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Abstract

In this paper, we are interested in approximating the solution to scalar conservation laws using systems of interacting stochastic particles. The scalar conservation law may involve a fractional Laplacian term of order $\alpha \in (0, 2]$. When $\alpha \leq 1$ as well as in the absence of this term (inviscid case), its solution is characterized by entropic inequalities. The probabilistic interpretation of the scalar conservation is based on a stochastic differential equation driven by an α-stable process and involving a drift nonlinear in the sense of McKean. The particle system is constructed by discretizing this equation in time by the Euler scheme and replacing the nonlinearity by interaction. Each particle carries a signed weight depending on its initial position. At each discretization time we kill the couples of particles with opposite weights and positions closer than a threshold since the contribution of the crossings of such particles has the wrong sign in the derivation of the entropic inequalities. We prove convergence of the particle approximation to the solution of the conservation law as the number of particles tends to ∞ whereas the discretization step, the killing threshold and, in the inviscid case, the coefficient multiplying the stable increments tend to 0 in some precise asymptotics depending on whether α is larger than the critical level 1.

Introduction

We are interested in providing a numerical probabilistic scheme for the fractional scalar conservation law of order α

$$\partial_t v(t, x) + \sigma^\alpha (-\Delta)^{\frac{\alpha}{2}} v(t, x) + \partial_x A(v(t, x)) = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \qquad (0.1)$$

where $-(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian operator of order $0 < \alpha \leq 2$ (defined in Section 2), and A is a function of class $\mathcal{C}^1$ from $\mathbb{R}$ to $\mathbb{R}$. We also consider the equation obtained by letting $\sigma \to 0$ in (0.2), namely the inviscid conservation law

$$\partial_t v(t, x) + \partial_x A(v(t, x)) = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}. \qquad (0.2)$$

In [9, 10], these equations are interpreted as Fokker-Planck equations associated to some stochastic differential equations nonlinear in the sense of McKean, which can be approximated by a particle

system. We introduce an Euler time discretization of this particle system and show the convergence of its empirical cumulative distribution function to the solution of (0.1). We also study its convergence to the solution of (0.2) as the parameter σ goes to 0.

Euler schemes for viscous conservation laws have already been studied in [3], [4], [5] or [6], where a convergence rate of $\frac{1}{\sqrt{N}} + \sqrt{\Delta t}$ is derived in the case $\alpha = 2$, N denoting the number of particles, and Δt being the time step.

To give the probabilistic interpretation to (0.1) we consider the space derivative $u = \partial_x v$ of a solution v to equation (0.1), which formally satisfies

$$\partial_t u_t = -\sigma^\alpha (-\Delta)^{\frac{\alpha}{2}} u_t - \partial_x \left(A'(H * u_t) u_t \right), \tag{0.3}$$

where $H = \mathbf{1}_{[0,\infty)}$ denotes the Heavyside function. When u_0 is a probability measure, that is, when the initial condition v_0 of Equation (0.1) is a cumulative distribution function, Equation (0.3) is the Fokker-Planck equation associated to the following nonlinear stochastic differential equation

$$\begin{cases} \mathrm{d}X_t &= \sigma \mathrm{d}L_t^\alpha + A'(H * u_t(X_t))\mathrm{d}t \\ u_t &= \text{law of } X_t \end{cases},$$

where L_t^α is a Markov process with generator $-(-\Delta)^{\frac{\alpha}{2}}$, namely $\sqrt{2}$ times a Brownian motion for $\alpha = 2$, and a stable Lévy process with index α in the case $\alpha < 2$, that is to say a pure jump Lévy process whose Lévy measure is given by $c_\alpha \frac{\mathrm{d}y}{|y|^{1+\alpha}}$, where c_α is some positive constant.

We can still give a probabilistic interpretation to Equation (0.1) if the initial condition v_0 has bounded variation, is right continuous and not constant. Indeed, in that case v_0 can be written as $v_0(x) = a + \int_{-\infty}^x \mathrm{d}u_0(y) = a + H * u_0(x)$ for some finite measure u_0. By replacing $v_0(x)$ by $(v_0(x) - a)(|u_0|(\mathbb{R}))^{-1}$ and $A(x)$ by $A(a + x|u_0|(\mathbb{R}))(|u_0|(\mathbb{R}))^{-1}$ in (0.1) ($|u_0|$ denoting the total variation of the measure u_0,), one can assume without loss of generality that $a = 0$ and that $|u_0|$ is a probability measure. We denote by $\gamma = \frac{\mathrm{d}u_0}{\mathrm{d}|u_0|}$ the Radon-Nikodym density of u_0 with respect to its total variation. Notice that γ takes values in $\{\pm 1\}$.

Then, Equation (0.3) is the Fokker-Planck equation associated to

$$\begin{cases} \mathrm{d}X_t &= \sigma \mathrm{d}L_t^\alpha + A'(H * \tilde{P}_t(X_t))\mathrm{d}t \\ P &= \text{law of } X \end{cases}, \tag{0.4}$$

where $\tilde{P}$ denotes the measure defined on the Skorokhod space $\mathcal{D}$ of càdlàg functions from $[0,\infty)$ to $\mathbb{R}$ by its Radon-Nikodym density $\frac{\mathrm{d}\tilde{P}}{\mathrm{d}P} = \gamma(f(0))$, with f the canonical process on $\mathcal{D}$, and $\tilde{P}_t$ denotes its time marginal at time t, *i.e* the measure defined by $\tilde{P}_t(B) = \int_{\mathcal{D}} \gamma(f(0))\mathbf{1}_B(f(t))\mathrm{d}P(f)$, for any B in the Borel $\sigma-$field of $\mathbb{R}$.

The rest of the paper is organized as follows:
In Section 1 we define the particle approximation for the stochastic differential equation (0.4).
Section 2 is devoted to the definition of the different notions of solutions used in the article.
In Section 3, we analyze the convergence of the time-discretized particle system to the solution of the conservation law in different settings : for both a constant or vanishing diffusion coefficient and any value of $0 < \alpha \le 2$.
Finally, we present some numerical simulations in Section 4. Those simulations are compared with the results of a deterministic method described in [7].

In the following, the letter K denotes some positive constant whose value can change from line to line.

1 The particle approximation

In this section we construct a discretization of (0.4) consisting of both a particle approximation in order to approximate the law of the solution and an Euler discretization to make the particles evolve in time. The idea is to introduce N particles $X^{N,1}, \ldots, X^{N,N}$ which are N interacting copies of the stochastic differential equation (0.4), where the actual law P of the process is replaced by the empirical distribution of the particles $\frac{1}{N} \sum_{i=1}^{N} \delta_{X^{N,i}}$.

In continuous time, those particles are driven by N independent Brownian motions or stable Lévy processes with index α and undergo a drift given by $A'(H * \tilde{\mu}_t^N(.))$, with $\tilde{\mu}_t^N = \frac{1}{N} \sum_{i=1}^{N} \gamma(X_0^{N,i}) \delta_{X_t^{N,i}}$. The natural way to introduce the measure $\tilde{\mu}_t^N$ in the dynamics is to give each particle a signed weight equal to the evaluation of γ at the initial position of the particle. Then, $H * \tilde{\mu}_t^N(x)$ is simply given by the sum of weights of particles situated left from x.

The entropy solution to (0.1) has a non-increasing total variation (see [2]), which can be interpreted probabilistically as a compensation of merging sample paths having opposite signs. For a more precise statement in the case $\alpha = 2$, see Lemma 2.1 in [9]. It is thus natural to adapt this behavior in our particle approximation by killing any merging couple of particles with opposite signs.

In [9] Jourdain proves, for $\alpha = 2$ in continuous time, the convergence of the particle system to the solution of the nonlinear stochastic differential equation through a propagation-of-chaos result. Moreover, the convergence of the signed cumulative distribution function $H * \tilde{\mu}_t^N$ to the solution to Equation (0.1) is also proved, as well as convergence to the solution to the inviscid equation as $\sigma \to 0$. In [10] the same results are generalized to the case $1 < \alpha < 2$, assuming $\gamma = 1$ in the case of a vanishig viscosity. However, to our knowledge there is even no existence result for the particle system in continuous time when $\alpha \leq 1$, since the driving Lévy process is somehow weaker than the nonlinear drift.

In discrete time, the probability of seeing two particles actually merging is 0. To adapt the murders from the continuous time setting, we thus kill, at each time step, any couple of particles with opposite signs separated by a distance smaller than a given threshold ε_N going to zero as N goes to ∞. Though, one has to be careful, since one can have more than two particles lying in a small interval of length ε_N. Precisely, the particles are killed in the following way: kill the leftmost couple of particles at consecutive positions separated by a distance smaller than the threshold ε_N and with opposite signs. Then, recursively apply the same algorithm to the remaining particles. This can be done with a computational cost of order $\mathcal{O}(N)$. The essential properties satisfied by this killing procedure are the following:

- to each killed particle is attached another killed particle, which has opposite signs and lies at a distance at most ε_N of the first particle.

- after the killing there is no couple of particles with opposite signs in a distance smaller than ε_N.

- the exchangeability of the particles is preserved.

- after the murder, the quantity $H * \tilde{\mu}_t^N(X_t^{N,i})$ remains the same for any surviving particle.

We are going to describe the killed processes by a couple (f, κ) in the space $\mathcal{K} = \mathcal{D} \times [0, \infty]$ of càdlàg functions f from $[0, \infty)$ to $\mathbb{R}$ endowed with a death time $\kappa \in [0, \infty]$. The space $\mathcal{K}$ is endowed with the product metric $d((f, \kappa_f), (g, \kappa_g)) = d_S(f, g) + |\arctan(\kappa_f) - \arctan(\kappa_g)|$, where d_S is the Skorokhod metric on $\mathcal{D}$, so that $(\mathcal{K}, d)$ is a complete metric space. It could seem more natural to consider the space $\mathcal{D}([0, \infty), \mathbb{R} \cup \{\partial\})$ of paths taking values in $\mathbb{R}$ endowed with a cemetery point ∂. However the corresponding topology is too strong to prove Proposition 3.4.

The precise description of the process is the following: each particle will be represented by a couple $(X^{N,i}, \kappa_i^N) \in \mathcal{K}$. Let $(X_0^i)_{i \in \mathbb{N}}$ be a sequence of independent random variables with common distribution $|u_0|$ and let $h_N > 0$ denote the time step of the Euler scheme. At time 0, kill the particles according to the preceding rules, that is to say, set $\kappa_i^N = 0$ for killed particles, which will

not move anymore. Those particles will not be taken into account anymore. Now, by induction, suppose that the particle system has been defined up to time kh_N, and kill the particles according to the preceding rules (*i.e.* set $\kappa_i^N = kh_N$ and $X_t^{N,i} = X_{kh_N}^{N,i}$ for all $t \geq kh_N$, if the particle with index i is one of those). Then let the particles still alive evolve up to time $(k+1)h_N$ according to

$$\mathrm{d}X_t^{N,i} = A'\left(\frac{1}{N}\sum_{\kappa_j^N > kh_N}\gamma(X_0^j)\mathbf{1}_{X_{kh_N}^{N,j} \leq X_{kh_N}^{N,i}}\right)\mathrm{d}t + \sigma_N\mathrm{d}L_t^i,$$

where $(L^i)_{i\in\mathbb{N}}$ is a sequence of independent α-stable Lévy processes for $\alpha < 2$, or a sequence of independent copies of $\sqrt{2}$ times Brownian motion, which are independent of the sequence $(X_0^i)_{i\in\mathbb{N}}$. The particle system is thus well-defined, by induction.

Let $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{(X^{N,i},\kappa_i^N)} \in \mathcal{P}(\mathcal{K})$ be the empirical distribution of the particles. For a probability measure Q on $\mathcal{K}$ and $t \geq 0$, we define a signed measure $\tilde{Q}_t$ on $\mathbb{R}$ by:

$$\tilde{Q}_t(B) = \int_{\mathcal{K}}\mathbf{1}_B(f(t))\mathbf{1}_{\kappa > t}\gamma(f(0))\mathrm{d}Q(f,\kappa),$$

for any B in the Borel σ−field of $\mathbb{R}$. With these notations, on the interval $[kh_N,(k+1)h_N)$, a particle, provided it is still alive, satisfies

$$\mathrm{d}X_t^{N,i} = A'\left(H * \tilde{\mu}_{kh_N}^N\left(X_{kh_N}^{N,i}\right)\right)\mathrm{d}t + \sigma_N\mathrm{d}L_t^i.$$

Notice that the sum of the weights of alive particles $\tilde{\mu}_t^N(\mathbb{R}) = \frac{1}{N}\sum_{\kappa_i^N > t}\gamma(X_0^i)$ is constant in time, since the particles are killed by couples of opposite signs.

2   Notion of solutions

In this section, we recall the different notions of solutions that are associated to the equations (0.1) and (0.2). Indeed, due to the shock-creating term $\partial_x(A(u_t))$, the notion of weak solution is too weak, and does not provide uniqueness when the diffusion term is not regularizing enough. The best suited notion in those cases is the notion of entropy solution.

In [11], Kruzhkov shows that for $v_0 \in \mathbb{L}^\infty((0,\infty))$ existence and uniqueness hold for entropy solutions to (0.2), defined as functions $v \in \mathbb{L}^\infty((0,\infty) \times \mathbb{R})$ satisfying, for any smooth convex function η, any nonnegative smooth function g with compact support on $[0,\infty) \times \mathbb{R}$ and any ψ satisfying $\psi' = \eta'A'$, the entropic inequality

$$\int_{\mathbb{R}}\eta(v_0)g_0 + \int_0^\infty\left(\int_{\mathbb{R}}\eta(v_t)\partial_t g_t + \psi(v_t)\partial_x g_t\right)\mathrm{d}t \geq 0. \tag{2.1}$$

It is well known that this entropy solution can be obtained as the limit of weak solutions to (0.1) as $\sigma \to 0$ in the case $\alpha = 2$.

Weak solutions to (0.1) (see [9]) are defined as functions $v \in \mathbb{L}^\infty((0,\infty) \times \mathbb{R})$ satisfying, for all smooth functions g with compact support in $[0,\infty) \times \mathbb{R}$,

$$\int_{\mathbb{R}}v_0 g_0 + \int_0^\infty\int_{\mathbb{R}}v_t\partial_t g_t\mathrm{d}t - \sigma^\alpha\int_0^\infty\int_{\mathbb{R}}v_t(-\Delta)^{\frac{\alpha}{2}}g_t\mathrm{d}t + \int_0^\infty\int_{\mathbb{R}}A(v_t)\partial_x g_t\mathrm{d}t = 0. \tag{2.2}$$

For $\alpha < 2$, we denote by $(-\Delta)^{\frac{\alpha}{2}}$ the fractional symmetric differential operator of order α, that can be defined through the Fourier transform:

$$\widehat{(-\Delta)^{\frac{\alpha}{2}}u}(\xi) = |\xi|^\alpha\hat{u}(\xi).$$

An equivalent definition for $(-\Delta)^{\frac{\alpha}{2}}$ uses an integral representation

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = c_\alpha \int_{\mathbb{R}} \frac{u(x+y) - u(x) - \mathbf{1}_{|y| \leq r} u'(x) y}{|y|^{1+\alpha}} \mathrm{d}y$$

for any $r \in (0, \infty)$ and some fixed constant c_α (see [8]), depending on the definition of the Fourier transform.

It has been proven in [9] and [10] that existence and uniqueness holds for weak solutions of (0.1), for $1 < \alpha \leq 2$. However, for $0 < \alpha \leq 1$, the diffusive term of order α in (0.1) is somehow dominated by the shock-creating term, which is of order 1, so that a weak formulation does not ensure uniqueness for the solution. We thus have to strengthen the notion of solution, and use entropy solutions to (0.1), defined in [2] as functions v in $\mathbb{L}^\infty((0, \infty) \times \mathbb{R})$ satisfying the relation

$$
\begin{aligned}
&\int_0^\infty \eta(v_0) g_0 + \int_0^\infty \int_{\mathbb{R}} (\eta(v_t) \partial_t g_t + \psi_t(v_t) \partial_x g_t) \, \mathrm{d}t \\
&+ c_\alpha \int_0^\infty \int_{\mathbb{R}} \int_{\{|y| > r\}} \eta'(v_t(x)) \frac{v_t(x + \sigma y) - v_t(x)}{|y|^{1+\alpha}} g_t(x) \mathrm{d}y \mathrm{d}x \mathrm{d}t \\
&+ c_\alpha \int_0^\infty \int_{\mathbb{R}} \int_{\{|y| \leq r\}} \eta(v_t(x)) \frac{g_t(x + \sigma y) - g_t(x) - \sigma y \partial_x g_t(x)}{|y|^{1+\alpha}} \mathrm{d}y \mathrm{d}x \mathrm{d}t \geq 0
\end{aligned}
\tag{2.3}
$$

for any $r > 0$, any nonnegative smooth function g with compact support in $[0, \infty) \times \mathbb{R}$, any smooth convex function $\eta : \mathbb{R} \to \mathbb{R}$ and any ψ satisfying $\psi' = \eta' A'$. Notice that from the convexity of η, the entropic formulation (2.3) for a parameter r implies the entropic formulation with parameter $r' > r$. Also notice, using the functions $\eta(x) = \pm x$ that an entropy solution to (0.1) is a weak solution to (0.1).

In [2], Alibaud shows that existence and uniqueness hold for entropy solutions of (0.1) provided that the initial condition v_0 lies in $\mathbb{L}^\infty(\mathbb{R})$. The entropy solution then lies in the space $\mathcal{C}([0, \infty), \mathbb{L}^1(\frac{\mathrm{d}x}{1+x^2}))$. He also proves that the entropy solution to (0.1) converges to the entropy solution to (0.2) in the space $\mathcal{C}([0, T], \mathbb{L}^1_{\mathrm{loc}}(\mathbb{R}))$ as $\sigma \to 0$.

3   Statement of the results

The aim of this article is to prove the three following convergence result, each one corresponding to a particular setting.

Theorem 3.1. *Assume $0 < \alpha \leq 1$. Let $\sigma_N \equiv \sigma$ be a constant sequence. Let ε_N and h_N be two vanishing sequences satisfying the inequalities*

$$N^{-\lambda} \leq 4 \sup_{[-1,1]} |A'| h_N \leq \varepsilon_N, \quad \text{and} \quad N^{-1/\alpha} \leq N^{-1/\lambda} \varepsilon_N$$

for some positive λ. For $\alpha = 1$, also assume $h_N \leq \varepsilon_N N^{-1/\lambda}$. It holds for any $T > 0$,

$$\lim_{N \to \infty} \int_0^T \mathbb{E} \left\| H * \tilde{\mu}_t^N - v_t \right\|_{\mathbb{L}^1\left(\frac{\mathrm{d}x}{1+x^2}\right)} \mathrm{d}t = 0,$$

where v_t denotes the entropy solution to the fractional conservation law (0.1).

Theorem 3.2. *Let ε_N, h_N and σ_N be three vanishing sequences such that*

$$N^{-\lambda} \leq 4 \sup_{[-1,1]} |A'| h_N \leq \varepsilon_N$$

for some $\lambda > 0$. If $\alpha > 1$, also assume $\sigma_N \leq \varepsilon_N^{1-\frac{1}{\alpha}} N^{-\frac{1}{\lambda}}$. Then, for any $T > 0$,

$$\lim_{N \to \infty} \int_0^T \mathbb{E} \left\| H * \tilde{\mu}_t^N - v_t \right\|_{\mathbb{L}^1\left(\frac{dx}{1+x^2}\right)} dt = 0,$$

where v_t denotes the entropy solution to the inviscid conservation law (0.2).

The additional assumption for $\alpha > 1$ comes from the fact that in this case, the dominant term is the diffusion, while in the limit there is no diffusion anymore. The assumption ensures that the diffusion is weak enough not to perturb the approximation. For $\alpha \leq 1$, the dominant term is the drift, as in the limit, so that no additional condition is needed.

Theorem 3.3. *Assume $1 < \alpha \leq 2$. Let $\sigma_N \equiv \sigma$ be a constant sequence, and let ε_N and h_N be two vanishing sequences. It holds for any $T > 0$,*

$$\lim_{N \to \infty} \int_0^T \mathbb{E} \left\| H * \tilde{\mu}_t^N - v_t \right\|_{\mathbb{L}^1\left(\frac{dx}{1+x^2}\right)} dt = 0,$$

where v_t denotes the weak solution to the fractional conservation law (0.1).

In order to prove those three theorems, we will have to control the probability of seeing particles merging. In the case $\alpha < 2$, this is mainly due to the conjunction of the small jumps of the stable process and the drift coefficient, while the large jumps of the stable term do not play an essential role. As a consequence, for $\alpha < 2$, we consider another family of evolutions coinciding with the Euler scheme on the time discretization grid, for which we consider differently the jumps which are smaller or larger than a given threshold r. The choice of this parameter has to be linked to the parameter r appearing in the entropic formulation (2.3), since they play a similar role: the third term in (2.3) corresponds to the effect of jumps larger than r in the driving Lévy process and the fourth term corresponds to jumps smaller than r. This evolution is designed so that on the first half of each time step, the process will evolve according to the drift and the small jumps, and on the second half of each time step, it will evolve according to the large jumps. More precisely, let

$$\nu^i(dy, dt) = \sum_{\Delta L_t^i \neq 0} \delta_{(\Delta L_t^i, t)}$$

be the jump measure associated to the Lévy process L^i and let

$$\tilde{\nu}^i(dy, dt) = \nu^i(dy, dt) - c_\alpha \frac{dy dt}{|y|^{1+\alpha}}$$

be the corresponding compensated measure, so that

$$L_t^i = \int_{(0,t] \times \{|y| > r\}} y \nu^i(dy, dt) + \int_{(0,t] \times \{|y| \leq r\}} y \tilde{\nu}^i(dy, dt),$$

where the right hand side does not depend on r. We define the process $X^{N,i,r}$ by

$$X^{N,i,r} = X_0^i + \sigma_N L^{N,i,r} + \sigma_N \Lambda^{N,i,r} + \mathcal{A}^{N,i},$$

where

- $L_t^{N,i,r}$ is the large jumps part defined by

$$L_t^{N,i,r} = \int_{(0,a(t)] \times \{|y| > r\}} y \nu^i(dy, ds),$$

6

where $a(t) = \begin{cases} kh_N & \text{for } t \in [kh_N, (k+1/2)h_N] \\ kh_N + 2(t - (k+1/2)h_N) & \text{for } t \in [(k+1/2)h_N, (k+1)h_N] \end{cases}$. This process is

constant on intervals $[kh_N, (k+1/2)h_N]$ and behaves like a Lévy process with jump measure $\mathbf{1}_{|y|>r} \frac{2c_\alpha \, \mathrm{d}y}{|y|^{1+\alpha}}$ on intervals $[(k+1/2)h_N, (k+1)h_N]$.

- $\Lambda_t^{N,i,r}$ is the small jumps part, defined by

$$\Lambda_t^{N,i,r} = \int_{(0,b(t)] \times \{|y| \leq r\}} \tilde{\nu}^i(\mathrm{d}y, \mathrm{d}s),$$

where $b(t) = \begin{cases} kh_N + 2(t - kh_N) & \text{for } t \in [kh_N, (k+1/2)h_N] \\ (k+1)h_N & \text{for } t \in [(k+1/2)h_N, (k+1)h_N] \end{cases}$. This term behaves like

a Lévy process with jump measure $\mathbf{1}_{|y| \leq r} \frac{2c_\alpha \, \mathrm{d}y}{|y|^{1+\alpha}}$ on intervals $[kh_N, (k+1/2)h_N]$ and is constant on intervals $[(k+1/2)h_N, (k+1)h_N]$. Notice that the process $\Lambda^{N,i,r}$ is a martingale.

- $\mathcal{A}^{N,i}$ is the drift part, which satisfies $\mathcal{A}_0^{N,i} = 0$, is constant over each interval $[(k+1/2)h_N, (k+1)h_N]$, and evolves as a piecewise affine process with derivative $2A'(H * \tilde{\mu}_{kh_N}^N(X_{kh_N}^{N,i}))$ on intervals $[kh_N, (k+1/2)h_N]$.

One can check that for any r, the process $(X^{N,1,r}, \ldots, X^{N,N,r})$ is equal to $(X^{N,1}, \ldots, X^{N,N})$ on the time discretization grid up to killing time. Conditionally on the positions of the particles at time kh_N, the particles evolve independently on $[kh_N, (k+1)h_N]$, and the evolution on $[kh_N, (k+1/2)h_N]$ is independent of the evolution on $[(k+1/2)h_N, (k+1)h_N]$. Since the entropic formulation (2.3) with parameter r is stronger than the one with parameter $r' \geq r$, we have to make the parameter r tend to zero in order to prove the entropic formulation for any parameter. However, this convergence has to satisfy some conditions with respect to N, h_N and ε_N. We will explain later why a suitable sequence r_N exists under the conditions given in the statement of Theorem 3.1.

In order to prove Theorems 3.1 and 3.2, we introduce $\mu^{N,r}$ the empirical distribution of the processes $(X^{N,i,r}, \kappa_i^N)$:

$$\mu^{N,r} = \frac{1}{N} \sum_{i=1}^{N} \delta_{(X^{N,i,r}, \kappa_i^N)} \in \mathcal{P}(\mathcal{K}),$$

and by $\pi^{N,r}$ the law of $\mu^{N,r}$.

The following proposition is the first step in the proof of Theorems 3.1, 3.2 and 3.3.

Proposition 3.4.
- *Assume $\alpha < 2$. For any bounded sequences (h_N), (σ_N) and (ε_N), and for any sequence (r_N), the family of probability measures $(\pi^{N,r_N})_{N \in \mathbb{N}}$ is tight in $\mathcal{P}(\mathcal{P}(\mathcal{K}))$.*

- *Denote by π^N the law of μ^N. For any bounded sequences (h_N), (σ_N) and (ε_N), the family of probability measures $(\pi^N)_{N \in \mathbb{N}}$ is tight in $\mathcal{P}(\mathcal{P}(\mathcal{K}))$.*

Proof. We first check the tightness of the family $(\pi^{N,r_N})_{N \in \mathbb{N}}$.

As stated in [13], checking the tightness of the sequence π^{N,r_N} boils down to checking the tightness of the sequence $(\text{Law}(X^{N,1,r_N}, \kappa_1^N))$. Owing to the product-space structure, we can check tightness for $X^{N,1,r_N}$ and κ_1^N separately.

Of course, tightness for κ_1^N is straightforward since it lies on the compact space $[0, \infty]$, and it is enough to check tightness for the laws of the path $(X^{N,1,r_N})$. For simplicity, we will assume that $A = 0$, which is not restrictive since A' is a bounded function so that the perturbation induced by A belongs to a compact subset of the space of continuous functions, from Ascoli's theorem (also notice that the addition functional from $\mathcal{D} \times \mathcal{C}([0, \infty))$ to $\mathcal{D}$ is continuous). We use Aldous' criterion to prove tightness (see [1]). First, the sequences $(X_0^{N,1,r_N})_{N \in \mathbb{N}}$ and $(\sup_{[0,T]} |\Delta X^{N,1,r_N}|)_{N \in \mathbb{N}}$ are

tight, since $(X_0^{N,1,r_N})$ is constant in law and $\left(\sup_{[0,T]}|\Delta X^{N,1,r_N}|\right)_N$ is dominated by the identically distributed sequence

$$\left(\left(\sup_N \sigma_N\right)\sup_{[0,T+\sup_N h_N]}|\Delta L^1|\right)_N.$$

Then let τ_N be a stopping time of the natural filtration of $X^{N,1,r_N}$ taking finitely many values, and let $(\delta_N)_{N\in\mathbb{N}}$ be a sequence of positive numbers going to 0 as $N\to\infty$. One can write

$$
\begin{aligned}
\mathbb{P}\left(\left|X_{\tau_N+\delta_N}^{N,1,r_N}-X_{\tau_N}^{N,1,r_N}\right|\geq\varepsilon\right)\leq{}&\mathbb{P}\left(\sigma_N\left|\Lambda_{\tau_N+\delta_N}^{N,1,r_N}-\Lambda_{\tau_N}^{N,1,r_N}\right|\geq\varepsilon/2\right)\\
&+\mathbb{P}\left(\sigma_N\left|L_{\tau_N+\delta_N}^{N,1,r_N}-L_{\tau_N}^{N,1,r_N}\right|\geq\varepsilon/2\right)\\
\leq{}&\mathbb{P}\left(\sup_{t\in[0,\delta_N]}\sigma_N|\mathcal{L}_t^{\leq r_N}|\geq\varepsilon/2\right)+\mathbb{P}\left(\sup_{t\in[0,\delta_N]}\sigma_N|\mathcal{L}_t^{>r_N}|\geq\varepsilon/2\right),
\end{aligned}
\tag{3.1}
$$

where

$$\mathcal{L}_t^{\leq r}=\int_{(0,t]\times\{|y|\leq r\}}y\tilde{\nu}(\mathrm{d}y,\mathrm{d}t)\quad\text{and}\quad\mathcal{L}_t^{>r}=\int_{(0,t]\times\{|y|>r\}}y\nu(\mathrm{d}y,\mathrm{d}t),$$

the measure ν being the jump measure of some Lévy process $\mathcal{L}$ with Lévy measure $\frac{2c_\alpha\,\mathrm{d}y}{|y|^{1+\alpha}}$, and $\tilde{\nu}$ is the compensated measure of ν. Now, using the maximal inequality for the martingale $(\mathcal{L}_t^{\leq r_N})_{t\in[0,\delta_N]}$, noticing that $(\mathcal{L}_{\delta_N}^{\leq r})_{r\in[0,1]}$ is also a martingale, we deduce

$$
\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,\delta_N]}|\mathcal{L}_t^{\leq r_N}|\geq\varepsilon/2\sigma_N\right)&\leq\sup_{r\in[0,\sup_N r_N]}\mathbb{P}\left(\sup_{t\in[0,\delta_N]}|\mathcal{L}_t^{\leq r}|\geq\varepsilon/2\sigma_N\right)\\
&\leq 2\sigma_N\varepsilon^{-1}\sup_{r\in[0,\sup_N r_N]}\mathbb{E}\left(|\mathcal{L}_{\delta_N}^{\leq r}|\right)\\
&=2\sigma_N\varepsilon^{-1}\mathbb{E}\left(|\mathcal{L}_{\delta_N}^{\leq\sup_N r_N}|\right)\\
&\xrightarrow[N\to\infty]{}0.
\end{aligned}
$$

For the large jumps parts, one writes,

$$
\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,\delta_N]}|\mathcal{L}_t^{>r_N}|\geq\varepsilon/2\sigma_N\right)&\leq\mathbb{P}\left(\sup_{t\in[0,\delta_N]}|\mathcal{L}_t|+\sup_{t\in[0,\delta_N]}|\mathcal{L}_t^{\leq r_N}|\geq\varepsilon/2\sigma_N\right)\\
&\leq\mathbb{P}\left(\sup_{t\in[0,\delta_N]}|\mathcal{L}_t|\geq\varepsilon/4\sigma_N\right)+\mathbb{P}\left(\sup_{t\in[0,\delta_N]}|\mathcal{L}_t^{\leq r_N}|\geq\varepsilon/4\sigma_N\right)\\
&\xrightarrow[N\to\infty]{}0.
\end{aligned}
$$

As a consequence, the family $(\mathrm{Law}(X^{N,1,r_N}))_{N\in\mathbb{N}}$ is tight in $\mathcal{D}$.

Thus, the family $(\pi^{N,r_N})_{N\in\mathbb{N}}$ is tight.

The proof is essentially the same for the tightness of $(\pi^N)_{N\in\mathbb{N}}$, with a few simplifications, since we do not treat separately large and small jumps. It also adapts in the case $\alpha=2$, since the Gaussian distribution has thinner tails than the $\alpha-$stable distribution for $\alpha<2$. $\qquad\square$

The use of the path space $\mathcal{K}$ instead of $\mathcal{D}([0,\infty),\mathbb{R}\cup\{\partial\})$ for a cemetery point ∂ is crucial in the proof of Proposition 3.4, since in the latter case, we need to control the jumps occuring close to the death time in order to prove tightness. The following example is illustrative: if we consider a

sequence f_n of paths starting at 0, jumping to 1 at time $1 - 1/n$, and being killed at time 1, then f_n does not converge in $\mathcal{D}([0, \infty), \mathbb{R} \cup \{\partial\})$, while it does in $\mathcal{K}$.

The following lemma deals with the initial condition of the particle system.

Lemma 3.5. *If π^∞ is the limit of some subsequence of π^N or π^{N,r_N}, then for π^∞-almost all Q, for all A in the Borel σ−field of $\mathbb{R}$,*

$$Q_0(A) := \int_\mathbb{R} \mathbf{1}_{\kappa > 0} \mathbf{1}_{f(0) \in A} \mathrm{d}Q(f, \kappa) = |u_0|(A). \tag{3.2}$$

In particular, κ is Q−almost surely positive for π^∞-almost all Q.

Proof. In a first time, we control the probability of seeing a partincle dying within a short time.

Let us write the Hahn decomposition $u_0^+ - u_0^-$ of the measure u_0, the measures u_0^+ and u_0^- being positive measures supported by two disjoint sets B^+ and B^-. From the inner regularity of the measure u_0^+, for any $\delta > 0$, one can find a closed set $F^+ \subset B^+$ such that $u_0^+(F^+) \geq u_0^+(B^+) - \delta$. The complement set $O^- = (F^+)^c$ is then an open subset of $\mathbb{R}$, which can thus be decomposed as a countable union of disjoint open intervals $O^- = \bigcup_{m=1}^\infty]a_m, b_m[$. For a large enough integer M, and for $\varepsilon_\delta > 0$ small enough, the set $O^\delta = \bigcup_{m=1}^M]a_m + \varepsilon_\delta, b_m - \varepsilon_\delta[$ is such that $u_0^-(O^\delta) \geq u_0^-(O^-) - \delta$. Consequently, we can write $\mathbb{R}$ as a partition

$$\mathbb{R} = F^+ \cup (B^- \cap O^\delta) \cup \mathcal{B}^\delta,$$

where $\mathcal{B}^\delta = (F^+ \cup (B^- \cap O^\delta))^c$ has small measure $|u_0|(\mathcal{B}^\delta) \leq 2\delta$, particles starting in F^+ have a positive sign, and particles starting in $(B^- \cap O^\delta)$ have a negative sign. Let N be large enough to ensure $\varepsilon_N \leq \varepsilon_\delta/3$. The distance between any element of F^+ and any element of O^δ is larger than ε_δ. As a consequence, if the particles with index i and j kill each other before time τ, then either one of them started in $\mathcal{B}^\delta$, or one of the particles i and j moved by a distance larger than $\varepsilon_\delta/3$. This writes

$$\sharp\left\{i, \kappa_i^N < \tau_\delta\right\} = 2\sharp\left\{(i, j), i < j, X^{N,i,r_N} \text{ and } X^{N,j,r_N} \text{ kill each other}\right\}$$

$$\leq 2\sharp\left\{i, X_0^{N,i,r_N} \in \mathcal{B}^\delta \text{ or } \sup_{t \in [0,\tau]} |X_t^{N,i,r_N} - X_0^i| \geq \varepsilon_\delta/3\right\}.$$

As a consequence, if $\tau_\delta > 0$ is small enough so that $\mathbb{P}(\sup_{t \in [0,\tau_\delta]} |X_t^{N,i,r_N} - X_0^{N,i,r_N}| \geq \varepsilon_\delta/3) \leq \delta$ (this can be achieved using an adaptation of (3.1)), it holds

$$\mathbb{P}(\kappa_1^N < \tau_\delta) = \frac{1}{N}\mathbb{E}\left(\sharp\left\{i, \kappa_i^N < \tau_\delta\right\}\right) \leq \frac{2}{N}\mathbb{E}\left(\sharp\left\{i, X_0^{N,i,r_N} \in \mathcal{B}^\delta \text{ or } \sup_{t \in [0,\tau_\delta]} |X_t^{N,i,r_N} - X_0^i| \geq \varepsilon_\delta/3\right\}\right)$$

$$\leq 2\mathbb{P}\left(X_0^i \in \mathcal{B}^\delta\right) + 2\mathbb{P}\left(\sup_{t \in [0,\tau_\delta]} |X_t^{N,i,r_N} - X_0^{N,i,r_N}| \geq \varepsilon_\delta/3\right)$$

$$\leq 6\delta.$$

Consequently,

$$\mathbb{E}^{\pi^\infty}(Q(\kappa < \tau)) \leq \liminf_N \mathbb{E}^{\pi^N}(Q(\kappa < \tau)) = \liminf_N \mathbb{P}(\kappa_1^N < \tau) \underset{\tau \to 0}{\to} 0.$$

Thus for π^∞-almost all Q, κ is Q−almost surely positive. As a consequence, for any bounded continuous function φ,

$$\mathbb{E}^{\pi^\infty}\left|\int_\mathcal{K} \mathbf{1}_{\kappa > 0}\varphi(f(0))\mathrm{d}Q(\kappa, f) - \int_\mathbb{R} \varphi \mathrm{d}|u_0|\right| = \mathbb{E}^{\pi^\infty}\left|\int_\mathcal{K} \varphi(f(0))\mathrm{d}Q(\kappa, f) - \int_\mathbb{R} \varphi \mathrm{d}|u_0|\right|$$

$$= \lim_N \mathbb{E}^{\pi^N}\left|\int_\mathcal{K} \varphi(f(0))\mathrm{d}Q(\kappa, f) - \int_\mathbb{R} \varphi \mathrm{d}|u_0|\right| = 0,$$

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from the law of large numbers.

$\square$

The main step in the proof of Theorems 3.1, 3.2 and 3.3 is the following proposition:

Proposition 3.6. *Let ε_N and h_N be vanishing sequences.*

- *If σ_N is a constant sequence and $0 < \alpha \leq 1$, suppose $N^{-1/\alpha} \leq N^{-1/\lambda}\varepsilon_N$ and $N^{-\lambda} \leq 4\sup_{[-1,1]}|A'|h_N \leq \varepsilon_N$ for some positive λ. If $\alpha = 1$, also assume $h_N \leq N^{-1/\lambda}\varepsilon_N$. Then, there exists a sequence (r_N) of positive real numbers, such that the limit of any converging subsequence of π^{N,r_N} gives full measure to the set*

$$\{Q \in \mathcal{P}(\mathcal{K}), H * \tilde{Q}_t(x) \text{ is the entropy solution to } (0.1)\}.$$

- *Let σ_N be a vanishing sequence and assume $N^{-\lambda} \leq 4\sup_{[-1,1]}|A'|h_N \leq \varepsilon_N$ for some positive λ. If $1 < \alpha \leq 2$, also assume $\sigma_N \leq \varepsilon_N^{1-\frac{1}{\alpha}}N^{-1/\lambda}$. Then*

$$\{Q \in \mathcal{P}(\mathcal{K}), H * \tilde{Q}_t(x) \text{ is the entropy solution to } (0.2)\}$$

is given full measure by any limit of a converging subsequence of π^{N,r_N}, for a well chosen sequence (r_N), in the case $\alpha < 2$, and by any limit of a converging subsequence of π^N if $\alpha = 2$.

- *If σ_N is a constant sequence and $1 < \alpha \leq 2$, the limit of any converging subsequence of π^N gives full measure to the set*

$$\{Q \in \mathcal{P}(\mathcal{K}), H * \tilde{Q}_t(x) \text{ is the weak solution to } (0.1)\}.$$

Proposition 3.6 will be proved in Section 3.1. We first admit it to end the proofs of Theorems 3.1, 3.2 and 3.3. We need the following lemma.

Lemma 3.7. *Let $\alpha < 2$ and r_N be a sequence of positive numbers going to zero. Then it holds, for any $T > 0$,*

$$\lim_{N\to\infty} \int_0^T \mathbb{E}\|H * \tilde{\mu}_t^N - H * \tilde{\mu}_t^{N,r_N}\|_{\mathbb{L}^1\left(\frac{\mathrm{d}x}{1+x^2}\right)}\mathrm{d}t = 0.$$

Proof. It holds, by exchangeability of the particles,

$$\int_0^T \mathbb{E}\|H * \tilde{\mu}_t^N - H * \tilde{\mu}_t^{N,r_N}\|_{\mathbb{L}^1\left(\frac{\mathrm{d}x}{1+x^2}\right)}\mathrm{d}t \leq \mathbb{E}\int_0^T \int_{\mathbb{R}} \frac{1}{N}\sum_{\kappa_i^N>t}\left|\mathbf{1}_{X_t^{N,i}\leq x} - \mathbf{1}_{X_t^{N,i,r_N}\leq x}\right|\frac{\mathrm{d}x\mathrm{d}t}{x^2+1}$$

$$\leq \int_0^T \mathbb{E}\left(\mathbf{1}_{\kappa_1^N\geq t}\left|X_t^{N,1} - X_t^{N,1,r_N}\right| \wedge \pi\right)\mathrm{d}t.$$

This last quantity goes to zero, since the processes $X^{N,1}$ and $X^{N,1,r_N}$ coincide on the discretization grid, whose mesh vanishes. Indeed, for $t \in [kh_N, (k+1)h_N)$

$$\mathbb{E}\left(\mathbf{1}_{\kappa_1^N>t}|X_t^{N,1,r_N} - X_t^{N,1}| \wedge \pi\right) \leq \mathbb{E}\left(\mathbf{1}_{\kappa_1^N>t}|X_t^{N,1,r_N} - X_{kh_N}^{N,1}| \wedge \pi\right) + \mathbb{E}\left(\mathbf{1}_{\kappa_1^N>t}|X_t^{N,1} - X_{kh_N}^{N,r_N}| \wedge \pi\right)$$

$$\leq Kh_N^{1/2}. \tag{3.3}$$

For this last estimate, we used, for an $\alpha-$stable Lévy process L, the inequality

$$\mathbb{E}\left(|L_t| \wedge 1\right) \leq K\mathbb{E}\left(|L_t|^{\alpha/2}\right) = Kt^{1/2}.$$

$\square$

From Lemma 3.7, it is sufficient to show $\lim_{N\to\infty} \int_0^T \mathbb{E}\|H * \tilde{\mu}_t^{N,r_N} - v_t\|_{\mathbb{L}^1(\frac{\mathrm{d}x}{1+x^2})}\mathrm{d}t$ in order to prove Theorems 3.1 and 3.2.

Proof of Theorems 3.1-3.2-3.3. We write the proof for Theorems 3.1 and 3.2 in the case $\alpha < 2$. The proof of Theorem 3.2 with $\alpha = 2$ and Theorem 3.3 is the same, with π^N replacing π^{N,r_N}.

Let γ^k be a Lipschitz continuous approximations of γ, with $\mathbb{P}(\gamma(X_0^1) \neq \gamma^k(X_0^1)) \leq 1/k$ (see [9], Lemma 2.5, for a construction of such a γ^k). We have, by exchangeability of the particles,

$$
\begin{aligned}
&\mathbb{E}\int_0^T \int_{\mathbb{R}} \left| H * \tilde{\mu}_t^{N,r_N}(x) - v_t(x) \right| \frac{\mathrm{d}x\mathrm{d}t}{x^2+1} \\
&\leq \mathbb{E}\int_0^T \int_{\mathbb{R}} \mathbf{1}_{\kappa_1^N > t} H(x - X_t^{N,1,r_N}) \left| \gamma(X_0^{N,1,r_N}) - \gamma^k(X_0^{N,1,r_N}) \right| \frac{\mathrm{d}x\mathrm{d}t}{x^2+1} \\
&\quad + \mathbb{E}^{\pi^N}\left(\int_0^\infty \int_{\mathbb{R}} \left| \int_{\mathcal{K}} \mathbf{1}_{\kappa > t} H(x - f(t))\gamma^k(f(0))\mathrm{d}Q(f,\kappa) - v_t(x) \right| \frac{\mathrm{d}x}{x^2+1} \right).
\end{aligned}
$$
(3.4)

From the assumption on γ^k, the first term in the right hand side of (3.4) is smaller than $2\pi/k$ which vanishes as k goes to ∞. The bounded function

$$
Q \mapsto \int_0^T \int_{\mathbb{R}} \left| \int_{\mathcal{K}} \mathbf{1}_{\kappa > t} H(x - f(t))\gamma^k(f(0))\mathrm{d}Q(f,\kappa) - v_t(x) \right| \frac{\mathrm{d}x\mathrm{d}t}{x^2+1}
$$

is continuous. From Proposition 3.6, the second term in the right hand side of (3.4) converges, as N goes to ∞ to

$$
\mathbb{E}^{\pi^\infty}\left(\int_0^T \int_{\mathbb{R}} \left| \int_{\mathcal{K}} \mathbf{1}_{\kappa > t} H(x - f(t))\left(\gamma^k(f(0)) - \gamma(f(0))\right)\mathrm{d}Q(f,\kappa) \right| \frac{\mathrm{d}x}{x^2+1} \right).
$$

This terms goes to zero as k tend to infinity using the argument of the begining of the proof with $X^{N,1,r_N}$ replaced by the canonical process y.

□

3.1 Proof of Proposition 3.6

This section is devoted to the proof of Proposition 3.6. Since the hardest part of this proof is the first two items, we do not give all details for the third item and for the second one in the case $\alpha = 2$. Indeed, for these two last settings, the separation of small jumps and large jump is not necessary for the proof.

Let r_N be a sequence of positive real numbers, going to zero as $N \to \infty$, which will be explicited later. Let $r > 0$ and c be reals numbers, η a smooth convex function, ψ a primitive of $A'\eta'$ and g a smooth compactly supported nonnegative function. We define the function $\varphi_t(x) = \int_{-\infty}^x g_t(y)\mathrm{d}y$. Note that φ is smooth, and nondecreasing with respect to the space variable. We consider a subsequence of π^{N,r_N}, still denoted π^{N,r_N} for simplicity, which converges to a limit π^∞. We want to prove that, for π^∞−almost all Q, the function $H * \tilde{Q}_t$ satisfies the entropy formulation associated to the corresponding case.

One can write, for any $k \geq 0$ and $t \in]kh_N, (k+1)h_N]$

$$
\mathbb{P}\left(\exists i,j, \kappa_i^N \wedge \kappa_j^N > t, X_t^{N,i,r} = X_t^{N,j,r}\right) = \mathbb{E}\left(\mathbb{P}\left(\exists i,j, \kappa_i^N \wedge \kappa_j^N > t, X_t^{N,i,r} = X_t^{N,j,r}\middle|(X_{kh_N}^{N,q})_q\right)\right)
$$
$$
= \mathbb{E}\left(\mathbb{P}\left(\exists i,j, \kappa_i^N \wedge \kappa_j^N > t, \sigma_N Z_t^{i,j,k,N} = X_{kh_N}^{N,j} - X_{kh_N}^{N,i} + \mathcal{A}_t^{N,j} - \mathcal{A}_t^{N,i}\middle|(X_{kh_N}^{N,q})_q\right)\right),
$$

where we denote

$$Z_t^{i,j,N,k} = \Lambda_t^{N,i,r} - \Lambda_{kh_N}^{N,i,r} - \Lambda_t^{N,j,r} + \Lambda_{kh_N}^{N,j,r} + L_t^{N,i,r} - L_{kh_N}^{N,i,r} - L_t^{N,j,r} + L_{kh_N}^{N,j,r}.$$

From the conditional independence of the processes $L^{N,i,r}$, $L^{N,j,r}$, $\Lambda^{N,i,r}$ and $\Lambda^{N,j,r}$, the random variable $Z_t^{i,j,N,k}$ has a density. As a consequence, since the process $\mathcal{A}_t^{N,j} - \mathcal{A}_t^{N,i}$ is deterministic on $[kh_N, (k+1)h_N]$ conditionally to $(X_{kh_N}^{N,q})_q$, the above probability is zero, meaning that for all time $t > 0$, the alive particles X_t^{N,i,r_N} almost surely have distinct positions. As a consequence, the function $\eta\left(H * \tilde{\mu}_t^{N,r_N}(x)\right)$ is the cumulative distribution function of the signed measure

$$\xi_t^N = \sum_{\kappa_i^N > t} w_t^i \delta_{X_t^{N,i,r_N}},$$

where

$$w_t^i = \mathbf{1}_{\kappa_i^N > t}\left(\eta\left(\frac{1}{N}\sum_{\substack{\kappa_j^N > t \\ X_t^{N,j,r_N} \leq X_t^{N,i,r_N}}} \gamma(X_0^j)\right) - \eta\left(\frac{1}{N}\sum_{\substack{\kappa_j^N > t \\ X_t^{N,j,r_N} < X_t^{N,i,r_N}}} \gamma(X_0^j)\right)\right)$$

$$= \mathbf{1}_{\kappa_i^N > t}\left(\eta\left(H * \tilde{\mu}_t^{N,r_N}\left(X_t^{N,i,r_N}\right)\right) - \eta\left(H * \tilde{\mu}_t^{N,r_N}\left(X_t^{N,i,r_N}-\right)\right)\right).$$

Let $(\zeta_m)_{m\in\mathbb{N}}$ be the increasing sequence of times which are either a jump time for some L^{N,i,r_N} (i.e. a jump of size $> r_N$ for X^{N,i,r_N}) or either a time of the form $kh_N/2$. One has

$$-\left\langle \xi_0^N, \varphi_0 \right\rangle = \sum_{m=1}^{\infty} \left\langle \xi_{\zeta_m}^N, \varphi_{\zeta_m} \right\rangle - \left\langle \xi_{\zeta_{m-1}}^N, \varphi_{\zeta_{m-1}} \right\rangle$$

$$= \sum_{\kappa_i^N > 0} \sum_{m=1}^{\infty} w_{\zeta_{m-1}}^i \left(\varphi_{\zeta_m}\left(X_{\zeta_m-}^{N,i,r_N}\right) - \varphi_{\zeta_{m-1}}\left(X_{\zeta_{m-1}}^{N,i,r_N}\right)\right) \qquad (3.5)$$

$$+ \sum_{\kappa_i^N > 0} \sum_{m=1}^{\infty} \left(w_{\zeta_m}^i \varphi_{\zeta_m}\left(X_{\zeta_m}^{N,i,r_N}\right) - w_{\zeta_{m-1}}^i \varphi_{\zeta_m}\left(X_{\zeta_m-}^{N,i,r_N}\right)\right).$$

Notice that these infinite sums are actually finite, since the function φ_t is identically zero when t is large enough, and since the process $(L^{N,1,r_N}, \ldots, L^{N,N,r_N})$ has a finite number of jumps on bounded intervals.

We consider the first term in the right hand side of (3.5). Denote by $\nu^{i,r} = \sum_{\Delta X_t^{N,i,r} \neq 0} \delta_{(\Delta L_t^{N,i,r} + \Delta \Lambda_t^{N,i,r}, t)}$ the jump measure associated to $L^{N,i,r} + \Lambda^{N,i,r}$, and by

$$\tilde{\nu}^{i,r}(\mathrm{d}y, \mathrm{d}t) = \nu^{i,r}(\mathrm{d}y, \mathrm{d}t) - 2c_\alpha\left(\chi_t^N \mathbf{1}_{|y| \leq r} + (1 - \chi_t^N)\mathbf{1}_{|y| > r}\right)\frac{\mathrm{d}y\mathrm{d}t}{|y|^{1+\alpha}}$$

its compensated measure, where $\chi_t^N = \sum_{k=0}^{\infty} \mathbf{1}_{[kh_N, (k+1/2)h_N)}(t)$. Let us apply Itō's Formula on the interval (ζ_{m-1}, ζ_m). If $\zeta_{m-1} = kh_N$ for some integer k, then $\zeta_m = (k+1/2)h_N$, and almost surely

$X^{N,i,r}_{(k+\frac{1}{2})h_N-} = X^{N,i,r}_{(k+\frac{1}{2})h_N}$ holds. As a consequence

$$\varphi_{(k+\frac{1}{2})h_N}\left(X^{N,i,r}_{(k+\frac{1}{2})h_N-}\right) - \varphi_{kh_N}\left(X^{N,i,r}_{kh_N}\right)$$

$$= \int_{kh_N}^{(k+\frac{1}{2})h_N} \partial_t\varphi_t(X^{N,i,r}_t)\mathrm{d}t + 2\int_{kh_N}^{(k+\frac{1}{2})h_N} \partial_x\varphi_t(X^{N,i,r}_t)A'\left(H*\tilde{\mu}^{N,r}_{kh_N}(X^{N,i,r}_{kh_N})\right)\mathrm{d}t.$$

$$+ \int_{(kh_N,(k+1/2)h_N)} \int_{\{|y|\le r\}} \left(\varphi_t(X^{N,i,r}_{t-}+\sigma_N y) - \varphi_t(X^{N,i,r}_{t-}) - \sigma_N y\partial_x\varphi_t(X^{N,i,r}_{t-})\right)\nu^{i,r}(\mathrm{d}y,\mathrm{d}t)$$

$$+ \sigma_N \int_{(kh_N,(k+1/2)h_N)} \partial_x\varphi_t(X^{N,i,r}_{t-})\left(\int_{\{|y|\le r\}} y\tilde{\nu}^{i,r}(\mathrm{d}y,\mathrm{d}t)\right).$$

If ζ_{m-1} is not of the form kh_N, then the process $X^{N,i,r}$ is constant on the interval $[\zeta_{m-1},\zeta_m)$, and one has $\varphi_{\zeta_m}(X^{N,i,r}_{\zeta_m-}) - \varphi_{\zeta_{m-1}}(X^{N,i,r}_{\zeta_{m-1}}) = \int_{\zeta_{m-1}}^{\zeta_m} \partial_t\varphi_t(X^{N,i,r}_t)\mathrm{d}t$. Summing over all the intervals (ζ_{m-1},ζ_m), Equation (3.5) writes, denoting $\tau_t = \max\{\zeta_m,\zeta_m \le t\}$,

$$-\left\langle \xi^N_0,\varphi_0\right\rangle = \sum_{\kappa^N_i>0} \int_0^\infty w^i_{\tau_t}\left(\partial_t\varphi_t(X^{N,i,r_N}_t) + 2\chi^N_t\partial_x\varphi_t(X^{N,i,r_N}_t)A'\left(H*\tilde{\mu}^{N,r_N}_{\tau_t}(X^{N,i,r_N}_{\tau_t})\right)\right)\mathrm{d}t$$

$$+ c_\alpha\sum_{\kappa^N_i>0} \int_0^\infty w^i_{\tau_t}\chi^N_t\int_{\{|y|\le r_N\}} \left(\varphi_t(X^{N,i,r_N}_t+\sigma_N y) - \varphi_t(X^{N,i,r_N}_t) - \sigma_N y\partial_x\varphi_t(X^{N,i,r_N}_t)\right)\frac{2\mathrm{d}y\mathrm{d}t}{|y|^{1+\alpha}}$$

$$+ \sum_{\kappa^N_i>0}\sum_{\substack{\text{large jump}\\\text{at }\zeta_m}} w^i_{\zeta_m}\varphi_{\zeta_m}(X^{N,i,r_N}_{\zeta_m}) - w^i_{\zeta_{m-1}}\varphi_{\zeta_m}(X^{N,i,r_N}_{\zeta_m-})$$

$$+ \sum_{\kappa^N_i>0}\sum_{\substack{\zeta_m\text{ of the}\\\text{form }kh_N}} (w^i_{\zeta_m}-w^i_{\zeta_{m-1}})\varphi_{\zeta_m}(X^{N,i,r_N}_{\zeta_m}) \qquad (3.6)$$

$$+ \sum_{\kappa^N_i>0}\sum_{\substack{\zeta_m\text{ of the}\\\text{form }(k+1/2)h_N}} (w^i_{\zeta_m}-w^i_{\zeta_{m-1}})\varphi_{\zeta_m}(X^{N,i,r_N}_{\zeta_m})$$

$$+ M_N.$$

Here, the third, fourth and fifth terms correspond to the second term in the right hand side of (3.5), and M_N is a martingale term given by

$$M_N = \sum_{\kappa^N_i>0} \int_0^\infty w^i_{\tau_t}\chi^N_t\int_{\{|y|\le r_N\}} \left(\varphi_t(X^{N,i,r_N}_{t-}+\sigma_N y) - \varphi_t(X^{N,i,r_N}_{t-})\right)\tilde{\nu}^{i,r_N}(\mathrm{d}y,\mathrm{d}t).$$

Equation (3.6) can be rewritten

$$T^1_N = T^2_N + T^3_N + T^4_N + T^5_N + M_N,$$

where $T^1_N = -\left\langle \xi^N_0,\varphi_0\right\rangle$, T^2_N is the sum of the two first terms in the right-hand-side of (3.6), T^3_N is the third one, T^4_N the fourth one and T^5_N the fifth one.

The four following Lemmas, whose proofs are postponed to Section 3.2 deal with the asymptotic behavior of the terms M_N, $T^2_N - T^1_N$, T^3_N and T^4_N.

Lemma 3.8. *It holds*

$$\mathbb{E}|M_N|^2 \le \frac{K\sigma^2_N r^{2-\alpha}_N}{N}$$

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for some positive constant K. The equivalent term in the case $\alpha = 2$,

$$M_N = \sigma_N \sum_{\kappa_i^N > 0} \int_0^\infty w_{\tau_t}^i \partial_x \varphi(X_t^{N,i}) \mathrm{d}L_t^i,$$

satifies the same estimate :

$$\mathbb{E}|M_N|^2 \leq K \frac{\sigma_N^2}{N}.$$

Lemma 3.9. • *It holds*

$$\mathbb{E}\left| -T_N^1 + T_N^2 + \int_0^\infty \int_{\mathbb{R}} \left(\eta(H * \tilde{\mu}_t^{N,r_N}) \partial_t g_t + 2\chi_t^N \psi(H * \tilde{\mu}_t^{N,r_N}) \partial_x g_t \right) \mathrm{d}t + \int_{\mathbb{R}} g_0 \eta(H * \tilde{\mu}_0^{N,r_N}) \mathrm{d}x \right.$$

$$\left. +2c_\alpha \int_0^\infty \chi_t^N \int_{\mathbb{R}} \int_{\{|y| \leq r_N\}} \eta(H * \tilde{\mu}_t^{N,r_N}(x)) \left(g_t(x + \sigma_N y) - g_t(x) - \sigma_N y \partial_x g_t(x) \right) \frac{\mathrm{d}y \mathrm{d}x \mathrm{d}t}{|y|^{1+\alpha}} \right| \underset{N \to \infty}{\to} 0.$$

• *If $r_N \leq 1/\sigma_N$, then*

$$\left| 2c_\alpha \int_0^\infty \chi_t^N \int_{\mathbb{R}} \int_{\{|y| \leq r_N\}} \eta(H * \tilde{\mu}_t^{N,r_N}(x)) \left(g_t(x + \sigma_N y) - g_t(x) - \sigma_N y \partial_x g_t(x) \right) \frac{\mathrm{d}y \mathrm{d}x \mathrm{d}t}{|y|^{1+\alpha}} \right| \leq K \sigma_N^\alpha.$$

The following lemma gives two estimates for the term T_N^3, the first being useful for a constant viscosity $\sigma_N \equiv \sigma$, and the second for vanishing viscosity $\sigma_N \to 0$.

Lemma 3.10. • *The error term*

$$\mathbb{E}\left| T_N^3 + 2c_\alpha \int_0^\infty (1 - \chi_t^N) \int_{\mathbb{R}} \int_{\{|y| > r_N\}} \eta'(H * \tilde{\mu}_t^{N,r_N}(x)) \left(H * \tilde{\mu}_t^{N,r_N}(x + \sigma_N y) - H * \tilde{\mu}_t^{N,r_N}(x) \right) g_t(x) \frac{\mathrm{d}y \mathrm{d}x \mathrm{d}t}{|y|^{1+\alpha}} \right|$$

vanishes if $N^{-1} r_N^{-\alpha}$ goes to 0.

• *It holds*

$$\mathbb{E}|T_N^3| \leq K(\sigma_N r_N^{1-\alpha} + \sigma_N^\alpha).$$

Lemma 3.11. *One has $\mathbb{E}|T_N^4| \underset{N \to \infty}{\to} 0$.*

We now have to control the probability for the last remaining term T_N^5 to be negative. If there is no crossing of particles with opposite signs between kh_N and $(k + 1/2)h_N$, for any k, then $T_N^5 \geq 0$. Indeed, let $X_{(k+1/2)h_N}^{N,i_1,r_N} \leq \ldots \leq X_{(k+1/2)h_N}^{N,i_q,r_N}$ be a maximal sequence of consecutive particles with same sign. The sequence $\left(\varphi_{(k+1/2)h_N}(X_{(k+1/2)h_N}^{N,i_l,r_N}) \right)_{l=1,\ldots,q}$ is thus a nondecreasing sequence, and from the convexity of η and the fact that no particles with opposite signs cross, $(w_{(k+1/2)h_N}^{i_l})_{l=1,\ldots,q}$ is the nondecreasing reordering of $(w_{kh_N}^{i_l})_{l=1,\ldots,q}$. Thus, from Lemma 3.13 below, $\sum_{\kappa_i^N > kh_N} (w_{(k+1/2)h_N}^i - w_{kh_N}^i) \varphi_{(k+1/2)h_N}(X_{(k+1/2)h_N}^{N,i,r_N})$ is nonnegative. It is thus sufficient to control the probability that two particles with opposite signs cross between kh_N and $(k + 1/2)h_N$. Since after the murder there is no couple of particles with opposite signs separated by a smaller distance than ε_N, this does not happen as soon as no particle drift by more than $\varepsilon_N/4$ and no particle is moved by more than $\varepsilon_N/4$ by the small jumps. The drift on half a time step is smaller than $\sup_{[-1,1]} |A'| h_N$ which is assumed to be smaller than $\varepsilon_N/4$. We control the contribution of the small jumps in the following lemma:

Lemma 3.12. *Let B_N be the event*

$$B_N = \left\{ \forall k \leq T/h_N, \forall i, \sigma_N \left| \Lambda^{i,r_N}_{(k+1/2)h_N} - \Lambda^{i,r_N}_{kh_N} \right| \leq \varepsilon_N/4 \right\},$$

so that no crossing of particles with opposite signs between kh_N and $(k+1/2)h_N$ occurs on B_N. One has, for $\alpha < 2$,

$$\mathbb{P}(B_N) \geq \left(1 - e^{Kh_N r_N^{-\alpha} - \varepsilon_N/4\sigma_N r_N} \right)^{NT/h_N},$$

For $\alpha = 2$, we define the event B_N by

$$B_N = \left\{ \forall k \leq T/h_N, \forall i, \sigma_N \left| L^i_{(k+1)h_N} - L^i_{kh_N} \right| \leq \varepsilon_N/4 \right\}.$$

It holds

$$\mathbb{P}(B_N) \geq \left(1 - Ke^{-\varepsilon_N^2/(32h_N \sigma_N^2)} \right)^{NT/h_N}.$$

The proof will be given in Section 3.2.

We now gather all the previous information to prove that, depending on the considered case, the entropic formulation or the weak formulation holds almost surely.

1. *Constant viscosity $\sigma_N \equiv \sigma$, with index $0 < \alpha \leq 1$.*

   Define, for $Q \in \mathcal{P}(\mathcal{K})$,

$$
\begin{aligned}
F^r_N(Q) = &\int_{\mathbb{R}} \eta(H * \tilde{Q}_0)g_0 + \int_0^\infty \int_{\mathbb{R}} \left(\eta(H * \tilde{Q}_t)\partial_t g + 2\chi_t^N \psi(H * \tilde{Q}_t)\partial_x g \right) \mathrm{d}t \\
&+ 2c_\alpha \int_0^\infty (1 - \chi_t^N) \int_{\mathbb{R}} \int_{\{|y|>r\}} \eta'(H * \tilde{Q}_t(x))(H * \tilde{Q}_t(x + \sigma_N y) - H * \tilde{Q}_t(x))g_t(x)\frac{\mathrm{d}y\mathrm{d}x\mathrm{d}t}{|y|^{1+\alpha}} \\
&+ 2c_\alpha \int_0^\infty \chi_t^N \int_{\mathbb{R}} \int_{\{|y|\leq r\}} \eta(H * \tilde{Q}_t(x))(g_t(x + \sigma_N y) - g_t(x) - \sigma_N y \partial_x g_t(x))\frac{\mathrm{d}y\mathrm{d}x\mathrm{d}t}{|y|^{1+\alpha}}
\end{aligned}
$$

   and

$$
\begin{aligned}
F^r(Q) = &\int_{\mathbb{R}} \eta(H * \tilde{Q}_0)g_0 + \int_0^\infty \int_{\mathbb{R}} \left(\eta(H * \tilde{Q}_t)\partial_t g + \psi(H * \tilde{Q}_t)\partial_x g \right) \mathrm{d}t \\
&+ c_\alpha \int_0^\infty \int_{\mathbb{R}} \int_{\{|y|>r\}} \eta'(H * \tilde{Q}_t(x))(H * \tilde{Q}_t(x + \sigma y) - H * \tilde{Q}_t(x))g_t(x)\frac{\mathrm{d}y\mathrm{d}x\mathrm{d}t}{|y|^{1+\alpha}} \\
&+ c_\alpha \int_0^\infty \int_{\mathbb{R}} \int_{\{|y|\leq r\}} \eta(H * \tilde{Q}_t(x))(g_t(x + \sigma y) - g_t(x) - \sigma y \partial_x g_t(x))\frac{\mathrm{d}y\mathrm{d}x\mathrm{d}t}{|y|^{1+\alpha}}.
\end{aligned}
$$

   Notice that from the convexity of η, one has

$$\eta'(H * \tilde{Q}_t(x))(H * \tilde{Q}_t(x + \sigma y) - H * \tilde{Q}_t(x)) \leq \eta(H * \tilde{Q}_t(x + \sigma y)) - \eta(H * \tilde{Q}_t(x)),$$

   so that for any $0 < r \leq r'$, it holds $F^r \leq F^{r'}$ and $F^r_N \leq F^{r'}_N$.

   From Equation (3.6), it holds, for N large enough so that $r_N \leq r$,

$$F^r_N(\mu^{N,r_N}) \geq F^{r_N}_N(\mu^{N,r_N}) = T^5_N + \left(-T^1_N + T^2_N + T^3_N + T^4_N + M_N + F^{r_N}_N(\mu^{N,r_N}) \right).$$

   From the assumptions made on ε_N and h_N one can construct a sequence r_N such that $N^{-1/\alpha} = o(r_N)$, $h_N r_N^{-\alpha} = o\left(\varepsilon_N r_N^{-1} \right)$ and $\frac{N}{h_N}e^{-\varepsilon_N/4\sigma r_N} \to 0$. Indeed, set $r_N = \varepsilon_N N^{-1/2\lambda}$. Then it holds $N^{-1/\alpha}r_N^{-1} \leq KN^{-1/2\lambda}$ and $\frac{h_N}{\varepsilon_N}r_N^{1-\alpha} = h_N \varepsilon_N^{-\alpha} N^{(\alpha-1)/2\lambda}$, which vanishes for any value of α.

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Then $\frac{N}{h_N}$ goes to infinity at the rate of a power of N, and $\varepsilon_N/r_N = N^{1/2\lambda}$ as well. Thus, $\frac{N}{h_N}e^{-\varepsilon_N/4\sigma r_N}$ vanishes.

As a consequence, from Lemmas 3.8, 3.9, 3.10 and 3.11, $\mathbb{E}\left|-T_N^1 + T_N^2 + T_N^3 + T_N^4 + M_N + F_N^{r_N}(\mu^{N,r_N})\right|$ vanishes as N tends to infinity, and the event B_N defined in Lemma 3.12 is such that $\mathbb{P}(B_N) \to 1$. On the event B_N, T_N^5 is almost-surely nonnegative, so that, from the uniform boundedness of F_N^r with respect to N, $\mathbb{E}^{\pi^{N,r_N}}(F_N^r(Q)^-) = \mathbb{E}(F_N^r(\mu^{N,r_N})^-)$ goes to 0. To show that the entropic formulation holds almost surely, we need a continuous approximation of F_N^r and F^r. We define $F^{r,\delta}$ and $F_N^{r,\delta}$ by replacing every occurrence of $H * \tilde{Q}_t$ in the definitions of F^r and F_N^r by $\int_{\mathcal{K}} \mathbf{1}_{\kappa>t} H(. - f(t))\gamma^\delta(f(0))\mathrm{d}Q(f,\kappa)$, where γ^δ is a Lipschitz continuous approximation of γ, with $\mathbb{P}(\gamma(X_0^1) \neq \gamma^\delta(X_0^1)) \leq \delta$ (see [9], Lemma 2.5, for the construction of γ^δ). Then, for any fixed δ and r, the family $\{F^{r,\delta}\} \cup \{F_N^{r,\delta}, N \in \mathbb{N}\}$ is equicontinuous for the topology of weak convergence. Indeed, let Q^k be a sequence of probability measures on $\mathcal{K}$ converging to Q as k goes to infinity. From the continuity of the application $\mathcal{K} \to \mathbb{R}, (f,\kappa) \mapsto \mathbf{1}_{\kappa>0}f(0)$, Q_0^k converges weakly to Q_0 (where Q_0 and Q_0^k are defined as in (3.2)), and from the continuity of the applications $\mathcal{K} \to \mathbb{R}, (f,\kappa) \mapsto \mathbf{1}_{\kappa>t}\gamma^\delta(f(0))\mathbf{1}_{f(t)\leq y}$ on the set $\{(f,\kappa) \in \mathcal{K}, f(t) = f(t-), f(t) \neq y\}$, for all t in the complement of the countable set $\{t \in [0,\infty), Q(\{f(t) \neq f(t-)\} \cup \{\kappa = t\}) > 0\}$, the quantity $\int_{\mathcal{K}} \mathbf{1}_{\kappa>t} H(. - f(t))\gamma^\delta(f(0))\mathrm{d}Q^k(f,\kappa)$ converges almost everywhere to $\int_{\mathcal{K}} \mathbf{1}_{\kappa>t} H(.-f(t))\gamma^\delta(f(0))\mathrm{d}Q(f,\kappa)$. From Lebesgue's bounded convergence theorem, we deduce that

$$\sup_N |F_N^{r;\delta}(Q^k) - F_N^{r;\delta}(Q)| + |F^{r,\delta}(Q^k) - F^{r,\delta}(Q)| \underset{k\to\infty}{\to} 0$$

yielding equicontinuity for $\{F^{r,\delta}\} \cup \{F_N^{r,\delta}, N \in \mathbb{N}\}$. Moreover, since the sequence χ_t^N converges $*$-weakly to $1/2$ in the space $\mathbb{L}^\infty((0,\infty))$, $F_N^{r,\delta}$ converges pointwise to $F^{r,\delta}$ as N goes to infinity. Ascoli's theorem thus implies that $F_N^{r,\delta}$ converges uniformly on compact sets to $F^{r,\delta}$. From the weak convergence of π^{N,r_N} to π^∞, one thus deduces

$$\mathbb{E}^{\pi^{N,r_N}}[F_N^{r;\delta}(Q)^-] \underset{N\to\infty}{\to} \mathbb{E}^{\pi^\infty}[F^{r,\delta}(Q)^-].$$

Moreover, for any $t > 0$, any y, and any probability measure Q satisfying $Q_0 = |u_0|$ (with Q_0 defined as in (3.2)), which holds true for $\pi^\infty-$almost all Q from Lemma 3.5, it holds

$$\left|H * \tilde{Q}_t(y) - \int_{\mathcal{K}} \mathbf{1}_{\kappa>t} H(y - f(t))\gamma^\delta(f(0))\mathrm{d}Q(f,\kappa)\right| \leq \int_{\mathbb{R}} |\gamma - \gamma^\delta|\mathrm{d}|u_0| \leq \delta,$$

yielding convergence to 0 for $\mathbb{E}^{\pi^\infty}|F^r(Q)^- - F^{r,\delta}(Q)^-| + \mathbb{E}^{\pi^{N,r_N}}|F_N^r(Q)^- - F_N^{r;\delta}(Q)^-|$ as δ goes to 0, uniformly in N. As a consequence, writing

$$\mathbb{E}^{\pi^\infty}(F^r(Q)^-) \leq \mathbb{E}^{\pi^\infty}|F^r(Q)^- - F^{r,\delta}(Q)^-| + \left|\mathbb{E}^{\pi^\infty}(F^{r,\delta}(Q)^-) - \mathbb{E}^{\pi^{N,r_N}}(F_N^{r;\delta}(Q)^-)\right|$$
$$+ \mathbb{E}^{\pi^{N,r_N}}|F_N^{r;\delta}(Q)^- - F_N^r(Q)^-| + \mathbb{E}^{\pi^{N,r_N}}(F_N^r(Q)^-)$$

we deduce that $F^r(Q)$ is nonnegative for $\pi^\infty-$almost all Q. We just have to notice that Lemma 3.5 yields that, $\pi^\infty-$almost surely, $H * \tilde{Q}_0 = v_0$ to conclude that the entropy formulation holds $\pi^\infty-$almost surely.

2. *Vanishing viscosity $\sigma_N \to 0$.*

We define

$$F_N^r(Q) = \int_{\mathbb{R}} \eta(H * \tilde{Q}_0)g_0 + \int_0^\infty \int_{\mathbb{R}} \left(\eta(H * \tilde{Q}_t)\partial_t g + 2\chi_t^N \psi(H * \tilde{Q}_t)\partial_x g\right)\mathrm{d}t$$

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and
$$F(Q) = \int_{\mathbb{R}} \eta(H * \tilde{Q}_0)g_0 + \int_0^\infty \int_{\mathbb{R}} \left(\eta(H * \tilde{Q}_t)\partial_t g + \psi(H * \tilde{Q}_t)\partial_x g \right) \mathrm{d}t.$$

Regularized versions $F_N^{r,\delta}$ and F^δ of F_N^r and F are also considered using the function γ^δ instead of γ. In the case $\alpha < 2$, the same arguments as above, using the second parts of Lemmas 3.9 and 3.10 will show that the entropy formulation holds $\pi^\infty-$almost surely for $H * \tilde{Q}_t$, provided there exists a sequence r_N such that $\frac{\sigma_N^2 r_N^{2-\alpha}}{N}$ and $\sigma_N r_N^{1-\alpha}$ vanish, $r_N \le \sigma_N^{-1}$, $h_N r_N^{-\alpha} = o(\varepsilon_N(\sigma_N r_N)^{-1})$ and $\frac{N}{h_N}e^{-\varepsilon_N/4\sigma_N r_N} \to 0$.

- For $\alpha \le 1$, any sequence r_N vanishing at a very quick rate will fit.
- For $\alpha > 1$, since we assumed $\sigma_N \le \varepsilon_N^{1-\frac{1}{\alpha}} N^{-1/\lambda}$ these conditions are satisfied by the sequence $r_N = \frac{\varepsilon_N}{\sigma_N} N^{-\frac{\alpha}{2\lambda(\alpha-1)}}$.

In the case $\alpha = 2$, Itō's formula writes

$$\begin{aligned}
\varphi_{(k+1)h_N}\left(X_{(k+1)h_N}^{N,i}\right) - \varphi_{kh_N}\left(X_{kh_N}^{N,i}\right) &= \int_{kh_N}^{(k+1)h_N} \partial_t \varphi_t(X_t^{N,i})\mathrm{d}t \\
&+ 2\int_{kh_N}^{(k+1)h_N} \partial_x \varphi_t(X_t^{N,i}) A'\left(H * \tilde{\mu}_{kh_N}^N(X_{kh_N}^{N,i})\right)\mathrm{d}t. \\
&+ \sigma_N^2 \int_{(kh_N,(k+1)h_N)} \int_{\{|y| \le r\}} \partial_x^2 \varphi_t(X_t^{N,i})\mathrm{d}t \\
&+ \sigma_N \int_{(kh_N,(k+1)h_N)} \partial_x \varphi_t(X_t^{N,i})\mathrm{d}L_t^i.
\end{aligned}$$

The three first terms are treated as in the case $\alpha < 2$, and the stochastic integral is dealt with using Lemma 3.8. For the entropic inequality to holds, we need to control the crossing of particles with opposite sign. From Lemma 3.12, if $\frac{N}{h_N}e^{-\frac{\varepsilon_N^2}{32\sigma_N^2 h_N}}$ goes to zero, then no crossing occurs. Since our assumptions yield $h_N \sigma_N^2 \le \varepsilon_N^2 N^{-1/\lambda}$ and $N/h_N \le K N^{1+\lambda}$, this condition holds true.

3. *Constant viscosity $\sigma_N \equiv \sigma$, with index $1 < \alpha \le 2$.*
   In this case, since we want to derive a weak formulation, we do not need to consider separately large and small jumps. As a consequence it is enough to study the process $X_t^{N,i}$.
   Let g be a smooth function with compact support, and define for $Q \in \mathcal{P}(\mathcal{K})$,

$$F(Q) = \int_{\mathbb{R}} H * \tilde{Q}_0 g_0 + \int_0^\infty \int_{\mathbb{R}} H * \tilde{Q}_t \partial_t g_t \mathrm{d}t - \sigma^\alpha \int_0^\infty \int_{\mathbb{R}} H * \tilde{Q}_t (-\Delta)^{\frac{\alpha}{2}} g_t \mathrm{d}t + \int_0^\infty \int_{\mathbb{R}} A(H * \tilde{Q}_t)\partial_x g_t.$$

Let $\varphi_t(x) = \int_{-\infty}^x g_t(y)\mathrm{d}y$. One has

$$\begin{aligned}
-\frac{1}{N}\sum_{\kappa_i^N > 0} \gamma(X_0^{N,i})\varphi_0(X_0^{N,i}) &= -\frac{1}{N}\sum_{k=0}^\infty \sum_{\kappa_i^N = (k+1)h_N} \gamma(X_0^{N,i})\varphi_{(k+1)h_N}(X_{(k+1)h_N}^{N,i}) \\
&+ \frac{1}{N}\sum_{k=0}^\infty \sum_{\kappa_i^N > kh_N} \gamma(X_0^{N,i})\left(\varphi_{(k+1)h_N}(X_{(k+1)h_N}^{N,i}) - \varphi_{kh_N}(X_{kh_N}^{N,i})\right).
\end{aligned}$$

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From Itō's formula, in the case $\alpha < 2$, when $\kappa_i^N > kh_N$,

$$\varphi_{(k+1)h_N}(X_{(k+1)h_N}^{N,i}) - \varphi_{kh_N}(X_{kh_N}^{N,i}) \tag{3.7}$$

$$= \int_{kh_N}^{(k+1)h_N} \partial_t \varphi_t(X_t^{N,i})\mathrm{d}t + \int_{kh_N}^{(k+1)h_N} \partial_x \varphi_t(X_t^{N,i})A'\left(H * \tilde{\mu}_{kh_N}(X_{kh_N}^{N,i})\right)\mathrm{d}t$$

$$+ c_\alpha \int_{(kh_N,(k+1)h_N)} \int_{\mathbb{R}} \left(\varphi_t(X_t^{N,i} + \sigma y) - \varphi_t(X_t^{N,i}) - \mathbf{1}_{\{|y|\le r\}}\sigma y \partial_x \varphi_t(X_t^{N,i})\right)\frac{\mathrm{d}y\mathrm{d}t}{|y|^{1+\alpha}}$$

$$+ \int_{(kh_N,(k+1)h_N)} \int_{\mathbb{R}} \left(\varphi_t(X_{t-}^{N,i} + \sigma y) - \varphi_t(X_{t-}^{N,i})\right)\tilde{\nu}^i(\mathrm{d}y,\mathrm{d}t).$$

We denote $\tau_t = \max\{kh_N, kh_N \le t\}$. Multiplying (3.7) by $\frac{1}{N}\mathbf{1}_{\kappa_i^N > kh_N}\gamma(X_0^{N,i})$, summing over i and k, and integrating by parts, one obtains

$$\int_{\mathbb{R}} g_0 H * \tilde{\mu}_0^N = -\int_0^\infty \int_{\mathbb{R}} \partial_t g_t H * \tilde{\mu}_t^N \mathrm{d}t + \int_0^\infty \int_{\mathbb{R}} (-\Delta)^{\frac{\alpha}{2}} g_t H * \tilde{\mu}_t^N \mathrm{d}t$$

$$+ \frac{1}{N}\int_0^\infty \sum_{\kappa_i^N > \tau_t} \gamma(X_0^{N,i})\partial_x \varphi_t(X_t^{N,i})A'\left(H * \tilde{\mu}_{kh_N}(X_{kh_N}^{N,i})\right)\mathrm{d}t$$

$$+ \frac{1}{N}\int_{(0,\infty)\times\mathbb{R}} \sum_{\kappa_i^N > \tau_t} \gamma(X_0^{N,i})\left(\varphi_t(X_{t-}^{N,i} + \sigma y) - \varphi_t(X_{t-}^{N,i})\right)\tilde{\nu}^i(\mathrm{d}y,\mathrm{d}t)$$

$$- \frac{1}{N}\sum_{k=0}^\infty \sum_{\kappa_i^N = (k+1)h_N} \gamma(X_0^{N,i}))\varphi_{(k+1)h_N}(X_{(k+1)h_N}^{N,i}), \tag{3.8}$$

Combining an adaptation of Lemma 3.14, stated in Section 3.2, with A replacing η, and integrating by parts, the difference

$$\frac{1}{N}\int_0^\infty \sum_{\kappa_i^N > \tau_t} \gamma(X_0^{N,i})\partial_x \varphi_t(X_t^{N,i})A'\left(H * \tilde{\mu}_{kh_N}(X_{kh_N}^{N,i})\right)\mathrm{d}t + \int_0^\infty \int_{\mathbb{R}} \partial_x g_t A(H * \tilde{\mu}_t^N)\mathrm{d}t$$

vanishes in $\mathbb{L}^1$. Using an adaptation Lemma 3.8, the the fourth term in the right hand side of (3.8) vanishes in $\mathbb{L}^2$. The fifth term vanishes in $\mathbb{L}^1$ since

$$\left|\frac{1}{N}\sum_{k=0}^\infty \sum_{\kappa_i^N = (k+1)h_N} \gamma(X_0^{N,i})\varphi_{(k+1)h_N}(X_{(k+1)h_N}^{N,i})\right|$$

$$\le \frac{1}{N}\sum_{k=0}^\infty \sum_{\substack{\text{pairs }\{i,j\}\text{ killled}\\\text{at time }(k+1)h_N}} \left|\varphi_{(k+1)h_N}\left(X_{(k+1)h_N}^{N,i}\right) - \varphi_{(k+1)h_N}\left(X_{(k+1)h_N}^{N,j}\right)\right|$$

$$\le K\varepsilon_N.$$

As a consequence, $\mathbb{E}^{\pi^N}|F(Q)| = \mathbb{E}|F(\mu^N)|$ vanishes. We conclude by regularizing the function γ as in the two first points, that $\mathbb{E}^{\pi^\infty}|F(Q)| = 0$. Thus, $F(Q) = 0$ almost surely, so that $H * \tilde{Q}$ almost surely satisfies the weak formulation.

The case $\alpha = 2$ is treated in the same way, the only difference lying in the nature of the stochastic integral.

3.2 Proofs of Lemmas 3.8 to 3.12

In this section, we give the proofs of the previously admitted lemmas of Section 3.1.

Proof of Lemma 3.8. Since the particles are driven by independent stable processes and since the inequality $|w_t^i| \leq \frac{K}{N}$ holds for some constant K not depending on t, i and N,

$$
\begin{aligned}
\mathbb{E}M_N^2 =& \mathbb{E}\left|\sum_{\kappa_i^N>0}\int_0^\infty w_{\tau_t}^i \chi_t^N \left(\int_{\{|y|\leq r_N\}}\left(\varphi_t(X_{t-}^{N,i,r_N}+\sigma_N y)-\varphi_t(X_{t-}^{N,i,r_N})\right)\tilde{\nu}^{i,r_N}(\mathrm{d}y,\mathrm{d}t)\right)\right|^2 \\
\leq& 2\sigma_N^2 c_\alpha \mathbb{E}\left(\sum_{\kappa_i^N>0}\int_0^\infty (w_{\tau_t}^i)^2 \chi_t^N \int_{\{|y|\leq r_N\}}(y\|g_t\|_\infty)^2\frac{\mathrm{d}y\mathrm{d}t}{|y|^{1+\alpha}}\right) \\
\leq& K\frac{\sigma_N^2 r_N^{2-\alpha}}{N}\int_0^\infty\|g_t\|_\infty^2\mathrm{d}t.
\end{aligned}
$$

A similar proof with stochastic integrals against Brownian motion yields the result for $\alpha = 2$. $\qquad\square$

Proof of Lemma 3.9. Integrating by parts, one finds

$$
\sum_{i=1}^N\int_0^\infty w_t^i\partial_t\varphi_t\left(X_t^{N,i,r_N}\right)\mathrm{d}t = -\int_0^\infty\int_\mathbb{R}\eta(H*\tilde{\mu}_t^{N,r_N})\partial_t g_t\mathrm{d}t + \int_0^\infty\int_\mathbb{R}\eta(\tilde{\mu}_t^{N,r_N}(\mathbb{R}))\partial_t g_t\mathrm{d}t
$$

yielding, from Lemma 3.14 below,

$$
\mathbb{E}\left|\sum_{i=1}^N\int_0^\infty w_{\tau_t}^i\partial_t\varphi_t\left(X_t^{N,i,r_N}\right)\mathrm{d}t + \int_0^\infty\int_\mathbb{R}\eta(H*\tilde{\mu}_t^{N,r_N})\partial_t g_t\mathrm{d}t - \int_0^\infty\int_\mathbb{R}\eta(\tilde{\mu}_t^{N,r_N}(\mathbb{R}))\partial_t g_t\mathrm{d}t\right|\underset{N\to\infty}{\longrightarrow} 0.
$$

From the constancy of $\tilde{\mu}_t^{N,r_N}(\mathbb{R})$ and an integration by parts, one has

$$
-T_N^1 + \int_0^\infty\int_\mathbb{R}\eta(\tilde{\mu}_t^{N,r_N}(\mathbb{R}))\partial_t g_t\mathrm{d}t = -\int_\mathbb{R}g_0\eta(H*\tilde{\mu}_0^{N,r_N}).
$$

Another integration by parts yields

$$
\begin{aligned}
&2c_\alpha\sum_{i=1}^N\int_0^\infty w_t^i\chi_t^N\int_{\{|y|\leq r_N\}}\varphi_t\left(X_t^{N,i,r_N}+\sigma_N y\right)-\varphi_t\left(X_t^{N,i,r_N}\right)-\sigma_N y\partial_x\varphi_t\left(X_t^{N,i,r_N}\right)\frac{\mathrm{d}y\mathrm{d}t}{|y|^{1+\alpha}} \\
=&-2c_\alpha\int_0^\infty\chi_t^N\int_{\{|y|\leq r_N\}}\int_\mathbb{R}(g_t(x+\sigma_N y)-g_t(x)-\sigma_N y\partial_x g_t(x))\eta(H*\tilde{\mu}_t^{N,r_N}(x))\frac{\mathrm{d}x\mathrm{d}y\mathrm{d}t}{|y|^{1+\alpha}} \\
&+2c_\alpha\int_0^\infty\chi_t^N\eta(\tilde{\mu}_t^{N,r_N}(\mathbb{R}))\int_{\{|y|\leq r_N\}}\int_\mathbb{R}(g_t(x+\sigma_N y)-g_t(x)-\sigma_N y\partial_x g_t(x))\frac{\mathrm{d}x\mathrm{d}y\mathrm{d}t}{|y|^{1+\alpha}} \\
=&-2c_\alpha\int_0^\infty\chi_t^N\int_{\{|y|\leq r_N\}}\int_\mathbb{R}(g_t(x+\sigma_N y)-g_t(x)-\sigma_N y\partial_x g_t(x))\eta(H*\tilde{\mu}_t^{N,r_N}(x))\frac{\mathrm{d}x\mathrm{d}y\mathrm{d}t}{|y|^{1+\alpha}}.
\end{aligned}
$$

Moreover, from the regularity of A and η, it holds

$$
w_{\tau_t}^i A'\left(H*\tilde{\mu}_{\tau_t}^{N,r_N}(X_{\tau_t}^{N,i,r_N})\right) = \psi\left(H*\tilde{\mu}_{\tau_t}^{N,r_N}(X_{\tau_t}^{N,i,r_N})\right)-\psi\left(H*\tilde{\mu}_{\tau_t}^{N,r_N}(X_{\tau_t}^{N,i,r_N}-)\right)+o\left(\frac{1}{N}\right),
$$

so that

$$\mathbb{E}\left|2\sum_{i=1}^{N}\int_0^\infty w_{\tau_t}^i \chi_t^N \partial_x\varphi_t\left(X_t^{N,i,r_N}\right) A'\left(H*\tilde\mu_{\tau_t}^{N,r_N}(X_{\tau_t}^{N,i,r_N})\right) \mathrm{d}t + 2\int_0^\infty \chi_t^N \int_{\mathbb{R}} \partial_x g_t \psi(H*\tilde\mu_t^{N,r_N})\mathrm{d}t\right| \underset{N\to\infty}{\to} 0,$$

from an adaptation of Lemma 3.14 (replacing η by ψ in the definition of w_t^i). This concludes the proof of the first item of Lemma 3.9.

To prove the second item, observe that the change of variable $z = \sigma_N y$ yields, for $r_N \le \frac{1}{\sigma_N}$,

$$\left|2c_\alpha \int_0^\infty \chi_t^N \int_{\mathbb{R}} \int_{\{|y|\le r_N\}} \eta(H*\tilde\mu_t^{N,r_N}(x))\left(g_t(x+\sigma_N y) - g_t(x) - \sigma_N y \partial_x g_t(x)\right) \frac{\mathrm{d}y\mathrm{d}x\mathrm{d}t}{|y|^{1+\alpha}}\right|$$

$$\le 2c_\alpha \sigma_N^\alpha \int_0^\infty \chi_t^N \int_{\mathbb{R}} \int_{\{|z|\le 1\}} \left|\eta(H*\tilde\mu_t^{N,r_N}(x))\left(g_t(x+z) - g_t(x) - z\partial_x g_t(x)\right)\right| \frac{\mathrm{d}z\mathrm{d}x\mathrm{d}t}{|z|^{1+\alpha}}.$$

$\square$

Proof of Lemma 3.10. First notice that

$$T_N^3 = \sum_{\kappa_i^N>0} \int_0^\infty (1-\chi_t^N) \int_{\{|y|>r_N\}} \left(\int_{\mathbb{R}} \varphi_t \mathrm{d}\rho_{t-}^{y,i}\right) \nu^{i,r_N}(\mathrm{d}y,\mathrm{d}t),$$

with ρ defined by the following formula: ($\bar\mu_t^{y,i,N,r_N}$ being the measure obtained by moving in the expression of $\tilde\mu_t^{N,r_N}$ the particle X_t^{N,i,r_N} to the position $X_t^{N,i,r_N} + \sigma_N y$)

$$\rho_t^{y,i} = \partial_x\left(\eta(H*\bar\mu_t^{y,i,N,r_N}) - \eta(H*\tilde\mu_t^{N,r_N})\right).$$

To prove the second item in Lemma 3.10, we integrate by parts, and, using the definition of $\bar\mu^{y,i,N,r_N}$ and the compactness of the support of g, it holds

$$\left|\int_{\mathbb{R}} \varphi_t \mathrm{d}\rho_t^{y,i}\right| = \left|\int_{\mathbb{R}} g_t\left(\eta(H*\bar\mu_t^{y,i,N,r_N}) - \eta(H*\tilde\mu_t^{N,r_N})\right)\right| \le K\frac{(\sigma_N y)\wedge 1}{N}, \tag{3.9}$$

so that

$$\mathbb{E}|T_N^3| \le K\int_0^\infty (1-\chi_t^N) \int_{\{|y|>r_N\}} (\sigma_N y)\wedge 1\frac{\mathrm{d}y\mathrm{d}t}{|y|^{1+\alpha}} \le K(\sigma_N^\alpha + \sigma_N r_N^{1-\alpha}).$$

Now let us prove the first item of Lemma 3.10. Applying the same martingale argument as the one used to prove $\mathbb{E}|M_N| \to 0$, and using the upper bound K/N in (3.9), one has

$$\mathbb{E}\left|T_N^3 - 2c_\alpha \int_0^\infty (1-\chi_t^N) \int_{\{|y|>r_N\}} \left(\sum_{\kappa_i^N>t} \int_{\mathbb{R}} \varphi_t \mathrm{d}\rho_t^{y,i}\right) \frac{\mathrm{d}y\mathrm{d}t}{|y|^{1+\alpha}}\right|^2 \le \frac{K}{r_N^\alpha N}.$$

Let us give a more explicit expression for $\rho_t^{y,i}$. For simplicity, we denote

$$\tilde w_t^i = \mathbf{1}_{\kappa_i^N>t}\left[\eta\left(\frac{1}{N}\sum_{\substack{j\ne i\\ \kappa_j^N>t}} \gamma(X_0^j)\mathbf{1}_{X_t^{N,j,r_N}\le X_t^{N,i,r_N}+\sigma_N y} + \frac{\gamma(X_0^i)}{N}\right) - \eta\left(\frac{1}{N}\sum_{\substack{j\ne i\\ \kappa_j^N>t}} \gamma(X_0^j)\mathbf{1}_{X_t^{N,j,r_N}\le X_t^{N,i,r_N}+\sigma_N y}\right)\right]$$

and for $i \neq j$,

$$\tilde{w}_t^{i,j,\pm} = \mathbf{1}_{\kappa_i^N > t} \mathbf{1}_{\kappa_j^N > t} \left[\eta \left(\frac{1}{N} \sum_{\substack{k \neq j \\ \kappa_k^N > t}} \gamma(X_0^k) \mathbf{1}_{X_t^{N,k,r_N} \leq X_t^{N,i,r_N}} \pm \frac{\gamma(X_0^j)}{N} \right) - \eta \left(\frac{1}{N} \sum_{\substack{k \neq j \\ \kappa_k^N > t}} \gamma(X_0^k) \mathbf{1}_{X_t^{N,k,r_N} \leq X_t^{N,i,r_N}} \right) \right].$$

One can write

$$\rho_t^y := \sum_{\kappa_i^N > t} \rho_t^{y,i} = \sum_{\kappa_i^N > t} \tilde{w}_t^i \delta_{X_t^{N,i,r_N} + \sigma_N y} - \sum_{\kappa_i^N > t} w_t^i \delta_{X_t^{N,i,r_N}}$$

$$+ \sum_{\kappa_i^N > t} \left(\sum_{\kappa_j^N > t} \left(\tilde{w}_t^{i,j,+} - w_t^i \right) \mathbf{1}_{X_t^{N,i,r_N} < X_t^{N,j,r_N}} \mathbf{1}_{X_t^{N,j,r_N} + \sigma_N y < X_t^{N,i,r_N}} \right) \delta_{X_t^{N,i,r_N}}$$

$$+ \sum_{\kappa_i^N > t} \left(\sum_{\kappa_j^N > t} \left(\tilde{w}_t^{i,j,-} - w_t^i \right) \mathbf{1}_{X_t^{N,j,r_N} < X_t^{N,i,r_N}} \mathbf{1}_{X_t^{N,i,r_N} < X_t^{N,j,r_N} + \sigma_N y} \right) \delta_{X_t^{N,i,r_N}}. \tag{3.10}$$

In this expression, the two first terms deal with particles jumping from the site X_t^{N,i,r_N} to the site $X_t^{N,i,r_N} + \sigma_N y$, while the third term corresponds to the jump from right to left of the particle labelled j above the particle labelled i and, conversely, the fourth term corresponds to the jumps of particle j from left to right over particle i. Notice that this last equality, as well as (3.11) below, only holds when each $X_t^{N,i,r_N} + \sigma_N y$ is distinct from all X_t^{N,j,r_N}. However, for all t, this condition holds dy-almost everywhere, which is enough for our purpose.

In the entropic formulation (2.3), the term that should appear for large jumps is given by

$$2c_\alpha \int_0^\infty \int_{\{|y| > r_N\}} \left(\int_{\mathbb{R}} \varphi_t \mathrm{d}\sigma_t^y \right) \frac{\mathrm{d}y \mathrm{d}t}{|y|^{1+\alpha}},$$

where

$$\sigma_t^y = \partial_x \left(\eta'(H * \tilde{\mu}_t^{N,r_N}) \left(H * \tilde{\mu}_t^{N,r_N}(\cdot - \sigma_N y) - H * \tilde{\mu}_t^{N,r_N} \right) \right)$$

$$= \frac{1}{N} \sum_{\kappa_i^N > t} \gamma(X_0^i) \eta' \left(H * \tilde{\mu}_t^{N,r_N}(X_t^{N,i,r_N} + \sigma_N y) \right) \delta_{X_t^{N,i,r_N} + \sigma_N y} - \frac{1}{N} \sum_{\kappa_i^N > t} \gamma(X_0^i) \eta' \left(H * \tilde{\mu}_t^{N,r_N}(X_t^{N,i,r_N}) \right) \delta_{X_t^{N,i,r_N}}$$

$$+ \sum_{\kappa_i^N > t} \left(H * \tilde{\mu}_t^{N,r_N}(X_t^{N,i,r_N} - \sigma_N y) - H * \tilde{\mu}_t^{N,r_N}(X_t^{N,i,r_N}) \right)$$

$$\times \left(\eta' \left(H * \tilde{\mu}_t^{N,r_N}(X_t^{N,i,r_N}) \right) - \eta' \left(H * \tilde{\mu}_t^{N,r_N}(X_t^{N,i,r_N}-) \right) \right) \delta_{X_t^{N,i,r_N}}. \tag{3.11}$$

When computing the difference $\rho_t^y - \sigma_t^y$ integrated against some bounded function, using Taylor expansions for η, one can check that, up to an error term of order $\mathcal{O}\left(\frac{1}{N} \right)$ the first terms in the right hand side of (3.10) and (3.11) cancel each other, the second terms as well, and so does the sum of the two last term in (3.10) with the last one in (3.11). Consequently,

$$\left| \int_0^\infty (1 - \chi_t^N) \int_{\{|y| > r_N\}} \left(\int_{\mathbb{R}} \varphi_t \mathrm{d}\rho_t^y \right) \frac{\mathrm{d}y \mathrm{d}t}{|y|^{1+\alpha}} - \int_0^\infty (1 - \chi_t^N) \int_{\{|y| > r_N\}} \left(\int_{\mathbb{R}} \varphi_t \mathrm{d}\sigma_t^y \right) \frac{\mathrm{d}y \mathrm{d}t}{|y|^{1+\alpha}} \right| \leq \frac{K}{N r_N^\alpha}.$$

This concludes the proof. $\qquad \square$

Proof of Lemma 3.11. For a time ζ_m of the form kh_N, no particle moved in the interval (ζ_{m-1}, ζ_m), so that $w_{\zeta_m}^i - w_{\zeta_{m-1}}^i = 0$, unless the particle labelled i has been killed at time ζ_m. Hence,

$$T_N^4 = \sum_{i=1}^N \sum_{\substack{\zeta_m \text{ of the} \\ \text{form } kh_N}} (w_{\zeta_m}^i - w_{\zeta_{m-1}}^i)\varphi_{\zeta_m}(X_{\zeta_m}^{N,i,r_N})$$

$$= -\sum_{\substack{\zeta_m \text{ of the} \\ \text{form } kh_N}} \sum_{\kappa_i^N = \zeta_m} w_{\zeta_{m-1}}^i \varphi_{\zeta_m}(X_{\zeta_m}^{N,i,r_N}).$$

This sum is actually a sum over pairs of close particles with opposite signs, thus

$$|T_N^4| = \left| \sum_{\substack{\zeta_m \text{ of the} \\ \text{form } kh_N}} \sum_{\substack{\text{pairs } \{i,j\} \text{ of particles} \\ \text{killed at time } \zeta_m}} \left(w_{\zeta_{m-1}}^i \varphi_{\zeta_m}(X_{\zeta_m}^{N,i,r_N}) + w_{\zeta_{m-1}}^j \varphi_{\zeta_m}(X_{\zeta_m}^{N,j,r_N}) \right) \right|$$

$$\leq \sum_{\substack{\zeta_m \text{ of the} \\ \text{form } kh_N}} \sum_{\substack{\text{pairs } \{i,j\} \text{ of particles} \\ \text{killed at time } \zeta_m}} \left| w_{\zeta_{m-1}}^i + w_{\zeta_{m-1}}^j \right| \|\varphi\|_\infty + \left| w_{\zeta_{m-1}}^j \right| \left| \varphi_{\zeta_m}(X_{\zeta_m}^{N,i,r_N}) - \varphi_{\zeta_m}(X_{\zeta_m}^{N,j,r_N}) \right|$$

$$\leq K\left(\frac{1}{N} + \varepsilon_N \right).$$

Indeed, a couple (i,j) of killed particles is such that $|X_{\zeta_m}^{N,i,r_N} - X_{\zeta_m}^{N,j,r_N}| \leq \varepsilon_N$ and is made of particles with opposite signs, so that

$$|w_{\zeta_{m-1}}^i + w_{\zeta_{m-1}}^j| = \left| \left(\gamma(X_0^i) + \gamma(X_0^j) \right) \eta'(H * \tilde{\mu}_t^{N,r_N}(X_t^{N,i,r_N})) + \mathcal{O}\left(\frac{1}{N^2} \right) \right| \leq \frac{K}{N^2}.$$

□

Proof of Lemma 3.12. Notice that from independence of the increments, denoting by $\mathcal{L}^{\leq r}$ a Lévy process with Lévy measure $c_\alpha \mathbf{1}_{|y| \leq r} \frac{dy}{|y|^{1+\alpha}}$, it holds

$$\mathbb{P}(B_N) = \mathbb{P}(\sigma_N |\mathcal{L}_{h_N}^{\leq r_N}| \leq \varepsilon_N/4)^{NT/h_N}$$

$$= \left(1 - \mathbb{P}\left(\sigma_N r_N |\mathcal{L}_{h_N r_N^{-\alpha}}^{\leq 1}| \geq \varepsilon_N/4 \right) \right)^{NT/h_N}.$$

Since the Lévy measure $c_\alpha \mathbf{1}_{|y| \leq 1} \frac{dy}{|y|^{1+\alpha}}$ has compact support, the random variables $\mathcal{L}_t^{\leq 1}$ have exponential moments, and Chernov's inequality yields

$$\mathbb{P}\left(\sigma_N r_N |\mathcal{L}_{h_N r_N^{-\alpha}}^{\leq 1}| \geq \varepsilon_N/4 \right) \leq \mathbb{E}\left(e^{\left| \mathcal{L}_{h_N r_N^{-\alpha}}^{\leq 1} \right|} \right) e^{-\varepsilon_N/4\sigma_N r_N} = e^{Kh_N r_N^{-\alpha} - \varepsilon_N/4\sigma_N r_N},$$

where the constant K does not depend on N.

In the Brownian case $\alpha = 2$, we use the tail estimate $\int_M^\infty e^{-x^2} dx \leq Ke^{-M^2}$ for positive M. □

Lemma 3.13. *Let $a_1 \leq \ldots \leq a_N$ and $b_1 \leq \ldots \leq b_N$ be two nondecreasing sequences of reals numbers. Then the quantity $\sum_{i=1}^N a_i b_{\sigma(i)}$ for some permutation σ is maximal when $\sigma(i) = i$ for all i.*

Proof. From optimal transportation theory (see [14, page 75]), the quantity $\sum_{i=1}^{N}(a_i - b_{\sigma(i)})^2$ is minimal when σ is the identity. Expanding the square, we see that $\sum_{i=1}^{N}(a_i - b_{\sigma(i)})^2 = \sum_{i=1}^{N}(a_i^2 + b_i^2) - 2\sum_{i=1}^{N} a_i b_{\sigma(i)}$. Thus, $\sum_{i=1}^{N} a_i b_{\sigma(i)}$ is maximal if and only if $\sum_{i=1}^{N}(a_i - b_{\sigma(i)})^2$ is minimal, concluding the proof. $\square$

Lemma 3.14. *Let f be some bounded function with compact support on $[0, \infty) \times \mathbb{R}$ which is smooth with respect to the space variable. If h_N vanishes and σ_N is bounded, it holds*

$$\lim_{N \to \infty} \mathbb{E} \left| \sum_{\kappa_i^N > 0} \int_0^\infty \left(w_t^i - w_{\tau_t}^i \right) f_t \left(X_t^{N,i,r_N} \right) \mathrm{d}t \right| = 0.$$

Proof. First notice that when t is not in an interval $[kh_N, (k+1/2)h_N]$, it holds $w_t^i = w_{\tau_t}^i$, since no particle moved between τ_t and t. Then, one can write, from the assumptions on f,

$$\left| \sum_{\kappa_i^N > 0} \int_0^\infty \left(w_t^i - w_{\tau_t}^i \right) f_t \left(X_t^{N,i,r_N} \right) \mathrm{d}t \right| \leq \left| \sum_{\kappa_i^N > 0} \int_0^T \chi_t^N \left(w_t^i f_t(X_t^{N,i,r_N}) - w_{\tau_t}^i f_t(X_{\tau_t}^{N,i,r_N}) \right) \mathrm{d}t \right|$$

$$+ \frac{K}{N} \sum_{\kappa_i^N > 0} \int_0^T \chi_t^N \left| X_t^{N,i,r_N} - X_{\tau_t}^{N,i,r_N} \right| \wedge 1 \mathrm{d}t.$$

Integrating by parts, it holds:

$$\left| \sum_{\kappa_i^N > 0} \int_0^T \chi_t^N \left(w_t^i f_t(X_t^{N,i,r_N}) - w_{\tau_t}^i f_t(X_{\tau_t}^{N,i,r_N}) \right) \mathrm{d}t \right|$$

$$= \left| \int_0^T \chi_t^N \int_{\mathbb{R}} \left(\eta \left(H * \tilde{\mu}_t^{N,r_N}(x) \right) - \eta \left(H * \tilde{\mu}_{\tau_t}^{N,r_N}(x) \right) \right) \partial_x f_t(x) \mathrm{d}x \mathrm{d}t \right|$$

$$\leq \frac{K}{N} \int_0^T \chi_t^N \int_{\mathbb{R}} \left(\sum_{\kappa_i^N > 0} \mathbf{1}_{X_t^{N,i,r_N} \leq x < X_{\tau_t}^{N,i,r_N}} + \mathbf{1}_{X_{\tau_t}^{N,i,r_N} \leq x < X_t^{N,i,r_N}} \right) \partial_x f_t(x) \mathrm{d}x \mathrm{d}t$$

$$\leq \frac{K}{N} \int_0^T \chi_t^N \sum_{\kappa_i^N > 0} \left| X_t^{N,i,r_N} - X_{\tau_t}^{N,i,r_N} \right| \wedge 1 \mathrm{d}t$$

We conclude the proof by writing

$$\mathbb{E} \frac{1}{N} \int_0^T \chi_t^N \sum_{\kappa_i^N > 0} \left| X_t^{N,i,r_N} - X_{\tau_t}^{N,i,r_N} \right| \wedge 1 \mathrm{d}t = \mathbb{E} \int_0^T \chi_t^N \mathbf{1}_{\kappa_1^N > 0} \left| X_t^{N,1,r_N} - X_{\tau_t}^{N,1,r_N} \right| \wedge 1 \mathrm{d}t$$

$$\leq T \left(h_N \sup_{[-1,1]} |A'| + \mathbb{E} \left((\sigma_N |\Lambda_{h_N}^{N,1,r_N}|) \wedge 1 \right) \right).$$

This last quantity vanishes when h_N goes to 0. $\square$

4 Numerical results

In this section, we illustrate our convergence results by some numerical simulations. We simulated the solution to the fractional and the inviscid Burgers equations

$$\partial_t u + \frac{1}{2} \partial_x (u^2) + \sigma^\alpha (-\Delta)^{\frac{\alpha}{2}} = 0 \ \ \text{and} \ \ \partial_t u + \frac{1}{2} \partial_x (u^2) = 0,$$

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corresponding to the choice $A(x) = x^2/2$, with different values for the parameter α.

One can find an explicit exact solution to the inviscid Burgers equation (see [12]) and we compare the result of the simulation to this exact solution in the vanishing viscosity setting. However, to our knowledge, no explicit solutions exist in the case of a positive viscosity coefficient for $\alpha < 2$, so that we have to compare the result of our simulation with the one given by another numerical method. Here, we use a deterministic method, introduced by Droniou in [7].

4.1 Constant viscosity ($\sigma_N = \sigma$)

We give three examples of approximation to the viscous conservation law. On Figures 1, 2 and 3, we show the approximation of the viscous conservation law with respective index $\alpha = 1.5$, $\alpha = 1$ and $\alpha = 0.1$ and diffusion coefficient $\sigma = 1$ using $N = 1000$ particles, with parameters $h = 0.01$ and $\varepsilon = 0.04$ at simulation times 0.25, 0.5, 0.75 and 1. The continuous line is the simulated solution, and the dotted line is the "exact" solution obtained with the determistic scheme of [7] using small time and space steps.

We now investigate the vanishing rate of the error, that is the Riemann sum on the discretization grid associated to the integral in Theorems 3.1, 3.2 and 3.3. On Figure 4 is depicted the logarithmic plot of the error as a function of N where we used the relation $h_N = 10/N$, and $\varepsilon_N = 40/N$, with N ranging from 10 to 10000, in the three cases $\alpha = 0.5$, 1 and 1.5. In the case $\alpha < 1$, this relation between N, h_N and ε_N satisfies the condition of Theorem 3.1. These pictures make us expect a convergence rate of $\frac{1}{\sqrt{N}}$, corresponding to the optimal rate analyzed theoretically in [5, 6], in the case $\alpha = 2$, without killing.
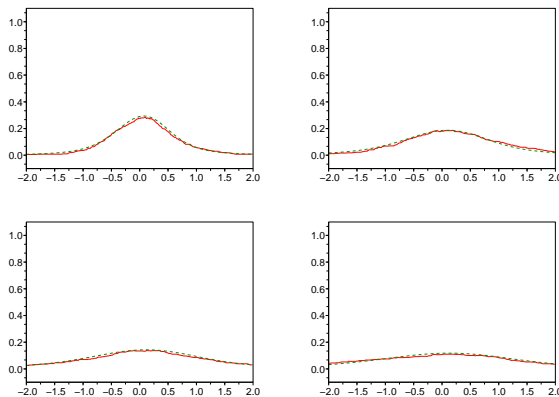

Figure 1: Approximation of the conservation law with index $\alpha = 1.5$.

4.1.1 Behaviour as $h \to 0$

We give in Figure 5 the approximation error at fixed number of particle, with a vanishing time step h, in logarithmic plot. We set the parameter ε to be equal to $4h$ so that the condition of Theorem 3.1 is satified. We took $N = 340000$ and $\sigma = 1$. We set $\alpha = 0.5$, $\alpha = 1$ and $\alpha = 1.5$ respectively. The different parameters h range from 1 to 2^{-8}. In [5, 6] it is shown, in case $\alpha = 2$ and the initial condition is monotonic, that the error is of order h. In view of Figure 5, it seems that the convergence rate is still of order h, even for $\alpha < 2$ and any initial condition with bounded variation.
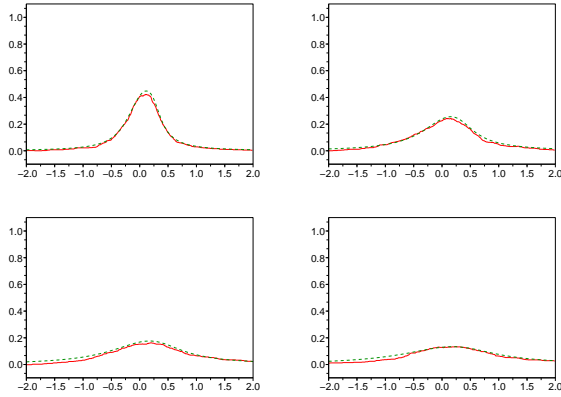
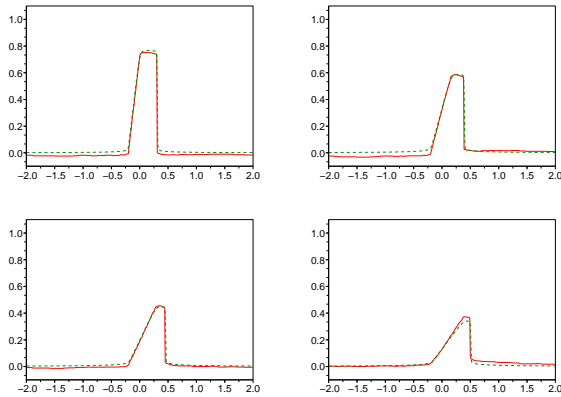Figure 2: Approximation of the conservation law with index $\alpha = 1$.

Figure 3: Approximation of the conservation law with index $\alpha = 0.1$.

4.2 Vanishing viscosity $(\sigma_N \to 0)$

We consider the Burgers equation

$$\partial_t v = \partial_x(u^2/2)$$

with initial condition $u_0(x) = \mathbf{1}_{[-3,-2]} - \mathbf{1}_{[2,3]}$, which is the cumulative distribution function of the measure $\delta_{-3} - \delta_{-2} + \delta_2 - \delta_3$. In that case, the solution of the Burgers equation is explicit and given by the expression

$$u(t,x) = \min\left(\frac{x+3}{t}, 1\right) \mathbf{1}_{[-3,\min(-2+\frac{t}{2},-3+\sqrt{2t},0)]} + \max\left(\frac{x-3}{t}, -1\right) \mathbf{1}_{[\max(2-\frac{t}{2},3-\sqrt{2t},0),3]}.$$

We compare the function u to the function obtained by running the Euler scheme with a small diffusion coefficient σ. One can expect the approximation to be better for large values of α. Indeed, for small values of α, the particles tend to jump very far away, and subsequently "disappear" from
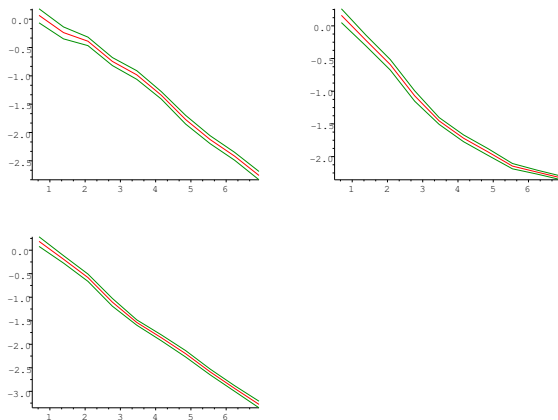
25

Figure 4: Logarithmic error in the approximation of the conservation law with index $\alpha = 0.5$, 1 and 1.5. The respective slopes are -0.46, -0.41 and -0.56.
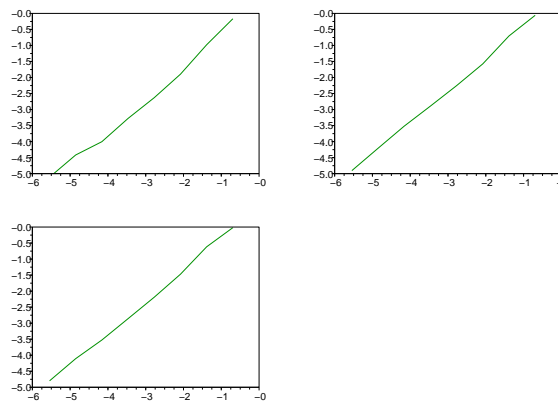

Figure 5: Logarithmic plot of the error as h vanishes, with a fixed number of particles, at respectively $\alpha = 0.5$, 1 and 1.5. The slopes are equal to 1 up to an error of 0.01.

the simulation. The consequence of this behavious is that the solution is somehow decreased by a multiplicative constant.

For large values of α, the approximation is quite good, even for not so small diffusion coefficients. Figure 6 gives the result of the simulation of the Euler scheme with parameters $\alpha = 1.5$, $\varepsilon = 0.04$, $\sigma = 0.1$ and $h = 0.01$, at the different times 2, 4, 6 and 8 for $N = 10000$ particles. Figure 7 gives the same simulation for $\alpha = 1$. In the case $\alpha < 1$, and especially when α is small, one need to take a very small value for the diffusion coefficient in order to have a reasonable approximation of the solution. Indeed, the approximation depicted on the Figure 8 is the appproximation of the solution at times 2, 4, 6 and 8 for diffusion coefficient $\sigma = 10^{-4}$. Here, we used 10000 particles killed at a distance $\varepsilon = 0.01$, the time step being $h = 0.01$. On Figure 9 we show the same simulation, with

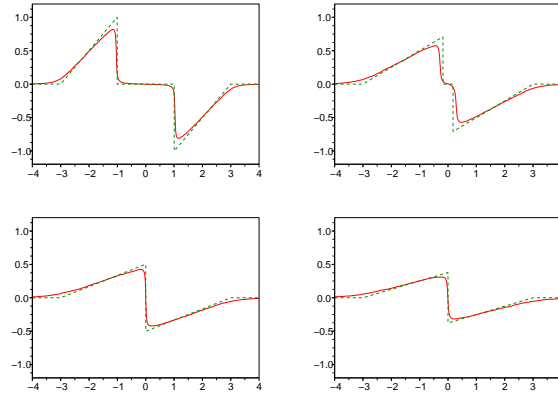diffusion coefficient changed to $\sigma = 10^{-12}$.

Figure 6: Approximation of the inviscid conservation law by a fractional Euler scheme with index $\alpha = 1.5$ and diffusion coefficient 0.1.
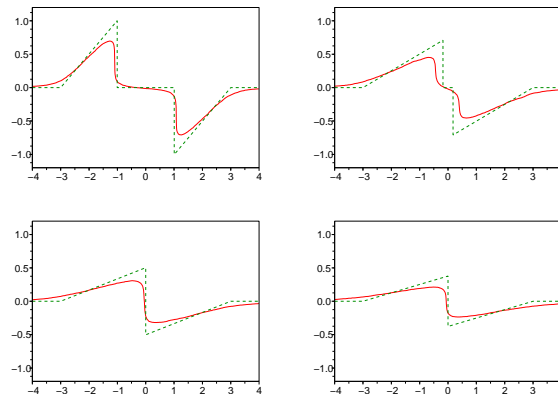

Figure 7: Approximation of the inviscid conservation law by a fractional Euler scheme with index $\alpha = 1$ and diffusion coefficient 0.1.

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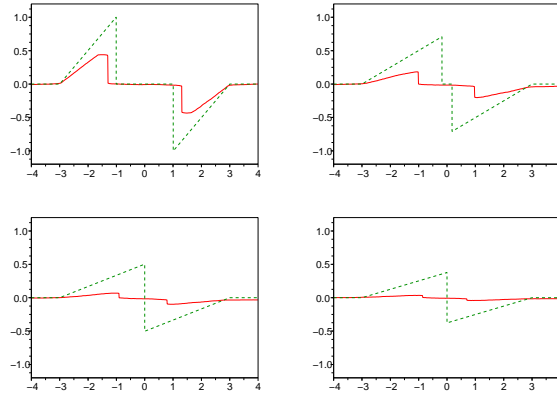
Figure 8: Approximation of the inviscid conservation law by a fractional Euler scheme with index $\alpha = 0.1$ and diffusion coefficient 10^{-4}.
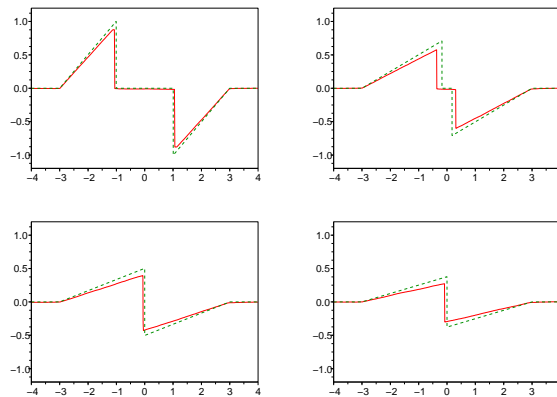

Figure 9: Approximation of the inviscid conservation law by a fractional Euler scheme with index $\alpha = 0.1$ and diffusion coefficient 10^{-12}.

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