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Pointwise Approximation of Piecewise Convex Function in L_p, Quasi Normed Spaces

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Abstract

Some authors introduced direct inequalities for the constrained approximation of convex and piecewise convex functions in C[-1,1] with restricted degree of approximation. Here we study the pointwise constrained approximation of piecewise convex functions by pointwise polynomials .

Key words:

pointwise, $L_p$space, convex approximation, quasi normed spaces

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1. Introduction and Notations:

Let $\mathcal{F} \in L_p(I)$ change its convexity finitely many, say $\mathbb{s} \geq 0$, time in the interval.
define $L_p(I) = \{\mathcal{F}: I \to R : \mathcal{F} \in L_p\}$,where $I = [-1,1]$,

$L_p^r(I) = \{\mathcal{F}: I \to R : \mathcal{F}^r \in L_p\}$, $\|\mathcal{F}\|_{Lp} = (\int_{-1}^{1} |\mathcal{F}(x)|^p)^{\frac{1}{p}}$, For κϵN and interval I, $\mathbb{s} \geq 0$, times interval.

We are interested in pointwise estimates on the approximation of $\mathcal{F}$ by algebraic polynomials there are coconvex with it.

The set of all collections $Y_{\mathbb{s}} := \{y_i\}_{i=1}^{\mathbb{s}}$ such that $-1 < y_{\mathbb{s}} < \cdots < y_1 < 1$, denoted by $Y_{\mathbb{s}}$, $\mathbb{s} \in N$.

Let $\Delta^{(2)}(Y_{\mathbb{s}})$ denoted the collection of all function $\mathcal{F} \in L_p[-1,1]$ that change convexity at the set $Y_{\mathbb{s}}$ and are convex in $[y_1, 1]$. Namely in the interval $[y_{i+1}, y_i]$, $\mathcal{F}$ is convex when i is even, and is concave when i is odd. We also use the notation $y_0 = 1$ and $y_{\mathbb{s}+1} = -1$. Denote

$$\Pi(x) = \prod_{i=1}^{\mathbb{s}} (x - y_i) \qquad (1.1)$$

Then for example, if $\mathcal{F} \in {L_p}^2[-1,1] \cap L_p[-1,1]$ then $\mathcal{F} \in \Delta^{(2)}(Y_{\mathbb{s}})$ if and only if $\Delta_h^{(2)}(\mathcal{F}) > 0$ in (-1,1)

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The convex function as the case $\mathfrak{s} = 0$, where we write $Y_0 := \{\emptyset\}$
$\mathcal{P}_n$ the set of algebraic polynomials of degree $< n$
Norm estimates on the degree of approximation of $\mathcal{F} \in \Delta^{(2)}(Y_s)$ by
$\mathcal{P}_n \in \mathbb{P}_n \cap \Delta^{(2)}(Y_s)$,namely, estimates of $E_n^{(2)}(\mathcal{F}, Y_s) := \inf_{\mathcal{P}_n \in \mathbb{P}_n \cap \Delta^{(2)}(Y_s)} \|\mathcal{F} - \mathcal{P}_n\|_p$

$$\Delta_h^\kappa \mathcal{F}(x) := \sum_{i=0}^{\kappa} (-1)^{\kappa-i} \binom{\kappa}{i} \mathcal{F}\left(x + \frac{\kappa}{2}h + ih\right), \text{if } x \mp \frac{\kappa}{2}h \in I \text{ and } := 0 \text{ otherwise.}$$

$w_\kappa(\mathcal{F}, t) := \sup_{h\in[0,t]} \|\Delta_h^\kappa \mathcal{F}\|_p = \sup_{h\in[0,t]} max_{x\in[-1,1]} |\Delta_h^\kappa \mathcal{F}(x)|$, $t > 0$ is the kth modulus of smoothness of $\mathcal{F}$ on I.
Given $n \in N, \kappa \in N, r \in N_0, \mathfrak{s} \in N_0$ and $Y_s \in \mathbb{Y}_s$. We wish to estimate

$$E_{n,\kappa,r}^{(2)}(\mathcal{F}, Y_s) := \inf_{\mathcal{P}_n \in \mathbb{P}_n \cap \Delta^{(2)}(Y_s)} \left\| \frac{\mathcal{F} - \mathcal{P}_n}{\Omega_n^r w_\kappa(\mathcal{F}^{(r)}, \Omega_n)} \right\|$$

$$= \inf_{\mathcal{P}_n \in \mathbb{P}_n \cap \Delta^{(2)}(Y_s)} max_{x\in[-1,1]} \left| \frac{\mathcal{F}(x) - \mathcal{P}_n(x)}{\Omega_n^r(x) w_\kappa(\mathcal{F}^{(r)}, \Omega_n(x))} \right|,$$

Where $\vartheta(x) = \sqrt{1 + x^2}$ and $\Omega_n(x) = \vartheta(x)n^{-1} + n^{-2}, x \in [-1,1]$,
In [1] by Nikolskii, Timan, Dzyadyk, Frend and Brudnyi study the classical pointwise estimates for unconstrained approximation which is also true for coconvex approximation .
In [5, 6] the authors restricted them attention to the following three cases:
a) $r \ge 3$ and $\mathfrak{s} = 1$, b) $r = 2, \kappa \le 3$ and $\mathfrak{s} = 1$, c) $r = 0$ and , $\kappa = 3$. All other cases also have been investigated .
For instance, for $\mathfrak{s} \ge 2$, it introduced by [3,8] that for no $\kappa \ge 1$, and $r \ge 0$. is it possible to have constants $C = C(\kappa, r, \mathfrak{s})$ and $N = N(\kappa, r, \mathfrak{s})$, depending only on κ, r and $\mathfrak{s}$, such that the inequality

$$E_{n,\kappa,r}^{(2)}(\mathcal{F}, Y_s) \le c, \qquad (1.2)$$

Is satisfied for all $n \ge \mathbb{N}$ and $Y_s \in \mathbb{Y}_s$, and for all $\mathcal{F} \in \Delta^{(2)}(Y_s) \cap L_p^r(I)$.
Furthermore, for $\mathfrak{s} = 1$, it follow by [7,Theorem 2] that no $\kappa \ge 1$ and $r \ge 0$ such that $\kappa + r > 2$, and $Y_1 \in \mathbb{Y}_1$, is possible to have constants $c = c(\kappa, r, Y_1)$ and $N = N(\kappa, r, Y_1)$, depending only on κ, r and Y_1, such that (1.2) hold for all $n \ge N$ and for all $\mathcal{F} \in \Delta^{(2)}(Y_s) \cap L_p^r(I)$.
In [4] and [5] the authors studied the pointwise approximation of special polynomials. In this chapter we study the general pointwise approximation by any piecewise polynomial for function in L_p[-1,1].
$\sum_\kappa := \sum_{\kappa,n}$ the x_j, $1 \le j \le n - 1$ piecewise polynomials of degree not exceeding $\kappa - 1$,that are continuous.
$\sum_{\kappa,n}^1$ is the space of all piecewise polynomials $\mathcal{P}$ with $\|\mathcal{P}'\|_p < \infty$,
That is, $\mathbb{S} \in \sum_{\kappa,n}$, if

$$\mathbb{S}|I_j = \mathcal{P}_{j;n} =: \mathcal{P}_j, \qquad j = 1, \dots, n,$$

Where $\mathcal{P}_j \in \mathbb{P}_\kappa$, and $\mathcal{P}_j(x) =: \mathcal{P}_{j+1}(x)$. $j = 1, \dots, n$.
Given $Y_s \in \mathbb{Y}_s$, let
$O_i := O_{i,n}(Y_s) := (x_{j+1}, x_{j-2})$, if $y_i \in [x_j, x_{j-1})$, denote

$$O := O(n, Y_s) := \cup_{i=1}^{s} O_i, \ O(n, \emptyset) := \emptyset,$$

And write $j \in H = H(n, Y_s)$, if $I_j \cap O = \emptyset$,

Finally, denote by $\sum_{\kappa,n}(Y_s) \subseteq \sum_{\kappa,n}$, the subset of those piecewise polynomials for which $\mathcal{P}_j \equiv \mathcal{P}_{j+1}$, whenever both $j, j+1 \notin H$.
For $\Gamma \in \Phi^\kappa, \Gamma \not\equiv 0$. And $\mathbb{S} \in \sum_{\kappa;n}(Y_s)$, denote

$$b_{i,j}(\mathbb{S}) := b_{i,j:n}(\mathbb{S}) = \frac{\|p_i - p_j\|_p}{\Gamma(h_j)}\left(\frac{h_j}{h_{i,j}}\right)^\kappa, \quad 1 \le i,j \le n.$$

Also, denote

$$b_\kappa(\mathbb{S}) := b_{\kappa;n}(\mathbb{S}) := max_{1\le i,j\le n} b_{i,j}(\mathbb{S}).$$

2. Auxiliary Lemma

In this section we introduce the result that we need to prove our Theorems.

Lemma 2.1 [1]

For any $\in I_i$, $\mathcal{F}: [-1,1] \to \mathbb{R}$,

$$|\mathcal{F}(x) - \mathcal{P}_j(x)| \le c\left(\frac{h_{i,j}}{h_j}\right)^\kappa (\Gamma(h_j) + |\mathcal{F} - \mathcal{P}_j|)$$

Lemma 2.2 [6]

($|B_v| \le c\Gamma(h_j)(\frac{h_{i,j}}{h_j})^\kappa, x \in I_i, v = 1,2,3)$, and $|B_3| \le c\Gamma(h_j)(\frac{h_{i,j}}{h_j})^\kappa$, $|B_2| \le c\Gamma(h_j)(\frac{h_{i,j}}{h_j})^{\kappa-1}$ and $|B_1| := h_{i,j}A_1$, $A_1 \le c\Gamma(h_j)\frac{h_{i,j}{}^{\kappa-1}}{h_j{}^\kappa}$, where $B_1(x) = \int_{x_i}^{x_j}(x-u)\mathbb{S}''du$, $B_2(x) = \int_{x_j}^{x}(x-u)\mathcal{P}_j''du$, $B_3(x) = -\int_{x_j}^{x}(x-u)\mathcal{P}_i''du$, and $\mathbb{S}, \mathcal{P}$ are polynomials.

Lemma 2.3 [6]

$|(x-x_j)_+ - \tau_j(x)| \le c(b,s)h_j\mathfrak{T}_j^b(x)$, where $\mathfrak{T}_j^b(x) = (\frac{h_j}{|x-x_j|+h_j})^b$, τ_j are polynomials with $b = \kappa + 3$, and

$$(x-x_j)_+ := \begin{cases} x - x_j, & \text{if } x \ge x_j, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.4 [6]

$\tau_j''\Pi(x)\Pi(x_j) \ge 0, x \in [-1,1], x_j \in I_j$.

Lemma 2.5 [6]

$\Omega_n^2(y) < 4\Omega(|x-y| + \Omega)$,
$(|x-y| + \Omega)/2 < |x-y| + \Omega_n(y) < 2(|x-y| + \Omega)$, $\quad x, y \in [-1,1]$.

3. Main Results

We begin this section to prove some Lemmas by properties of the $b_{i,j}$'s .

Proposition (3.1)

Let $\Gamma \in \Phi^\kappa, \kappa \in N, \mathcal{F} \in L_p(I)$and $\mathbb{S} \in \sum_{\kappa,n}$, If $w_\kappa(\mathcal{F},t)_p \le c(p)\Gamma(t)$ and

$$(3.2) \qquad \|\mathcal{F} - \mathbb{S}\|_p \le c(p)\Gamma(\|\Omega_n(x)\|_p), x \in [-1,1]$$

Then

$$b_\kappa(s,\Gamma) \le c(\kappa, p)$$

where $c(p,\kappa)$ depends on p and κ.

Proof.

Recalled that $\Gamma \not\equiv 0$,so that $\Gamma(x) > 0, x > 0$. For $1 \le i, j \le n$,

we have $b_{i,j}(\mathbb{S}) = \frac{\|p_i - p_j\|_p}{\Gamma(h_j)} (\frac{h_j}{h_{i,j}})^\kappa$

$$b_{i,j}(\mathbb{S}, \Gamma) \le 2^{\frac{1}{p}} \left(\frac{\|\mathcal{P}_i - \mathcal{F}\|_p}{\Gamma(h_j)} \left(\frac{h_j}{h_{ij}}\right)^\kappa + \frac{\|\mathcal{F} - \mathcal{P}_j\|_p}{\Gamma(h_j)} \left(\frac{h_j}{h_{i,j}}\right)^\kappa \right) := \mathcal{J}_1 + \mathcal{J}_2 ,$$

where $\|\mathcal{P}_i - \mathcal{F}\|_p = (\int_{-1}^{1} (\mathcal{P}_i - \mathcal{F})^p)^{\frac{1}{p}}$

$$\|\mathcal{F} - \mathcal{P}_j\|_p = \left(\int_{-1}^{1} (\mathcal{F} - \mathcal{P}_j)^p\right)^{\frac{1}{p}}$$

now by(3.1) and(3.2)

$\|p_i - \mathcal{F}\|_p \le c(p)\Gamma(\|\Omega_n(x)\|_p) \le c(p)\Gamma(h_i), x \in [-1,1]$,

Whence $\mathcal{J}_1 = \frac{\|\mathcal{P}_i - \mathcal{F}\|_p}{\Gamma(h_j)} \left(\frac{h_j}{h_{ij}}\right)^\kappa$, $\mathcal{J}_1 \le \frac{\Gamma(h_i)}{\Gamma(h_j)} \left(\frac{h_j}{h_{i,j}}\right)^\kappa \le 1$,

Hence , $\mathcal{J}_1 \le 1$, where we used the fact that if $h_i \le h_j$,

then $\Gamma(h_i) \le \Gamma(h_j)$,and if $h_i > h_j$ then $\Gamma(h_i)/\Gamma(h_j) {h_j}^k \le {h_i}^k$.

So as for $\mathcal{J}_2$, we observe that

$w_\kappa(\mathcal{F} - \mathcal{P}_j, t)_p = w_\kappa(\mathcal{F}, t)_p \le \Gamma(t)$.

So that by Lemma (2.1), taking $x_0 := x_j$ and $h := \frac{h_j}{\kappa - 1}$, we obtain for each $x \in I_i$,

$\|\mathcal{F}(x) - \mathcal{P}_j(x)\|_p \le c(p) \left(\frac{h_{i,j}}{h_j}\right)^\kappa (\Gamma(h_j) + \|\mathcal{F} - \mathcal{P}_j\|_p)$and so,

$\|\mathcal{F} - \mathcal{P}_j\|_p \le c(p)(\frac{h_{i,j}}{h_j})^k \Gamma(h_j)$. Applying(3.2)

$\frac{\|\mathcal{F} - \mathcal{P}_j\|_p}{\Gamma(h_j)} (\frac{h_j}{h_{i,j}})^\kappa \le c(p)$, therefor $\mathcal{J}_2 \le c$.

<u>**Proposition**</u> **3.3**

Let $\kappa \ge 3, \Gamma \in \Phi^\kappa$and $\mathbb{S} \in \Sigma^1_{\kappa;n}$ Then $b_\kappa(\mathbb{S}) \le c(p) \left\| \frac{\Omega^2 s''}{\Gamma(\Omega)} \right\|_p$,

Proof.

since $\mathcal{P}_j(x) = s(-1) + s'(-1)(x + 1) + \int_{-1}^{x_j} (x - u) s''(u) du + \int_{x_j}^{x} (x - u)\mathcal{P}''_j(u) du$. and

$$\mathcal{P}_i(x) = s(-1) + s'(-1)(x + 1) + \int_{-1}^{x_i} (x - u)s''(u)du + \int_{x_i}^{x} (x - u)\mathcal{P}''_i(u)du.$$

We have $\mathcal{P}_j - \mathcal{P}_i = \int_{x_i}^{x_j} (x - u)s''(u)du + \int_{x_j}^{x} (x - u)p''_j(u)du - \int_{x_i}^{x} (x - u)p''_i(u)du. =: B_1(x) + B_2(x) + B_3(x)$,then

$\|\mathcal{P}_j - \mathcal{P}_i\|_p \le c(p)(\|B_1(x)\|_p + \|B_2(x)\|_p + \|B_3(x)\|_p)$.

Clearly, our proposition readily follows from the inequalities

(3.4) $(\|B_v\|_p \le c\Gamma(h_j)(\frac{h_{i,j}}{h_j})^\kappa, x \in I_i, v = 1,2,3$

Now to prove (3.4) for $v = 1,2,3$

So by Lemma(2.2) $(|B_v| \le c\Gamma(h_j)(\frac{h_{i,j}}{h_j})^\kappa, x \in I_i, v = 1,2,3$

$\|B_3(x)\|_p \le c(p)\Gamma(h_j)(\frac{h_{i,j}}{h_j})^\kappa$, $\|B_2(x)\|_p \le c(P)\Gamma(h_j)\frac{(h_{i,j})^{\kappa-1}}{{h_j}^\kappa}$.

$\|B_1(x)\|_p \le h_{i,j}B_1$, where $B_1 \le c(p)\Gamma(h_j)\frac{(h_{i,j})^{k-1}}{h_j^k}$

So $max_{1\le i,j\le n}\{b_{i,j}(s,\Gamma)\} = b_\kappa(s,\Gamma)$

So $\frac{\|\mathcal{P}_j-\mathcal{P}_i\|_p}{\Gamma(h_j)}(\frac{h_i}{h_{i,j}})^\kappa =: b_{i,j}(s,\Gamma)$,we have $b_\kappa(s) \le \|B_v(x)\|_p \le c(p)\Gamma(h_j)(\frac{h_{i,j}}{h_j})^\kappa, x \in I_j, v = 1,2,3$.

(i. e$\|\mathcal{P}_j - \mathcal{P}_i\|_p \le c(p)\Gamma(h_j)(\frac{h_{i,j}}{h_j})^k$. This prove is complete.

Theorem (3.5)

Let $\kappa \ge 3, \Gamma \in \Phi^\kappa, \mathbb{S} \in \sum_{\kappa;n/2}(Y_s) \cap \Delta^{(2)}(Y_s)$, where n is an even number, and

$\|\mathcal{P}_{\iota;n/2} - \mathcal{P}_{\iota-1;n/2}\|_{p(I_{\iota;\frac{n}{2}}\cup I_{\iota-1;\frac{n}{2}})} \le c(p)\Gamma\left(x_{\iota-2;\frac{n}{2}} - x_{\iota;\frac{n}{2}}\right), 2 \le \iota \le n/2$

where $\mathbb{S}|I_\iota;\frac{n}{2} =: \mathcal{P}_{\iota;\frac{n}{2}}, \iota = 1, \dots, n/2$, then there is an $\tilde{\mathbb{S}} \in \sum^1_{\kappa;n}(Y_s) \cap \Delta^{(2)}(Y_s)$, such that

(3.6) $\|\mathbb{S}(x) - \tilde{\mathbb{S}}(x)\|_p \le c(p)\Gamma(\Omega), x \in [-1,1]$

Proof.

For $2 \le j \le n$, set

$$a_j(x) := \frac{1}{2}\frac{x_{j-1} - x_{j-2}}{x_{j-1} - x_j}\frac{\mathbb{S}'(x_{j-1}+0) - \mathbb{S}'(x_{j-1}-0)}{x_j - x_{j-2}}(x - x_j)^2$$

And $a_1(x) :\equiv 0$, and for $1 \le j \le n-1$, set

$b_j(x) := \frac{1}{2}\frac{x_j-x_{j+1}}{x_j-x_{j-1}}\frac{\mathbb{S}'(x_j+0)-\mathbb{S}'(x_j-0)}{x_{j+1}-x_{j-1}}(x - x_{j-1})^2$, and $b_n(x) :\equiv 0$.

Then

$\tilde{\mathbb{S}}(x) = \mathbb{S}(x) + a_j(x) + b_j(x), x \in I_j$,

Is the required function, simple computation, give

$a_{j+1}(x_j) = b_j(x_j), 1 \le j \le n-1$, while

$b_{j+1}(x_j) := \frac{1}{2}\frac{x_{j+1}-x_{j+2}}{x_{j+1}-x_j}\frac{\mathbb{S}'(x_{j+1}+0)-\mathbb{S}'(x_{j+1}-0)}{x_{j+2}-x_j}(x - x_j)^2 =$

$a_j(x_j) := \frac{1}{2}\frac{x_{j-1}-x_{j-2}}{x_{j-1}-x_j}\frac{\mathbb{S}'(x_{j-1}+0)-\mathbb{S}'(x_{j-1}-0)}{x_j-x_{j-2}}(x - x_j)^2 = 0, 1 \le j \le n-1$.

Hence $\tilde{\mathbb{S}}(x_j + 0) = \tilde{\mathbb{S}}(x_j - 0), 1 \le j \le n-1$.

Also $\tilde{\mathbb{S}}'(x_j + 0) = \frac{x_{j+1}-x_j}{x_{j+1}-x_{j-1}}\mathbb{S}'(x_j - 0) + \frac{x_j-x_{j-1}}{x_{j+1}-x_{j-1}}\mathbb{S}'(x_j + 0) =$

$\frac{x_{j+1}-x_j}{x_{j+1}-x_{j-1}}\mathbb{S}'(x_j + 0) + \frac{x_j-x_{j-1}}{x_{j+1}-x_{j-1}}\mathbb{S}'(x_j - 0) = \tilde{\mathbb{S}}'(x_j - 0), 1 \le j \le n-1$

So that $\tilde{\mathbb{S}} \in \sum^1_{\kappa;n}$.

We need to prove that

(3.7) $\tilde{\mathbb{S}} \in \sum^1_{\kappa;n}(Y_s) \cap \Delta^{(2)}(Y_s)$,

In order to prove that $\tilde{\mathbb{S}} \in \sum^1_{\kappa;n}(Y_s)$, we have to show, with $\mathcal{P}_j := \tilde{\mathbb{S}}|I_j, 1 \le j \le n-1$, that if j, $j+1 \notin H(n, Y_s)$, then $\mathcal{P}_j = \mathcal{P}_{j+1}$, rec all that

$\mathcal{P}_{\iota;n/2} := \tilde{\mathbb{S}}|I_{\iota,\frac{n}{2}}, \iota = 1, \dots, n/2$ and distinguish between two cases, either j is even, in which case $j/2 \notin H(n/2, Y_s)$ and $\frac{j}{2} + 1 \notin H(n/2, Y_s)$

Hence $\mathcal{P}_{\frac{j}{2},n/2} = \mathcal{P}_{\frac{j}{2}+1,n/2}$, which implies that

$\mathbb{S}'(x_{j+v}+0)=\mathbb{S}'(x_{j+v}-0)$, for $v=0,\pm1$, that is, $a_j=a_{j+1}=b_j=b_{j+1}=0$.Thus $\mathcal{P}_j=\mathcal{P}_{j+1}$ as required.
Or j is odd, which case x_j is not a node of $\mathbb{S}$, and it follows that there is an inflection point y_i in the interval $[x_{\frac{j+1}{2},\frac{n}{2}}, x_{\frac{j-1}{2},\frac{n}{2}})$, so that $\frac{j+v}{2}\notin H\left(\frac{n}{2},Y_s\right)$, for $v=3,\pm1$.
Hence $\mathbb{S}'\left(x_{\frac{j+1}{2},\frac{n}{2n}}+0\right)=\mathbb{S}'\left(x_{\frac{j+1}{2},\frac{n}{2n}}-0\right)$ and $\mathbb{S}'\left(x_{\frac{j-1}{2},\frac{n}{2n}}+0\right)=\mathbb{S}'\left(x_{\frac{j-1}{2},\frac{n}{2n}}-0\right)$,
which in turn imply that $a_j=b_{j+1}\equiv 0$, also $\mathbb{S}'(x_j+0)=\mathbb{S}'(x_j-0)$, since x_j is not a node of $\mathbb{S}$,which implies that $a_{j+1}=b_j\equiv 0$
Thus, we conclude that $\tilde{\mathbb{S}}(x)=\mathbb{S}(x)=\mathcal{P}_{\frac{j+1}{2},\frac{n}{2}}(x)$ for $x\in[x_{\frac{j+1}{2},\frac{n}{2}}, x_{\frac{j-1}{2},\frac{n}{2}}]$.
This completes the proof that
(3.8) $\tilde{\mathbb{S}}\in\sum^1_{\kappa;n}(Y_s)$, Finally, $a''_j(x)=\frac{x_{j-2}-x_{j-1}}{(x_{j-2}-x_j)(x_{j-1}-x_j)}(\mathbb{S}'(x_{j-1}+0)-\mathbb{S}'(x_{j-1}-0))$ and
$b''_j=\frac{x_j-x_{j+1}}{(x_{j-1}-x_{j+1})(x_{j-1}-x_j)}(\mathbb{S}'(x_j+0)-\mathbb{S}'(x_j-0))$
So that we readily conclude that
(3.9) $\tilde{\mathbb{S}}''(x)\Pi(x)\geq 0, \in[-1,1]\backslash\{x_j\}_{j=1}^{n-1}$,
Combining (3.8) and (3.9) , we obtain (3.7).
In order to conclude the proof, recall that for odd j , $\mathbb{S}'(x_j+0)-\mathbb{S}'(x_j-0)=0$ and if j is even, then by Markovs inequality

$$\|\mathbb{S}'(x_j+0)-\mathbb{S}'(x_j-0)\|_p=\left\|\mathcal{P}'_{\frac{j}{2},\frac{n}{2}}(x_j)-\mathcal{P}'_{\frac{j}{2}+1,\frac{n}{2}}(x_j)\right\|_p$$

$$\leq c(p)\frac{2\kappa^2}{h_{\frac{j}{2}+1,\frac{n}{2}}+h_{\frac{j}{2},\frac{n}{2}}}\left\|\mathcal{P}_{\frac{j}{2},\frac{n}{2}}-\mathcal{P}_{\frac{j}{2}+1,\frac{n}{2}}\right\|_{p(I_{\frac{j}{2}+1,\frac{n}{2}}\cup I_{\frac{j}{2},\frac{n}{2}})}$$

$$\leq c(p)\frac{2\kappa^2}{h_{\frac{j}{2}+1,\frac{n}{2}}+h_{\frac{j}{2},\frac{n}{2}}}\Gamma(h_{\frac{j}{2}+1,\frac{n}{2}}+h_{\frac{j}{2},\frac{n}{2}})$$

$\leq c(p)\frac{\Gamma(h_j)}{h_j}$. This implies (3.6) and conclude our proof

<u>Theorem</u> (3.10)
For each $\mathbb{S}\in\sum_{\kappa;n/2}(Y_s)\cap\Delta^{(2)}(Y_s)$ satisfying
(3.11) $b_\kappa(\mathbb{S})\leq 1$,
There is $\tilde{\mathbb{S}}\in\sum^1_{\kappa;n}(Y_s)\cap\Delta^{(2)}(Y_s)$, such that
(3.12) $b_{\kappa;2n}(\tilde{\mathbb{S}})\leq c$.
Proof.
It follows by $b_{i,j}(\mathbb{S}):=b_{i,j;n}(\mathbb{S}):=\frac{\|p_i-p_j\|_p}{\Gamma(h_j)}\left(\frac{h_j}{h_{i,j}}\right)^\kappa$, $1\leq i,j\leq n$, and $b_\kappa(\mathbb{S})\leq 1$, that $\|p_j-p_{j-1}\|_{p(I_j\cup I_{j-1})}\leq c(p)\Gamma(x_{j-2}-x_j), 2\leq j\leq n$
Hence, by Theorem (3.5) implies the existence of a piecewise polynomial $\tilde{\mathbb{S}}\in\sum^1_{\kappa;n}(Y_s)\cap\Delta^{(2)}(Y_s)$, such that
(3.13) $\|\mathbb{S}(x)-\tilde{\mathbb{S}}(x)\|_p\leq c(p)\Gamma(\Omega(x)), x\in[-1,1]$

And we only have to prove (3.10) for $\mathscr{v},\mu = 1,\dots,2\mathfrak{n}$, set $\mathfrak{j} := \left[\frac{\mathscr{v}}{2}\right]$, and $\mathfrak{i} := [\mu/2]$.
Then by Lemma (3.1) and (1.3.11),

(3.14) $$b_{\mu,\mathscr{v};2\mathfrak{n}}(\mathbb{S}) := \frac{\|p_{\mu;2\mathfrak{n}}-p_{\mathscr{v};2\mathfrak{n}}\|_{p(I_{\mu;2\mathfrak{n}})}}{\Gamma(h_{\mathscr{v};2\mathfrak{n}})}\left(\frac{h_{\mathscr{v};2\mathfrak{n}}}{h_{\mu;2\mathfrak{n}}}\right)^{\kappa} \le c(p)\frac{\|p_{\mu;2\mathfrak{n}}-p_{\mathscr{v};2\mathfrak{n}}\|_{p(I_{\mu;2\mathfrak{n}})}}{\Gamma(h_{\mathfrak{j}})}\left(\frac{h_{\mathfrak{j}}}{h_{\mathfrak{i},\mathfrak{j}}}\right)^{\kappa}$$

$$\le c(p)\frac{\|p_{\mu;2\mathfrak{n}}-p_{\mathscr{v};2\mathfrak{n}}\|_{p(I_{\mu;2\mathfrak{n}})}}{\Gamma(h_{\mathfrak{j}})}\left(\frac{h_{\mathfrak{j}}}{h_{\mathfrak{i},\mathfrak{j}}}\right)^{\kappa} + c(p)b_{\mathfrak{i},\mathfrak{j}}(\mathbb{S}) - c(p)b_{\mathfrak{i},\mathfrak{j}}(\mathbb{S})$$

$$\le c(p)b_{\mathfrak{i},\mathfrak{j}}(\mathbb{S}) + c(p)\frac{\|p_{\mu;2\mathfrak{n}}-p_{\mathfrak{i}}\|_{p(I_{\mu;2\mathfrak{n}})}}{\Gamma(h_{\mathfrak{j}})}\left(\frac{h_{\mathfrak{j}}}{h_{\mathfrak{i},\mathfrak{j}}}\right)^{\kappa} + c(p)\frac{\|p_{\mathscr{v};2\mathfrak{n}}-p_{\mathfrak{j}}\|_{p(I_{\mu;2\mathfrak{n}})}}{\Gamma(h_{\mathfrak{j}})}\left(\frac{h_{\mathfrak{j}}}{h_{\mathfrak{i},\mathfrak{j}}}\right)^{\kappa}$$

$$=: c(p) + \mathcal{J}_1 + \mathcal{J}_2$$

Now, by(3.12)

(3.15) $$\mathcal{J}_1 \le c(p)\frac{\Gamma(h_{\mathfrak{i}})}{\Gamma(\mathrm{h}_{\mathfrak{j}})}\left(\frac{\mathrm{h}_{\mathfrak{j}}}{\mathrm{h}_{\mathfrak{i},\mathfrak{j}}}\right)^{\kappa} \le c,$$

Where we used the fact that if $h_{\mathfrak{i}} < h_{\mathfrak{j}}$,
then $\Gamma(\mathrm{h}_{\mathfrak{i}}) \le \Gamma(\mathrm{h}_{\mathfrak{j}})$, and if $\mathrm{h}_{\mathfrak{i}} > \mathrm{h}_{\mathfrak{j}}$ then $\Gamma(\mathrm{h}_{\mathfrak{i}})/\Gamma(\mathrm{h}_{\mathfrak{j}})\mathrm{h}_{\mathfrak{j}}^{\mathrm{k}} \le \mathrm{h}_{\mathfrak{i}}^{\mathrm{k}}$
Finally

(3.16) $$\mathcal{J}_2 \le c(p)\frac{\|p_{\mathscr{v};2\mathfrak{n}}-p_{\mathfrak{j}}\|_{p(I_{\mu,\mathscr{v};2\mathfrak{n}})}}{\Gamma(h_{\mathfrak{j}})}\left(\frac{h_{\mathfrak{j}}}{h_{\mathfrak{i},\mathfrak{j}}}\right)^{\kappa} \le$$

$$c(p)\left(\frac{h_{\mathfrak{i},\mathfrak{j}}}{h_{\mathfrak{j}}}\right)^{\kappa}\frac{\|p_{\mathscr{v};2\mathfrak{n}}-p_{\mathfrak{j}}\|_{p(I_{\mu,\mathscr{v};2\mathfrak{n}})}}{\Gamma(h_{\mathfrak{j}})}\left(\frac{h_{\mathfrak{j}}}{h_{\mathfrak{i},\mathfrak{j}}}\right)^{\kappa} \le c,$$

the second inequality we used the fact that both polynomials are of degree $< \kappa$, and for the last inequality we again applied(3.13). Where for Substituting (3.15) and (3.16) in to (3.14) we complete the prove of Theorem 3.10.

<u>Theorem</u> (3.17)
Let $Y_{\mathbb{s}} \in \mathbb{Y}_{\mathbb{s}}, \kappa \in N, \Gamma \in \Phi^{\kappa}$ and $G \in \Delta^{(2)}(Y_{\mathbb{s}}) \cap L_p^1[-1,1]$ such that $G \in L_p^2(x_{\mathfrak{j}}, x_{\mathfrak{j}-1}), \mathfrak{j} = 1,\dots,\mathbb{s}$, and be given. If G is a linear function $l_{\mathfrak{i}}$ on each $O_{\mathfrak{i}}, \mathfrak{i} = 1,\dots,\mathbb{s}$, and

(3.18) $$\|G''(x)\|_p \le c(p)\left\|\frac{\Gamma(\Omega)}{\Omega^2}\right\|_p, x \in [-1,1]\backslash\{x_{\mathfrak{j}}\}_{\mathfrak{j}=1}^{\mathfrak{n}-1}$$, then there is a polynomial $P_{\tilde{\mathfrak{n}}} \in \Delta^{(2)}(Y_{\mathbb{s}})$ of degree $\tilde{\mathfrak{n}} \le c\mathfrak{n}$ such that

(3.19) $$\|G''(x) - P_{\tilde{\mathfrak{n}}}\|_p \le c(p)\,\Gamma(\Omega), x \in [-1,1]$$

Proof.
Without loss of generality we may assume that $\mathfrak{n}$ is even and that the assumptions of the Lemma hold with $\mathfrak{n}/2$ in place of , that is , we assume that $G \in L_p^2(x_{2\mathfrak{j}}, x_{2\mathfrak{j}-2}), \mathfrak{j} = 1,\dots,\mathfrak{n}/2$, and that G is a linear function $l_{\mathfrak{i}}$ on each $O_{\mathfrak{i},\frac{\mathfrak{n}}{2}}(Y_{\mathbb{s}}), \mathfrak{i} = 1,\dots,\mathbb{s}$
Let L be the polygonal line interpolating G at all points $x_{\mathfrak{j}},, \mathfrak{j} = 0,\dots,\mathfrak{n}$.
Clearly $L \in \Delta^{(2)}(Y_{\mathbb{s}})$, since G is a linear function on each $O_{\mathfrak{i},\frac{\mathfrak{n}}{2}}(Y_{\mathbb{s}})$.Then, with $G_{\mathfrak{j}} = [x_{\mathfrak{j}+1}, x_{\mathfrak{j}}, x_{\mathfrak{j}-1}, G](x_{\mathfrak{j}-1} - x_{\mathfrak{j}+1})$

$$= \frac{\mathrm{G}(x_{\mathfrak{j}-1}) - \mathrm{G}(x_{\mathfrak{j}})}{x_{\mathfrak{j}-1} - x_{\mathfrak{j}}} - \frac{\mathrm{G}(x_{\mathfrak{j}+1}) - \mathrm{G}(x_{\mathfrak{j}})}{x_{\mathfrak{j}+1} - x_{\mathfrak{j}}}$$

The polygonal line L may be represented in the form

(3.20) $L(x) = G(-1) + [x_n, x_{n-1}, G](x+1) + \sum_{j=1}^{n-1} G_j(x - x_j)_+$
$= G(-1) + [x_n, x_{n-1}, G](x+1) + \sum_{j\in H} G_j(x - x_j)_+$
Where in the last equality we used the assumption that G is a linear function on each $O_{i,\frac{n}{2}}(Y_s)$.
Since $G \in L_p^1[x_{j+1}, x_{j-1}]$ and $G \in L_p^2(x_{j+1}, x_j) \cup (x_j, x_{j-1})$, we have by (3.18),
(3.21) $\|G_j\|_p \le c(p)\|G''(x)\|_p(x_{j-1} - x_{j+1}) \le c(p)\frac{\Gamma(h_j)}{h_j}$
Similarly, it readily follows by (3.18) and by Lemma 2.5 that for all $x \in [x_j, x_{j-1}], 1 \le j \le n$,
$\|G(x) - L(x)\|_p = \|[x, x_j, x_{j-1}, G](x - x_j)(x - x_{j-1})\|_p$
(3.22) $\le c(p)\frac{1}{2}(x - x_j)(x - x_{j-1})\|G''(\theta)\|_p \le c(p)\Gamma(h_j) \le c(p)\Gamma(\Omega),$
Where $\in (x_j, x_{j-1})$. let
(3.23) $P_{\tilde{n}} := G(-1) + [x_n, x_{n-1}, G](x+1) + \sum_{j\in H} G_j\, \mathcal{T}_j(x)$
Where τ_j are the polynomials guaranteed by Lemma 2.3 with $b = \kappa + 3$.since $\Pi(x_j)G_j \ge 0, j \in H$, it readily follows by Lemma 2.4 that $P_{\tilde{n}} \in \Delta^{(2)}(Y_s)$.
Now, by Lemma 2.3 and Proposition 3.3

$$\|(x - x_j)_+ - \tau_j(x)\|_p \le c(p)h_j\left\|\left(\frac{h_j}{|x - x_j| + h_j}\right)^{\kappa+3}\right\|_p$$

$$\le c(p){h_j}^{2-\kappa}\left\|\frac{{h_j}^{2\kappa+2}}{(|x - x_j| + h_j)^{\kappa+3}}\right\|_p$$

$$\le c(p)\left\|\frac{{h_j}^{2-\kappa}}{(|x - x_j| + \Omega)^2}\min\{\Omega^{\kappa+1}, {h_j}^{\kappa+1}\}\right\|_p, x \in [-1,1]$$

Since $\Gamma \in \Phi^\kappa$, it follows that $\Gamma(h_j) \le \frac{{h_j}^\kappa}{\min\{\Omega^\kappa, {h_j}^\kappa\}}\Gamma(\Omega)$,
We obtain
(3.24) $\|L(x) - P_{\tilde{n}}(x)\|_p \le c(p)\|\sum_{j\in H} G_j(x - x_j)_+ - \mathcal{T}_j(x)\|_p$

$$\le c(p)\left\|\sum_{j\in H}\Gamma(h_j)\frac{{h_j}^{1-\kappa}}{(|x - x_j| + \Omega)^2}\min\{\Omega^{\kappa+1}, {h_j}^{\kappa+1}\}\right\|_p$$

$$\le c(p)\left\|\Gamma(\Omega)\sum_{j\in H}\frac{h_j}{(|x - x_j| + \Omega)^2}\min\{\Omega, h_j\}\right\|_p$$

$$\le c(p)\left\|\Omega\,\Gamma(\Omega)\sum_{j=1}^{n}\frac{h_j}{(|x - x_j| + \Omega)^2}\right\|_p \le c(p)\Omega\,\Gamma(\Omega)\int_\Omega^\infty \frac{dt}{t^2} = c(p)\,\Gamma(\Omega)$$

In order to complete the proof we have to estimate $G - P_{\tilde{n}}$ for all $x \in [-1,1]$.
We have $\|G(x) - P_{\tilde{n}}(x)\|_p \le c(p)\|G(x) - L(x)\|_p + \|L(x) - P_{\tilde{n}}(x)\|_p$
$\le c(p)\Gamma(\Omega) + \|L(x) - P_{\tilde{n}}(x)\|_p \le c(p)\Gamma(\Omega)$.

4. Conclusions and Recommendations

In spite of the shape preserving constraints we can approximate a piecewise convex function by a pointwise polynomial in terms of higher orders moduli of smoothness.

Conflict of interests.

There are non-conflicts of interest.

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الخلاصة

قدم بعض الباحثين المبرهنات المباشرة للتقريب المحدب والمتغير التحدب للدوال المستمرة المعرفة على الفترة $[-1,1]$, والذي بدوره قيد درجة التقريب الافضل.

في هذا البحث قمنا بدراسة التقريب المحدب للدوال متغيرة التحدب في الفضاءات L_p عندما $0 < p < 1$.

الكلمات الدالة:

إتجاه النقطة ، تقريب محدب ، مسافات شبه معيارية