers

TAXATION, MONETARY POLICY, AND THE EQUITY-EFFICIENCY TRADEOFF

Ronald Morris Harstad, Assistant Professor
Department of Economics

#585

College of Commerce and Business Administration
University of Illinois at Urbana-Champaign

FACULTY WORKING PAPERS

College of Commerce and Business Administration

University of Illinois at Urbana-Champaign

July 27, 1979

TAXATION, MONETARY POLICY, AND THE EQUITY-EFFICIENCY TRADEOFF

Ronald Morris Harstad, Assistant Professor
Department of Economics

#585

Summary and Acknowledgements -- See Next Page

Comments welcomed, please consult before quoting.

Abstract

The set of government policies consistent with monetary competitive equilibrium is modelled. Given assumptions conventional for the optimal taxation literature, it is demonstrated that no loss in attainable welfare results if, in response to an arbitrary monetary policy, attention is limited to taxation and administrative policies which may be productively efficient. Antecedent efficiency theorems are shown to be corollaries.

Acknowledgements

The research presented here was begun in my Ph.D. thesis at the University of Pennsylvania. Progress of the research has been aided by support from the National Science Foundation (grant SOC 74-03974-A01), the Center for Analytic Research in Economics and the Social Sciences of the University of Pennsylvania, the Mathematical Social Sciences Board (attendance at MSSB Holderness, N.H., May 1977 conference), and the University of Illinois Foundation. I have benefitted from comments by David Cass, Walter P. Heller, James S. Jordan, John A. Weymark, and especially Karl Shell, whose patient criticism has been encouraging. These people should not share responsibility for the shortcomings of the paper.

1. Introduction

Koopmans [1957] initiated the equilibrium analysis of social trade-offs between equity and efficiency when the initial distribution of endowments in an economy is deemed inequitable. He demonstrated the decentralizability of any Pareto-efficient allocation as a general competitive equilibrium, through the use of costlessly administrable lump-sum taxes and transfers.

When the government does not have such potent redistributive capabilities, the extent of efficiency sacrifice, for the sake of equity, which is embodied in an optimal tax regime depends crucially upon the nature of the governmental choice mechanism. Useful implications can be drawn, however, from characterizations holding for every tax system which is optimal for a government maximizing some continuous Paretian welfare function. These are the implications derivable solely from the presumption that no welfare loss results should an economy restrict attention to allocations which are consumptively efficient.

Diamond and Mirrlees [1971] analyzed a government limited to the selection of quantity taxes on individual commodities. Given a mild aggregate demand assumption,[1] they have shown that no sacrifice of aggregate productive efficiency for the sake of equity obtains. Heller and Shell [1974] allowed for endogenous selection of redistributive instruments, with the resources required for administration of tax policy dependent upon the type and complexity of tax instruments used. Adding a stringent continuity assumption for administrative costs,[2] they also characterized productive efficiency as desirable.

The next section of this paper interprets the Diamond-Mirrlees and Heller-Shell results by examining the types of governmental policies which are compatible with an Arrow-Debreu-McKenzie general equilibrium model, and the meaning of efficiency in such a model. Section 3 outlines an equilibrium model with money, transactions constraints, and a full set of equilibrium-compatible policies: taxation, a reasonably general formulation of monetary policy, and administrative policies. The model is of some interest in its rather straightforward compilation of these features.

Section 4 uses this model to demonstrate a sort of qualitative separability of taxation and monetary policies. Specifically, no loss in attainable welfare results if, in response to an arbitrary monetary policy, attention is limited to taxation and administrative policies which may be productively efficient. The general equilibrium analogues to Diamond-Mirrlees and Heller-Shell results set out in Section 2 are corollaries of this efficiency theorem.

2. Equilibrium Modelling of Policies

The Diamond-Mirrlees and Heller-Shell approaches to equilibrium modelling of redistributive policy employ simplifications which omit a fundamental complexity of general competitive equilibrium. Both model the government as directly selecting allocations and prices for sectors of the economy. The models are decentralized in the sense that the government is constrained to select allocations from a "decentralizable" set.[3]

The decision to use competitive equilibrium tools does impose some limits on the types of government policies which can be analyzed.

Presumably moral suasion is not appropriately analyzed in terms of how equilibria are perturbed by jawboning. Nor is a model where firms are price-taking profit or value maximizers useful to examine antitrust policy.

What set of policies are consistent with the precepts of equilibrium models? Basically, policies whose impact can be expressed in the equations which define equilibrium, materials balance and decentralization. A (possibly non-zero) vector of excess supplies which are delivered to government agencies can be entered into the materials balance equation:

$$y - x \geq g \tag{1}$$

where y is aggregate net productive supply, x is aggregate net consumptive demand, g is this vector of administrative inputs, and the inequality is not strict for any commodity which is a productive resource (non-zero productive price).

The remaining equations defining an equilibrium specify that the prices agents face support the allocations included in the materials balance equation. Government policies can specify wedges or distortions by which the price vector any agent faces differs from that faced by the base agent (say, firm 1). For convenience, this paper will follow notationally the simplification of Diamond-Mirrlees, and collapse these policies into the equation

$$q - p = \tau \tag{2}$$

where q is the price vector facing all consumers, p the productive sector prices, and τ a vector of taxes (including subsidies).

An extensive set of policies can be so modelled, including agricultural surplus and price support programs. If, as below, a monetary equilibrium is being modelled, the monetary components of (1) allows specification of currency printing and open market policies, and the monetary components of (2) allows differential taxation of bonds.

The fundamental complexity of equilibrium theory is that any (p,q) satisfying (2) and supporting allocations satisfying (1) is an equilibrium for an economy where the government has selected policies (g,τ). Equilibrium is not unique, except under unusually heroic assumptions, even given a specification of government policies which is as complete as consistency allows. Government in competitive equilibrium does not have specifiable policies powerful enough to enable the direct selection of prices or allocations.

Recognition of this fact complicates the formulation of the productive efficiency question. Diamond-Mirrlees and Heller-Shell characterized optimal policies as exhibiting productive efficiency, in that public sector activity is efficient when evaluated at producer prices. This will not in general be true for any government policy package, that prespecified policies will be efficient at every set of prices which are equilibrium prices. The only known exception is laissez-faire policies ($(g,\tau) = 0$), which will not be feasible if, for example, contract enforcement is a costly activity.

Full equilibrium analogues to the Diamond-Mirrlees and Heller-Shell theorems are well-defined, however, in the sense of attainable welfare, which is the highest level of social welfare attainable in any equilibrium associated with a given policy package. The Diamond-Mirrlees and Heller-Shell assumptions apply directly in full equilibrium models, and in each, their theorem implies:

> Equilibrium Productive Efficiency: Limiting attention to policies for which there exists a productively efficient equilibrium does not reduce attainable welfare.

3. The Model Outlined

3.1. Markets and prices

The economy to be analyzed has open spot markets for $j = 1,\dots,J$ commodities in each time period $t = 1,\dots,\Omega$. Futures markets, extensively analyzed elsewhere, are eliminated here for simplicity. On all markets, goods exchange solely for fiat money, identified as a quantity of greenbacks, one unit of which has value p_{0t} (to producers) or q_{0t} (to consumers) in period $t = 1,\dots,\Omega$.

This notational convention is maintained throughout: for $t = 1,\dots,\Omega$, any market statistic v_t is $(J + 1)$-dimensional, with a zero component, v_{0t}, relating to money. v_t' is v_t absent v_{0t}, $v = (v_1,\dots,v_\Omega)$ has dimensionality $\theta = \Omega(J + 1)$, and $v' = (v_1',\dots,v_\Omega')$ is ΩJ-dimensional. Where reference is made to an arbitrary component v_{jt}, the possibility of reference to v_{0t} is disallowed unless explicit.

All agents are presumed to know with certainty prices for all periods at the beginning of the economy, and to show no concern for occurrences after period Ω. Uncertainty is simply beyond the scope of the present analysis.

Firms face producer prices $p = (p_1,\dots,p_t,\dots,p_\Omega)$, and households face consumer prices $q = (q_1,\dots,q_\Omega)$, with $q = p + \tau$, where $\tau = (\tau_1,\dots,\tau_\Omega)$ is the vector of quantity taxes.

Producer prices are chosen from

$$P = \{p \in \mathbb{R}_+^\theta \mid \sum_{j=0}^{J} p_{jt} = 1,\ t = 1,\dots,\Omega\}^4, \tag{3}$$

and the candidate space for consumer prices is

$$Q = \{q \in P \mid q_{0t} \in (0,1),\ t = 1,\dots,\Omega\}. \tag{4}$$

Reference will frequently be made to prices associated with an arbitrary terminal horizon price for money, which is labelled μ. Define $P(\cdot) : (0,1) \to o(P)^5$ by

$$P(\mu) = \{p \in P \mid p_{0\Omega} = \mu\}. \tag{5}$$

As p and q are identically normed, τ lies in

$$T = \{\tau \in \mathbb{R}^\theta \mid \sum_{j=0}^{J} \tau_{jt} = 0,\ -1 \leq \tau_{jt} \leq 1,\ j = 0,..J,\ t = 1,..,\Omega\} . \tag{6}$$

Let $\varepsilon_L(\cdot), \varepsilon_U(\cdot) : (0,1) \to \mathbb{R}^1$ be defined arbitrarily, subject to, for all $\mu \in (0,1)$,

$$0 < \varepsilon_U(\mu) < 1 - \mu < 1 + \varepsilon_L(\mu) < 1. \tag{7}$$

Define $T(\cdot) : (0,1) \to o(T)$ by

$$T(\mu) = \{\tau \in T \mid \varepsilon_L(\mu) \leq \tau_{0\Omega} \leq \varepsilon_U(\mu)\}. \tag{8}$$

When $p \in P(\mu)$, choosing $\tau \in T(\mu)$ will satisfy $p + \tau = q \in Q$.

Placing p_t in the simplex by (3) is familiar. Much less common is the simplicial restriction of q_t. As modelled below, firm supply correspondences are homogeneous of degree zero in p_t, and household demand correspondences exhibit zero homogeneity in q_t. Thus, if an equilibrium existed where $\sum_{j=0}^{J} p_{jt} = 1$ and $\sum_{j=0}^{J} q_{jt} = 3$, then p_t and $q_t/3$ would yield the same allocations, thus still being an equilibrium and allowing $\sum_{j=0}^{J} \tau_{jt} = 0$. So the norm restrictions in (4) and (6) constitute a permissible numeraire choice.

With this numeraire convention, τ_{0t} may be interpreted as the "average" excise tax level on goods in period t, and τ_{jt} as the commodity-specific deviation from this average.

3.2. Households

A superscript h represents any of the H households, which differ substantively from usual equilibrium treatments only in the requirement that purchases be covered by beginning-of-period money inventories, and in the monetary tax liability at the end of the economy.

Notation:

for $h = 1,\ldots,H$, $j=1,\ldots,J$, $t = 1,\ldots,\Omega$:

x_{jt}^{h}: net purchase by household h of good j in period t.

$x_{jt}^{hB} = \max(0,x_{jt}^{h})$: gross purchase by h of j in t.

ω_{jt}^{h}: endowment of h of good j at beginning of t.

c_{jt}^{h}: consumption by h of j during t.

As mentioned, zero components of (J + 1)-dimensional vectors refer to money. Money is not consumed, so c_{0t}^{h} is never used. ω_{0t}^{h} is not used, as the scalar m_t^h will represent the closing money inventory (cash on hand after trading and compensation) for $t = 0,1,2,\ldots,\Omega$, and thus m_0^h represents the initial money endowment. In this way, households are modelled as receiving no automatic money endowment in period $t = 2,3,\ldots,\Omega$, but rather as using m_{t-1}^h, the previous closing inventory, to cover period t purchases. x_{0t}^h is the net amount of money received on markets in $t = 1,\ldots,\Omega$. For notational convenience, we adopt $x_{0t}^{hB} = -m_{t-1}^h$, for $t = 1,\ldots,\Omega$.

The following combinations are designated: $x_t^h = (x_{0t}^h, x_{1t}^h, \ldots, x_{Jt}^h)$, $x^h = (x_1^h,\ldots,x_\Omega^h)$, $x_t^{hB} = (x_{0t}^{hB}, x_{1t}^{hB},\ldots,x_{Jt}^{hB})$, $x = \sum_h x^h$, $\tilde{x} = X_h x^h$, $c_t^{h'} = (c_{1t}^h,\ldots,c_{Jt}^h)$, $c^{h'} = (c_1^{h'},\ldots,c_\Omega^{h'})$, $\omega_t^{h'} = (\omega_{1t}^h,\ldots,\omega_{Jt}^h)$.

Finally, a representation is needed of a component vector of a consumptive sector allocation $\tilde{x}$: $\tilde{x}_k = (x_{jt}^1,\ldots,x_{jt}^h,\ldots,x_{jt}^H)$, $k = j + J(t-1) = 1 + J(t-1), 2 + J(t-1),\ldots,Jt$, $t = 1,\ldots,\Omega$.

Households face four types of constraints, for $h = 1,\ldots,H$, $t = 1,\ldots,\Omega$:

Nonnegativity: $$x_t^{h'} + \omega_t^{h'} \geq c_t^{h'} \in C^h, \qquad (9)$$

where $C^h \subset \mathbb{R}_+^J$ is the feasible consumption set. This is a standard constraint: nonnegative, physically consumable consumption out of stocks.

Fidelity: $$q_t x_t^h \leq 0. \qquad (10)$$

This is merely a definition of x_{0t}^h, satisfied whenever the value of goods received does not exceed the value of cash dispersed.

Solvency: $$0 \leq m_t^h \leq m_{t-1}^h + x_{0t}^h. \qquad (11)$$

A household raising or lowering its money holdings as purchases and sales are (potentially) dissynchronized is constrained to be solvent, that is, to maintain at all times a nonnegative money inventory.[6]

$m_\Omega^h \geq 0$ is perhaps the closest analogue in this model to a budget constraint. The right-hand inequality is merely an accounting balance: closing inventory cannot exceed opening inventory plus net receipts.

Liquidity: $$q_t x_t^{hB} \leq 0 \tag{12}$$

This constraint codifies the timing convention chosen as perhaps the simplest modelling of a critical role for money in economizing on transactions.[7] During each period, purchases must be covered with cash before compensation for sales during the period is realized. Thus period t purchases are limited in value by (12) to the value of m_{t-1}^h, the previous period's closing money inventory. Period t sales will raise m_t^h and cover purchases in t + 1.

Denote the household's feasible or budget correspondence $X^h(\cdot)$: $Q \to \wp(\mathbb{R}^\theta)$, defined by

$$X^h(q) = \{x^h \in \mathbb{R}^\theta \mid x^h \text{ satisfies (9)-(12)}\}. \tag{13}$$

Household behavior is then specified by

$$\max U^h(c^{h'}) \text{ subject to } x^h \in X^h(q),\ m_0^h \leq m_\Omega^h. \tag{14}$$

The convention specified by the final inequality in (14) is separated from the constraints of (13) to draw attention to the treatment of terminal money stock. In positing a terminal horizon of the economy known with certainty from the beginning by all agents, an anomaly not characteristic of real-world economies is introduced. As money is inherently valueless upon termination of the economy, all agents will attempt to rid themselves of it for the sake of greater last-minute consumption, making a positive price of money impossible in period Ω. But if valueless then, money must also have a zero price in $\Omega - 1$, and by induction, in all periods.

Following Heller [1974] and Okuno [1976] in adopting a suggestion of Lerner [1947], an artificial contrivance is used to respond to this artificially created anomaly. The government is presumed to collect in cash a tax liability from each household, upon the close of period Ω trading, in nominal amount equal to the initial cash endowment of the household, m_0^h. Contrained by (14) to have at least this much money on hand, the household is willing to supply commodities for money in the final period. The possibility of a positive terminal money price being substainable is ensured.

The (net) demand correspondence $x^h(\cdot) : Q \to \wp(\mathbb{R}^\theta)$ is defined by:

$$x^h(q) = \{x^h \in X^h(q) \mid x^h \text{ is a solution to } (14)\}. \qquad (15)$$

Maintained assumptions, for $h = 1,\ldots,H$:

h.1. C^h is closed, convex, bounded below, and contains $\omega_t^{h'}$ in its interior, for $t = 1,\ldots,\Omega$,[8] $m_0^h > 0$.

Clearly $\text{int}X^h(q) \neq \emptyset$ by these assumptions, which then have the familiar purpose of ensuring that $X^h(\cdot)$ is continuous, compact- and convex-valued on Q (Heller [1974], Lemma 2, p. 101). $m_0^h > 0$ guarantees the possibility of nonautarkic behavior, allowing some credulity for the presumed inefficiency of barter transactions.

h.2. $u^h(\cdot)$ is a continuous, nondecreasing, real-valued function which exhibits convex epigraphs and local nonsatiation.

Presumably familiar, this guarantees that $x^h(\cdot)$ is upper hemi-continuous (uhc) and compact- and convex-valued on Q (Berge [1963], p. 116).

h.3. (Diamond-Mirrlees) For all $\tilde{x} \in \tilde{x}(q)$ for any $q \in Q$, either

(1) $\tilde{x}_k < 0$ for some $k \in \{1,2,\ldots,J\Omega\}$, or

(2) $\tilde{x}_k > 0$ for some $k \in \{1,2,\ldots,J\Omega\}$ for which $q_{jt} > 0$, $k = j + J(t-1)$.[9]

3.3. Firms

Extensive exploration of firm behavior is well outside the focus of this study. Consequently, modelling of firms has been simplified as much as possible, even at the expense of treatment asymmetric with household modelling.

Any of the F firms is represented by superscript f. Net supply by firm f of good j in period t is y^f_{jt}, $f = 1,\ldots,F$, $j = 1,\ldots,J$, $t = 1,\ldots,\Omega$. y^f_{0t} designates net units of money dispersed on markets in $t = 1,\ldots,\Omega$. Thus, for $t = 1,\ldots,\Omega$, if $p_{0t} > 0$,

$$y^f_{0t} \equiv -p'_t\, y^{f'}_t / p_{0t}. \qquad (16)$$

By rearrangement, $p_t y^f_t \equiv 0$. As before, $y^{f'}_t = (y^f_{1t},\ldots,y^f_{Jt})$, $y^f_t = (y^f_{0t}, y^{f'}_t)$, $y^f = (y^f_1,\ldots,y^f_\Omega)$, $y = \sum_f y^f$, $\tilde{y} = X_f y^f$.

A production plan $y^{f'}_t$ may be interpreted as the production of outputs $\max(0, y^{f'}_t)$ from inputs $\max(0, -y^{f'}_t)$. The set of feasible production plans is $Y^f \subset \mathbb{R}^J$.

Futures markets have been excluded as cumbersome, and analysis of their import can be found elsewhere. Production is modelled as intra-period with possibilities unchanged over the course of the economy.

Firm behavior, presumed to be price-taking profit maximization, can thus be described intraperiodically, as the distinction between nominal profits in t ($p_t' \, y_t^{f'}$) and the discounted value of profits in t ($-y_{0t}^f$) has no behavioral significance.[10,11]

The (net) supply correspondence $y^f(\cdot) : P \to \wp(\mathbb{R}^\theta)$, f = 1,...,F, is defined by:

$$y^f(p) = \{y^f \in \mathbb{R}^\theta \mid y_t^{f'} \in Y^f,\ p_t' y_t^{f'} \geq p_t' \, \hat{y}_t' \text{ for all } \hat{y}_t' \in Y^f,$$

$$p_t y_t^f = 0,\ p_{0t} = 0 \text{ implies } \mid y_{0t}^f \mid \leq k_y,\ t = 1,\ldots,\Omega\}, \qquad (17)$$

where k_y is an arbitrary, large bound.

Assumptions for firms:

$\mathfrak{F}$.1. For all f = 1,...,F : Y^f is a closed, convex cone containing $\mathbb{R}_-^J$.

Convexity is a standard assumption, and Cass [3] and McKenzie [17] establish the use of a fictitious factor of production to represent decreasing returns production, with profits distribution to share-holders, as choice over a cone technology.[12]

$\mathfrak{F}$.2. Production is irreversible.

For this modelling of firm behavior, $y^f(\cdot)$ is uhc on P (Harstad [1977], Lemma 2.1).

3.4. Governmental activity

The government selects a vector of quantity taxes $\tau \in T$, and a vector g of inputs to administration, constrained by

$$(g,\tau) \in G \subset \{(g,\tau) \in \mathbb{R}^{2\theta} \mid \tau \in T,\ g_t' \geq 0,\ |g_{0t}| \leq k_g,\ t = 1,\ldots,\Omega\}. \quad (18)$$

where G is the administrative feasibility set, and k_g is some arbitrary, large bound.

It is useful to define the feasible correspondence $G(\cdot) : T \to \wp(\mathbb{R}^{\theta})$,

by $$G(\tau) = \{g \in \mathbb{R}^{\theta} \mid (g,\tau) \in G\}. \quad (19)$$

The following assumption is maintained:

$\mathcal{g}$.1. G is closed and exhibits limited free disposal $((g,\tau) \in G$ and $\hat{g}' \geq g'$ imply $(\hat{g},\tau) \in G)$. $G(\tau) \neq \emptyset$ for all $\tau \in T$.

This ensures the feasibility of an active commodity policy which could purchase more than the minimal requirement of any administrative input. Additionally, the assumption opens a full role for monetary policy, as only physical commodities are required for administration--any inflow or outflow of money is administratively feasible. So the stream $(g_{01}, g_{02}, \ldots, g_{0\Omega})$ will represent the governmentally selected monetary policy, the net inflows of money to the treasury.

$\mathcal{g}$.2. (Heller-Shell) $G(\cdot)$ is a continuous correspondence on T.

3.5. Equilibrium for given policies

For ease of notation, let $a = (\tilde{x}, \tilde{y}, p, q) \in \mathbb{R}^{4\theta}$, and $A = [\tilde{x}(Q) \times \tilde{y}(P) \times P \times Q] \subset \mathbb{R}^{4\theta}$.

For given $(g,\tau) \in G$, an equilibrium is a vector $a \in A$ satisfying

$$q - p = \tau \tag{20}$$

$$x - y + g \leq 0, \; p(x - y + g) = 0, \tag{21}$$

$$x^h \in x^h(q) \quad \text{for all } h = 1,\ldots,H, \tag{22}$$

and

$$y^f \in y^f(p) \quad \text{for all } f = 1,\ldots,F.\text{[13]} \tag{23}$$

Monetary equilibrium models generally are ensured of the existence of equilibrium, as setting all spot market money prices at zero leaves autarky the only possibility for firms and households. Such a position, however, constitutes excessive reliance on the assumption that barter transactions are unused since less efficient than monetary transactions. This assumption is quite tenable when money has a positive price, but barter transactions, not necessarily in balancing amounts, would undoubtedly be desired in the event of monetary collapse.

Autarky has also been ignored in the mechanisms used to model administrative costs, which are deemed to arise from the need to monitor economic activity. This is in the spirit of the endogenous policy approach of Heller and Shell [1974], for upon finding desired redistribution impossible or prohibitively costly in a barter economy, a government could be predicted to encourage and support the development and maintenance of monetary modes of exchange.

For these reasons, equilibrium has been defined so as to include the requirement ($q \in Q$) that monetary methods of exchange are being used by consumers. The possibility that $P_{0t} = 0$ in equilibrium is included only due to the simplified representation of firm behavior which has made this variable insignificant, as a component of a.

The (possibly empty) equilibrium correspondence $E(\cdot) : G \to \wp(A)$ is defined by:

$$E(g,\tau) = \{a \in A \mid a \text{ satisfies (20)-(23)}\} \tag{24}$$

Let $$D = \{(g,\tau) \in G \mid E(g,\tau) \neq \emptyset\}. \tag{25}$$

D is the domain for policy selection in an equilibrium model of endogenous policy theory.

It would be convenient to have a guarantee that D is nonempty. Only minor adaptations are needed to apply the equilibrium theorem of Heller [1974] to show that an equilibrium exists for $(g,\tau) = (0,0)$. However, the administrative feasibility set may not include the zero vector.

Mantel [1975], in a barter model, demonstrates the existence of equilibrium for any tax package satisfying a complex nonexcessive subsidies assumption. He does not show that the set of taxes meeting this restriction is nonempty. As nonexcessive subsidies roughly requires that the government satisfy its budget constraint at any price vector (not just at equilibrium prices), I have not succeeded in constructing an argument that any nontrivial (g,τ) meets this condition. The nonemptiness of D remains a technical difficulty.

3.6. The equilibrium-policy set

It is convenient to designate $z = (a, g, \tau) = (\tilde{x}, \tilde{y}, p, q, g, \tau)$, and to define the graph of the correspondence $E(\cdot)$ as the equilibrium-policy set:

$$Z = \{z \in A \times D \mid a \in E(g,\tau)\}. \tag{26}$$

Let the correspondence $Z(\cdot) : (0,1) \to o(Z)$ be defined by:

$$Z(\mu) = \{z \in Z \mid \tau \in T(\mu),\ p_{0\Omega} = \mu\}. \tag{27}$$

Then Z is bounded, and $Z(\mu)$ is compact for $\mu \in (0,1)$ (Harstad [1977], Lemmata 2.2, 2.3, Theorem 2.4).[14]

3.7. Optimal policies

Let $D(\cdot) : (0,1) \to o(D)$ be defined by

$$D(\mu) = \{(g,\tau) \in D \mid \tau \in T(\mu)\}. \tag{28}$$

The relationships among terminal price levels, policy packages, and attainable consumption sector allocations are represented by $e(\cdot) : (0,1) \times D \to o[\tilde{x}(Q)]$, defined by

$$e(\mu,g,\tau) = \begin{cases} \{\tilde{x} \mid (\tilde{x},\tilde{y},p,q) \in E(g,\tau), \text{ for some } q,\tilde{y},p \in P(\mu)\}, & \tau \in T(\mu) \\ \emptyset, & \text{otherwise.} \end{cases} \tag{29}$$

Let the welfare index $w(\cdot) : \mathbb{R}^{H\theta} \to \mathbb{R}$ be a continuous function of allocations, and Paretian in the sense that it is derivable from a welfare function which is an increasing function of utility levels. Designate the attainable welfare function, $W(\cdot) : (0,1) \times D \to \mathbb{R}$, defined by

$$W(\mu,g,\tau) = \begin{cases} [\max w(\tilde{x}) \text{ subject to } \tilde{x} \in e(\mu,g,\tau)], & (g,\tau) \in D(\mu) \\ \emptyset, & \text{otherwise.} \end{cases} \tag{30}$$

Existence of this maximum is guaranteed by compactness of $Z(\mu)$.

Theorem 1. For any $\mu \in (0,1)$, attainable welfare $W(\mu,\cdot)$ achieves a maximum on $D(\mu)$.

Proof: Pick arbitrary $\mu \in (0,1)$. Define $b(\cdot) : Z(\mu) \to \mathbb{R}$ by $b(z) \equiv w(\tilde{x})$. As $Z(\mu)$ is compact, and $b(\cdot)$ is continuous by construction, there exists $z^* \in Z(\mu)$ satisfying $b(z^*) \geq b(z)$ for all $z \in Z(\mu)$. By construction, $W(\mu,g^*,\tau^*) \geq W(\mu,g,\tau)$ for all $(g,\tau) \in D(\mu)$.

The existence of optimal policies in the face of price indeterminacy cannot be guaranteed.[15]

4. Productive Efficiency

For convenience, represent g as (g_0,g'), where $g_0 = (g_{01},\ldots,g_{0\Omega})$, $g' = (g'_1,\ldots,g'_\Omega)$, and define $\Gamma(\cdot) : (0,1) \to o(\mathbb{R}^\Omega)$ by

$$\Gamma(\mu) = \{g_0 \in \mathbb{R}^\Omega \mid (g_0,g',\tau) \in \overline{D}(\mu) \text{ for some } g' \geq 0,\ \tau \in T(\mu)\} \quad (31)$$

where $\overline{D}(\mu)$ is $D(\mu)$ adjusted for component reordering. Let

$$\Gamma = \{g_0 \in \mathbb{R}^\Omega \mid g_0 \in \Gamma(\mu) \text{ for some } \mu \in (0,1)\}. \quad (32)$$

Define $D^0(\cdot) : (0,1) \times \Gamma \to o(D)$ by

$$D^0(\mu,\gamma) = \begin{cases} \{(g,\tau) \in D(\mu) \mid g_0 = \gamma\}, & \gamma \in \Gamma(\mu) \\ \emptyset, & \text{otherwise,} \end{cases} \quad (33)$$

and $Z^0(\cdot) : (0,1) \times \Gamma \to o(Z)$ by

$$Z^0(\mu,\gamma) = \begin{cases} \{z \in Z \mid (g,\tau) \in D^0(\mu,\gamma)\}, & \gamma \in \Gamma(\mu) \\ \emptyset, \text{ otherwise.} \end{cases} \quad (34)$$

$D^0(\mu,\gamma)$ is the set of policies which are decentralizable given an arbitrary resolution of price indeterminacy and an arbitrary choice of monetary policy. Next, the set of consumer allocations attainable given these restrictions must be specified. Let $e^0(\cdot) : (0,1) \times \Gamma \to o[\tilde{x}(Q)]$ be defined by

$$e^0(\mu,\gamma) = \begin{cases} \{\tilde{x} \in \tilde{x}(Q) \mid \tilde{x} \in e(\mu,g,\tau) \text{ for some } (g,\tau) \in D^0(\mu,\gamma)\}, & \gamma \in \Gamma(\mu) \\ \emptyset, \text{ otherwise.} \end{cases} \quad (35)$$

Efficiency will be evaluated in terms of whether aggregate production net of administrative inputs is on the boundary of the net possibilities set. Designating $Y = \sum_f (Y^f \times \ldots\ldots \times Y^f) \subset \mathbb{R}^{J\Omega}$ and defining $G'(\cdot) : T \to o(\mathbb{R}_+^{J\Omega})$ by

$$G'(\tau) = \{g' \in \mathbb{R}_+^{J\Omega} \mid (g_0,g') \in G(\tau) \text{ for some } g_0 \in \mathbb{R}^{\Omega}\}, \quad (36)$$

represent the net possibilities correspondence as $N(\cdot) : T \to o(\mathbb{R}^{J\Omega})$, defined by

$$N(\tau) = \{n' \in \mathbb{R}^{J\Omega} \mid n' \in [Y - G'(\tau)]\}. \quad (37)$$

The two critical sets for efficiency analysis can now be defined. Let $Z^1(\cdot) : (0,1) \times \Gamma \to o(Z)$ be defined by

$$Z^1(\mu,\gamma) = \begin{cases} \{z \in Z^0(\mu,\gamma) \mid w(\tilde{x}) \geq w(\hat{\tilde{x}}) \text{ for all } \hat{\tilde{x}} \in e^0(\mu,\gamma)\}, & \gamma \in \Gamma(\mu) \\ \emptyset, & \text{otherwise,} \end{cases} \quad (38)$$

and $Z^2(\cdot) : (0,1) \times \Gamma \to o(Z)$ by

$$Z^2(\mu,\gamma) = \begin{cases} \{z \in Z^0(\mu,\gamma) \mid y' - g' = n' \in \text{bdy } N(\tau)\}, & \gamma \in \Gamma(\mu) \\ \emptyset, & \text{otherwise.} \end{cases} \quad (39)$$

Given μ,γ, $Z^1(\mu,\gamma)$ is the set of equilibrium-policy relations for which the maximum level of attainable welfare is reached, and $Z^2(\mu,\gamma)$ is the productively efficient subset of Z.

Theorem 2. Given any $\mu \in (0,1)$, any $\gamma \in \Gamma(\mu)$, $Z^1(\mu,\gamma) \cap Z^2(\mu,\gamma) \neq \emptyset$. That is, for arbitrary monetary policy, optimal tax and administrative policy responses are productively efficient.[16]

The proof of the theorem, which is presented in section 5, handles several complications which do not arise in the Diamond-Mirrlees model, but essentially the same logic is used. Diamond-Mirrlees presume an inefficient optimum. Given their demand assumption, there exists a Pareto-improving price change. If the presumed optimal allocation is interior to the aggregate net production possibilities set, there exists a nearby feasible allocation supported by the Pareto-superior price vector. This contradicts the original presumption. Notice that their argument held net tax receipts constant at zero while showing feasibility of a dominating allocation. In this model, zero monetary inflows have no special role, so the Diamond-Mirrlees argument can be replicated holding monetary inflows at an arbitrary level.

5. Proof of Theorem 2

1) Suppose the contrary: $Z^1(\mu,\gamma) \cap Z^2(\mu,\gamma) = \emptyset$. For any $z^* \in Z^1(\mu,\gamma)$, $n^* \in \text{int } N(\tau^*)$.

2) By h.3., there exists $\bar{q} \in \mathbb{R}^\theta$, $\bar{q} \neq 0$, so that $\hat{q} = (q^* + \Delta\bar{q}) \in Q$ is Pareto-superior to q^* for any $\Delta \in (0,1]$.

For arbitrary $\varepsilon_1 > 0$, suppose $\tilde{x}(\tilde{q}) \cap n_{\varepsilon_1}(\tilde{x}^*) = \emptyset$ for every $\Delta \in (0,1]$.[17] As $Z^0(\mu,\gamma)$ is compact by construction (Harstad [1977], Thm. 2.4), and $\tilde{x}(\cdot)$ is uhc on Q, there exists $z^{**} \in Z^1(\mu,\gamma)$ such that either:

a) $z^{**} \in Z^2(\mu,\gamma)$, as desired, or

b) there exists $\hat{q}$, Pareto-superior to q^{**} and arbitrarily close, so that $\tilde{x}(\hat{q}) \cap n_{\varepsilon_1}(x^{**}) \neq \emptyset$ for any $\varepsilon_1 > 0$.

3) Assume, then, without loss of generality that for any $(\varepsilon_1, \varepsilon_2) >> 0$, there exists $(\hat{\tilde{x}},\hat{q})$ satisfying:

a) $\hat{q} \in Q$ is Pareto-superior to q^*,

b) $\hat{q} \in n_{\varepsilon_2}(q^*)$, and

c) $\hat{\tilde{x}} \in \tilde{x}(\hat{q})$, $\hat{\tilde{x}} \in n_{\varepsilon_1}(\tilde{x}^*)$.

4) As $n^{*\prime} \in \text{int } N(\tau^*)$ and $G'(\cdot)$ is continuous on T by g.2., $n^{*\prime} \in N(\hat{\tau})$ for any $\hat{\tau} \in n_{\varepsilon_3}(\tau^*)$. This in turn implies $\hat{n}' \in N(\hat{\tau})$ for any $\hat{n}' \in n_{\varepsilon_4}(n^{*\prime})$, and $\varepsilon_3 > 0$, any $\varepsilon_4 > 0$ for suitably small $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$.

5) Therefore, there exists $(\hat{g},\hat{\tilde{y}})$ satisfying:

a) $\hat{y}' - \hat{g}' = \hat{n}'$,

b) $\hat{g} \in G(\hat{t})$, as $G(\cdot)$ is continuous,

c) $\hat{g}_0 = \gamma$, by (18), and

d) $\hat{\tilde{y}} \in \tilde{y}(\hat{p})$ for some $\hat{p} \in \mathcal{N}_{\varepsilon_5}(p^*)$, for any $\varepsilon_5 > 0$, for suitably small $(\varepsilon_1, \varepsilon_2, \varepsilon_4)$.

6) As $\hat{q} \in \mathcal{N}_{\varepsilon_2}(q^*)$ and $\hat{p} \in \mathcal{N}_{\varepsilon_5}(p^*)$, $(\hat{q} - \hat{p}) = \hat{\tau} \in \mathcal{N}_{\varepsilon_3}(\tau^*)$ for any $\varepsilon_3 > 0$ for suitably small $(\varepsilon_2, \varepsilon_3)$.

7) If for any component jt, $x^*_{jt} < n^*_{jt}$, then $p^*_{jt} = 0$, and $\hat{p}$ could have been chosen above with $\hat{p}_{jt} = 0$, while satisfying $\hat{p} \in \mathcal{N}_{\varepsilon_5}(p^*)$, $\hat{n}' \in \mathcal{N}_{\varepsilon_4}(n^*)$, guaranteeing $\hat{x}_{jt} \leq \hat{n}_{jt}$.

8) For any component jt with $x^*_{jt} = n^*_{jt}$, $\hat{x}_{jt} \in \mathcal{N}(n^*_{jt})$. Thus, for suitably small $(\varepsilon_1, \ldots, \varepsilon_5)$, $\hat{z}$ can be chosen to guarantee $\hat{a} \in E(\hat{g}, \hat{\tau})$. As $\hat{q}$ was chosen Pareto-superior to q^*, $w(\hat{\tilde{x}}) > w(\tilde{x})$, contradicting supposition that $z^* \in Z^1(\mu, \gamma)$.

Footnotes

1. Assumption h.3. below, used to guarantee the existence of Pareto-improving price changes.

2. Assumption g.2. below, used to guarantee that small changes in tax rates can be feasibly administered given small changes in administrative resources. Heller-Shell present a realistic example of an administrative technology which does not satisfy this assumption.

3. Diamond-Mirrlees model the government as selecting consumer prices and firm allocations, subject to being on the boundary of the private sector possibilities set. Heller-Shell have the government choosing firm and household allocations, subject to the existence of a vector of buying and selling prices, each for producers and consumers, which supports the allocations, given household-specific lump-sum taxes and transfers, and firm-specific profits taxes and license fees.

4. $\mathbb{R}^\theta$ is the θ-dimensional real space. Throughout, $\mathbb{R}^\theta_+ = \{v \in \mathbb{R}^\theta \mid v \geq 0\}$, $\mathbb{R}^\theta_{++} = \text{int}\,\mathbb{R}^\theta_+$. Vector inequalities: $v \geq 0$ implies $v_j \geq 0$, all j including 0; $v > 0$ implies $v \geq 0 \neq v$; $v >> 0$ implies $v_j > 0$ all j including 0.

5. o(S) is read "the set of subsets of S" and is the power set for S. A mapping $\phi: S \to \hat{S}$ is point-valued, $\phi : S \to o(\hat{S})$ is set-valued.

6. An obvious generalization, avoided for simplicity, is to require nonnegative money holdings in a model with bond markets. If the sale of bonds includes the incurrence of transactions costs, possibly through a timing mechanism similar to (12) (some feature is needed to make bonds and money distinct assets in equilibrium), earlier drafts and various futures markets literatures convince me that the qualitative results below are maintained.

7. More complicated modellings of transactions costs may well leave households with nonconvex budget sets, which would create substantial mathematical difficulties best avoided here. If impact of transactions costs is restricted to maintain convexity, and money still plays a critical role, the structure of household behavior should be qualitatively unaltered, and the analysis to follow should apply.

8. $\omega_t^{h'} \in \text{int } C^h$ implies $\omega_t^{h'} >> 0$. Heller [1974], p. 103: "This is a familiar if absurd assumption. I believe it can be replaced in the usual manner at the usual cost of a substantial increase in the complexity of the proofs." Heller does not vary underlying parameters and examine the resultant variation in equilibria. I cannot be certain the assumption is not crucial here. Weymark [1978], p. 4: "In the context of the optimal taxation literature, where no lump-sum taxation is possible, this is a highly restrictive assumption." On the one hand, with administrative costs incumbent upon all tax instruments, lump-sum taxation no longer enjoys such a unique position. On the other hand, (12) is more readily defended when time periods are interpreted as short, making $\omega_t^{h'} >> 0$, all t, even less tenable.

9. This assumes the existence of either a common demand or a common supply good. Weymark [1978] shows in essence that this can be weakened to being able to find a basis for the commodity space such that some composite good is in common demand or common supply. No primitive assumptions (that is, no assumptions about individual characteristics of agents, rather than combined characteristics of sets of agents) suffice to guarantee the existence of common demand or supply goods.

10. Note that if $p_{0t} = 1$ for any $t \in \{1,\ldots,\Omega\}$, any feasible $y_t^{f'}$ is profit-maximal, and $y_{0t}^f = 0$. If $p_{0t} = 0$, any t, nominal profit maximization is well-defined, and the supply correspondence defined in (17) includes any y^f which is optimal in other periods, and for which $y_t^{f'}$ maximizes nominal profits (that is, it includes all bounded values of y_{0t}^f).

11. Readers may be interested in whether the analysis to follow extends to a model where firms are treated symmetrically to households, and must pay cash for period t purchases before receiving payment for period t sales. First, the model would have to allow for firms having initial endowments of cash, and a terminal tax would be required to provide an incentive for firms to supply goods for money in period Ω. Second, in such a model, the distinction between nominal and discounted profits would be critical. Without additional features, the objective of price-taking firms would be unclear. If a full set of future markets were added, and provision was made for an equilibrium distinction between bonds and money as assets, and money still economized on transactions, firms would maximize the first-period discounted value of profits streams. Analysis virtually identical to that presented below could be applied by similar argument to such a model, with the candidate set for producer prices restricted to Q. A significantly dissimilar argument would be required at one step, noted in footnote 14, below.

12. Household behavior has been modelled consistent with the possibility of a commodity in which utility is degenerate. The strict positivity of endowments, discussed in footnote 8, implies that each household holds a positive ownership share of each firm. $\mathbb{R}^J_- = \{v \in \mathbb{R}^J \mid v \leq 0\}$.

13. A direct implication of the equilibrium conditions is the exact balance of the government budget at the level of the selected monetary policy in each period where $p_{0t} > 0$ (if $p_{0t} = 0$, y_{0t} is indeterminate--footnote 10). From (21),

$$g_{0t} = y_{0t} - x_{0t} = q_t'x_t' / q_{0t} - p_t'y_t' / p_{0t}$$

$$= \frac{q_t'x_t'}{\tau_{0t}} + \frac{\tau_t'x_t' + p_t'(x_t' - y_t')}{p_{0t}}, \text{ so}$$

$$g_{0t} = \frac{q_t'x_t'}{\tau_{0t}} + \frac{\tau_t'x_t'}{p_{0t}} - \frac{p_t'g_t'}{p_{0t}} .$$

The last line states that the net inflow of money to the government in period t equals the net receipts from the average tax rate plus the net receipts from the commodity-specific deviations from this average rate (either of these two terms may be negative) minus the money spent to purchase administrative inputs.

14. Lemma 2.2: Z is bounded. Lemma 2.3: Given any $\mu \in (0,1)$, any sequence $\{q^n\} \to q^0$ with $q^n \in Q$ for all n, and $q^0 \in [P(\mu)/Q]$; for any $\{x^{hn}\}$ with $x^{hn} \in x^h(q^n)$, $\{x^{hn}\}$ is unbounded, for any $h = 1,\ldots,H$. Theorem 2.4: For any $\mu \in (0,1)$, $Z(\mu)$ is compact.
    To extend the present analysis to the model discussed in footnote 11, a lemma similar to Lemma 2.3 would be required, showing that firm supplies explode when $\{p^n\} \to p^0$ with $p^n \in Q$ for all n, $p^0 \in [P(\mu)/Q]$.

15. Harstad [1977] presents an example. It assumes a convergent sequence of policies increasing in attainable welfare. The associated welfare-attaining sequence of allocations converges to an allocation which is not feasible, because the corresponding sequence of equilibrium prices converges to monetary collapse.

16. Resolving price level indeterminacy does not directly enter the efficiency argument, but is needed to ensure the existence of an optimum, that is, to ensure $Z^1(\mu,\gamma) \neq \emptyset$.

17. $n_\varepsilon(v) = \{\hat{v} \in S \mid \|\hat{v}-v\| < \varepsilon\}$ for any $v \in S$.

References

1. Kenneth J. Arrow and Gerard Debreu, Existence of equilibrium for a competitive economy, Econometrica 22 (1954), 265-290.

2. Claude Berge, "Topological Spaces," Macmillan, New York, 1963.

3. David Cass, Duality: a symmetric approach from the economist's vantage point, J. Econ. Theory 7 (1974), 272-295.

4. Gerard Debreu, Economies with a finite set of equilibria, Econometrica 38 (1970), 387-392.

5. Peter A. Diamond and James A. Mirrlees, Optimal taxation and public production, Amer. Econ. Rev. 61 (1971), 8-27 and 261-278.

6. Ronald Morris Harstad, "Elements of an Equilibrium Theory of Taxation," Ph.D. dissertation, University of Pennsylvania, 1977.

7. Walter Perrin Heller, The holding of money balances in general equilibrium, J. Econ. Theory 7 (1974), 93-108.

8. Heller and Karl Shell, On optimal taxation with costly administration, Amer. Econ. Rev. 64 (1974), 338-345.

9. Werner Hildenbrand, "Core and Equilibria of a Large Economy," Princeton University Press, 1974.

10. Tjalling C. Koopmans, "Three Essays on the State of Economic Science," McGraw-Hill, New York, 1957.

11. Abba P. Lerner, Money as a creature of the state, Proc. Amer. Econ. Assoc. 37 (1947), 2.

12. Rolf R. Mantel, General equilibrium and optimal taxes, J. Math. Econ. 2 (1975), 187-200.

13. Lionel McKenzie, On the existence of a general equilibrium for a competitive market, Econometrica 27 (1959), 54-71.

14. Masahiro Okuno, General equilibrium with money: indeterminacy of price level and efficiency, J. Econ. Theory 12 (1976), 402-415.

15. John A. Weymark, On Pareto-improving price changes, J. Econ. Theory 19 (1978), 338-346.