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YU-TING CHEN AND JEAN-FRANÇOIS DELMAS

ABSTRACT. We present an elementary model of random size varying population given by a stationary continuous state branching process. For this model we compute the joint distribution of: the time to the most recent common ancestor, the size of the current population and the size of the population just before the most recent common ancestor (MRCA). In particular we show a natural mild bottleneck effect as the size of the population just before the MRCA is stochastically smaller than the size of the current population. We also compute the number of old families which corresponds to the number of individuals involved in the last coalescent event of the genealogical tree. By studying more precisely the genealogical structure of the population, we get asymptotics for the number of ancestors just before the current time. We give explicit computations in the case of the quadratic branching mechanism. In this case, the size of the population at the MRCA is, in mean, less by 1/3 than size of the current population size. We also provide in this case the fluctuations for the renormalized number of ancestors.

1. INTRODUCTION

A large literature is devoted to constant size population models. It goes back to Wright [47] (1930) and Fisher [21] (1931) in discrete time, and Moran [39] (1958) in continuous time. Models for constant infinite population in continuous time with spatial motion were introduced by Fleming and Viot [22] (1979). On the other hand, the study of the genealogical tree of constant size population was initiated by Kingman [29] (1982), and described in a more general setting by Pitman [43] (1999) and Sagitov [46] (1999). The complete description of the genealogy of the Fleming-Viot process can be partially done using the historical super-process by Dawson and Perkins [13] (1991) and precisely by using the look-down process developed by Donnelly and Kurtz [14, 15] (1999) or the stochastic flows from Bertoin and Le Gall [9, 10, 11] (2003).

It is however natural to consider random size varying population models. Branching population models, for which sizes of the population are random, goes back to Galton and Watson [23] (1873) in discrete time and with finite mass individual. Jirina [26] (1958) considered continuous state branching process (CB) models corresponding to individuals with infinitesimal mass. The genealogy of those processes can be partially described through the historical super-process. However the continuum Lévy tree introduced by Le Gall and Le Jan [34] (1998) and developed later by Duquesne and Le Gall [16] (2002) allows to give a complete

description of the genealogy in the critical and sub-critical cases. See the approach of Abraham and Delmas [1] (2008) or Berestycki, Kyprianou and Murillo [7] (2009) for a description of the genealogy in the super-critical cases.

The two families of models: constant size population and branching populations are, in certain cases, linked. The case of a quadratic branching corresponds to the fact that only two genealogical lines of the population genealogical tree can merge together. In this particular case, it is possible to establish links between the constant size population model and CB models. Thus, conditionally on having a constant population size, the Dawson-Watanabe super-process is a Fleming-Viot process, see Etheridge-March [18] (1991). On the other hand, using a time change (with speed proportional to the inverse of the population size), it is possible to recover a Fleming-Viot process from a Dawson-Watanabe super-process, see Perkins [41] (1992). Birkner, Blath, Capaldo, Etheridge, Möhle, Schweinsberg and Wakolbinger [12] (2005) have given similar results for stable branching mechanism. In the same spirit, Kaj and Krone [27] (2003) studied the genealogical structure of models of random size varying population models and recover the Kingman coalescent with a random time change.

Recently, some authors studied the coalescent process (or genealogical tree) of random size varying population, in this direction see Möhle [38] (2002), Lambert [30] (2003) for branching process and Jagers and Sagitov [25] (2004) for stationary random size varying population.

Our primary interest is to present an elementary model of random size varying population and exhibit some interesting property which could not be observed in constant size model. The most striking example is the natural mild bottleneck effect: in a stationary regime, the size of the population just before the most recent common ancestor (MRCA) is stochastically smaller than the current population size. Our second goal is to give some properties of the coalescent tree such as: time to the most recent common ancestor (TMRCA), number of individuals involved in the last coalescent event, asymptotic behavior of the number of recent ancestors.

One of the major drawback of the branching population models is that either the population becomes extinct or decreases to 0, which happens with probability 1 in the (sub)critical cases, or blows up exponentially fast with positive probability in the super-critical case. In particular there is no stationary regime, and the study of the genealogy of a current population depends on the arbitrary original size and time of the initial population. To circumvent this problem, we consider a sub-critical CB, $Y = (Y_t, t \geq 0)$, with branching mechanism ψ given by (1). We get the Q-process by conditioning Y to non-extinction (which is an event of zero probability), see [35] and [31]. The Q-process can also be seen as a CB with immigration, see [45]. We take the opportunity to present a probabilistic construction of independent interest for the Q-process in Corollary 3.5 which relies on a Williams' decomposition of CB described in [2]. A first study of the genealogical tree of the Q-process can be found in [30].

We consider the Q-process under its stationary distribution and defined on the real line: $Z = (Z_t, t \in \mathbb{R})$. Its Laplace transform, see (3.6), is given by

$$\mathbb{E}\left[e^{-\lambda Z_t}\right] = \exp\left(-\int_0^\infty ds\; \tilde{\psi}'(u(\lambda, s))\right), \quad \lambda \geq 0, \quad t \in \mathbb{R},$$

where $\tilde{\psi}(\lambda) = \psi(\lambda) - \lambda\psi'(0)$. In order for Z_t to be finite, we shall assume condition (A2):

$$\int_0^1 \left(\frac{1}{v\psi'(0)} - \frac{1}{\psi(v)}\right) dv < +\infty.$$

In order for the TMRCA to be finite, we assume condition (A1):

$$\int_1^\infty \frac{dv}{\psi(v)} < +\infty.$$

Notice a very similar condition exists to characterize coalescent processes which descent from infinity, see [6].

As in the look-down representation for constant size population, we shall represent the process Z using the picture of an immortal individual which gives birth to independent sub-populations or families. For fixed time $t_0 = 0$ (which we can indeed choose to be equal to 0 by stationarity), we consider A the TMRCA of the population living at time 0, $Z^A = Z_{(-A)-}$ the size of the population just before the MRCA, Z^I the size of the population at time 0 which has been generated by the immortal individual over the time interval $(-A, 0)$ and $Z^O = Z_0 - Z^I$ the size of the population at time 0 which has been generated by the immortal individual at time $-A$. In Theorem 4.1, we give the joint distribution of (A, Z^A, Z^I, Z^O). One interesting phenomenon is Corollary 4.3.

Corollary 1. *Conditionally on A; Z^A, Z^I and Z^O are independent.*

In particular, conditionally on A, Z^A and Z are independent. Conditionally on A, Z^A depends on the past before $-A$ of the process Z and has to die at time 0, Z^O corresponds to the size of the population at time 0 generated at time $-A$ and Z^I corresponds to the size of the population at time 0 generated by the immortal individual over the time interval $(-A, 0)$. Then, as the immortal individual gives birth to independent populations, the Corollary is then intuitively clear.

One of the most striking result, the natural mild bottleneck effect, is stated in Proposition 4.5:

Proposition 2. *Z^A is stochastically smaller than Z_0.*

Thus just before the MRCA, the population size is unusually small. Notice this result is not true in general if one considers the size of the population at the MRCA instead of just before, see Remark 4.6. We get nice quantitative results for the quadratic branching mechanism case, see Corollary 7.2.

Corollary 3. *Assume ψ is quadratic (and given by (42)). We have: a.s.*

$$\mathbb{P}(Z^A < Z_0 | A) = \frac{11}{16} \quad and \quad \mathbb{E}[Z^A | A] = \frac{2}{3} \mathbb{E}[Z_0 | A]$$

an in particular:

$$\mathbb{P}(Z^A < Z_0) = \frac{11}{16} \quad and \quad \mathbb{E}[Z^A] = \frac{2}{3} \mathbb{E}[Z_0].$$

Notice that even is Z^A is stochastically smaller than Z_0 it is not a.s. smaller.

We also give in Theorem 4.7 the joint distribution of Z_0 and the TMRCA of the immortal individual and n individuals picked at random in the population at time 0. See also related results in [30].

We investigate in Proposition 5.2 the joint distribution of A, Z_0 and N^A, where $N^A + 1$ represents the number of individuals involved in the last coalescent event of the genealogical tree. Under a first moment condition on Z, we get that if the TMRCA is large, then the last coalescent event is likely to involve only two individuals. In the stable case, this first moment condition is not satisfied, and the last coalescent event does not depend on the TMRCA, see Remark 5.5. This suggests a result similar to the one obtained in [12]: in the stable case,

the topology of the genealogical tree (which does not take into account the length of the branches) may not depend on its depth given by the TMRCA.

After giving a more precise description of the genealogy of Z using continuum Lévy trees, we compute in Theorem 6.9 the asymptotic behavior of the number of ancestors at time $-s$, M_s, of the population at time 0.

Theorem 4. *The following convergence holds in probability:*

$$\lim_{s\downarrow 0} \frac{M_s}{c(s)} = Z_0,$$

where $c(s)$ is related to the extinction probability of Y and defined by $\int_{c(t)}^{\infty} \frac{dv}{\psi(v)} = t$.

This result is very similar to the one obtained on coalescent process in [6] (notice the convergence is a.s. in [6]). We can precise the fluctuations in the quadratic case, see Theorem 7.8.

Theorem 5. *Assume ψ is quadratic (and given by (42)). We have*

$$\sqrt{c(s)\mathbb{E}[Z_0]} \left(\frac{M_s}{c(s)} - Z_0 \right) \xrightarrow[s\downarrow 0+]{(d)} (Z_0 - Z_0'),$$

where Z_0' is distributed as Z_0 and independent of Z_0.

The paper is organized as follows. We first recall well known facts on CB in Section 2. We introduce in Section 3 the corresponding stationary CB, which is related to the Q-process of the CB, and give its first properties. We give the joint distribution of (A, Z^A, Z^I, Z^O) in Section 4 and prove the natural bottleneck effect, that is Z^A is stochasitcally smaller than Z_0. We compute the number of old families (or number of individuals involved in the last coalescent event) in Section 5 and the asymptotics of the number of ancestors in Section 6. A first consequent part of this latter Section is devoted to the introduction of the genealogy of CB processes using continuum random Lévy trees. We give more detailed results in the quadratic branching setting of Section 7.

2. CONTINUOUS-STATE BRANCHING PROCESS (CB)

We recall some well-known fact on continuous-state branching process (CB), see for example [36] and references therein. We consider a **sub-critical** branching mechanism ψ: for $\lambda \geq 0$,

$$(1) \qquad \psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,+\infty)} \pi(d\ell) \left[e^{-\lambda\ell} - 1 + \lambda\ell \right],$$

where $\alpha = \psi'(0) > 0$, $\beta \geq 0$ and π is a Radon measure on $(0, +\infty)$ such that $\int_{(0,+\infty)}(\ell \wedge \ell^2)\, \pi(d\ell) < +\infty$. We consider the non trivial case that is either $\beta > 0$ or $\pi((0,1)) = +\infty$. Notice that ψ is convex, of class $\mathcal{C}^1$ on $[0, +\infty)$ and of class $\mathcal{C}^\infty$ on $(0, +\infty)$ and $\psi''(0+) \in (0, +\infty]$.

Let P_x be the law of a CB $Y = (Y_t, t \geq 0)$ started at mass $x \geq 0$ and with branching mechanism ψ, and let E_x be the corresponding expectation. The process Y is a càd-làg $\mathbb{R}_+$-valued Feller process and 0 is a cemetery point. The process Y has no fixed discontinuities. For every $\lambda > 0$, for every $t \geq 0$, we have

$$(2) \qquad\qquad E_x \left[e^{-\lambda Y_t} \right] = e^{-xu(\lambda, t)},$$

where the function u is the unique non-negative solution of

$$(3) \qquad u(\lambda, t) + \int_0^t \psi\big(u(\lambda, s)\big)ds = \lambda, \quad \lambda \geq 0, \quad t \geq 0.$$

Note that the function u is equivalently characterized as the unique non-negative solution of

$$(4) \qquad \int_{u(\lambda,t)}^{\lambda} \frac{dr}{\psi(r)} = t \quad \lambda \geq 0, \quad t \geq 0.$$

or as the unique non-negative solution of: for $\lambda \geq 0$,

$$(5) \qquad \begin{cases} \partial_t u + \psi(u) &= 0 \quad t > 0, \\ u(\lambda, 0) &= \lambda. \end{cases}$$

Markov property of Y implies that for all $\lambda, s, t \geq 0$:

$$(6) \qquad u(u(\lambda, t), s) = u(\lambda, t + s).$$

Let $\mathbb{N}$ be the canonical measure (we shall also call it excursion measure) associated to Y. It is a σ-finite measure which intuitively describe the distribution of Y started at an infinitesimal mass. We recall that if

$$\sum_{i \in I} \delta_{x_i, Y^i}(dx, dY)$$

is a Poisson point measure with intensity $\mathbf{1}_{[0,+\infty)}(x) \, dx \mathbb{N}[dY]$, then

$$(7) \qquad \sum_{i \in I} \mathbf{1}_{\{x_i \leq x\}} Y^i$$

is distributed as Y under P_x. In particular, we have: for $\lambda \geq 0$

$$\mathbb{N}\left[1 - \mathrm{e}^{-\lambda Y_t}\right] = \lim_{x \downarrow 0} \frac{1}{x} \mathrm{E}_x\left[1 - \mathrm{e}^{-\lambda Y_t}\right] = u(\lambda, t).$$

For convenience, we shall put $Y_t = 0$ for $t < 0$.

Let $\zeta = \inf\{t; Y_t = 0\}$ be the extinction time of Y. We consider the function:

$$(8) \qquad c(t) = \mathbb{N}[\zeta > t] = \mathbb{N}[Y_t > 0] = \lim_{\lambda \to \infty} \uparrow u(\lambda, t).$$

We shall assume throughout this paper, but for Sections 3.1 and 3.3, that the strong extinction property holds:

$$(A1) \qquad \int_1^{\infty} \frac{dv}{\psi(v)} < +\infty.$$

It follows from (4) and (8) that c is the unique non-negative solution of:

$$(9) \qquad \int_{c(t)}^{\infty} \frac{dv}{\psi(v)} = t, \quad t > 0.$$

Thanks to (A1), we get that $c(t)$ is finite for all $t > 0$ and $\mathbb{N}[\zeta = +\infty] = 0$. We also get that c is continuous decreasing and thus one-to-one from $(0, +\infty)$ to $(0, +\infty)$. Letting λ goes to infinity in (6) yields that for $s, t \geq 0$

$$(10) \qquad u(c(t), s) = c(t + s).$$

3. STATIONARY CB

In contrast to Wright-Fisher population models, CB models do not exhibit stationary distributions. However, by conditioning sub-critical CB to non-extinction (see [45], [20] and [31] for details), one get the so-called Q-process, which we denotes by Y''. This process is also a CB process with immigration introduced in [28] and may have a stationary distribution. This process, as pointed out in [3] see also [19], has a heuristic interpretation by introducing a fixed ancestral lineage, namely it is an independent sum of the process Y and the population thrown off by an "immortal individual" whose laws coincide with the law of a generic population Y.

We introduce the process Y'' in Section 3.1 as well as its stationary version Z. Then we check in Section 3.2, that under (A1) the process Y'' is indeed the Q-process associated to Y. This gives then a natural interpretation of Z. We give preliminary results on the process Z in Sections 3.3 and 3.4.

3.1. **Poisson point measure of CB.** We consider the following Poisson point measures.

- Let $\mathcal{N}_0(dr, dt) = \sum_{i \in I} \delta_{(r_i, t_i)}(dr, dt)$ be a Poisson point measure on $(0, +\infty) \times \mathbb{R}$ with intensity

$$r \, \pi(dr) dt.$$

- Conditionally on $\mathcal{N}_0$, let $(\mathcal{N}_{1,i}, i \in I)$, where $\mathcal{N}_{1,i}(dt, dY) = \sum_{j \in J_{1,i}} \delta_{t_j, Y^j}(dt, dY)$, be independent Poisson point measures with respective intensity

$$r_i \delta_{t_i}(dt) \mathbb{N}[dY].$$

  Notice that for all $j \in J_{1,i}$, we have $t_j = t_i$. We set $J_1 = \bigcup_{i \in I} J_{1,i}$ and $\mathcal{N}_1(dt, dY) = \sum_{j \in J_1} \delta_{t_j, Y^j}(dt, dY)$.

- Let $\mathcal{N}_2(dt, dY) = \sum_{j \in J_2} \delta_{t_j, Y^j}(dt, dY)$ be a Poisson point measure independent of $(\mathcal{N}_0, \mathcal{N}_1)$ and with intensity

$$2\beta \, dt \, \mathbb{N}[dY].$$

We set $\mathcal{J} = J_1 \bigcup J_2$. We shall call Y^j, with $j \in \mathcal{J}$ a **family** and t_j its birth time.

We will consider the two following processes $Y'' = (Y_t'', t \geq 0)$ and its stationary version $Z = (Z_t, t \in \mathbb{R})$:

$$\tag{11} Y_t'' = \sum_{j \in \mathcal{J}, t_j > 0} Y_{t-t_j}^j,$$

$$\tag{12} Z_t = \sum_{j \in \mathcal{J}} Y_{t-t_j}^j.$$

We will denote by $\mathbb{P}$ the probability under which Y'' and Z are defined and $\mathbb{E}$ the corresponding expectation.

At this stage, let us emphasize there is another natural decomposition of Y'' and Z. For $i \in I$, set $Y^i = \sum_{j \in J_{1,i}} Y^j$ and $\mathcal{I} = I \bigcup J_2$. The random measure

$$\tag{13} \mathcal{N}_3(dt, dY) = \sum_{i \in \mathcal{I}} \delta_{t_i, Y^i}(dt, dY)$$

is a Poisson point measure with intensity $dt \mu(dY)$ and

$$\tag{14} \mu(dY) = 2\beta \mathbb{N}[dY] + \int_{(0, +\infty)} \ell \pi(d\ell) \, \mathrm{P}_\ell(dY).$$

And we have:

$$(15) \qquad Y_t'' = \sum_{i \in \mathcal{I}; t_i > 0} Y_{t-t_i}^i,$$

$$(16) \qquad Z_t = \sum_{i \in \mathcal{I}} Y_{t-t_i}^i.$$

We shall call Y^i, with $i \in \mathcal{I}$ a **clan** and t_i its birth time. For $j \in J_2$, Y^j is a clan and a family. Notice that a.s. two clans have different birth time, but families in the same clan have the same birth time.

The presentation with clans is simpler than the representation with families and most of the results can be obtained using the former representation. We will use the family representation in Sections 5 and 6.

We define $\tilde{\psi}$ by:

$$(17) \qquad \tilde{\psi}(\lambda) = \psi(\lambda) - \lambda \psi'(0) = \psi(\lambda) - \alpha \lambda.$$

We first give a Lemma on the family representation.

Lemma 3.1. *Let F be a non-negative measurable function. We have*

$$(18) \qquad \mathbb{E}\left[e^{-\sum_{j \in \mathcal{J}} F(t_j, Y^j)}\right] = \exp\left(-\int_{\mathbb{R}} dt \ \tilde{\psi}'(\mathbb{N}[1 - e^{F(t,Y)}])\right).$$

Proof. Using Poisson point measure properties, we get:

$$\mathbb{E}\left[e^{-\sum_{j \in \mathcal{J}} F(t_j, Y^j)}\right] = \mathbb{E}\left[e^{-\sum_{j \in J_1} F(t_j, Y^j)}\right] \mathbb{E}\left[e^{-\sum_{j \in J_2} F(t_j, Y^j)}\right]$$

$$= \mathbb{E}\left[e^{-\sum_{i \in I} r_i \mathbb{N}[1 - F(t_i, Y)]}\right] e^{-2\beta \int dt \ \mathbb{N}[1 - e^{F(t,Y)}]}$$

$$= e^{-\int dt \int_{(0,+\infty)} \ell \pi(d\ell) \ (1 - \exp(-\ell \mathbb{N}[1 - F(t,Y)]))} e^{-2\beta \int dt \ \mathbb{N}[1 - e^{F(t,Y)}]}$$

$$= e^{-\int dt \ \tilde{\psi}'(\mathbb{N}[1 - e^{F(t,Y)}])}.$$

$\square$

Proposition 3.2. *The process Y'' is a CB with branching mechanism ψ and immigration function $\tilde{\psi}'$:*

$$\tilde{\psi}'(\lambda) = 2\beta\lambda + \int_{(0,+\infty)} \ell \pi(d\ell) \ (1 - e^{-\lambda \ell})$$

started at $Y_0'' = 0$.

Proof. This is a direct consequence of Lemma 3.1 and results from [28]. $\square$

In particular Y'' is a strong Markov process started at 0 and its transition kernel is characterized by: for $\lambda \geq 0$, $t \geq 0$, $r \geq 0$

$$\mathbb{E}[e^{-\lambda Y_t''} | Y_0'' = r] = \exp\left(-ru(\lambda, t) - \int_0^t \tilde{\psi}'(u(\lambda, s)) \ ds\right).$$

The next result is then straightforward.

Corollary 3.3. *For each $t \in \mathbb{R}$, $\{Z_s; s \geq t\}$ has the same law as a CB with branching mechanism ψ and immigration function $\tilde{\psi}'$ started at the invariant distribution $\mathbb{P}(Z_t \in \cdot)$.*

3.2. **Q-process.** We check the process Y'' is indeed the Q-process for CB using Williams' decomposition.

Let $m > 0$ and $\nu_m(dt) = \sum_{i \in I} r_i \delta_{t_i}(dt)$, where $\sum_{i \in I} \delta_{(r_i, t_i)}(dr, dt)$ is a Poisson point measure with intensity

$$\mathbf{1}_{[0,m]}(t)\, \mathrm{e}^{-rc(m-t)}\ r\, \pi(dr) dt.$$

Conditionally on ν_m, let $\sum_{j \in \mathcal{J}^m} \delta_{t_j, Y^j}(dt, dY)$ be a Poisson point measure with intensity

$$\left(\nu_m(dt) + 2\beta \mathbf{1}_{[0,m]}(t)\, dt \right)\ \mathbb{N}[dY, \zeta < m - t].$$

The next Proposition is a consequence of Theorem 3.3 in [2].

Proposition 3.4. *Assume (A1) holds. Under $\mathbb{N}$, conditionally on $\{\zeta = m\}$, Y is distributed as $(Y'_t, t \geq 0)$ where*

$$Y'_t = \sum_{j \in \mathcal{J}^m} Y^j_{t-t_j}.$$

It is then easy to deduce the following Corollary using representation (15) of Y''.

Corollary 3.5. *Assume (A1) holds. The limit distribution of Y under $\mathbb{N}$, conditionally on $\{\zeta = m\}$, as m goes to infinity, is the distribution of Y'' from Proposition 3.2.*

Corollary 3.5 readily implies that the Q-process associated to Y, that is the limit distribution of Y under $\mathbb{N}$, conditionally on $\{\zeta \geq m\}$, as m goes to infinity, is the distribution of Y'' from Proposition 3.2.

3.3. **Stationary CB.** We first give an interpretation of Z in population terms. At time t, Z_t correspond to the size of a population generated by an immortal individual (with zero mass) which gives birth at rate 2β to clans (or families) which sizes evolve independently as Y under $\mathbb{N}$ and at rate 1 with intensity $r\, \pi(dr)$ to clans with initial size r which evolve independently as Y under P_r.

By construction the process Z is stationary. The next Lemma which gives the Laplace transform of Z is a direct consequence of the construction of Z.

Lemma 3.6. *For all $t \in \mathbb{R}$ and $\lambda \geq 0$, the Laplace transform of Z_t is given by:*

$$(19) \qquad \mathbb{E}\left[\mathrm{e}^{-\lambda Z_t} \right] = \exp\left(-\int_0^\infty ds\ \tilde{\psi}'(u(\lambda, s)) \right).$$

Proof. Using Lemma 3.1, we have:

$$\mathbb{E}\left[\mathrm{e}^{-\lambda Z_t} \right] = \exp\left(-\int_{\mathbb{R}} ds\ \tilde{\psi}'(\mathbb{N}[1 - \mathrm{e}^{-\lambda Y_{t-s}}]) \right) = \exp\left(-\int_0^\infty ds\ \tilde{\psi}'(u(\lambda, s)) \right).$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We shall consider the following assumption

$$(A2) \qquad\qquad \int_1^{+\infty} \ell \log(\ell)\, \pi(d\ell) < +\infty.$$

The next Lemma is well known (notice condition (A1) is not assumed).

Lemma 3.7. *In the sub-critical case, the following conditions are equivalent:*

  (i) *(A2) holds.*
  (ii) $\int_0^1 \left(\frac{1}{\alpha v} - \frac{1}{\psi(v)} \right) dv < +\infty.$
  (iii) $\mathbb{E}_r[Y_t \log(Y_t)] < +\infty$ *for some $t > 0$ and $r > 0$.*

(iv) $\mathbb{E}_r[Y_t \log(Y_t)] < +\infty$ *for all $t > 0$ and $r > 0$.*

Proof. For (i) $\Leftrightarrow$ (ii) see [24] proof of Theorem 4a, and for (ii) $\Leftrightarrow$ (iii) (or (iv)) use Lemma 1 p.25 of [5]. $\qquad\square$

The next Proposition gives a condition for finiteness of Z, see also [42] in a more general framework.

Proposition 3.8. *We have $\mathbb{P}(Z_0 < +\infty) = 1$ if and only if (A2) holds.*

Proof. Thanks to (19), we get $\mathbb{P}(Z_0 < +\infty) = 1$ if and only if $\lim_{\lambda \to 0} \int_0^\infty ds\ \tilde{\psi}'(u(\lambda, s)) = 0$. As $\lambda \mapsto u(\lambda, s)$ decreases to 0 as λ goes down to 0 for all $s \geq 0$, we deduce by dominated convergence that $\mathbb{P}(Z_0 < +\infty) = 1$ if and only if $\int_0^\infty ds\ \tilde{\psi}'(u(\lambda, s)) < +\infty$ for at least one $\lambda > 0$.

Notice that $\partial_t u + \psi(u) = 0$ implies $\psi'(u) = -\partial_t^2 u / \partial_t u$, and hence for every $0 \leq t < T < +\infty$ we have

$$(20) \qquad \int_t^T \tilde{\psi}'(u(\lambda, s))\ ds = \log\left(\frac{\psi(u(\lambda, t))\, \mathrm{e}^{\alpha t}}{\psi(u(\lambda, T))\, \mathrm{e}^{\alpha T}}\right).$$

We deduce that $T \mapsto \psi(u(\lambda, T))\, \mathrm{e}^{\alpha T}$ is decreasing. We also get that $\int_0^\infty ds\ \tilde{\psi}'(u(\lambda, s)) < +\infty$ if and only if $\lim_{T \to +\infty} \psi(u(\lambda, T))\, \mathrm{e}^{\alpha T} > 0$ or equivalently $\lim_{T \to +\infty} u(\lambda, T)\, \mathrm{e}^{\alpha T} > 0$ as $\lim_{T \to +\infty} u(\lambda, T) = 0$ thanks to (4).

We deduce from (4) that

$$(21) \qquad u(\lambda, T)\, \mathrm{e}^{\alpha T} = \lambda \exp\left(\alpha \int_{u(\lambda, T)}^\lambda dr\ \left(\frac{1}{\psi(r)} - \frac{1}{\alpha r}\right)\right).$$

Thus we deduce from Lemma 3.7 that $\mathbb{P}(Z_0 < +\infty) = 1$ if and only if (A2) holds. $\qquad\square$

Corollary 3.9. *Assume (A2) holds. We have for $\lambda > 0$, $t \in \mathbb{R}$:*

$$(22) \qquad \mathbb{E}\left[Z_t\, \mathrm{e}^{-\lambda Z_t}\right] = \frac{\tilde{\psi}'(\lambda)}{\psi(\lambda)} \mathbb{E}\left[\mathrm{e}^{-\lambda Z_t}\right].$$

In particular, we have:

$$(23) \qquad \mathbb{E}[Z_t] = \frac{\psi''(0+)}{\psi'(0)} \in (0, +\infty].$$

Proof. We deduce from (19) that:

$$\mathbb{E}\left[Z_t\, \mathrm{e}^{-\lambda Z_t}\right] = \mathbb{E}\left[\mathrm{e}^{-\lambda Z_t}\right] \partial_\lambda \int_0^\infty \tilde{\psi}'(u(\lambda, s))\ ds.$$

We deduce from (4) that $\lambda \mapsto u(\lambda, s)$ is increasing and of class $\mathcal{C}^\infty$ on $(0, +\infty)$ and that

$$(24) \qquad \partial_\lambda u(\lambda, s) = \frac{\psi(u(\lambda, s))}{\psi(\lambda)} = \frac{-\partial_s u(\lambda, s)}{\psi(\lambda)}.$$

Thus, we get:

$$\partial_\lambda \int_0^\infty \tilde{\psi}'(u(\lambda, s))\ ds = \int_0^\infty \psi''(u(\lambda, s)) \partial_\lambda u(\lambda, s)\ ds$$

$$= -\frac{1}{\psi(\lambda)} \int_0^\infty \psi''(u(\lambda, s)) \partial_s u(\lambda, s)\ ds$$

$$= \frac{\tilde{\psi}'(\lambda)}{\psi(\lambda)}.$$

The last part of the Corollary is immediate. $\qquad\square$

Remark 3.10. Assumption (A1) is not needed to define the process Y'' or the stationary process Z. However the study of MRCA for Z is not relevant if (A1) does not hold.

Notice, we will introduce a complete genealogical structure for Z in Section 6 by using a genealogical structure of the families $(Y^j, j \in \mathcal{J})$.

From now on, we shall assume that (A1) and (A2) are in force.

3.4. **Further property for stationary CB.** By construction, we deduce that for all $t \in \mathbb{R}$, the process $(Z_{s+t}, s \geq 0)$ is a CB with branching mechanism ψ and immigration function $\tilde{\psi}'$ started as the stationary distribution whose Laplace transform is given by (19). Then Proposition 1.1 in [28] implies that Z is a Hunt process and in particular it is càd-làg and strongly Markov taking values in $[0, +\infty]$. By stationarity and since $+\infty$ is a cemetery point for Z, we deduce that a.s. for all $t \in \mathbb{R}$, Z_t is finite.

Next, we recall some asymptotic properties of the functions u and c given in Lemma 3.1 of [30].

Lemma 3.11. *For every $\lambda \in (0, \infty)$, we have*

$$(25) \qquad \lim_{t \to \infty} \frac{u(\lambda, t)}{c(t)} = e^{-\alpha c^{-1}(\lambda)},$$

and there exists $\kappa_* \in (0, \infty)$ such that

$$(26) \qquad \lim_{t \to \infty} c(t) e^{\alpha t} = \kappa_*.$$

We also compute some integral of $\tilde{\psi}'$.

Proposition 3.12. *The followings hold for every $0 \leq t < \infty$:*

$$(27) \qquad \int_t^\infty \tilde{\psi}'(u(\lambda, s)) ds = \log \left(\frac{\psi(u(\lambda, t)) \, e^{\alpha t + \alpha c^{-1}(\lambda)}}{\kappa_* \alpha} \right), \quad \lambda > 0,$$

$$(28) \qquad \int_t^\infty \tilde{\psi}'(c(s)) ds = \log \left(\frac{\psi(c(t)) e^{\alpha t}}{\kappa_* \alpha} \right),$$

where the constant κ_* is defined in Lemma 3.11.

Proof. We deduce from (20), (25) and (26) that:

$$\lim_{T \to \infty} \psi(u(\lambda, T)) \, e^{\alpha T} = \lim_{T \to \infty} \frac{\psi(u(\lambda, T))}{u(\lambda, T)} \frac{u(\lambda, T)}{c(T)} c(T) \, e^{\alpha T} = \alpha \ e^{-\alpha c^{-1}(\lambda)} \kappa_*,$$

and (27) follows by letting $T \longrightarrow \infty$ for both sides of (20). Then, let λ goes to infinity in (27) to get (28) and use the monotone convergence theorem. $\qquad\square$

As a consequence of (27) with $t = 0$ and Lemma 3.6, we get the following Corollary.

Corollary 3.13. *For all $t \in \mathbb{R}$ and $\lambda \geq 0$, the Laplace transform of Z_t is given by:*

$$(29) \qquad \mathbb{E}\left[e^{-\lambda Z_t}\right] = \exp\left(-\int_0^\infty ds \, \tilde{\psi}'(u(\lambda, s))\right) = \frac{e^{-\alpha c^{-1}(\lambda)} \kappa_* \alpha}{\psi(\lambda)}.$$

Eventually, we check that Z is non-zero. Recall notations from Section 3.1. Let $\zeta_i = \inf\{t > 0; Y_t^i = 0\}$ be the duration of the family or clan Y^i and $t_i + \zeta_i$ its extinction time, with i in I, J_1 or J_2.

Proposition 3.14. *We have*

$$\mathbb{P}\left(\sum_{i\in\mathcal{I}}\mathbf{1}_{(t_i,t_i+\zeta_i)}(t) > 0, \ \forall t\in\mathbb{R}\right) = 1,$$

In particular, we have $\mathbb{P}(\exists t\in\mathbb{R}; Z_t = 0) = 0$.

For $-\infty < a < b < +\infty$, we will consider in the forthcoming proof

$$(30) \qquad N_{a,b} = \sum_{i\in\mathcal{I}} \mathbf{1}_{\{t_i<a;b<t_i+\zeta_i\}},$$

the number of clans born before a and still alive at time b. Notice $N_{a,b}$ is a Poisson random variable with parameter

$$\Lambda(b-a) := \int dr\,\mu(dY)\,\mathbf{1}_{(-\infty,a)}(r)\mathbf{1}_{\{\zeta+r>b\}}$$

$$= \int_{b-a}^{\infty} dr\,\tilde{\psi}'(c(r))$$

$$(31) \qquad = \log\left(\frac{\psi(c(b-a))\,\mathrm{e}^{\alpha(b-a)}}{\kappa_*\alpha}\right),$$

where we used (14) the definition of μ for the first equality and (28) for the last equality.

Proof. Observe that no clan surviving at time $t\in(a,b)$ implies that there are no clan surviving on any non-degenerate interval containing t. Hence, for any $n\geq 1$, we have:

$$\left\{\exists t\in(a,b),\ \sum_{i\in\mathcal{I}}\mathbf{1}_{(t_i,t_i+\zeta_i)}(t)=0\right\} \subset \bigcup_{j=1}^{n}\left\{N_{u_{j-1},u_j}=0\right\} \cup \bigcup_{j=1}^{n+1}\left\{N_{v_{j-1},v_j}=0\right\}$$

where $u_j = a + j(b-a)/n$ and $v_j = a + (2j-1)(b-a)/2n$. Notice that N_{u_{j-1},u_j} and N_{v_{j-1},v_j} are Poisson random variables with parameter $\theta_n = \Lambda((b-a)/n)$. We deduce that

$$(32) \qquad \mathbb{P}\left(\exists t\in(a,b),\ \sum_{i\in\mathcal{I}}\mathbf{1}_{(t_i,t_i+\zeta_i)}(t)=0\right) \leq (2n+1)\,\mathrm{e}^{-\theta_n}.$$

Therefore the first part of the Proposition will be proved as soon as $\lim_{n\to+\infty} n\exp(-\theta_n) = 0$ which, thanks to formula (31), will be implied by $\lim_{t\to 0} t\psi(c(t)) = +\infty$ and thus by

$$(33) \qquad \lim_{\lambda\to+\infty}\int_{\lambda}^{+\infty}\frac{dr}{\psi(r)}\psi(\lambda) = +\infty.$$

Hypothesis on β and π imply there exists a constant $c_0 > 0$ such that

$$\alpha\lambda \leq \psi(\lambda) \leq c_0\lambda^2 \quad\text{and}\quad \lim_{\lambda\to+\infty}\psi(\lambda)/\lambda = +\infty.$$

Therefore (33) is in force.

The second part of the Proposition is clear by definition of ζ_i and representation (16). $\quad\square$

4. TMRCA and populations sizes

We consider the coalescence of the genealogy at a fixed time t_0. Thanks to stationarity, we may assume that $t_0 = 0$ and we write Z instead of Z_0. There are infinitely many clans contributing to the population at time 0. The Poisson random variable introduced in (30), with $b = 0$, gives the number of clans born before a and still alive at time 0. Notice its parameter is finite, see (31). Therefore, there are only finitely many clans born before a and alive at time 0. In particular, this implies that there is one unique oldest clan alive at time 0. We denote by $-A$ the birth time of this unique oldest clan at time 0:

$$A = -\inf\{t_i \le 0; Y^i_{-t_i} > 0, i \in \mathcal{I}\}.$$

We set Z^O the population size of this clan at time 0:

$$Z^O := Y^i_{-t_i}, \qquad \text{if } A = -t_i.$$

The time A is also the time to the most recent common ancestor (TMRCA) of the population at time 0. The size of all the clans alive at time 0 with birth time in $(-A, 0)$ is given by

$$Z^I := Z - Z^O.$$

We are also interested in the size of the population just before the most recent common ancestor (MRCA):

$$Z^A := Z_{(-A)-} = \sum_{i \in \mathcal{I}} Y^i_{(-A-t_i)} \mathbf{1}_{\{t_i < -A\}}.$$

Theorem 4.1. *The joint distribution of (A, Z^A, Z^I, Z^O) is characterized by the following: for $\lambda, \gamma, \eta \ge 0$ and $t \ge 0$,*

$$(34) \quad \mathbb{E}\left[e^{-\lambda Z^A - \gamma Z^I - \eta Z^O}; A \in dt \right]$$

$$= dt \left(\tilde{\psi}'(c(t)) - \tilde{\psi}'(u(\eta, t)) \right) \times \exp\left(-\int_0^t ds\, \tilde{\psi}'(u(\gamma, s)) - \int_0^\infty ds\, \tilde{\psi}'(u(\lambda + c(t), s)) \right).$$

Proof. Given f a non-negative Borel measurable function defined on $\mathbb{R}$, we have

$$\mathbb{E}\left[e^{-\lambda Z^A - \gamma Z^I - \eta Z^O} f(A) \right]$$

$$= \mathbb{E}\left[\sum_{j \in \mathcal{I}} \exp\left(-\lambda \sum_{i \in \mathcal{I}, t_i < t_j} Y^i_{(t_j - t_i)} - \gamma \sum_{i \in \mathcal{I}, t_i > t_j} Y^i_{-t_i} - \eta Y^j_{-t_j} \right) \right.$$

$$\left. f(-t_j) \, \mathbf{1}_{\left\{ Y^j_{-t_j} > 0, \sum_{i \in \mathcal{I}, t_i < t_j} \mathbf{1}_{\{Y^i_{-t_i} > 0\}} = 0 \right\}} \right]$$

$$= \int_0^\infty dt\, \mu\left(e^{-\eta Y_t}; Y_t > 0 \right) f(t) \, \mathbb{E}\left[\exp\left(-\gamma \sum_{i \in \mathcal{I}, t_i > -t} Y^i_{-t_i} \right) \right]$$

$$\lim_{K \to \infty} \mathbb{E}\left[\exp\left(-\lambda \sum_{t_i < -t} \left(Y^i_{(-t-t_i)} + K \mathbf{1}_{\{Y^i_{-t_i} > 0\}} \right) \right) \right],$$

where we used that Poisson point measures over disjoint sets are independent. We have:

$$\mu\left(e^{-\eta Y_t}; Y_t > 0 \right) = \mu\left(\mathbf{1}_{\{Y_t > 0\}} - (1 - e^{-\eta Y_t}) \right) = \tilde{\psi}'(c(t)) - \tilde{\psi}'(u(\eta, t)).$$

Using Lemma 3.1, we get:

$$\mathbb{E}\left[\exp\left(-\gamma \sum_{i\in\mathcal{I},t_i>-t} Y^i_{-t_i}\right)\right] = \exp\left(-\int_0^t ds\ \tilde{\psi}'(u(\gamma,s))\right).$$

We also have:

$$\lim_{K\to\infty} \mathbb{E}\left[\exp\left(-\lambda \sum_{i\in\mathcal{I},t_i<-t} \left(Y^i_{(-t-t_i)} + K\mathbf{1}_{\{Y^i_{-t_i}>0\}}\right)\right)\right]$$

$$= \exp\left(-\int ds\ \mathbf{1}_{\{s>0\}}\ \mu\left(1 - e^{-\lambda Y_s}\ \mathbf{1}_{\{Y_{s+t}=0\}}\right)\right)$$

$$= \exp\left(-\int ds\ \mathbf{1}_{\{s>0\}}\ \mu\left(1 - e^{-\lambda Y_s}\ \mathrm{P}_{Y_s}\left(Y_t = 0\right)\right)\right)$$

$$= \exp\left(-\int ds\ \mathbf{1}_{\{s>0\}}\ \mu\left(1 - e^{-(\lambda+c(t))Y_s}\right)\right)$$

$$= \exp\left(-\int_0^\infty ds\ \tilde{\psi}'(u(\lambda+c(t),s))\right),$$

where we used exponential formulas for Poisson point measure in the first equality and the Markov property of Y for the second equality. Putting things together, we then get (34). $\square$

It is then easy to derive the distribution of the TMRCA A.

Corollary 4.2. *The distribution function of A is given by*

$$\mathbb{P}(A \leq t) = \mathbb{E}[e^{-c(t)Z}] = \exp\left(-\int_t^\infty ds\ \tilde{\psi}'(c(s))\right),$$

and A has density, f_A, with respect to the Lebesgue measure given by:

$$(35) \qquad f_A(t) = \tilde{\psi}'(c(t)) \exp\left(-\int_t^\infty ds\ \tilde{\psi}'(c(s))\right) \mathbf{1}_{\{t>0\}} = \frac{\tilde{\psi}'(c(t))}{\psi(c(t))}\ e^{-\alpha t}\ \kappa_*\alpha\mathbf{1}_{\{t>0\}}.$$

Proof. This is a direct consequence of Theorem 4.1 and (10). Use Lemma 3.6 to get (35). $\square$

The next result is a direct consequence of Theorem 4.1.

Corollary 4.3. *Conditionally on A, the three random variables Z^I, Z^A and Z^O are independent.*

We can also give the mean of the population size just before the most recent common ancestor (MRCA) (to be compared to the mean size of the current population given by (23)).

Corollary 4.4. *Let $t > 0$. We have*

$$(36) \qquad \mathbb{E}\left[e^{-\lambda Z^A}|A = t\right] = \frac{\mathbb{E}\left[e^{-(\lambda+c(t))Z}\right]}{\mathbb{E}\left[e^{-c(t)Z}\right]} \quad and \quad \mathbb{E}[Z^A|A = t] = \frac{\tilde{\psi}'(c(t))}{\psi(c(t))}.$$

Proof. This is a direct consequence of Theorem 4.1 and of (22). $\square$

We deduce from (36) that the distribution of Z^A conditionally on $\{A = t\}$ converges, as t goes to infinity, to the distribution of Z.

As another application of Theorem 4.1, we get that the population just before the MRCA, Z^A, is stochastically smaller than the current population, Z. Note that strong inequality, namely inequality in the almost-surely sense, does not hold in general (see Section 7).

Proposition 4.5. *We have* $\mathbb{P}(Z^A \leq z | A = t) \geq \mathbb{P}(Z \leq z)$ *for all* $z \geq 0$ *and* $t \geq 0$. *Hence, the population size* Z^A *is stochastically smaller than* Z: $\mathbb{P}(Z^A \leq z) \geq \mathbb{P}(Z \leq z)$ *for all* $z \geq 0$. *In particular, we have*

$$\mathbb{E}[Z^A | A] \leq \mathbb{E}[Z] \quad a.s.$$

Proof. The first equality of (36) implies that for any non-negative measurable function F defined on $\mathbb{R}$,

$$\mathbb{E}\left[F(Z^A) | A = t\right] = \frac{\mathbb{E}\left[F(Z) \, \mathrm{e}^{-c(t)Z}\right]}{\mathbb{E}\left[\mathrm{e}^{-c(t)Z}\right]}.$$

Note that $\mathrm{e}^{-c(t)Z} - \mathbb{E}\left[\mathrm{e}^{-c(t)Z}\right]$ is non-negative for Z less than $\frac{1}{-c(t)} \log\left(\mathbb{E}\left[\mathrm{e}^{-c(t)Z}\right]\right)$ and non-positive otherwise, and that $\lim_{z \to \infty} \mathbb{E}[\mathrm{e}^{-c(t)Z}; Z \leq z] - \mathbb{E}[\mathrm{e}^{-c(t)Z}]\mathbb{P}(Z \leq z) = 0$. We deduce that:

$$\mathbb{P}(Z^A \leq z | A = t) = \frac{\mathbb{E}\left[\mathrm{e}^{-c(t)Z}; Z \leq z\right]}{\mathbb{E}\left[\mathrm{e}^{-c(t)Z}\right]} \geq \mathbb{P}(Z \leq z).$$

For the last assertion, recall that for any non-negative random variable, we have $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x)dx$. $\square$

Remark 4.6. Instead of considering Z^A, the size of the population just before the MRCA, we could consider the size of the population at the MRCA, Z_+^A, which is formally given by

$$Z_+^A = Z^A + \sum_{i \in I} Y_0^i \, \mathbf{1}_{\{t_i = -A\}}.$$

Notice we don't take into account the contribution of $i \in J_2$ as for those indices we have $Y_0^i = 0$. (In particular if $\pi = 0$, then Z is continuous and $Z^A = Z_+^A$.) Similar computations as those in the proof of Theorem 4.1 yield: for $\lambda, t > 0$

$$\mathbb{E}[\mathrm{e}^{-\lambda Z_+^A} | A = t] = \mathbb{E}[\mathrm{e}^{-\lambda Z^A} | A = t] \frac{\psi'(\lambda + c(t)) - \psi'(\lambda)}{\psi'(c(t)) - \psi'(0)}.$$

If $\psi''(0) = +\infty$, then we get that $\lim_{t \to +\infty} \mathbb{E}[\mathrm{e}^{-\lambda Z_+^A} | A = t] = 0$. Thus, conditionally on $\{A = t\}$, for t large, we have that Z_+^A is likely to be very large. (Intuitively, a clan is born at time $-t$ which has survive up to time 0; and if t is large, it is very likely to have a large initial size.) Therefore, Z_+^A is not stochastically smaller than Z in the general case.

We may also consider the TMRCA of the immortal individual and individuals taken independently and uniformly among the current population living at time t. Let $J_t^n \subset \mathcal{I}$ be the indices of the clans of the randomly chosen n individuals alive at time t. (One individual chosen at random in the population at time t belongs to the clan, i with probability $Y_{t-t_i}^i / Z_t$.) Notice that $\mathrm{Card}\,(J_t^n) \leq n$. The TMRCA for the n individuals alive at time t and the immortal individual is given by:

$$A_t^n := -\inf\{t_i; i \in J_t^n, i \in \mathcal{I}\}.$$

Because of the stationarity, we shall focus on $t = 0$ and write A^n for A_t^n. The joint law of Z and A^n can be characterized by the following result.

Theorem 4.7. *For any $n \geq 1$ and any $\lambda, T \geq 0$, we have*

$$\mathbb{E}\left[Z^n \, e^{-\lambda Z} \, \mathbf{1}_{\{A^n \leq T\}}\right] = \frac{e^{-\alpha c^{-1}(\lambda)} \, \kappa_* \alpha}{\psi(u(\lambda, T))} (-1)^n \frac{\partial^n}{\partial^n \eta} \left(\frac{\psi(u(\lambda + \eta, T))}{\psi(\lambda + \eta)}\right)\bigg|_{\eta=0}.$$

Proof. By definition, we have:

$$\mathbb{E}\left[Z^n \, e^{-\lambda Z} \, \mathbf{1}_{\{A^n \leq T\}}\right]$$

$$= \mathbb{E}\left[Z^n \sum_{i_1, \cdots, i_n} \frac{Y^{i_1}_{-t_{i_1}}}{Z} \cdots \frac{Y^{i_n}_{-t_{i_n}}}{Z} \prod_{k=1}^n \mathbf{1}_{\{-t_i \leq T\}} \, e^{-\lambda Z}\right]$$

$$= \mathbb{E}\left[\left(\int \mathcal{N}_3(ds, dY) \, Y_{-s} \mathbf{1}_{\{-s \leq T\}}\right)^n \exp\left(-\lambda \int \mathcal{N}_3(ds, dY) \, Y_{-s}\right)\right]$$

$$= (-1)^n \frac{\partial^n}{\partial^n \eta} \mathbb{E}\left[\exp\left(-\int \mathcal{N}_3(ds, dY) \, \left(\eta Y_{-s} \mathbf{1}_{\{-s \leq T\}} + \lambda Y_{-s}\right)\right)\right]\bigg|_{\eta=0}$$

$$= (-1)^n \frac{\partial^n}{\partial^n \eta} \exp\left(-\int_T^\infty ds \, \tilde{\psi}'(u(\lambda, s)) - \int_0^T ds \, \tilde{\psi}'(u(\lambda + \eta, s))\right)\bigg|_{\eta=0},$$

where $\mathcal{N}_3$ in the second equality is defined by (13). The result then follows from (20) and (27). □

Remark 4.8. Following almost the same lines as the proof of Theorem 4.7, one can characterize explicitly the joint distribution of $\left\{\left(Z_{r_j}, A_{r_j}^{n_j}\right); 1 \leq j \leq m\right\}$ for any $m, n_1, \cdots, n_m \in \mathbb{N}^*$ and $-\infty < r_1 < r_2 < \cdots < r_m < \infty$.

5. NUMBER OF OLD FAMILIES

We now consider the number families in the oldest clan alive at time 0. This correspond to the number of individuals involved in the last coalescent event of the genealogical tree. To this end, we take the representation (12) for Z.

Definition 5.1. *The number of oldest families alive at time 0 (excluding the immortal particle) is defined by:*

$$(37) \qquad N^A = \sum_{j \in \mathcal{J}} \mathbf{1}_{\{A = -t_j, \, Y^j_{-t_j} > 0\}} = \sum_{j \in \mathcal{J}} \mathbf{1}_{\{A = -t_j, \, \zeta_j > -t_j\}}.$$

We have $N^A \geq 1$. In the particular case $\pi = 0$ and $\beta > 0$, we have $\mathcal{J} = J_2$ and $N^A = 1$.

The following proposition give the joint law of A, N^A and Z.

Proposition 5.2. *We have for $a \in [0, 1]$, $\lambda \geq 0$, $t \geq 0$,*

$$\mathbb{E}\left[a^{N^A} e^{-\lambda Z} | A = t\right] = \frac{\psi'(c(t)) - \psi'((1-a)c(t) + au(\lambda, t))}{\tilde{\psi}'(c(t))} e^{-\int_0^t \tilde{\psi}'(u(\lambda, r)) \, dr}.$$

and

$$\mathbb{E}\left[a^{N^A} | A = t\right] = \frac{\psi'(c(t)) - \psi'((1-a)c(t))}{\tilde{\psi}'(c(t))} = 1 - \frac{\tilde{\psi}'\left((1-a)c(t)\right)}{\tilde{\psi}'\left(c(t)\right)}.$$

Proof. Recall notations from Section 3.1. For $i \in \mathcal{I}$, we set $J_i^* = J_{1,i}$ if $i \in I$ and $J_i^* = \{i\}$ if $i \in J_2$. Given any non-negative function f, we have, using (12) and (16):

$$\mathbb{E}\left[a^{N^A} \mathrm{e}^{-\lambda Z} f(A)\right]$$

$$= \mathbb{E}\left[\mathrm{e}^{-\lambda \sum_{k \in \mathcal{I}} Y_{-t_k}^k} \sum_{i \in \mathcal{I}} a^{\sum_{j \in J_i^*} \mathbf{1}_{\{\zeta_j > -t_i\}}} f(-t_i) \, \mathbf{1}_{\{Y_{-t_i}^i \neq 0\}} \mathbf{1}_{\left\{\sum_{k' \in \mathcal{I}, t_{k'} < t_i} \mathbf{1}_{\{Y_{-t_i}^{k'} > 0\}} = 0\right\}}\right]$$

$$= \int_0^\infty ds \, f(s) \, \mathbb{E}\left[\mathrm{e}^{-\lambda \sum_{k \in \mathcal{I}} Y_{-t_k}^k \mathbf{1}_{\{t_k > -s\}}}\right] \, \mathbb{P}\left(\sum_{k \in \mathcal{I}} \mathbf{1}_{\{t_k < -s, \, Y_s^k > 0\}} = 0\right)$$

$$\times \left(2\beta \mathbb{N}\left[a \, \mathrm{e}^{-\lambda Y_s} \mathbf{1}_{\{Y_s > 0\}}\right] + \int_{(0,+\infty)} \ell \pi(d\ell) \, \mathbb{E}_\ell\left[a^{\sum_{j \in J_3} \mathbf{1}_{\{Y_s^j > 0\}}} \mathrm{e}^{-\lambda \sum_{j \in J_3} Y_s^j} \mathbf{1}_{\{\sum_{j \in J_3} Y_s^j > 0\}}\right]\right),$$

where $\sum_{j \in J_3} \delta_{Y^j}(dY)$ is under $\mathbb{E}_\ell$ a Poisson point measure with intensity $\ell \mathbb{N}[dY]$. We have

$$\mathbb{E}\left[\mathrm{e}^{-\lambda \sum_{k \in \mathcal{I}} Y_{-t_k}^k \mathbf{1}_{\{t_k > -s\}}}\right] \, \mathbb{P}\left(\sum_{k \in \mathcal{I}} \mathbf{1}_{\{t_k < -s, \, Y_s^k > 0\}} = 0\right) = \mathrm{e}^{-\int_0^s dr \, \tilde{\psi}'(u(\lambda, r)) - \int_s^\infty dr \, \tilde{\psi}'(c(r))}.$$

We also have

$$\mathbb{N}\left[\mathrm{e}^{-\lambda Y_s} \mathbf{1}_{\{Y_s > 0\}}\right] = \mathbb{N}[Y_s > 0] - \mathbb{N}[1 - \mathrm{e}^{-\lambda Y_s}] = c(s) - u(\lambda, s).$$

and

$$\mathbb{E}_\ell\left[a^{\sum_{j \in J_3} \mathbf{1}_{\{Y_s^j > 0\}}} \mathrm{e}^{-\lambda \sum_{j \in J_3} Y_s^j} \mathbf{1}_{\{\sum_{j \in J_3} Y_s^j > 0\}}\right]$$

$$= \mathbb{E}_\ell\left[a^{\sum_{j \in J_3} \mathbf{1}_{\{Y_s^j > 0\}}} \mathrm{e}^{-\lambda \sum_{j \in J_3} Y_s^j}\right] - \mathbb{P}_\ell\left(\sum_{j \in J_3} Y_s^j = 0\right)$$

$$= \exp\left(-\ell \mathbb{N}[(1 - a \, \mathrm{e}^{-\lambda Y_s}) \mathbf{1}_{\{Y_s > 0\}}]\right) - \exp\left(-\ell \mathbb{N}[Y_s > 0]\right)$$

$$= \exp\left(-\ell \mathbb{N}[Y_s > 0] + \ell a \mathbb{N}[\mathrm{e}^{-\lambda Y_s}] \mathbf{1}_{\{Y_s > 0\}}]\right) - \exp\left(-\ell \mathbb{N}[Y_s > 0]\right)$$

$$= \exp\left(-\ell\left((1 - a)c(s) - au(\lambda, s)\right)\right) - \exp\left(-\ell c(s)\right).$$

Thus, we get:

$$2\beta \mathbb{N}\left[a \, \mathrm{e}^{-\lambda Y_s} \mathbf{1}_{\{Y_s > 0\}}\right] + \int_{(0,+\infty)} \ell \pi(d\ell) \, \mathbb{E}_\ell\left[a^{\sum_{j \in J_3} \mathbf{1}_{\{Y_s^j > 0\}}} \mathrm{e}^{-\lambda \sum_{j \in J_3} Y_s^j} \mathbf{1}_{\{\sum_{j \in J_3} Y_s^j > 0\}}\right]$$

$$= \psi'(c(s)) - \psi'((1 - a)c(s) + au(\lambda, s)).$$

Putting things together, we obtain:

$$\mathbb{E}\left[a^{N^A} \mathrm{e}^{-\lambda Z} f(A)\right]$$

$$= \int_0^\infty ds \, f(s) \, \mathrm{e}^{-\int_0^s dr \, \tilde{\psi}'(u(\lambda, r)) - \int_s^\infty dr \, \tilde{\psi}'(c(r))} \left[\psi'(c(s)) - \psi'((1 - a)c(s) + au(\lambda, s))\right].$$

Then, use (35) for the density of A to get the result. $\square$

Corollary 5.3. *We have:*

$$(38) \qquad \mathbb{P}(N^A = n | A = t) = (-1)^{n+1} \frac{c(t)^n \psi^{(n+1)}(c(t))}{n! \, \tilde{\psi}'(c(t))}, \quad n \in \mathbb{N}^*.$$

Suppose that $\psi''(0+) < \infty$ (that is $\mathbb{E}[Z] < +\infty$). Then, we have

$$\mathbb{E}[N^A | A = t] = \psi''(0) \frac{c(t)}{\tilde{\psi}'(c(t))}.$$

Furthermore the function $t \longmapsto \mathbb{E}[N^A | A = t]$ is non-increasing.

Proof. The first two assertions are straightforward consequences of Proposition 5.2. To get the monotonicity of $t \longmapsto \mathbb{E}[N^A | A = t]$, we simply notice that both $t \longmapsto c(t)$ and

$$x \longmapsto \frac{\tilde{\psi}'(x)}{x} = 2\beta + \int_0^\infty \pi(d\ell)\ell \, \frac{1 - e^{-x\ell}}{x}$$

are non-increasing. $\qquad \square$

Remark 5.4. Suppose that $\psi''(0+) < \infty$. We deduce from (38) that

$$\lim_{t \to +\infty} \mathbb{P}(N^A = 1 | A = t) = 1.$$

Thus, the distribution of N^A conditionally on $\{A = t\}$ converges as t goes to infinity to 1. So roughly speaking N^A is likely to be equal to 1 if the TMRCA (or age of the oldest clan alive) is large. Notice that if $\psi''(0+) = +\infty$, this result may be false (see the next Remark).

Remark 5.5. Let us consider the stable cases, $\psi(\lambda) = \alpha\lambda + c_0\lambda^{1+\alpha_0}$, with $c_0 > 0$ and $\alpha_0 \in (0, 1]$. We deduce from Corollary 5.3 that

$$\mathbb{E}[a^{N^A} | A = t] = 1 - (1-a)^{\alpha_0}.$$

In particular N^A is independent of A. The case $\alpha_0 = 1$ correspond to the quadratic branching mechanism and we get that a.s. $N^A = 1$. For $\alpha_0 \in (0, 1)$, we deduce from (38) that: for $n \in \mathbb{N}^*$

$$\mathbb{P}(N^A = n | A = t) = \frac{1}{n!} \alpha_0 \prod_{k=1}^{n-1} (k - \alpha_0).$$

For $\alpha_0 \in (0, 1)$, we have $\psi''(0+) = +\infty$ and the result of Remark 5.4 does not hold.

6. Asymptotics for the number of ancestors

The number $N_{-s,0}$ defined by (30) of clans born before time $-s$ and alive at time 0 is non-decreasing and is distributed as a Poisson random variable with parameter $\Lambda(s)$ given by (31). As $\Lambda(s)$ goes to infinity as s goes down to 0, we deduce that $N_{-s,0}$ tends to infinity almost surely as $s \downarrow 0+$. A natural question is then how fast the numbers $N_{-s,0}$ tend to infinity. It follows from the definition of the Poisson random measure $\mathcal{N}_3$ in (13) that $\{N_{-\Lambda^{-1}(s),0}; s \geq 0\}$ is Poisson process with parameter 1, and by the strong law of large numbers for Lévy processes (see [8]), we deduce that

$$\lim_{s \downarrow 0+} \frac{N_{-s,0}}{\Lambda(s)} = 1 \quad \text{almost surely.}$$

One can also ask how fast the number M_s of ancestors at time $-s$ of the current population living at time 0 tends to infinity. To answer this question, we need to introduce the genealogy of the families. Notice the genealogy of a CB is a richer structure than the CB itself.

6.1. **Genealogy of CB.** We recall here the construction of the Lévy continuum random tree (CRT) introduced in [34, 33] and developed later in [16] for critical or sub-critical branching mechanism. The results of this section are mainly extracted from [16], except for the next subsection which is extracted from [32].

6.1.1. *Real trees and their coding by a continuous function.* Let us first recall the definition of real trees.

Definition 6.1. *A metric space $(\mathcal{T}, d)$ is a real tree if the following two properties hold for every $v_1, v_2 \in \mathcal{T}$.*
- *(Unique geodesic.) There is a unique isometric map f_{v_1, v_2} from $[0, d(v_1, v_2)]$ into $\mathcal{T}$ such that*
$$f_{v_1, v_2}(0) = v_1 \qquad and \qquad f_{v_1, v_2}(d(v_1, v_2)) = v_2.$$
- *(No loop.) If q is a continuous injective map from $[0, 1]$ into $\mathcal{T}$ such that $q(0) = v_1$ and $q(1) = v_2$, we have*
$$q([0, 1]) = f_{v_1, v_2}([0, d(v_1, v_2)]).$$

A rooted real tree is a real tree $(\mathcal{T}, d)$ with a distinguished vertex $v_\emptyset$ called the root.

Let $(\mathcal{T}, d)$ be a rooted real tree. The range of the mapping f_{v_1, v_2} is denoted by $[\![v_1, v_2]\!]$ (this is the line between v_1 and v_2 in the tree). In particular, for every vertex $v \in \mathcal{T}$, $[\![v_\emptyset, v]\!]$ is the path going from the root to v which we call the ancestral line of vertex v. More generally, we say that a vertex v is an ancestor of a vertex v' if $v \in [\![v_\emptyset, v']\!]$. If $(v_k \in K)$ is a set of vertex of $\mathcal{T}$, there is a unique $a \in \mathcal{T}$ such that $[\![v_\emptyset, a]\!] = \bigcap_{k \in K} [\![v_\emptyset, v_k]\!]$. We call a the most recent common ancestor of $(v_k \in K)$. A leaf is a vertex which is the ancestor of itself only. We say that $d(\emptyset, v)$ is the level (or generation) of the vertex v.

We now recall the coding of a compact real tree by a continuous function $g : [0, +\infty) \longrightarrow [0, +\infty)$ with compact support and such that $g(0) = 0$. We also assume that g is not identically 0. For every $0 \leq s \leq t$, we set
$$m_g(s, t) = \inf_{u \in [s,t]} g(u) \quad and \quad d_g(s, t) = g(s) + g(t) - 2m_g(s, t).$$

We then introduce the equivalence relation $s \sim t$ if and only if $d_g(s, t) = 0$. Let $\mathcal{T}_g$ be the quotient space $[0, +\infty)/ \sim$. It is easy to check that d_g induces a distance on $\mathcal{T}_g$. Moreover, $(\mathcal{T}_g, d_g)$ is a compact real tree (see [17], Theorem 2.1). We say that g is the height process of the tree $\mathcal{T}_g$.

For instance, when g is a normalized Brownian excursion, the associated real tree is Aldous' CRT [4].

6.1.2. *The underlying Lévy process.* We present now how to define a height process that codes a random real trees describing the genealogy of a CB using a Lévy process with Laplace exponent given by the branching mechanism ψ. We shall consider only the case of the sub-critical branching mechanism ψ given by (1).

Let $X = (X_t, t \geq 0)$ be a $\mathbb{R}$-valued Lévy process with no negative jumps, starting from 0 and with Laplace exponent ψ under the probability measure $\mathbf{P}$ (and $\mathbf{E}$ the corresponding expectation): for $\lambda \geq 0$, $\mathbf{E}\left[\mathrm{e}^{-\lambda X_t}\right] = \mathrm{e}^{t\psi(\lambda)}$. Since we assume that $\beta > 0$ or $\pi((0, 1)) = +\infty$, we get that a.s. X is of infinite variation.

We introduce some processes related to X. Let $I = (I_t, t \geq 0)$ be the infimum process of X, $I_t = \inf_{0 \leq s \leq t} X_s$, and let $S = (S_t, t \geq 0)$ be the supremum process, $S_t = \sup_{0 \leq s \leq t} X_s$. We

will also consider for every $0 \leq s \leq t$ the infimum of X over $[s,t]$:

$$I_t^s = \inf_{s \leq r \leq t} X_r.$$

The point 0 is regular for the Markov process $X - I$, and $-I$ is the local time of $X - I$ at 0 (see [8], chap. VII). Let $\mathbf{N}$ be the associated excursion measure of the process $X - I$ away from 0. Let $\sigma = \inf\{t > 0; X_t - I_t = 0\}$ be the length of the excursion of $X - I$ under $\mathbf{N}$. We have $X_0 = I_0 = 0$ $\mathbf{N}$-a.e.

Since X is of infinite variation, 0 is also regular for the Markov process $S - X$. The local time, $L = (L_t, t \geq 0)$, of $S - X$ at 0 will be normalized so that

$$\mathbf{E}[\mathrm{e}^{-\lambda S_{L_t^{-1}}}] = \mathrm{e}^{-t\psi(\lambda)/\lambda},$$

where $L_t^{-1} = \inf\{s \geq 0; L_s \geq t\}$ (see also [8] Theorem VII.4 (ii)).

6.1.3. *The height process and the Lévy CRT.* For each $t \geq 0$, we consider the reversed process at time t, $\hat{X}^{(t)} = (\hat{X}_s^{(t)}, 0 \leq s \leq t)$ by:

$$\hat{X}_s^{(t)} = X_t - X_{(t-s)-} \quad \text{if} \quad 0 \leq s < t,$$

and $\hat{X}_t^{(t)} = X_t$. The two processes $(\hat{X}_s^{(t)}, 0 \leq s \leq t)$ and $(X_s, 0 \leq s \leq t)$ have the same law. Let $\hat{S}^{(t)}$ be the supremum process of $\hat{X}^{(t)}$ and $\hat{L}^{(t)}$ be the local time at 0 of $\hat{S}^{(t)} - \hat{X}^{(t)}$ with the same normalization as L. As assumption (A1) is in force, there exists a continuous modification $H = (H_t, t \geq 0)$ of the process $(\hat{L}^{(t)}, t \geq 0)$, see Theorem 1.4.3 in [16]. The process H is the so-called height-process and $(\mathcal{T}_H, d_H)$ is the corresponding Lévy tree. Notice that $\mathbf{N}$-a.e. we have $H_t = 0$ for $t \geq \sigma$.

6.1.4. *Local time for the height process and CB.* We now check that $\mathcal{T}_H$ represents the genealogy of a CB with branching mechanism ψ.

The local time of the height process is defined through the next result, see [16], Lemma 1.3.2 and Proposition 1.3.3.

Proposition 6.2. *There exists a jointly measurable process $(L_s^a, a \geq 0, s \geq 0)$ which is continuous and non-decreasing in the variable s such that:*

- *For every $t \geq 0$, $\lim_{\varepsilon \to 0} \sup_{a \geq 0} \mathbf{E}\left[\sup_{s \leq t} \left|\varepsilon^{-1} \int_0^s \mathbf{1}_{\{a < H_r \leq a+\varepsilon\}} \, dr - L_s^a\right|\right] = 0.$*

- *For every $t \geq 0$, $\lim_{\varepsilon \to 0} \sup_{a \geq \varepsilon} \mathbf{E}\left[\sup_{s \leq t} \left|\varepsilon^{-1} \int_0^s \mathbf{1}_{\{a-\varepsilon < H_r \leq a\}} \, dr - L_s^a\right|\right] = 0.$*

- *$\mathbf{P}$-a.s., for every $t \geq 0$, $L_t^0 = -I_t$.*

- *The occupation time formula holds: for any non-negative measurable function g on $\mathbb{R}_+$ and any $s \geq 0$, $\int_0^s g(H_r) \, dr = \int_{(0,+\infty)} g(a) L_s^a \, da.$*

Let $T_x = \inf\{t \geq 0; I_t \leq -x\}$. We have the following Ray-Knight theorem which explains why the Lévy CRT can be viewed as the genealogical tree of a CB.

Proposition 6.3 ([16], Theorem 1.4.1)**.** *The process $(L_{T_x}^a, a \geq 0)$ is distributed under $\mathbf{P}$ as Y under P_x (i.e. is a CB with branching mechanism ψ starting at x).*

We then get the following Corollary.

Corollary 6.4. *The process $L(H) = (L_\sigma^a, a \geq 0)$ is distributed under the excursion measure $\mathbf{N}$ as Y under its excursion measure $\mathbb{N}$.*

Informally, L_σ^a counts the number of vertices (in fact leaves) of $\mathcal{T}_H$ at level a under $\mathbf{N}$.

6.1.5. *Poissonian representation of the height process above a level.* Let $a > 0$ be fixed. We consider the excursions of the height process H above a under the excursion measure $\mathbb{N}$. Precisely, let (u_k, v_k), $k \in \mathcal{K}$ be the excursions of H above a over the time interval $[0, \sigma]$. We set $H^k = (H_{u_k + s \wedge v_k} - a, s \geq 0)$.

The next result is a consequence of Proposition 4.2.3 in [16].

Proposition 6.5. *Conditionally on $(L_\sigma^r, r \leq a)$, the measure $\sum_{k \in \mathcal{K}} \delta_{H^k}(dH)$ is a Poisson point measure with intensity $L_\sigma^a \mathbb{N}[dH]$.*

We give a definition for the number of ancestors, which will be used in the next section.

Definition 6.6. *The number of ancestors at time a of the population (coded by H) living at time b is the number of excursions of H above level a which reach level $b > a$:*

$$R_{a,b}(H) = \sum_{k \in \mathcal{K}} \mathbf{1}_{\{\zeta_k \geq b - a\}},$$

where $\zeta_k = \max\{H_s^k, s \geq 0\}$.

6.2. **Genealogy of Z.** Recall notations from Section 6.1.5. In order to simplify notations, we shall write $\mathbb{N}$ for $\mathbf{N}$.

We use formulation (12) to construct the genealogy of Z. Recall notation $\mathcal{N}_0$ from Section 3.1.

- Conditionally on $\mathcal{N}_0$, let $\tilde{\mathcal{N}}_1(dt, dH) = \sum_{j \in J_1} \delta_{t_j, H^j}(dt, dH)$ be a Poisson point measure with intensity $\nu(dt)\,\mathbb{N}[dH]$ with $\nu(dt) = \sum_{i \in I} r_i \delta_{t_i}(dt)$.
- Let $\tilde{\mathcal{N}}_2(dt, dH) = \sum_{j \in J_2} \delta_{t_j, H^j}(dt, dH)$ be a Poisson point measure independent of $(\mathcal{N}_0, \tilde{\mathcal{N}}_1)$ and with intensity $2\beta\, dt\, \mathbb{N}[dH]$.

We will write Y^j for $L(H^j)$ for $j \in \mathcal{J} = J_1 \bigcup J_2$. Thus notation (12) is still consistent with the previous Sections, thanks to Corollary 6.4. And the process $\sum_{j \in \mathcal{J}} \delta_{t_j, H^j}$ allows to code for the genealogy of the families of Z.

Let $s > 0$. Following Definition 6.6, we consider M_s the number of ancestors at time $-s$ of the current population living at time 0, not including the immortal individual:

$$M_s = \sum_{j \in \mathcal{J}} \mathbf{1}_{\{t_j < -s\}} R_{-s - t_j, -t_j}(H^j).$$

6.3. **Asymptotics for the number of ancestors.** We first give a technical Lemma, which proof is postponed to the end of this Section.

Lemma 6.7. *The joint distribution of M_s and Z_0 is characterized by the following equation: for $\eta, \lambda \geq 0$ $s > 0$,*

$$(39) \qquad \mathbb{E}\left[\mathrm{e}^{-\eta M_s - \lambda Z_0}\right] = \mathrm{e}^{-\int_0^s dr\, \tilde{\psi}'(u(\lambda, r))} \mathbb{E}\left[\mathrm{e}^{-Z_{-s}[(1 - \mathrm{e}^{-\eta})c(s) + \mathrm{e}^{-\eta} u(\lambda, s)]}\right].$$

In particular, M_s has the same distribution as $V_{Z_{-s}}^s$, where V^s is a Poisson process with parameter $c(s)$ independent of $(Z_t, t \in \mathbb{R})$.

Remark 6.8. Note that one can replace Z_{-s} by Z_0 for the right hand side of (39) thanks to stationarity. The effect of our presentation is to emphasize the branching property: conditionally on Z_{-s}, the number of families with lifetime larger than s is a Poisson random variable with parameter the product of population size Z_{-s} and the rate $c(s) = \mathbb{N}(\zeta > s)$ that one family has lifetime lager than s.

The next result is the analogue of the result on the number of ancestors for coalescent process given in [6] and [37].

Theorem 6.9. *The following convergence holds in probability:*

$$\lim_{s \to 0} \frac{M_s}{c(s)} = Z_0.$$

Proof. Let $\rho > 0$. We take $\eta = \rho/c(s)$. We deduce from (39) that:

$$\lim_{s \to 0} \mathbb{E}\left[e^{-\rho\frac{M_s}{c(s)} - \lambda Z_0}\right] = \mathbb{E}\left[e^{-Z_0(\rho+\lambda)}\right].$$

This implies that $\left(\dfrac{M_s}{c(s)}, Z_0\right)$ converges in distribution to (Z_0, Z_0), which gives the result. $\square$

Remark 6.10. Suppose in addition that $\int_0^\infty x^2 \pi(dx) < \infty$. Set $\tilde{\pi}(dx) = x^2 \pi(dx)$. Then the $\tilde{\pi}$-coalescent N^μ defined in [6] comes down from infinity by the assumption (A2) (see [6] and the references therein). It was shown in [6] that the speed of coming down from infinity satisfies

$$(40) \qquad\qquad \lim_{t \downarrow 0+} \frac{N_t^\mu}{c(t)} = 1 \quad \text{almost surely.}$$

From the heuristic duality between coalescence and branching processes, our result in Theorem 6.9 can be seen as a duality to (40).

Proof of lemma 6.7. For any $\eta, \lambda \geq 0$, we have:

$$\mathbb{E}\left[e^{-\eta M_s - \lambda Z_0}\right] = \mathbb{E}\left[e^{-\lambda \sum_{j \in \mathcal{J}} \mathbf{1}_{\{-s \leq t_j \leq 0\}} Y^j_{-t_j}}\right] \mathbb{E}\left[\exp\left(-\eta M_s - \lambda \sum_{j \in \mathcal{J}} \mathbf{1}_{\{t_j < -s\}} Y^j_{-t_i}\right)\right]$$

$$= \exp\left(-\int_0^s dr\, \tilde{\psi}'(u(\lambda, r))\right)$$

$$\mathbb{E}\left[\exp\left(-\sum_{j \in \mathcal{J}} \mathbf{1}_{\{t_j < -s\}}\left(\eta R_{-s-t_j, -t_j}(H^j) + \lambda Y^j_{-t_i}\right)\right)\right]$$

$$(41) \qquad = \exp\left(-\int_0^s dr\, \tilde{\psi}'(u(\lambda, r))\right)$$

$$\exp\left(-\int_0^\infty da\, \tilde{\psi}'\left(\mathbb{N}[1 - \exp(-\eta R_{a, a+s}(H) - \lambda Y_{a+s})]\right)\right),$$

where we used that Poisson random measures over disjoint sets are independent in the first equality, Lemma 3.1 in the second equality and a immediate generalization of Lemma 3.1 to genealogies in the third equality.

Using notations from Section 6.1.5 on the Poissonian representation of the height process above level a from Proposition 6.5, we get

$$\mathbb{N}\left[1 - e^{-\eta R_{a,a+s}(H) - \lambda Y_{a+s}}\right] = \mathbb{N}\left[1 - e^{-\sum_{k \in \mathcal{K}} \eta \mathbf{1}_{\{\varsigma_k \geq s\}} + \lambda Y(H^k)_s}\right]$$

$$= \mathbb{N}\left[1 - e^{-Y_a \mathbb{N}[1 - \exp(-\eta \mathbf{1}_{\{\varsigma \geq s\}} - \lambda Y_s)]}\right].$$

As $1 - \exp\left(-\eta \mathbf{1}_{\{\varsigma \geq s\}} - \lambda Y_s\right) = (1 - e^{-\eta})\mathbf{1}_{\{\varsigma \geq s\}} + e^{-\eta}(1 - e^{-\lambda Y_s})$, we deduce that

$$\mathbb{N}\left[1 - e^{-\eta R_{a,a+s}(H) - \lambda Y_{a+s}}\right] = \mathbb{N}\left[1 - e^{-\lambda' Y_a}\right] = u\left(\lambda', a\right),$$

with $\lambda' = (1 - e^{-\eta})c(s) + e^{-\eta} u(\lambda, s)$. Then we use (3.6) to write

$$\exp\left(-\int_0^\infty da\ \tilde{\psi}'\left(\mathbb{N}[1 - \exp(-\eta R_{a,a+s}(H) + \lambda Y_{a+s})]\right)\right) = \exp\left(-\int_0^\infty da\ \tilde{\psi}'\left(u(\lambda', a)\right)\right)$$
$$= \mathbb{E}\left[e^{-\lambda' Z_{-s}}\right].$$

Plugging this in (41), we get (39). $\qquad\square$

7. The quadratic branching mechanism

Let $(\mathbf{e}_k; k \in \mathbb{N})$ be independent exponential random variables with mean 1.

7.1. **Preliminaries.**
In this Section we give some explicit distributions and more precise results for the case of quadratic branching mechanism:

$$\psi(\lambda) = \beta\lambda^2 + 2\beta\theta\lambda, \tag{42}$$

where $\beta > 0$ and $\theta > 0$. We have

$$u(\lambda, t) = \frac{2\theta\lambda}{(2\theta + \lambda)\,e^{2\theta\beta t} - \lambda}, \quad c(t) = \frac{2\theta}{e^{2\theta\beta t} - 1}, \quad \kappa_* = 2\theta.$$

For every $t \in \mathbb{R}$, it follows from Corollary 3.3 that the process $\{Z_{s+t}; s \geq 0\}$ has the same distribution as the strong solution of the following stochastic differential equation

$$dX_s = \sqrt{2\beta X_s}\,dW_s + 2\beta(1 - \theta X_s)\,ds,$$

with initial law $\mathbb{P}(Z_0 \in \cdot)$, where W is a standard Brownian motion (see [44] Section XI.3 for the existence of strong solution).

7.2. **Joint law of the TMRCA and populations sizes.**
We have the following representations.

Theorem 7.1. *Assume ψ is given by (42).*

(i) We have for $\lambda \geq 0$:

$$\mathbb{E}[e^{-\lambda Z}] = \left(\frac{2\theta}{2\theta + \lambda}\right)^2 \quad and \quad Z \overset{(d)}{=} \frac{1}{2\theta}(\mathbf{e}_1 + \mathbf{e}_2). \tag{43}$$

(ii) We have for $t \geq 0$:

$$\mathbb{P}(A \leq t) = (1 - e^{-2\theta\beta t})^2 \quad and \quad A \overset{(d)}{=} \frac{1}{2\theta\beta}\max(\mathbf{e}_1, \mathbf{e}_2). \tag{44}$$

(iii) Conditionally on $\{A = t\}$, we have the following distribution representation:

$$\left(Z^A, Z^I, Z^O\right) \overset{(d)}{=} \left(\frac{\mathbf{e}_1 + \mathbf{e}_2}{2\theta + c(t)}, \frac{\mathbf{e}_3 + \mathbf{e}_4}{2\theta + c(t)}, \frac{\mathbf{e}_5}{2\theta + c(t)}\right). \tag{45}$$

Proof. By Lemma 19, we have

$$\mathbb{E}[e^{-\lambda Z}] = \left(\frac{2\theta}{2\theta + \lambda}\right)^2.$$

This gives (i). Using Theorem 4.1, we obtain:

$$\mathbb{E}[e^{-\lambda Z^A - \gamma Z^I - \eta Z^O}; A \in dt] = \frac{2\beta(2\theta)^6\,e^{6\theta\beta t}(e^{2\theta\beta t} - 1)}{[(2\theta + \eta)\,e^{2\theta\beta t} - \eta][(2\theta + \gamma)\,e^{2\theta\beta t} - \gamma]^2[(2\theta + \lambda)\,e^{2\theta\beta t} - \lambda]^2}\,dt.$$

We then deduce (ii) and (iii). $\qquad\square$

We then are able to compare more precisely the size of the current population $Z = Z^I + Z^O$ with the size of the population Z^A just before the birth time of the MRCA. As $(Z_t, t \in \mathbb{R})$ is continuous, notice that that Z^A is also the size of the population at the birth time of the MRCA. Recall that Z^A is stochastically smaller than Z. The next Corollary indicates that Z^A is however not a.s. smaller than Z.

Corollary 7.2. *Assume ψ is given by (42). We have: a.s.*

$$\mathbb{P}(Z^A < Z|A) = \frac{11}{16} \quad and \quad \mathbb{E}[Z^A|A] = \frac{2}{3}\mathbb{E}[Z|A]$$

as well as

$$\mathbb{P}(Z^A < Z) = \frac{11}{16} \quad and \quad \mathbb{E}[Z^A] = \frac{2}{3}\mathbb{E}[Z].$$

Proof. We have

$$\mathbb{P}(Z^A < Z|A) = \mathbb{P}(\mathbf{e}_1 + \mathbf{e}_2 < \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5) = \frac{11}{16}.$$

The other equalities are obvious. $\qquad\square$

There is also an interesting result (which is not valid for general branching mechanism) which can be interpreted by time reversal. Recall ζ is the extinction time of Y.

Proposition 7.3. *Assume ψ is given by (42). Conditionally on Z, A is distributed as ζ under P_Z: for all $t \geq 0$*

(46) $$\mathbb{P}(A > t|Z) = \mathrm{e}^{-c(t)Z} = \mathrm{P}_Z(\zeta \leq t).$$

Proof. We deduce from (43) and (44) that the densities of Z and A are:

(47) $$f_A(t) = 4\theta\beta\,\mathrm{e}^{-2\theta\beta t}(1 - \mathrm{e}^{-2\theta\beta t})\mathbf{1}_{\{t>0\}} \quad and \quad f_Z(z) = (2\theta)^2 z\,\mathrm{e}^{-2\theta z}\,\mathbf{1}_{\{z>0\}}.$$

We also deduce from (45) the density of Z conditionally on $A = t$:

$$f_{Z|A=t}(z) = (2\theta + c(t))^3 z^2\,\mathrm{e}^{-(2\theta+c(t))z}\,\mathbf{1}_{\{z>0\}}.$$

Using Bayes' rule, we get the density of A conditionally on $Z = z$: for $z, t > 0$

$$f_{A|Z=z}(t) = f_{Z|A=t}(z)\frac{f_A(t)}{f_Z(z)} = \frac{z(2\theta)^2\beta}{(\mathrm{e}^{2\theta\beta t}-1)^2}\,\mathrm{e}^{2\theta\beta t}\exp\left(-\frac{2\theta z}{\mathrm{e}^{2\theta\beta t}-1}\right) = -c'(t)z\,\mathrm{e}^{-c(t)z}.$$

We obtain $\mathbb{P}(A \leq t|Z) = \mathrm{e}^{-c(t)Z}$. Then, we conclude as

$$\mathrm{P}_r(\zeta \leq t) = \mathrm{e}^{-r\mathbb{N}[\zeta \geq t]} = \mathrm{e}^{-rc(t)},$$

where we used the Poissonian representation of Y given by (7). $\qquad\square$

Notice that (46) implies that

$$\mathbb{P}(c(A)Z \geq c(t)Z|Z) = \mathbb{P}(A \leq t|Z) = \mathrm{e}^{-c(t)Z}.$$

We obtain that $c(A)Z$ is independent of Z and $c(A)Z \overset{(\mathrm{d})}{=} \mathbf{e}_1$. We thus deduce the following Corollary.

Corollary 7.4. *Assume ψ is given by (42). We have the following representation:*

$$(Z, c(A), Z^A) \overset{(\mathrm{d})}{=} \left(\frac{\mathbf{e}_1 + \mathbf{e}_2}{2\theta}, 2\theta\frac{\mathbf{e}_3}{\mathbf{e}_1 + \mathbf{e}_2}, \frac{1}{2\theta}\frac{\mathbf{e}_1 + \mathbf{e}_2}{\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3}(\mathbf{e}_4 + \mathbf{e}_5)\right).$$

Remark 7.5. It is also easy to check that conditionally on $\{Z = z\}$, A is distributed as $\frac{1}{2\beta\theta}\log\left(1 + \frac{2\theta z}{\mathbf{e}_3}\right)$. In particular, we deduce that A is distributed as $\frac{1}{2\beta\theta}\log\left(1 + \frac{\mathbf{e}_1 + \mathbf{e}_2}{\mathbf{e}_3}\right)$.

7.3. **TMRCA for n individuals.** Next, we consider the joint distribution of Z and A^n the TMRCA of the immortal individual and n individuals chosen at random among the current population. The next result is a direct application of Theorem 4.7.

Proposition 7.6. *Assume ψ is given by (42). We set $s = 1 - \mathrm{e}^{-2\beta\theta t}$. We have for $n \in \mathbb{N}^*$:*

$$\mathbb{E}\left[Z^n\, \mathrm{e}^{-\lambda Z}\, \mathbf{1}_{\{A^n \in [0,t]\}}\right] = \frac{(n+1)!s^n}{(2\theta + \lambda s)^n}\left(\frac{2\theta}{2\theta + \lambda}\right)^2,$$

and the size-biased distribution of A^n is the maximum of n independent exponential random variables with mean 1:

$$\mathbb{E}\left[Z^n \mathbf{1}_{\{A^n \in [0,t]\}}\right] = \mathbb{E}[Z^n](1 - \mathrm{e}^{-2\beta\theta t})^n.$$

We can compute explicitly the distribution of A^1. See also [30], section 3, for similar computations in a slightly different setting.

Proposition 7.7. *Assume ψ is given by (42). We set $s = 1 - \mathrm{e}^{-2\beta\theta t}$. We have:*

$$(48)\quad \mathbb{P}(A^1 \le t) = 2\frac{s}{1-s}\left(1 + \frac{s}{1-s}\log(s)\right) \quad and \quad \mathbb{P}(c(A^1)Z \ge x|Z) = \frac{2}{x} - \frac{2}{x^2}(1 - \mathrm{e}^{-x}).$$

In particular $c(A^1)Z$ is independent of Z.

Notice that $\mathbb{P}(A \le t) = s^2$ so that we recover from (48) the trivial inequality $\mathbb{P}(A^1 \le t) \ge \mathbb{P}(A \le t)$ as $A \ge A^1$.

Proof. Applying Theorem 4.7, we get

$$\mathbb{E}[\mathrm{e}^{-\lambda Z}\, \mathbf{1}_{\{A^1 \le t\}}] = \int_\lambda^\infty d\eta\, \mathbb{E}[Z\,\mathrm{e}^{-\eta Z}\, \mathbf{1}_{\{A^1 \le t\}}]$$

$$(49)\qquad = 2(\mathrm{e}^{2\theta\beta t} - 1)^2\left(\frac{1}{(\mathrm{e}^{2\theta\beta t} - 1)}\frac{2\theta}{2\theta + \lambda} - \log\left(1 + \frac{1}{(\mathrm{e}^{2\theta\beta t} - 1)}\frac{2\theta}{2\theta + \lambda}\right)\right).$$

In particular, the distribution of A^1 is given by

$$\mathbb{P}(A^1 \le t) = 2(\mathrm{e}^{2\theta\beta t} - 1)^2\left(\frac{1}{(\mathrm{e}^{2\theta\beta t} - 1)} - \log\left(1 + \frac{1}{(\mathrm{e}^{2\theta\beta t} - 1)}\right)\right).$$

Applying inverse Laplace transforms to (49) and using the density of Z given in (47), we get that the conditional law of A^1 given Z:

$$\mathbb{P}(A^1 \le t|Z) = \frac{2(\mathrm{e}^{2\theta\beta t} - 1)^2}{(2\theta)^2 Z}\left(\frac{2\theta}{\mathrm{e}^{2\theta\beta t} - 1} + \frac{\mathrm{e}^{-2\theta Z/(\mathrm{e}^{2\theta\beta t} - 1)} - 1}{Z}\right),$$

which implies that

$$\mathbb{P}(2\theta Z/(\mathrm{e}^{2\theta\beta A^1} - 1) > x) = \frac{2}{x} - \frac{2}{x^2}(1 - \mathrm{e}^{-x}).$$

$\square$

7.4. **Fluctuations for the renormalized number of ancestors.** Finally, we complete Theorem 6.9 by giving the fluctuations for the renormalized number of ancestors.

Theorem 7.8. *Assume ψ is given by (42). We have*

$$\sqrt{c(s)\mathbb{E}[Z]}\left(\frac{M_s}{c(s)} - Z\right) \xrightarrow[s\downarrow 0+]{(\mathrm{d})} (Z - Z'),$$

where Z' is distributed as Z and independent of Z.

Proof. We first note that for every $\lambda > 0$,

$$\int_0^s \tilde{\psi}'\left(u(\lambda\sqrt{c(s)}, r)\right) \, dr \leq s\psi'\left(\lambda\sqrt{c(s)}\right) \xrightarrow[s\downarrow 0+]{} 0,$$

and

$$\lim_{s\to 0}(1 - e^{\lambda/\sqrt{c(s)}})c(s) + e^{\lambda/\sqrt{c(s)}} u(\lambda\sqrt{c(s)}, s) = -\lambda^2/2.$$

Under the current assumption on the exponent ψ, $\mathbb{E}[e^{\lambda Z}] < \infty$ and $\mathbb{E}[e^{\lambda M_s}] < \infty$ for $\lambda > 0$ small enough. Hence, by an analytic continuation argument, we see that (39) implies that for $\lambda > 0$ small enough, the following holds for all small s:

(50)
$$\mathbb{E}\left[e^{-\lambda\sqrt{c(s)}\left(Z - \frac{M_s}{c(s)}\right)}\right] = e^{-\int_0^s \tilde{\psi}'(u(\lambda\sqrt{c(s)}, r)) \, dr} \mathbb{E}\left[e^{-Z\left((1 - e^{\lambda/\sqrt{c(s)}})c(s) + e^{\lambda/\sqrt{c(s)}} u(\lambda\sqrt{c(s)}, s)\right)}\right].$$

Hence, for all small $\lambda > 0$, we have

$$\lim_{s\to 0}\mathbb{E}\left[e^{-\lambda\sqrt{c(s)}(Z - \frac{M_s}{c(s)})}\right] = \mathbb{E}\left[e^{\lambda^2 Z/2}\right] = \left(\frac{2\theta}{2\theta - \lambda^2/2}\right)^2 = \mathbb{E}\left[e^{-\lambda(Z - Z')/\sqrt{\mathbb{E}[Z]}}\right],$$

since $\mathbb{E}[Z] = 1/\theta$. The result is then a consequence of [40]. $\square$

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