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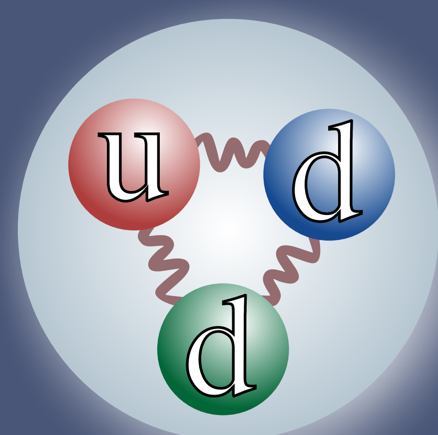

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Primera edición, 2009
Bogotá, Colombia

Publicación digital:
Editorial UN
direditorial@unal.edu.co
Bogotá, Colombia

Diseño de carátula: Andrea Kratzer M.

Construction of three-quark wave functions with definite symmetry

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Abstract

We present a pedagogical construction of three-quark wave functions making use of particle interchange symmetry properties. We consider spatial, spin and isospin degrees of freedom. Color is introduced to obtain completely antisymmetric wave functions. We also analyze the general structure of the spatial part of the wave functions both in coordinate and momentum space.

Contents

Nobody knows if the white is white,
neither if the red is red,
nobody knows in his intimate painful conscience
scared of his own inhuman transcendence
what is truth in what is really truth.

Fragment from "Dios, el alma la muerte",
Fernando Paz Castillo

1 Introduction

The general study of the wave functions for three identical spin 1/2 particles is of great interest both in nuclear and in subnuclear physics, more precisely, for investigating the properties of ^{3}H and ^{3}He and for constructing three-quark models for the nucleon and its resonances [1]. The latter point will be the main objective of these notes.

In this work we present a pedagogical introduction to the three-body, equal mass, quantum mechanical problem, by using, as starting point, standard concepts of nonrelativistic quantum mechanics. After studying this note, an *advanced* student should be able to construct complete three-quark wave functions and to calculate physical observables. In this way, he should be ready to face *actual* investigation problems in this field, as, for example, the *relativistic* calculation of approximate solutions for effective quark Hamiltonians and of electroweak matrix-elements (form factors) for baryonic particles. Many points analyzed in the present work can be also found in standard texbooks [2], [3], [4] and in specialized articles [1]. Aim of these notes is to gather in a concise and pedagogical text all the relevant information, writing explicitly the expressions of the wave functions that are needed for the calculations. As a specific application of the formalism, in the last section we study the calculation of the nucleon magnetic moment in the constituent quark model.

This work should be self-consistent. Cited papers are not strictly necessary for its comprehension but should help the reader to gain a deeper insight in this field of investigation.

Considering the physical problem at a dynamical level, we recall that one possible way to understand low-energy baryonic phenomenology is presently based on the Relativistic Hamiltonian Dynamics (RHD), that represents a consistent theoretical framework for the study of composite systems with a *fixed number* of strongly interacting particles [5, 6]. RHD satisfies both quantum mechanics and relativistic covariance introducing as generators of the space-time symmetries, quantum-mechanical operators (Hamiltonian, three-momentum, angular momentum and Lorentz boost), that fulfill the commutation rules of the Poincaré algebra. The form of these operators is different from that of the nonrelativistic ones but all the computational rules and the structure of the wave functions, remain the *same* as in the nonrelativistic quantum mechanics. This point explains the interest of the present study as an introduction to the investigation on baryonic systems.

At pedagogical level the construction of antisymmetric wave functions for interacting identical fermions is usually studied in standard courses of quantum mechanics, but, in that case, the interest is focussed on problems of atomic physics, in which the electrons mainly interact with the nucleus, considered as a *fixed* source of potential. On the other hand, the study of the few-body nuclear and subnuclear systems, in which the constituent identical particles only interact among them, strictly requires the separation of the Center of Mass motion from the intrinsic one.

To this aim, specific techniques must be introduced, in particular the definition of *mixed symmetric* (MS) an *mixed antisymmetric* (MA) functions, besides the *symmetric* (S) and *antisymmetric* (A) ones, that are used to construct the total, completely antisymmetric, wave functions, as required by the Pauli exclusion principle. We point out that, in the study of two-body systems, the MS and MA functions do not appear. In fact, they represent an original aspect of the three-body problems.

Furthermore, we give a phenomenological and historical introduction to the concepts of Isospin and Color, explicitly constructing the corresponding wave functions and studying their symmetry properties.

As for the spatial part of the wave functions, we shall initially consider, as the most transparent pedagogical introduction to our three-body problem, the nonrelativistic harmonic oscillator (NR HO) model. In this case, simple, analytic solutions can be obtained. We point out that the *completeness* of such set of states really allows to use it in research problems, also for

the variational solution of relativistic Hamiltonians. Furthermore, one can straightforwardly perform their Fourier Transform in order to obtain the corresponding momentum space wave functions, that are stricly necessary to calculate the electroweak form factors in the recent models based on the RHD.
We explain that other spatial wave functions can be also advantageously implemented for the variational calculations. These new spatial functions are obtained leaving *the same* symmetry structure of the NR HO ones, but introducing a different asymptotic behavior. In this way we suggest to the interested student a possible field of investigation.

2 General symmetry properties

We previously recall the fundamental properties of the operator $\hat{P}_{ij}$, that interchanges the particle i with the particle j (being obviously $i \neq j$). It determines if a state function is symmetric, or not, under particles interchange. This operator satisfies the property $\hat{P}_{ij}^2 = 1$ and therefore, $\hat{P}_{ij}^\dagger = \hat{P}_{ij} = \hat{P}_{ij}^{-1}$. In consequence, the eigenvalues of this operator are

$$\hat{P}_{ij} = \begin{cases} +1 & \text{for symmetric states,} \\ -1 & \text{for antisymmetric states.} \end{cases}$$

We now consider an operator $\hat{O}_{ij}$ for the composite system, that depends on the variables of the particles i and j. We have

$$\hat{P}_{ij}\hat{O}_{ij}|\Psi\rangle = \hat{O}_{ji}\hat{P}_{ij}|\Psi\rangle \tag{1}$$

that holds for an arbitrary $|\Psi\rangle$. Then, the operator transformation is

$$\hat{P}_{ij}\hat{O}_{ij}\hat{P}_{ij} = \hat{O}_{ji} \tag{2}$$

2.1 Jacobi variables and mixed-symmetry quantities

The definition of the spatial variables represents the starting point for the construction of the wave functions of a composite system [1]. Furthermore, in our case, the use of the Jacobi variables allows to introduce in a natural way the pairs of *mixed symmetry* quantities.

For three identical particles, one can choose as standard spatial variables the position vectors $\vec{r}_i$ or their conjugated momenta $\vec{p}_i$. Here and in the following $i = 1, 2, 3$. In order to separate the Center of Mass wave function (that, for a bound system without external interactions, is simply represented by a plane wave of momentum $\vec{P}$) from the intrinsic one and to display the symmetry properties of the latter, an adequate treatment is obtained introducing the so-called Jacobi variables. In our case they are defined as

$$\vec{\rho} = \frac{1}{\sqrt{2}}(\vec{r}_1 - \vec{r}_2) \tag{3a}$$

$$\vec{\lambda} = \frac{1}{\sqrt{6}}(\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3) \tag{3b}$$

$$\vec{R} = \frac{1}{3}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3) \tag{3c}$$

where $\vec{\rho}$ y $\vec{\lambda}$ represent variables for the internal motion, and $\vec{R}$ the position of the Center of Mass of the system. The transformations (3) can be inverted, giving

$$\vec{r}_1 = +\frac{1}{\sqrt{2}}\vec{\rho} + \frac{1}{\sqrt{6}}\vec{\lambda} + \vec{R} \tag{4a}$$

$$\vec{r}_2 = -\frac{1}{\sqrt{2}}\vec{\rho} + \frac{1}{\sqrt{6}}\vec{\lambda} + \vec{R} \tag{4b}$$

$$\vec{r}_3 = -\sqrt{\frac{2}{3}}\vec{\lambda} + \vec{R} \tag{4c}$$

In coordinate space, by using the standard chain rule

$$\frac{\partial}{\partial \vec{r}_i} = \frac{\partial \vec{\rho}}{\partial \vec{r}_i}\frac{\partial}{\partial \vec{\rho}} + \frac{\partial \vec{\lambda}}{\partial \vec{r}_i}\frac{\partial}{\partial \vec{\lambda}} + \frac{\partial \vec{R}}{\partial \vec{r}_i}\frac{\partial}{\partial \vec{R}} \tag{5}$$

we can express the particle momentum operators by means of the conjugated Jacobi momenta $\vec{p}_\rho$, $\vec{p}_\lambda$ and $\vec{P}$, as

$$\vec{p}_1 = +\frac{1}{\sqrt{2}}\vec{p}_\rho + \frac{1}{\sqrt{6}}\vec{p}_\lambda + \frac{1}{3}\vec{P} \tag{6a}$$

$$\vec{p}_2 = -\frac{1}{\sqrt{2}}\vec{p}_\rho + \frac{1}{\sqrt{6}}\vec{p}_\lambda + \frac{1}{3}\vec{P} \tag{6b}$$

$$\vec{p}_3 = -\sqrt{\frac{2}{3}}\vec{p}_\lambda + \frac{1}{3}\vec{P} \tag{6c}$$

being $\vec{p}_\rho = -i\hbar\frac{\partial}{\partial\vec{\rho}}$, $\vec{p}_\lambda = -i\hbar\frac{\partial}{\partial\vec{\lambda}}$ and $\vec{P} = -i\hbar\frac{\partial}{\partial\vec{R}}$.
Finally, inverting the previous equations (6), we obtain the definitions of the Jacobi momenta

$$\vec{p}_\rho = \frac{1}{\sqrt{2}}\vec{p}_1 - \frac{1}{\sqrt{2}}\vec{p}_2 \tag{7a}$$

$$\vec{p}_\lambda = \frac{1}{\sqrt{6}}\vec{p}_1 + \frac{1}{\sqrt{6}}\vec{p}_2 - \sqrt{\frac{2}{3}}\vec{p}_3 \tag{7b}$$

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 \tag{7c}$$

When studying relativistic quark models by means of RHD, the *rest* reference frame, is used for the calculations. In consequence, eqs.(6) with $\vec{P} = 0$ are taken to *define* the relation between the three rest frame (not independent) particle momenta and the two (independent) Jacobi momenta $\vec{p}_\rho$ and $\vec{p}_\lambda$.
The rest frame wave functions are determined as the eigenfunctions of the corresponding *mass operator* that will be discussed in sect. 8. Then, these wave functions are boosted (by means of the RHD boost operator) to any generic reference frame to study the scattering processes of the bound system.

We now explain some remarkable properties of the Jacobi variables with respect to the interchange operators. By using their definitions in eqs.(3), the reader can easily verify that

$$\hat{P}_{ij}\,\vec{R}\,\hat{P}_{ij} = \vec{R} \quad \text{for all } ij \text{ pairs} \tag{8}$$

and

$$\hat{P}_{12}\,\vec{\rho}\,\hat{P}_{12} = -\vec{\rho} \qquad \hat{P}_{13}\,\vec{\rho}\,\hat{P}_{13} = \frac{1}{2}\vec{\rho} - \frac{\sqrt{3}}{2}\vec{\lambda} \tag{9}$$

$$\hat{P}_{12}\,\vec{\lambda}\,\hat{P}_{12} = \vec{\lambda} \qquad \hat{P}_{13}\,\vec{\lambda}\,\hat{P}_{13} = -\frac{\sqrt{3}}{2}\vec{\rho} - \frac{1}{2}\vec{\lambda} \tag{10}$$

As it could be expected, the Center of Mass coordinate $\vec{R}$ is a completely symmetric (S) quantity. On the other hand, due to the transformation properties shown in eqs.(9) and (10), $\vec{\rho}$ and $\vec{\lambda}$ represent a *pair* of a mixed antisymmetric (MA) quantity and a mixed symmetric (MS) one, respectively.
Also, note that the conjugated momenta $\vec{p}_\rho$, $\vec{p}_\lambda$ and $\vec{P}$ given in eqs.(7) have the same symmetry properties as the corresponding Jacobi position variables.

The following point is of *fundamental* importance for the construction of three-quark wave functions: we shall introduce other pairs of quantities, (in particular, some factors that appear will in the total the wave functions) with the same transformation properties as $\vec{\rho}$ and $\vec{\lambda}$. Such pairs of quantities will be also denoted as MA MS.
Furthermore, two pairs of MA MS quantities can be combined, according to the rules of eq.(21), for constructing four new quantities with the following transformation properties: S, A and a pair MA MS.
Finally, wave functions with different symmetry properties are mutually orthogonal.

3 The Spin

The quarks are spin 1/2 particles. This property is strictly necessary to reproduce the observed values of the angular momenta of the nucleon and of the other resonances. The well-known spin composition rules will be used to construct the *total* spin functions. Their transformation properties will be analyzed.

3.1 Three-fermion spin functions and their symmetry properties

We now build the coupled spin functions for three spin 1/2 particles. We shall adopt the following standard angular momentum coupling notation:

$$[\varphi_{j_1} \otimes \varphi_{j_2}]_{J,M} = \sum_{m_1,m_2} \varphi_{j_1,m_1} \varphi_{j_2,m_2} \langle j_1 m_1; j_2 m_2 | JM \rangle$$

where the last factor is the appropriate the Clebsch-Gordan coefficient.
The standard procedure consists in coupling the spins of the particles 1 and 2 to S_{12} and then this last quantity to the spin of the particle 3. It gives

$$\chi^{S_{12}}_{S,M} = \left[\left[\chi_{\frac{1}{2}}(1) \otimes \chi_{\frac{1}{2}}(2) \right]_{S_{12}} \otimes \chi_{\frac{1}{2}}(3) \right]_{S,M} \tag{11}$$

Due to triangular inequality, the only possible values are $S_{12} = 0, 1$ and $S = 1/2$, $3/2$.

By using the standard notation $\chi_{\frac{1}{2},\frac{1}{2}} = \uparrow$ and $\chi_{\frac{1}{2},-\frac{1}{2}} = \downarrow$, for spin *up* and spin *down* one-particle functions, we can write in a more explicit way the previous equation (11) for all the possible values of S_{12}, S and M. It takes the form:

$$\chi^0_{\frac{1}{2},+\frac{1}{2}} = \frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \tag{12a}$$

$$\chi^0_{\frac{1}{2},-\frac{1}{2}} = \frac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow) \tag{12b}$$

$$\chi^1_{\frac{1}{2},+\frac{1}{2}} = -\frac{1}{\sqrt{6}}(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow) \tag{12c}$$

$$\chi^1_{\frac{1}{2},-\frac{1}{2}} = \frac{1}{\sqrt{6}}(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow - 2\downarrow\downarrow\uparrow) \tag{12d}$$

$$\chi^1_{\frac{3}{2},+\frac{3}{2}} = \uparrow\uparrow\uparrow \tag{12e}$$

$$\chi^1_{\frac{3}{2},+\frac{1}{2}} = \frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow) \tag{12f}$$

$$\chi^1_{\frac{3}{2},-\frac{1}{2}} = \frac{1}{\sqrt{3}}(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow) \tag{12g}$$

$$\chi^1_{\frac{3}{2},-\frac{3}{2}} = \downarrow\downarrow\downarrow \tag{12h}$$

In all the terms, the arrows represent, *in the order*, the spin projection (on the z-axis) of the particle 1, 2 and 3.
By applying the permutation operators $\hat{P}_{ij}$ to the previous spin functions (12), one can directly verify that, for a given spin projection M, $\chi^0_{\frac{1}{2},M}$ and $\chi^1_{\frac{1}{2},M}$ represent a pair of MA and MS quantities, respectively. Finally the $\chi^1_{\frac{3}{2},M}$ are all completely symmetric (S) functions.

Note the following general property: all the members of a spin *multiplet*, that is, all the functions with given values of S_{12} and S but different values of the third component M, have the *same* permutational symmetry.
The symmetry properties of eqs.(12) will be used in subsect. 2.1 to construct spin-isospin wave functions with definite symmetry.

4 The Isospin

Before introducing the isospin for the quarks, it is useful to recall its general features in the context of nuclear physics, where it was introduced for the first time. Historically, isospin represents the first example of a quantum number of an *internal symmetry*, not related to space-time transformations.

4.1 Isospin and nuclear interactions

To study the isospin formalism, we briefly recall some general phenomenological properties of the nuclear particle interactions.

1) The proton and the neutron have approximately the same mass, being $m_p c^2 = 938.27\ MeV$ and $m_n c^2 = 939.57\ MeV$.

2) In the nuclei their interactions give rise to binding energies of the order of $8\ MeV = 8 \times 10^6\ eV$ *per nucleon*, justifying the definition of *strong interactions*.

3) These interactions have a *short range* in the sense that if the distance d between the two nucleons is $d \geq 2\ fm = 2 \times 10^{-13} cm$ their interaction is vanishing. Note that the presence of this *cutoff* in the range is completely different with respect to the case of the electromagnetic interactions, where the Coulomb potential $V_C = e_1 e_2 / r$ is a *long range* one, that is, decreases continously with *no cutoff*.

4) For a given set of quantum numbers of the state of the system, the strong interaction is the same for all kinds of nucleons, no matter if they are protons or neutrons.

The properties 1) and 4) allow to introduce the formalism of isospin. As a first step, one has to consider the proton and the neutron as two (quasi) degenerate states of the same particle, the *nucleon*, with different electric charge. In this sense, the property 4) can be rephrased, saying that the strong interactions are *charge independent*.
Formally the two nucleon states can be represented by means of the following

spinors:

$$\begin{pmatrix}1\\0\end{pmatrix} \quad \text{for the proton and} \quad \begin{pmatrix}0\\1\end{pmatrix} \quad \text{for the neutron.}$$

These two states are eigenstates of the following operator

$$\tau_3 = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \tag{13}$$

so that the nucleon charge operator (without the numerical value of the elementary charge $\mid e|$) is defined as

$$\hat{Q}_N = \frac{1}{2}\tau_3 + \frac{1}{2} \tag{14}$$

with eigenvalues $Q_p = 1$ and $Q_n = 0$.
We can now introduce all the *three* Pauli isospin matrices $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ and define the isospin operator as $\vec{t} = \vec{\tau}/2$, satisfying the standard commutation rules

$$[t_i, t_j] = i\varepsilon_{ijk}t_k \tag{15}$$

where ε_{ijk} is the Levi-Civita tensor. This formalism is exactly the same as that of spin 1/2.
We can make the following question: is the isospin formalim really useful to describe the nuclear systems ?
The answer is *yes* if the strong interaction Hamiltonian is *invariant* under rotations in the *abstract* isospin space. In this case, defining for a nucleus with A nucleons the total isospin operator as

$$\hat{\vec{T}} = \sum_{i=1}^{A} \vec{t}(i) \tag{16}$$

one can diagonalize, simultaneously with the strong Hamiltonian H_{str}, $\hat{\vec{T}}^2$ and the third component $\hat{T}_3$, that are *conserved quantities*:

$$[H_{str}, \hat{T}_3] = 0 \quad [H_{str}, \hat{\vec{T}}^2] = 0$$

Furthermore, systems with the same T but different values of T_3 belong to degenerate multiplets, i.e. they have the same rest energy.

Formally, to satisfy the previous commutation rules, a two-body (i, j) strong interaction operator can only contain the following operators

$$\mathbf{1}, \qquad \vec{\tau}(i)\vec{\tau}(j)$$

The physical consequences of the isospin space rotation invariance are *experimentally* verified with great accuracy in nuclear physics and in the scattering processes of hadronic particles. The discrepancies (for example the breakdown of the multiplet degeneracy) are due to the presence of the electromagnetic (and weak) interaction. In fact one has

$$[H_{em}, \hat{T}_3] = 0 \quad \text{but} \quad [H_{em}, \hat{\vec{T}}^2] \neq 0$$

However, the effects due to H_{em} can be, in general, calculated perturbatively, obtaining a good agreement with the experimental data.

As for the presence of *multiplets* in nuclear physics, we recall that the nucleon is an iso-doublet ($T = 1/2$), the Deuteron ^{2}H is an iso-singlet ($T = 0$) because it is the *only* bound state of two nucleons (*pp* and *nn* are not bound), the ^{3}H and ^{3}He form an iso-doublet ($T = 1/2$) because they have almost the same binding energy (apart from the small repulsive electromagnetic contribution in the ^{3}He), the ^{4}He is an iso-singlet ....
As an exercise, the reader is suggested to obtain the values of T of the above multiplets by using the composition rules of isospin 1/2 objects. Recall that, due to the commutation rules of eq.(15), all the mathematical properties of isospin, in particular the *composition rule*, are exactly the same as those of the spin.
Finally, in the general case of A nucleons, note that T is *integer* for A even and *semi-integer* for A odd.

Up to this point we have considered strong interacting particles with semi-integer spin (in particular the nucleons). This kind of particles are called *baryons*. There are other strong interacting particles with *integer spin*. These particles are called *mesons*. Among them, the most important for the study of the nuclear processes, is the pion. Really, there are *three* pions: π^+, π^0 and π^- of charge $+1$, 0 and -1, respectively. Their masses are $m_{\pi^+}c^2 = m_{\pi^-}c^2 = 139.57 MeV$ and $m_{\pi^0}c^2 = 134.97 MeV$. For our study, it is relevant to recall that the strong interactions of the pions are invariant under isospin rotations and that the three pions (obviously) belong to an iso-triplet: $T = 1$ gives $2T + 1 = 3$ degenerate isospin states. The charge of the three pions can be expressed in terms of the third component of the isospin by means of the simple equation

$$\hat{Q}_\pi = \hat{T}_3 \tag{17}$$

We note that, with respect to eq.(14) the last term $1/2$ is missing. To explain in very simple words this difference between nucleons and pions (or, more generally, between baryons and mesons) we observe that, in a reaction, a single meson can appear or disappear, while a baryon cannot. A new *conserved* quantum number is introduced, that is the *baryon number* B. For the nucleon (and in general for a baryon) we have $B = 1$, while for the pion (and in general for a meson) we have $B = 0$.
Baryons and mesons, that is all the particles with strong interactions, are called *hadrons*.
By means of the baryon number we can unify eqs.(14) and (17) in the form

$$\hat{Q} = \hat{T}_3 + \frac{B}{2} \tag{18}$$

that works for both baryons and mesons.

4.2 Isospin of the quarks up and down

We can now make the following question. Are the hadrons really *elementary* particles or *composite* systems, like the atoms and the nuclei?
The answer relies on the experimental data about their static electromagnetic properties (in particular, the nucleon magnetic moment, that will be studied in sect. 9) and about the elastic and inelastic electron scattering. All the data clearly indicate that some *constituent* particles are present inside the hadrons.
The most successfull way to describe the experimental data of the hadronic particles is to introduce, as constituents, the *quarks* whose properties will be discussed in the following. Without entering into the details, we observe that, starting from the concept of quarks, it has been developed a *fundamental* quantum field theory for the study of the quarks and of their interactions. This theory is the Quantum Chromo-Dynamics (QCD). However, the same properties of this theory do not allow for a direct calculation of many low-energy observables of the hadronic particles, as, for example, their masses and

their form factors, that are measured by means of electron (or, in general, lepton) scattering processes.
For this reason, *effective* quark models (to which is directed the main interst of the present work) have been introduced. These models represent a practical and fruitful way to investigate the properties of the hadrons considered as bound systems of strongly interacting quarks. Quark models were firstly developed in the framework of the nonrelativistic quantum mechanics. Nowadays RHD is currently used.

Going to the quark properties, actually it is accepted the existence of six kinds of quarks (up, down, strange, charm, bottom, top). Only the quarks up (u) and down (d) are relevant (at least in a first approximation) for studying the nucleon and the pion.
We summarize here some important properties.
1) All the quarks have spin 1/2.
2) The nucleon and the baryons are made up of three quarks. The mesons are made up by a quark and an antiquark.
3) The charge of the quarks u and d are respectively $2/3$ and $-1/3$, so that the correct values for the nucleon charge are obtained. The quark content of the proton is *uud* and that of the neutron is *udd*.
4) The quarks u and d are assumed to have (almost) equal masses. In consequence, they can be assigned to an isodoublet and are represented by the same isospinors used for the proton and the neutron, respectively.
5) Their charge is obtained in terms of the isospin by means of eq.(18) setting $B_q = 1/3$. It means that, instead of eq.(14), one has

$$\hat{Q}_q = \frac{1}{2}\tau_3 + \frac{1}{6} \tag{19}$$

We stress that the choice $B_q = 1/3$ correctly gives the nucleon baryon number $B_N = 1$ simply *summing* the baryon numbers of the three quarks.
As for the mesons, their *vanishing* baryon number is obtained recalling that the antiquark baryon number is $B_{\bar{q}} = -1/3$.

Remembering that the spin and the isospin have the same algebraic properties, we can construct the isospin functions for three particles with isospin 1/2 exactly in the same way as the spin functions of eq.(11). We have

$$\Phi^{T_{12}}_{T,M_T} = \left[\left[\Phi_{\frac{1}{2}}(1) \otimes \Phi_{\frac{1}{2}}(2)\right]_{T_{12}} \otimes \Phi_{\frac{1}{2}}(3)\right]_{T,M_T} \tag{20}$$

Their explicit expressions are the same as those obtained in the equations (12), by replacing $\uparrow$ with u and $\downarrow$ with d. In passing, for the analysis of the three nucleon systems, as ^{3}He and ^{3}H, instead of u and d one has to replace p and n, respectively.

4.3 Linear combinations of the spin and isospin functions

The objective of this subsection is to construct spin-isospin wave functions with definite permutational symmetry. To this aim, consider the following general procedure. Take two pairs of mixed-symmetry functions: $f(\textsc{ms})$, $f(\textsc{ma})$ and $g(\textsc{ms})$, $g(\textsc{ma})$. One can build a symmetric function F_S, an anti-symmetric one F_A and a pair of mixed-symmetry functions F_{MA} F_{MS} in the following way:

$$F_S = +\frac{1}{\sqrt{2}}\left(f(\textsc{ms})g(\textsc{ms}) + f(\textsc{ma})g(\textsc{ma})\right) \tag{21a}$$

$$F_A = +\frac{1}{\sqrt{2}}\left(f(\textsc{ms})g(\textsc{ma}) - f(\textsc{ma})g(\textsc{ms})\right) \tag{21b}$$

$$F_{MA} = +\frac{1}{\sqrt{2}}\left(f(\textsc{ms})g(\textsc{ma}) - f(\textsc{ma})g(\textsc{ms})\right) \tag{21c}$$

$$F_{MS} = -\frac{1}{\sqrt{2}}\left(f(\textsc{ms})g(\textsc{ms}) - f(\textsc{ma})g(\textsc{ma})\right) \tag{21d}$$

The reader should check, as an exercise, the symmetry properties of the previous functions. Finally, for further applications, we recall that the symmetry properties of a quantity are not modified when this quantity is multiplied by a completely symmetric function.
Taking into account these rules, the following combinations can be made with spin and isospin functions. We write the result as

$$G^{\sigma}_{T,T_3;S,M_S}$$

where σ denotes the total symmetry (S,A,MS,MA); T, T_3 and S, M_S represent the total isospin and spin quantum numbers, respectively. We have

$$G^{S}_{\frac{3}{2},T_3;\frac{3}{2},M_S} = \Phi^{1}_{\frac{3}{2},T_3}\chi^{1}_{\frac{3}{2},M_S} \tag{22a}$$

$$G^{S}_{\frac{1}{2},T_3;\frac{1}{2},M_S} = \frac{1}{\sqrt{2}}\left[\Phi^{1}_{\frac{1}{2},T_3}\chi^{1}_{\frac{1}{2},M_S} + \Phi^{0}_{\frac{1}{2},T_3}\chi^{0}_{\frac{1}{2},M_S}\right] \tag{22b}$$

$$G^{MS}_{\frac{1}{2},T_3;\frac{1}{2},M_S} = -\frac{1}{\sqrt{2}}\left[\Phi^{1}_{\frac{1}{2},T_3}\chi^{1}_{\frac{1}{2},M_S} - \Phi^{0}_{\frac{1}{2},T_3}\chi^{0}_{\frac{1}{2},M_S}\right] \tag{22c}$$

$$G^{MA}_{\frac{1}{2},T_3;\frac{1}{2},M_A} = \frac{1}{\sqrt{2}}\left[\Phi^{1}_{\frac{1}{2},T_3}\chi^{0}_{\frac{1}{2},M_S} + \Phi^{0}_{\frac{1}{2},T_3}\chi^{1}_{\frac{1}{2},M_S}\right] \tag{22d}$$

$$G^{MS}_{\frac{3}{2},T_3;\frac{1}{2},M_S} = \Phi^{1}_{\frac{3}{2},T_3}\chi^{1}_{\frac{1}{2},M_S} \tag{22e}$$

$$G^{MA}_{\frac{3}{2},T_3;\frac{1}{2},M_S} = \Phi^{1}_{\frac{3}{2},T_3}\chi^{0}_{\frac{1}{2},M_S} \tag{22f}$$

$$G^{MS}_{\frac{1}{2},T_3;\frac{3}{2},M_S} = \Phi^{1}_{\frac{1}{2},T_3}\chi^{1}_{\frac{3}{2},M_S} \tag{22g}$$

$$G^{MA}_{\frac{1}{2},T_3;\frac{3}{2},M_S} = \Phi^{0}_{\frac{1}{2},T_3}\chi^{1}_{\frac{3}{2},M_S} \tag{22h}$$

$$G^{A}_{\frac{1}{2},T_3;\frac{1}{2},M_S} = \frac{1}{\sqrt{2}}\left[\Phi^{1}_{\frac{1}{2},T_3}\chi^{0}_{\frac{1}{2},M_S} - \Phi^{0}_{\frac{1}{2},T_3}\chi^{1}_{\frac{1}{2},M_S}\right] \tag{22i}$$

5 The Color

All the machinery discussed above, that is the quark spin and isospin, and the spatial part of the wave functions, that will be analyzed in detail in sect. 6, are not sufficient to describe baryons according to the fundamental principles of quantum mechanics. A new quantum number must be introduced for the quarks: the *Color*. In the following subsection we give a historical argument based on the symmetry properties of the Δ resonance wave function.

5.1 The puzzle of the Δ resonance wave function

Before facing the problem of the Δ resonance wave function, we must give some general explanations about the baryonic resonances.
We have seen that the nucleon can be modelized as a bound system made up of three quarks. In the same way as other composite bound systems (like atoms and nuclei, for example), it has excited states. These states are unstable: it means that, due to the strong interaction, they decay very fast

to the nucleon ground state. Denoting as Δt the lifetime of such a state, the so-called time-energy uncertainty relation

$$\Delta t \Delta E \simeq \hbar$$

says that these states, also called *resonances*, do not have a fixed energy but, rather, present an energy distribution of width ΔE. (The quark models are not able, up to now, to reproduce the width of the distribution but predict its peak value that is assumed to be the energy of the resonance).
In any case, by means of experimental measurements one determines the energy value of the resonance, its spin (that is the total angular momentum of the state in its rest frame), its parity and its isospin [7]. The same concepts also hold for the *mesonic* resonant states.
The lowest-lying nucleon resonance is the $\Delta_{3/2}^{3/2\ +}(1232)$ where the numerical value in parenthesis represents the peak value of its rest energy, that is $M_\Delta c^2 = 1232 MeV$. The other indices represent its spin and isospin values and the positive parity of the resonance. Due to the isospin value, *four* almost degenerate charge states are possible, see eq.(18) with $B = 1$, that are Δ_{++}, Δ_+, Δ_0 and Δ_-.
The Δ resonance can be excited by means of $\pi\ N \rightarrow \Delta$ scattering experiments. When the Center of Mass energy ($\sqrt{s}$) of the $\pi\ N$ system is equal to the peak value of the Δ mass, the scattering cross section reaches a maximum, highlighting to the resonant character of the process.
For the introduction of the color we are interested here in the wave function of the Δ resonance. For simplicity we consider initially the Δ_{++} charge state. Due to this charge value, the isospin part of its wave function, must be $|uuu>$. Analogously, taking the maximum projection angular momentum state $M_S = J = 3/2$ and assuming that the spatial part of the wave function has orbital angular momentum $L = 0$, one has that the spin function must be $|\uparrow\uparrow\uparrow>$.
We obtain the important result that the product of the spin and isospin parts of the wave fuction is *symmetric* under particle interchange.

Let us examine in more detail our assumption about the spatial part of the wave function. The data of the electromagnetic $N \rightarrow \Delta$ transitions indicate that this wave function must be very similar to that of the nucleon. Furthermore, the $N\ \Delta$ mass difference (relatively small with respect to the baryonic excitation energies) can be explained in terms of a spin-spin interaction, that will be shown in eq.(58) of subsect. 8.2. In fact, the matrix elements of this interaction are different for the Δ (aligned spins with $J = 3/2$) and for the Nucleon (non-aligned spins with $J = 1/2$) state. Summarizing, the spatial part of the Δ wave function must be very similar to the nucleon one, that, representing a ground state is, in general, a *symmetric* wave function with $L = 0$, as we shall see in eq.(39) for the harmonic oscillator model.

We obtain the result that the total wave function given by our quark model is *completely symmetric*, clearly violating the Pauli exclusion principle. One can repeat this construction for the general case of M_S and T_3 not equal to 3/2. The quantum numbers of the Δ with the spin and isospin wave functions of the previous sections require:

$$\Psi_\Delta(1232) = G^{S}_{\frac{3}{2},T_3;\frac{3}{2},M_S} \cdot \varphi_{space} \tag{23}$$

The reader can easily check that this result generalizes in a straightforward way the maximum projection case of $M_S = T_3 = 3/2$. Recall that in a given spin (or isospin) multiplet all the members have the same permutational symmetry!

5.2 The Introduction of Color

The color idea was suggested by O.W.Greenberg of the Maryland University almost to the same time that the quarks model appeared in 1964. One makes the hypothesis that the quarks have a new (hidden) quantum number, the color, that analogously to the isospin, gives rise to a new factor in the total wave function. To fulfill Pauli priciple, this factor is required to be *completely antisymmetric* under particle interchange. *Exact* invariance with respect to rotations in this new space is also assumed.
Looking at three-body spin functions of eqs.(12) one sees that no completely antisymmetric function can be found. It means that the SU(2) algebra (that is used for spin and isospin) is not able to solve our problem. It is necessary to introduce the SU(3) algebra, in which each quark can be found in three color states, conventionally denoted as *red*(r), *green* (g) and *blue* (b).
Without entering into the formal details, all the baryons are in a colorless (white) state, represented by the following antisymmetric wave function:

$$\Psi^{Bar}_{color} = \frac{1}{\sqrt{6}} \sum_{a,b,c} \epsilon_{abc} \psi_a(1) \psi_b(2) \psi_c(3) \tag{24}$$

where ϵ_{abc} is the (antisymmetric) Levi-Civita tensor and the sum is performed over the color indices of each quark. In this way the color wave function of the previous equation can be also written as

$$\Psi^{Bar}_{color} = \frac{1}{\sqrt{6}} \left[r(gb - bg) - b(gr - rg) - g(rb - br) \right] \tag{25}$$

For completeness we also give the meson color wave function, that has the form

$$\Psi^{Mes}_{color} = \frac{1}{\sqrt{3}} \left(r\bar{r} + g\bar{g} + b\bar{b} \right) \tag{26}$$

that also represents a colorless state. The notation $\bar{r}, \bar{g}, \bar{b}$ stands for the *anticolor* associated to the antiquarks of the meson.
We stress that the color represents an *exact* symmetry of the hadronic systems. For this reason, analogously to the isospin case, the quark-quark (i, j) interaction can only contain the operators

$$\mathbf{1}, \qquad \vec{\lambda}(i)\vec{\lambda}(j) = \sum_{a=1}^{8} \lambda_a(i)\lambda_a(j)$$

where the $\vec{\lambda}$ represent the eight 3×3 Gell-Mann matrices. These matrices, being the generators of SU(3), play the same role as the three Pauli matrices in SU(2).
For quark model calculations, the relevant matrix elements of $\vec{\lambda}(i)\vec{\lambda}(j)$ are are easily calculated for the baryonic and mesonic particles whose color states are represented by eq.(25) and eq.(26), respectively.

5.3 Final comments and remarks

Other proofs of the existence of color are given by the properties of the $\pi^0 \to 2\gamma$ decay and by the study of the cross-section ratio

$$R = \sigma(e^+e^- \to hadrons)/\sigma(e^+e^- \to \mu^+\mu^-)$$

that allow to *count* the number of the quark states.

The *global* invariance under rotations in color space is brought at *local* level constructing the gauge field theory denoted as Quantum Chromo-Dynamics

(QCD) that is the *fundamental* theory of strong interactions. In this theory, the Gluon exchange (analogously to the Photon exchange in QED), is responsible of the quark interactions.

All the *observed* hadrons are in a colorless state. In other words, colored particles, in particular free quarks, are not observed. This property is called color confinement.

As for the following developments, the total wave function of the nucleon and of all the baryonic resonances is given by a *symmetric* spin-space-isospin term multiplied by the *antisymmetric* color function of eq.(25). When not strictly necessary, the color function will not be explicitly written.

6 The spatial part of the wave function

In order to determine the spatial part of the wave functions it is necessary to to fix the *dynamics* of the system by means of a *model* for its Hamiltonian operator. The starting point is, for its simplicity, a nonrelativistic Hamiltonian with a Harmonic Oscillator potential, that will be denoted as NR HO model.

6.1 The NR HO model

This model is characterized by the following Hamiltonian

$$H = T + V \tag{27a}$$

$$T = T_{NR} = \sum_i \frac{p_i^2}{2m} \tag{27b}$$

$$V = V_{HO} = \frac{1}{2} k \sum_{i>j} (\vec{r}_i - \vec{r}_j)^2 \tag{27c}$$

where the sums are performed over the three constituent quarks.
According to the nonrelativistic picture, we introduce the nucleon mass $M = 3m$. For brevity, we shall not explicitly write its *additive* contribution to the quark nonrelativistic Hamiltonian.

By using eqs.(6) one can express the nonrelativistic kinetic energy in terms of the Jacobi momenta:

$$T_{NR} = \frac{P^{\,2}}{2M} + \frac{1}{2m}(p_\rho^{\,2} + p_\lambda^2) \tag{28}$$

this quantity contains two contributions: the first term represents the Center of Mass motion and the second one the *intrinsic* quark motion. The latter term is relevant to determine the masses of the nucleon and of the resonant excited states. Note that *only* in the nonrelativistic approximation one has three independent quadratic terms in P^2, p_ρ^2 and p_λ^2.
Analogously, by means of eqs.(4), it is possible to express the Harmonic Oscillator potential in terms of the Jacobi coordinates:

$$V_{HO} = \frac{3}{2}k(\rho^2 + \lambda^2) \tag{29}$$

Note that the use of the Jacobi variables allows to *separate* V_{HO} into two independent contributions in $\vec{\rho}$ and $\vec{\lambda}$. This property, that *only* holds for the Harmonic Oscillator potential, greatly simplifies the solution of the Hamiltonian eigenvalue problem. At phenomenological level, the potential V_{HO} is able to represent the quark confinement inside the baryon. We can write the total NR HO Hamiltonian in the form

$$H = \frac{P^{\,2}}{2M} + \hat{M}_{HO}^{NR} \tag{30}$$

with the NR HO mass operator written as

$$\hat{M}_{HO}^{NR} = \frac{1}{2m}(p_\rho^{\,2} + p_\lambda^2) + \frac{3}{2}k(\rho^{\,2} + \lambda^{\,2}) \tag{31}$$

Such definition is not usual in a nonrelativistic context but is intended to help the reader to pass to the study of relativistic quark models.
The first term of eq.(30) gives rise, in the total wave function of the system, to a standard plane wave factor (completely symmetric with respect to particle interchange) of the form

$$(2\pi\hbar)^{-3/2} \exp\left(\frac{i}{\hbar}\vec{P}\vec{R}\right)$$

normalized in a unitary volume. In general we shall neglect this factor and concentrate our attention on finding the eigenstates and the eigenvalues of

$\hat{M}^{NR}_{HO}$. This operator can be written as

$$\hat{M}^{NR}_{HO} = H^{(\lambda)}_{HO} + H^{(\rho)}_{HO} \quad (32)$$

with

$$H^{(x)}_{HO} = \frac{\vec{p}_x^{\,2}}{2m} + \frac{3}{2}k\vec{x}^{\,2} \quad (33)$$

The solutions for the single oscillator Hamiltonian have an analytic form

$$H^{(x)}_{HO}\Psi_{n,l,m}(\vec{x}) = \left(n + \frac{3}{2}\right)\omega\Psi_{n,l,m}(\vec{x}) \quad (34)$$

From now on, unless otherwise stated, we set $\hbar = 1$. In the previous equation $\omega = \sqrt{3k/m}$, n represents quantum number of the energy and l, m those of the orbital angular momentum. In more detail, the wave functions are

$$\Psi_{n,l,m}(\vec{x}) = \alpha^{3/2}Q_{n,l}(\xi)Y_{l,m}(\Omega_x)\exp(-\frac{1}{2}\xi^2) \quad (35)$$

with $\xi = \alpha x$ and $\alpha = \sqrt{m\omega}$.
The $Y_{l,m}(\Omega_x)$ are the standard spherical harmonics of the angles of $\vec{x}$. We introduce the following polinomials

$$Q_{n,l}(\xi) = \left[\frac{2[(n-l)/2]!}{\Gamma[(n+l+3)/2]}\right]^{1/2} \xi^l \; L^{l+1/2}_{(n-l)/2}(\xi^2) \quad (36)$$

The last factor of the previous equation represents the Laguerre polinomials defined as in the tables [8]. A thorough analysis of the quantum-mechanical HO is given by a specialized text [9]. We highlight here only some relevant properties of its solutions.

1) The asymptotic behaviour (the same for all the states) is given by the Gaussian factor of eq.(35).

2) The solutions of the single nonrelativistic harmonic oscillator eigenvalue problem present an *accidental degeneration*. In fact, for a given n, the angular momentum eigenvalue l takes all the values with the same parity of n, from 0, or 1, to n.

3) The behavior of the wave functions for $\xi \to 0$ is ξ^l, as shown in eq.(36).

Another important and very useful property of the HO wave functions is that their Fourier Transforms (that means: the wave functions in momentum space) can be calculated analytically and, apart from the phase factor $(-i)^n$, have the same form of eqs.(35) and (36). More precisely, they are

$$\Psi_{n,l,m}(\vec{p}) = (-i)^n \alpha^{-3/2} Q_{n,l}(\chi) Y_{l,m}(\Omega_p) \exp(-\frac{1}{2}\chi^2) \tag{37}$$

with $\chi = p/\alpha$

6.2 Spatial wave functions with definite symmetry

Given the *separated* form of $\hat{M}_{HO}^{NR}$ of eq.(30), we observe that any product of two HO wave functions of $\vec{\rho}$ and $\vec{\lambda}$ is an eigenfunction of $\hat{M}_{HO}^{NR}$ with the energy eigenvalue $E_N = (N+3)\omega$, being $N = n_\rho + n_\lambda$. The task is now to construct eigenfunctions *not only* of the rest energy , *but also* of the total orbital angular momentum $\vec{L} = \vec{l}_\rho + \vec{l}_\lambda$ (with eigenvalues L, M) and, moreover, with definite symmetry.
As for the angular momentum one has perform a standard tensor coupling of the spherical harmonics $Y_{l_\rho,m_\rho}(\Omega_\rho)$ and $Y_{l_\lambda,m_\lambda}(\Omega_\lambda)$.
The complete results for the spatial wave functions up to $N = 2$ have been found in different works [1] and take the form that is given in the following by using the compact notation

$$\Psi^{\sigma,\pi}_{N,L,M}$$

where σ, π respectively represent the symmetry and parity indices.
By means of the transformation properties of $\vec{\rho}$ and $\vec{\lambda}$, the reader should verify the symmetry properties of those wave functions. For convenience we also previously introduce the *completely symmetric* asymptotic factor

$$E_{HO} = \exp\left[-\frac{1}{2}\alpha^2(\rho^2 + \lambda^2)\right] \tag{38}$$

$\star$ For the ground state, that is $N = 0$, the wave function is completely symmetric with $L = 0$

$$\Psi^{S,+}_{0,00} = \frac{\alpha^3}{\pi^{3/2}} E_{HO} = \Psi_{00}(\vec{\rho})\Psi_{00}(\vec{\lambda}) \tag{39}$$

$\star$ For the level with $N = 1$ one has a pair of mixed symmetry wave functions with $L = 1$ and negative parity

$$\Psi^{MA,-}_{1,1M} = \sqrt{\frac{8}{3}}\frac{\alpha^4}{\pi}\rho Y_{1,M}(\Omega_\rho)E_{HO} = \Psi_{00}(\vec{\lambda})\Psi_{1,1M}(\vec{\rho}) \tag{40a}$$

$$\Psi^{MS,-}_{1,1M} = \sqrt{\frac{8}{3}}\frac{\alpha^4}{\pi}\lambda Y_{1,M}(\Omega_\lambda)E_{HO} = \Psi_{1,1M}(\vec{\lambda})\Psi_{0,0}(\vec{\rho}) \tag{40b}$$

For the level with $N = 2$ one has seven possible combinations. In particular:
$\star$ a symmetric wave function with $L = 0$

$$\begin{aligned}\Psi^{S,+}_{2,00} &= \frac{1}{\sqrt{3}}\frac{\alpha^5}{\pi^{3/2}}(\rho^2 + \lambda^2 - 3\alpha^{-2})E_{HO} \\ &= -\frac{1}{\sqrt{2}}[\Psi_{0,0}(\vec{\rho})\Psi_{2,0}(\vec{\lambda}) + \Psi_{0,0}(\vec{\lambda})\Psi_{2,0}(\vec{\rho})]\end{aligned} \tag{41}$$

$\star$ a mixed symmetry pair with $L = 0$

$$\Psi^{MA,+}_{2,00} = \frac{2}{\sqrt{3}}\frac{\alpha^5}{\pi^{3/2}}(\vec{\rho}\vec{\lambda})E_{HO} = -[\Psi_{1,1}(\vec{\rho}) \otimes \Psi_{1,1}(\vec{\lambda})]_{0,0} \tag{42a}$$

$$\Psi^{MS,+}_{2,00} = \frac{1}{\sqrt{3}}\frac{\alpha^5}{\pi^{3/2}}(\rho^2 - \lambda^2)E_{HO} = \frac{1}{\sqrt{2}}[\Psi_{0,0}(\vec{\rho})\Psi_{2,0}(\vec{\lambda}) - \Psi_{0,0}(\vec{\lambda})\Psi_{2,0}(\vec{\rho})] \tag{42b}$$

$\star$ a symmetric wave function with $L = 2$

$$\begin{aligned}\Psi^{S,+}_{2,2M} &= \frac{8}{\sqrt{15}}\frac{\alpha^5}{\pi}[\rho^2 Y_{2M}(\Omega_\rho) + \lambda^2 Y_{2M}(\Omega_\lambda)]E_{HO} \\ &= \frac{1}{\sqrt{2}}[\Psi_{2,2M}(\vec{\lambda})\Psi_{0,0}(\vec{\rho}) + \Psi_{2,2M}(\vec{\rho})\Psi_{0,0}(\vec{\lambda})]\end{aligned} \tag{43}$$

⋆ a mixed symmetry pair with $L = 2$

$$\Psi^{MA+}_{2,2M} = \frac{8}{3}\frac{\alpha^5}{\pi^{1/2}}\rho\lambda[Y_1(\Omega_\rho)\otimes Y_1(\Omega_\lambda)]_{2,M}E_{HO} = [\Psi_{1,1}(\vec{\rho})\otimes\Psi_{1,1}(\vec{\lambda})]_{2,M} \quad (44)$$

$$\begin{aligned}\Psi^{MS+}_{2,2M} &= \frac{8}{\sqrt{15}}\frac{\alpha^5}{\pi}[\rho^2 Y_{2M}(\Omega_\rho) - \lambda^2 Y_{2M}(\Omega_\lambda)]E_{HO} \\ &= \frac{1}{\sqrt{2}}[-\Psi_{2,2M}(\vec{\lambda})\Psi_{0,0}(\vec{\rho}) + \Psi_{2,2M}(\vec{\rho})\Psi_{0,0}(\vec{\lambda})] \quad (45)\end{aligned}$$

⋆ and finally a completely antisymmetric wave function with $L = 1$

$$\Psi^{A+}_{2,2M} = \frac{\sqrt{2}}{\pi^{3/2}}i\alpha^5[\vec{\lambda}\times\vec{\rho}]_{1,M}E_{HO} = [\Psi_{1,1}(\vec{\lambda})\otimes\Psi_{1,1}(\vec{\rho})]_{1,M} \quad (46)$$

It is very useful to determine the Fourier Transform of the previous wave functions. To this aim one can use eq.(37) for the $\vec{\rho}$ and $\vec{\lambda}$ oscillator wave functions.
Considering the expressions after the *first* " = " sign in eqs.(39)-(46), the corresponding Fourier Transformed functions can be obtained by inserting the phase factor

$$(-i)^N$$

and making *everywhere* the replacements

$$\vec{\rho} \to \vec{p}_\rho\ , \qquad \vec{\lambda} \to \vec{p}_\lambda \quad (47a)$$

$$\alpha \to \alpha^{-1} \quad (47b)$$

On the other hand, in the expressions after the *second* " = " sign, one has only to replace the corresponding momentum space HO wave functions of eq.(37).

6.3 Comments and developments

In the next section, with standard techniques, we shall construct the total wave functions with the spatial, spin, isospin (and color) terms.
Considering that the spin dependent interaction is usually considered as a perturbation and most models are isospin independent or, in any case, weakly isospin dependent, the values of the energy levels, (with the corresponding

angular momentum and parity quantum numbers) would be roughly given by the results of the previous subsection. Note that for this model the only free parameter is the HO *quantum* ω. A *constant* term is also frequently added to parametrize the unknown quantum effects for the bound states.
But such results are in *poor* agreement with the experimental data. The origin of this discrepancy is clearly due to the *unphysical* character of $\hat{M}_{HO}^{NR}$ of eq.(32) that is not able to represent in an effective way the complexity of the bound system quark dynamics.
Some details about more realistic quark Hamiltonians will be given in sect.8. We stress here that, for those Hamiltonians, an analytic solution cannot be found. One can use a variational diagonalization-minimization procedure possibly taking (as a starting point) the NR HO wave functions of the previous section as the *base* states.
In this respect one can obtain a better accuracy of the variational solutions by:
a) using a *larger* set of base states;
b) modifying the asymptotic behavior E_{HO}.

Let us first examine point (b). With the exception of $\Psi^{S,+}_{2,00}$ of eq.(41), all the other wave functions listed in the previous subsection are mutually orthogonal due to the different symmetry (σ), parity (π) and angular momentum (L) values. (On the other hand the ortogonality of $\Psi^{S,+}_{0,00}$ and $\Psi^{S,+}_{2,00}$, given by eqs. (39) and (41), is due to the factor $\rho^2 + \lambda^2 - 3\alpha^{-2}$ in the latter function). As a consequence, in those functions, whose orthogonality is due to the different values of σ, π and L, it is possible to replace E_{HO} with any symmetric, $L = 0$, normalizable, function. In this way one still obtains a set of mutually orthogonal wave functions with the same σ, π, L, M of the HO ones.
As for the point (a), one can construct new orthogonal wave functions with the *same* σ and L, by means of multiplicative symmetric terms. If E_{HO} is used, these terms are represented by standard Laguerre polynomials.
For example, $\rho^2 + \lambda^2 - 3\alpha^{-2}$ in $\Psi^{S,+}_{2,00}$ is the lowest degree polynomial, after the constant one, for the $\sigma = S$ (symmetric), $L = 0$ case.
The construction of these new spatial wave functions will be studied in a forthcoming work introducing the formalism of the *hyperspherical variables.*
Exactly the same arguments also hold for the wave functions in momentum space.
Finally, we recall that our wave functions (also the modified ones) do not represent in any case a *complete* set of states. For example, one inmediately

recognizes that states with $l_\rho, l_\lambda > 2$ and different symmetry structure are not included in the present analysis and should be also inserted to enlarge the base states and give a more accurate description of the baryonic structure.

7 Construction of the total wave function

The procedure for constructing the total wave functions consists in making combinations of the spatial functions of eqs.(39)-(46) with the spin-isospin ones of eqs.(22). Standard tensor coupling is performed to obtain the total angular angular momentum J of the baryonic state. The symmetry rules of eqs.(21) are also used. A completely symmetric wave function is obtained. We shall omit everywhere the antisymmetric color factor.
In passing, we recall that a similar procedure is used when studying the wave functions of the three-nucleon systems, i.e. ^{3}H and ^{3}He. But, in this case, there is no color factor and one has to construct a completely *antisymmetric* spin-isospin-spatial wave function.

7.1 The low-lying states wave functions

To explain the procedure, we give here four relevant examples for the nucleon and the first (low-lying) excited states of the resonance spectrum.
For simplicity we shall not write the angular momentum quantum numbers in the *l.h.s.* of the following equations.

Assuming *exactly* the NR HO model of the previous section, one has for the (ground) nucleon state

$$\Psi_{N(940)} = G^S_{\frac{1}{2}T_3,\frac{1}{2}M} \cdot \Psi^{S+}_{00} \tag{48}$$

and for the $N(1440)$, that is a resonance with the same quantum numbers of the nucleon

$$\Psi_{N(1440)} = G^S_{\frac{1}{2}T_3,\frac{1}{2}M} \cdot \Psi^{S+}_{20} \tag{49}$$

As discussed when introducing the color, the wave function of the $\Delta(1232)$ is

$$\Psi_{\Delta(1232)} = G^S_{\frac{3}{2}T_3,\frac{3}{2}M} \cdot \Psi^{S+}_{00} \tag{50}$$

Note that the nucleon and the $\Delta(1232)$ would be *degenerate* in the NR HO model. This degeneracy is removed by the spin-spin interaction that will be briefly discussed in the next section.

For the *negative parity* $N(1520)$ resonance, the wave function is

$$\Psi_{N(1520)} = \frac{1}{\sqrt{2}}[[\Psi_{1,1}^{MS,-} \otimes G^{MS}_{\frac{1}{2}T_3,\frac{1}{2}}]_{\frac{1}{2},M} + [\Psi_{1,1}^{MA,-} \otimes G^{MA}_{\frac{1}{2}T_3,\frac{3}{2}}]_{\frac{1}{2},M}] \tag{51}$$

where the spatial wave functions of eqs.(40) have been used. Note that, according to the NR HO model, this state would have an energy (ω) *lower* than that (2ω) of the state given eq.(49), in disagreement with the experimental data.

8 Quark model Hamiltonians

One of the most relevant constraint on the quark Hamiltonian is represented by special relativity. This issue is very important not only for the *spectrum* of the resonat states, but, even more, for the study of the *form factors* by means of scattering processes in which the struck hadronic particle adquires a relativistic velocity.
Given that it is not possible to solve QCD, that is the appropriate *relativistic quantum field theory* for these systems, one constructs effective relativistic models that reproduce the main symmetries of the underlying field theory.
As anticipated in the introduction, the so-called Relativistic Hamiltonian Dynamics (RHD)[5, 6], with different formulations, allows to construct relativistic models for systems with a fixed number of constituents. The starting point of this construction is the definition of a (relativistic) *mass operator* in the rest frame of the hadron. This mass operator $\hat{M}$ replaces the nonrelativistic one introduced in subsect. 6.1. The eigenfunctions of this operator are then *boosted* to a different reference frame to calculate the electroweak form factors.
As for the structure of $\hat{M}$ (analogously to the nonrelativistic models), it is given by the sum of a relativistic kinetic operator K and an interaction operator W:

$$\hat{M} = K + W \tag{52}$$

8.1 The relativistic kinetic energy

The relativistic kinetic energy term is of the form

$$K = \sum_i \sqrt{\vec{p_i}^{*2} + m^2} \tag{53}$$

where for the $\vec{p_i}^*$ one has to use the expressions (6) in terms of $\vec{p}_\rho$ and $\vec{p}_\lambda$ setting $\vec{P} = 0$, given that the invariant mass operator M is defined in the hadronic rest frame.
With such kinetic term it does not seem possible to find analytic eigenfunctions for $\hat{M}$. One can search for approximate solutions by means of a variational minimization-diagonalization procedure. In this context, using antisymmetric wave functions, it is possible to take the contribution of one quark (the #3 for convenience) multiplying by 3 the result:

$$< K > \rightarrow 3 < K_3 > = 3 < \sqrt{\frac{2}{3}\vec{p}_\lambda^{\,2} + m^2} > \tag{54}$$

In this way a remarkable simplification of the calculations is obtained.

8.2 The interaction operator

As for the interaction term W, it is conventionally divided in two contributions: a Confining (C) term and a Gluon Exchange (GE) one:

$$W = W^C + W^{GE} \tag{55}$$

The confining term, rather than *quadratic* in the interquark distance, as in the HO model, is taken as a *linear* function of $|\vec{r}_i - \vec{r}_j|$ as suggested by numerical approximate solutions of *lattice* QCD.
Moreover, introducing the hyperradius $x = \sqrt{\vec{\rho}^2 + \vec{\lambda}^2}$, an hyperlinear confining potential $W^C_{hl} = \alpha x$ has been also successfully proposed [10]. In fact,
a) this potential physically represents the three-body correlations of the quark strong interactions;
b) its matrix-elements are easily calculated with spatial wave functions of the kind of sect. 7;
c) it helps to put in the correct order the energies of the $N(1440)$ and of the $N(1520)$ (negative parity) resonances.

As for the GE term, it is usually *inspired* by the Fermi-Breit (FB) nonrelativistic reduction of the one-gluon exchange Feynman diagram for the case of quark-quark interaction. The total result is obtained summing over all the (i, j) quark pairs.
In principle, the procedure of this reduction is the same as those of Quantum Electro-Dynamics, where the interaction is produced by the photon exchange (see for example ref. [11]) but replacing the electromagnetic coupling constant $\alpha_{em} = e^2/\hbar c = 1/137.0359$ with the (phenomenological) strong one α_s and inserting the color operator $\vec{\lambda}(i)\vec{\lambda}(j)$

$$W^{GE} = \alpha_s \sum_{i>j} \vec{\lambda}(i)\vec{\lambda}(j) W_{ij}^{FB} \tag{56}$$

One finds the following contributions to W_{ij}^{FB}
⋆ A short distance, *Coulombic* term

$$W_{ij}^{Coul} = \frac{1}{|\vec{r}_i - \vec{r}_j|} \tag{57}$$

that is also replaced by a $1/x$ *hypercoulombic* interaction [10]. Phenomenologically, this term is important for the correct positioning of the low-lying resonances.
⋆ A spin-spin interaction of the kind

$$W_{ij}^{SS} = -\frac{\pi}{m^2}\frac{2}{3}\vec{\sigma}_i\vec{\sigma}_j\delta(\vec{r}_i - \vec{r}_j) \tag{58}$$

where the delta function can be conveniently replaced by a more physical and formally less pathological finite range spatial function. This spin-spin interaction is strictly necessary to remove the $N - \Delta$ degeneracy.
⋆ A tensor interaction of the form

$$W_{ij}^{T} = \frac{1}{4m^2}\left[\vec{\sigma}_i\vec{\sigma}_j - 3(\sigma_i\hat{r}_{ij})(\sigma_j\hat{r}_{ij})\right]\frac{1}{|\vec{r}_i - \vec{r}_j|^3} \tag{59}$$

A spin orbit contribution also appears in the Fermi-Breit reduction, but it is omitted in the quark model Hamiltonians because it is not beneficial for the reproduction of the resonance spectrum. It has been argued that the effective quark-quark interaction is produced not only by the gluon interchange but also by other (effective) mesons of pseudoscalar and scalar nature. A

reduction procedure, similar to the FB one, can be performed. A suitable choice of the parameters could finally cancel the spin-orbit terms.
Summarizing, from the previous equations, the FB interaction is

$$W_{ij}^{FB} = W_{ij}^{Coul} + W_{ij}^{SS} + W_{ij}^{T} \tag{60}$$

An important task for the current investigation is to implement in the quark model Hamiltonian a fully relativistic interaction operator obtained by saturating different Dirac current terms and then to solve, with the the best possible accuracy, the corresponding eigenvalue problem.

9 The magnetic moment of the nucleon

As an example of a very simple calculation that involves the quark model wave functions, we take the nucleon magnetic moment. The experimental value of this observable clearly shows that the nucleon is *not a point-like* Dirac particle. We show that quark model allows to predict its value with good accuracy. For clarity, in this section we explicitly write the Planck constant $\hbar$.

9.1 The magnetic moment of spin 1/2 particles

Recall that, according to the Dirac equation, the magnetic dipole operator, associated to the spin $\vec{s} = \frac{\hbar}{2}\vec{\sigma}$, is

$$\vec{\mu}_s = \frac{e}{mc}\vec{s} = \frac{e\hbar}{2mc}\ \vec{\sigma} \tag{61}$$

Here m represents the mass and e the charge of a (generic) spin 1/2 particle. If the particle is in motion and the orbital angular momentum is $\vec{l} \neq 0$ the total magnetic dipole (as well known in atomic physics) is

$$\vec{\mu}_t = \vec{\mu}_s + \vec{\mu}_l = \frac{e}{2mc}(2\vec{s} + \hbar\vec{l}) = \frac{e\hbar}{2mc}\ (\vec{\sigma} + \vec{l}) \tag{62}$$

where $\vec{l}$ is measured in units of $\hbar$.
In order to test the prediction of the Dirac equation, it is necessary to *measure* the spin magnetic dipole of a particle, or, more exactly, the factor that multiplies the Pauli matrices $\vec{\sigma}$ in eqs.(61) and (62).

First, recall that the interaction with an external magnetic field $\vec{B}$ is

$$H_{int} = -\vec{\mu}_t \vec{B} \tag{63}$$

Second, take a magnetic field of known intensity directed along the z-axis, that is $\vec{B} = (0, 0, B_z)$.
Third, consider a particle with $l = l_z = 0$ and the spin polarized along the z-axis, that is with $s_z = +\hbar/2$. One conveniently introduces the *Dirac spin magnetic moment*

$$\mu^D = <\uparrow |\mu_z| \uparrow> = <\uparrow |\frac{e\hbar}{2mc}\,\sigma_z| \uparrow> = \frac{e\hbar}{2mc} \tag{64}$$

so that

$$<\uparrow |H_{int}| \uparrow> = -B_z \mu^D \tag{65}$$

that shows how to obtain μ^D from an energy measurement.
Apart from very small radiative corrections, the numerical *measured* values of μ^D for the electron and for the muon (that are *point-like* particles) are in agreement with the prediction of the Dirac equation that is represented by the last expression of eq.(64).

In the nucleon case, the situation is completely different. For practical reasons, it has been introduced the numerical quantity denoted as the *nuclear magneton*: $\mu_N = |e|\hbar/(2Mc)$, being M the nucleon mass and $|e|$ the elementary charge. As shown by eq.(64), the prediction of the Dirac equation, using the nucleon mass and charge, would give

$$\mu_p^D = \mu_N \tag{66a}$$
$$\mu_n^D = 0 \tag{66b}$$

for the proton and the neutron, respectively.
On the contrary, the experimental values are

$$\mu_p^{Exp} = +2.793\mu_N \tag{67a}$$
$$\mu_n^{Exp} = -1.913\mu_N \tag{67b}$$

Dirac equation is not able to reproduce these experimental data because the nucleon is not a point-like particle.

9.2 The quark model calculation

We can calculate the magnetic moment of the nucleon by means of the quark model.
First, in order to write the magnetic dipole operator, we assume that the quarks are point-like Dirac particles, so that in the nonrelativistic approximation, we can sum up the contributions given by eq.(62) for each quark. We have

$$\vec{\mu} = \frac{|\, e|\hbar}{2mc} \sum_{i=1}^{3} \hat{Q}_i \left(\vec{\sigma}_i + \vec{l}_i^{\,*} \right) \tag{68}$$

where m is here the quark mass, $\hat{Q}_i$ is the *i-th* quark (fractionary) charge operator of eq.(19) and $\vec{l}_i^{\,*}$ represents the intrinsic orbital angular momentum (in units of $\hbar$) due to the movement of the *i-th* quark *inside* the nucleon. As before, the measurable quantity μ is defined as

$$\mu = <\uparrow |\mu_z| \uparrow> \tag{69}$$

(For brevity we do not write any index to distinguish proton and neutron). The result will be different from the Dirac one, due to the composite character of the system.
In the previous equation the *nucleon* state $|\uparrow>$ is represented by a completely antisymmetric three-quark state. Due to antisymmetry we can simplify the calculations taking three times the contribution of one quark, say the #3. By means of eq.(68) one finds

$$\mu = \frac{3|e|\hbar}{2mc} <\uparrow |\hat{Q}_3 \left(\sigma_3^z + l_3^{*z} \right) | \uparrow> \tag{70}$$

We take for the nucleon wave function the expression given by eq.(48), obviously with spin up and $T_3 = +/-\frac{1}{2}$ for p/n, respectively. As usual, the color wave functions give 1 when calculating the matrix element.
In what follows we shall not use the specific HO form of the spatial part of the wave function. Our result is more general and always holds when this spatial part is a *completely symmetric*, $L = 0$ state.
Due to the latter condition the contribution of l_3^{*z} to the matrix element is vanishing and, taking into account eq.(48), eq.(70) can be rewritten as

$$\mu = \frac{3|e|\hbar}{2mc} < G^S_{\frac{1}{2}T_3,\frac{1}{2}\frac{1}{2}} |\hat{Q}_3 \sigma_3^z| G^S_{\frac{1}{2}T_3,\frac{1}{2}\frac{1}{2}} > \tag{71}$$

Now $G^S_{\frac{1}{2}T_3,\frac{1}{2}\frac{1}{2}}$ of eq.(22b) must be expressed in terms of of the spin and isospin parts given in eq.(12). The relevant spin matrix elements are

$$< \chi^1_{\frac{1}{2},\frac{1}{2}} |\sigma^z_3| \chi^1_{\frac{1}{2},\frac{1}{2}} >= -\frac{1}{3} \tag{72a}$$

$$< \chi^0_{\frac{1}{2},\frac{1}{2}} |\sigma^z_3| \chi^0_{\frac{1}{2},\frac{1}{2}} >= +1 \tag{72b}$$

The charge matrix elements with isospin wave functions are

$$< \Phi^1_{\frac{1}{2},+\frac{1}{2}} |\hat{Q}_3| \Phi^1_{\frac{1}{2},+\frac{1}{2}} >= 0 \tag{73a}$$

$$< \Phi^1_{\frac{1}{2},-\frac{1}{2}} |\hat{Q}_3| \Phi^1_{\frac{1}{2},-\frac{1}{2}} >= \frac{1}{3} \tag{73b}$$

$$< \Phi^0_{\frac{1}{2},+\frac{1}{2}} |\hat{Q}_3| \Phi^0_{\frac{1}{2},+\frac{1}{2}} >= +\frac{2}{3} \tag{73c}$$

$$< \Phi^0_{\frac{1}{2},-\frac{1}{2}} |\hat{Q}_3| \Phi^0_{\frac{1}{2},-\frac{1}{2}} >= -\frac{1}{3} \tag{73d}$$

Obviously, the nondiagonal matrix elements are vanishing.
Using the previous results in eq.(71) one finally finds the following values for the nucleon magnetic moment

$$\mu_p = \frac{|e|\hbar}{2mc} = 3\mu_N \qquad \mu_n = -\frac{|e|\hbar}{3mc} = -2\mu_N \tag{74}$$

where the expressions in terms of μ_N have been obtained *assuming*, for the quark mass, the value $m = M/3$, according to the nonrelativistic approximation. Note that a good agreement with the experimental data of eq.(67) is obtained.
Moreover, the quark model gives the ratio $\mu_p/\mu_n = -3/2 = -1.5$ *independently* of the (unknown) value of the quark mass, to be compared with the experimental ratio obtained from eq.(67), that is $\mu_p^{Exp}/\mu_n^{Exp} = -1.460$.

⋆ ⋆ ⋆ ⋆

Concluding, we point out that the previous results for the nucleon magnetic moment μ (that can be measured in *static* experiments) have suggested to use quark model to investigate also the *dynamic* electromagnetic observables of the nucleon.

In more detail, the nucleon magnetic moment can be considered as the static limit of the magnetic form factor $G_M(Q^2)$, where Q^2 represents the squared momentum transfer in an elastic electron scattering process. It means that $G_M(Q^2 = 0) = \mu$.
The electric form factor $G_E(Q^2)$ and the magnetic one $G_M(Q^2)$, represent, roughly speaking, the electric and magnetic internal structure of the nucleon. These quantities are being measured with high accuracy by means of polarized electron scattering processes at the Jefferson Laboratory. From the theoretical point of view, their calculation strictly requires *relativity*, because the nucleon, in the initial and/or final state of the scattering process, is found in a *relativistic* motion with respect to a generic reference frame.
The form of the electromagnetic current and of the Lorentz boost operators is studied in the context of RHD, using the wave functions introduced in the present work.
Both theoretically and experimentally, interesting and unexpected results have been obtained, encouraging further investigations. In particular, accurate and reliable expressions for the wave functions are needed.

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