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Ascending auctions: some impossibility results and their resolutions with final price discounts*

Laurent Lamy†

Abstract

When bidders are not substitutes, we show that there is no standard ascending auction that implements a bidder-optimal competitive equilibrium under truthful bidding. Such an impossibility holds also in environments where the Vickrey payoff vector is a competitive equilibrium payoff and is thus stronger than de Vries, Schummer and Vohra's [*On ascending Vickrey auctions for heterogeneous objects*, J. Econ. Theory, 132, 95-118] impossibility result with regards to the Vickrey payoff vector under general valuations. Similarly to Mishra and Parkes [*Ascending price Vickrey auctions for general valuations*, J. Econ. Theory, 132, 335-366], the impossibility can be circumvented by giving price discounts to the bidders from the final vector of prices. Nevertheless, the similarity is misleading: the solution we propose satisfies a minimality information revelation property that fails to be satisfied in any ascending auction that implements the Vickrey payoffs for general valuations. We investigate related issues when strictly positive increments have to be used under general continuous valuations.

Keywords: ascending auctions, combinatorial auctions, bidder-optimal competitive equilibrium, non-linear pricing, Vickrey payoffs, increments

JEL classification: D44, D45, C70

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1 Introduction

In the same way as the English auction has desirable features compared to the second price auction, its sealed-bid counterpart, developing ascending counterparts of sealed-bid formats is of primer interest in the multi-object auction literature. A main issue in sealed-bid formats as the Vickrey auction where bidders are invited to reveal directly their preferences truthfully is their reluctance to disclose such an information that can have adverse effects in future interactions or if the auctioneer may cheat (see Rothkopf et al. [41]'s formalization and Moldovanu and Tietzel [36]'s historical example). A completely different argument in favor of ascending formats comes from the possibility for bidders to refine their valuations during the auction process as developed in Compte and Jehiel [8, 9]. Their argument seems especially relevant in those ascending formats, as the FCC spectrum auctions, that last several weeks or even months and where heterogenous goods are sold such that bidders have to decide on which bundle to acquire information.[1] Similarly, bidders' willingness to pay are endogenous with allocative externalities and ascending formats allow them to refine their beliefs about their relevant competitors. With some structure on the externality terms, Das Varma [12] shows that the English auction reveals more payoff-relevant information compared to standard sealed bid formats and raises thus a greater revenue. For the sale of a good with interdependent values and asymmetric bidders, Krishna [26] and Dubra et al. [16] establish the existence of an efficient ex post equilibrium under some mild assumptions while inefficiency would prevail in the second price auction if there are strictly more than two bidders. Finally, the 'Linkage Principle' developed by Milgrom and Weber [33] for single unit auctions would intuitively plead in favor of ascending multi-object formats in environments with interdependent valuations and correlated private signals.[2]

Under unit-demand, Demange et al. [15] consider an ascending auction with linear and anonymous prices that implements bidders' most preferred competitive equilibrium, which coincides with the Vickrey payoff vector as established by Leonard [30], and thus the efficient assignment in an incentive compatible way. When bid-

[1]On similar grounds, Ye [42] argues in favor of a multi-stage auction that leaves room for information acquisition in the middle of the auction process.

[2]See Ausubel [2] for an extension of the linkage principle to multi-unit environments with flat multi-demand. Note however that Perry and Reny [38] display an example where the linkage principle fails in a multi-unit auction.

ders are satisfying the gross substitutes condition, Gul and Stacchetti [22] generalize Demange et al. [15]: their ascending auction implements bidders' most preferred competitive equilibrium beyond the unit-demand framework. However, such an outcome may not coincide with the Vickrey payoffs. Gul and Stacchetti [22] actually establish the impossibility to implement the Vickrey payoffs through an ascending auction with linear and anonymous prices, even if bidders' preferences are satisfying the gross substitutes condition. However, the impossibility result is circumvented if non-linear and non-anonymous pricing is allowed as established by de Vries et al. [14] by using the primal-dual (PD) algorithm of a linear programming formulation of the efficient assignment problem developed previously by Bikhchandani and Ostroy [5]. Their ascending auction implements a point in the set of bidder-optimal competitive equilibrium payoffs when bidders are substitutes. In such a case, this set is a singleton and coincides with the Vickrey payoff vector.[3] However, de Vries et al. [14] show that the impossibility reappears when at least one bidder has a valuation function that does not satisfy the gross substitutes condition. Following the combinatorial auction literature, we consider from now on "classes" of auctions that allow non-linear and non-anonymous pricing systems. In this vein, we use the terminology 'competitive equilibrium' with regards to such general pricing systems.[4]

The starting point of the paper is an impossibility result faced by standard ascending auctions defined in the same way as in de Vries et al. [14]: no standard ascending auction yields a bidder-optimal competitive equilibrium payoff (or equivalently a bidder-optimal Core payoff) under truthful bidding for general valuations. Furthermore, the impossibility still holds under a restricted set of preferences such that the Vickrey payoff vector is a competitive equilibrium payoff vector and thus coincides with the set -which is then a singleton- of bidder-optimal competitive equilibrium payoffs. As a by-product, note that we fill a gap with respect to de Vries et al. [14]'s analysis of the implementation of the Vickrey payoff vector: when bidders are substitutes, the Vickrey payoff vector, which is then a bidder-optimal competitive equilibrium payoff vector, can be implemented with a standard ascending auction. On the contrary, if some bidders have complementary preferences, de Vries et al. [14]

[3]The seemingly contradiction with Gul and Stacchetti [22] (where bidders are substitutes since each bidder satisfies the gross substitutes condition) is that the latter consider bidders' most preferred competitive equilibrium with respect to the set of linear and anonymous price vectors and not the set of non-linear and non-anonymous price vector as in de Vries et al. [14].

[4]See the recent collection of papers in Cramton et al. [10].

build an example with a class of preferences where the Vickrey payoff vector is not a competitive equilibrium and which prevents the possibility to implement it with some standard ascending auction. Here we obtain that the Vickrey payoff vector being for sure a competitive equilibrium is not a sufficient condition to implement it through a standard ascending auction. This result comes from an intuition that already appeared as a comment in de Vries et al. [14]: "since the early rounds of a PD auction may force a bidder to compete against only a subset of other bidders [...], complementarities between that subset may drive prices too high. Intuitively, if the "wrong" subset of bidders is chosen to compete with itself, then prices on some bundles could be driven too high. In that case, VCG payments could not be reached monotonically" (p108). In other words, this means that prices may be pushed in an inappropriate way such that a bidder may obtain less than the minimum of his bidder-optimal competitive equilibrium payoffs. Since this intuition has not been formalized previously, it is carefully illustrated through a simple example in section 3 under a standard ascending format that corresponds to a subgradient algorithm with respect to Bikhchandani and Ostroy [5]'s formulation, as in Ausubel and Milgrom [4] and Parkes [37]. Proposition 4.1 then formalizes that such an intuition prevails in *any* standard ascending auction.

In a second step, we propose to add a price discount stage from the final vector of prices in order to circumvent our previous impossibility result. The idea of price discounts has been first introduced by Mishra and Parkes [34] to circumvent the impossibility to implement the Vickrey payoffs in an ascending way with general valuations. Price discounts per se do not break the desirable features of ascending formats that have been gathered in the first paragraph. Nevertheless, coupled with the general definition of ascending auctions considered in de Vries et al. [14] and also here, price discounts are allowing an implementation of the Vickrey payoffs which breaks the analogy with the English auction: first recover fully bidders' preferences by raising the prices and then implement the Vickrey payoffs with price discounts. Mishra and Parkes [34]'s proposal is less extreme than this stylized auction (however it may coincide with it in some generic cases) but still relies on the unappealing feature that the auctioneer may still raise the prices after a competitive equilibrium has been reached. At this stage when the final assignment is thus fixed in an efficient auction, the final payoff of a given bidder in Mishra and Parkes [34]'s Vickrey auction will not depend anymore on his own further reports but only on the further reports of his opponents. Consequently, once a competitive equilibrium has been reached, bidders are then indifferent between all possible reports.[5] Due to such an indifference and according to Rothkopf et al. [41]'s perspective of bidders' reluctance to reveal their true preferences, the incentives to report truthfully additional parts from their own preferences does not seem satisfactory. More generally, the price dynamic of an ascending auction according to Mishra and Parkes [34]'s definition may raise the price of provisionally winning bids. This invites us to reconsider what should be taken as an ascending auction by introducing a *minimality* property requiring that only the prices of provisional losing bidders according to the current set of bids can be raised. We show that the impossibility to implement a bidder-optimal competitive equilibrium with a standard ascending auction is circumvented with some minimal ascending auctions: contrary to the former, our solutions involve price discounts after the price dynamic has stopped. Since they satisfy the minimality property, our solutions have thus a completely different nature than Mishra and Parkes [34]'s.

Furthermore, we move in a third step to the issue of the implementation of the Vickrey payoff vector with a *minimal* ascending auction. On the one hand and as a by-product of our analysis of the implementation of a bidder-optimal competitive equilibrium, we obtain that if the Vickrey payoff vector is a competitive equilibrium payoff, then there exists ascending auctions (with price discounts) that implement it under truthful bidding. On the other hand, under general valuations, we extend de Vries et al. [14]'s impossibility result to minimal ascending auctions with price discounts. The failure of the minimality property is thus not specific to Mishra and Parkes [34]'s proposal but would prevail in *any* ascending auction that implements the Vickrey payoffs.

Up to this stage, the analysis has been mainly limited to environments with integer valuations. Finally, we move to more general environments with continuous valuations and investigate the robustness of our analysis to the introduction of a

[5] This unappealing feature has not been pointed by Mishra and Parkes [34] and is a corollary of the foundation of the Vickrey payoff for a given bidder that depends solely on the externality imposed on his opponents which does not depend anymore on his own preferences once the final efficient assignment for the entire coalition of bidders has been found. A similar unappealing feature occurs in Mishra and Parkes [35]'s descending Vickrey auctions once a competitive equilibrium of the main economy has been found and also in Ausubel [3]'s dynamic auction with multiple parallel auctions, e.g. once the Vickrey payoff of some bidders has been computed. In the same vein, Jehiel and Moldovanu [24] pointed a similar unappealing indifference in Mezzetti [31]'s efficient mechanism with interdependent valuations.

discrepancy between valuations and bid increments such that the price dynamic can not match exactly the indifference curves: we show that, under truthful bidding, no standard ascending auction with positive increments can guarantee to yield an assignment that generates a welfare as close as possible to the efficient assignment for any profile of valuations, however small the increment can be. The intuition is similar than the one that sustains our first impossibility result: a bidder may exit the auction though his contribution to the total welfare is not of the same order as the bidding increment. This impossibility stands in contrast with both Demange et al. [15] and Milgrom [32] who have shown in their respective auction models where bidders have substitutes preferences that bid increments add only a nuisance term that vanishes when the increments go to zero. The non-robustness to positive increments is then circumvented by means of price discounts of the size of the increments.

This paper is organized as follows. Section 2 introduces the model and the notation. Section 3 gives a simple example which illustrates a crucial underlying intuition in our subsequent analysis. Section 4 considers the implementation of a bidder-optimal competitive equilibrium. Section 5 revisits the implementation of the Vickrey payoff vector: de Vries et al. [14]'s impossibility result is extended to minimal ascending auctions with price discounts. Section 6 investigates the robustness of ascending auctions to increments and discusses the consequences for practical auction design. Section 7 concludes. Proofs are relegated in Appendices A-J.

2 The package model

2.1 The assignment problem

There is a finite set of bidders $\mathcal{N} = \{1 \ldots, N\}$, a single seller indexed as agent 0 and a finite set of indivisible goods G. The set of feasible assignments of the goods is denoted as follows:

$$\mathbf{A} = \{\mathcal{A} \in (2^G)^{N+1} : i \neq j \Rightarrow \mathcal{A}_i \cap \mathcal{A}_j = \emptyset \text{ and } \bigcup_{i=0}^{N} \mathcal{A}_i = G\}.$$

Each bidder $i \in \mathcal{N}$ has a non-negative valuation for each set of goods $H \subseteq G$, denoted by $v_{i,H}$, and with $v_{i,\emptyset} = 0$. We assume that $v_{i,\cdot}$ is nondecreasing for any $i \in \mathcal{N}$, i.e. $H \subseteq H'$ implies $v_{i,H} \leq v_{i,H'}$. Preferences are quasi-linear: a bidder i who consumes $H \subseteq G$ and makes a payment of $p_{i,H} \in \mathbb{R}_+$, which denotes the price of bundle H to bidder i, receives a net payoff of $v_{i,H} - p_{i,H}$.[6] For a given assignment $\mathcal{A}$, the welfare equals then $\sum_{i \in \mathcal{N}} v_{i,\mathcal{A}_i}$. The demand set of bidder i at price vector $p \in \mathbb{R}_+^{2^G \times N}$ and the supply set of the seller to a subset $S \subseteq \mathcal{N}$ of bidders are defined respectively as follows:

$$D_i(p; v) := \operatorname{Arg} \max_{H \subseteq G} v_{i,H} - p_{i,H}.$$

$$L_S(p) := \operatorname{Arg} \max_{\mathcal{A} \in \mathbf{A}} \sum_{i \in S} p_{i,\mathcal{A}_i}.$$

Definition 1 (Competitive equilibrium) Price vector p and assignment $\mathcal{A}$ are a competitive equilibrium (CE) of economy $E(S)$, for some $S \subseteq \mathcal{N}$, if $\mathcal{A}_i \in D_i(p; v)$ for every bidder $i \in S$ and $\mathcal{A} \in L_S(p)$. Price p is called a CE price vector of economy $E(S)$.

For a given price vector $p \in \mathbb{R}_+^{2^G \times N}$ of economy $E(\mathcal{N})$, let $\gamma(p) \in \mathbb{R}_+^{N+1}$ denote the corresponding payoff vector such that $\gamma_0(p) = \max_{\mathcal{A} \in \mathbf{A}} \sum_{i \in \mathcal{N}} p_{i,\mathcal{A}_i}$ and $\gamma_i(p) = \max_{H \subseteq G} v_{i,H} - p_{i,H}$ for $i \in \mathcal{N}$. If p is a CE price vector then $\gamma(p)$ is called a CE payoff (of economy $E(\mathcal{N})$). In the following, let $CEP(\mathcal{N}, v)$ denote the set of CE payoffs.

Definition 2 The set of bidder-optimal CE payoffs [respectively of weak bidder-optimal CE payoffs] is the set containing the elements $\gamma \in CEP(\mathcal{N}, v)$ such that there exists no CE payoff $\gamma' \in CEP(\mathcal{N}, v)$ with $\gamma'_i \geq \gamma_i$ for all $i \in \mathcal{N}$ and such that at least one inequality is strict [respectively with $\gamma'_i > \gamma_i$ for all $i \in \mathcal{N}$].

In the following, those sets are respectively denoted by $BOCE(\mathcal{N}, v)$ and $wBOCE(\mathcal{N}, v)$. A bidder-optimal CE price vector is a CE price vector $p \in \mathbb{R}_+^{2^G \times N}$ such that $\gamma(p)$ is a bidder-optimal CE payoff.

We consider the characteristic function associated to this assignment problem which is defined by $w(S) := \max_{\mathcal{A} \in \mathrm{A}} \sum_{i \in S} v_{i,\mathcal{A}_i}$ for any $S \subseteq \mathcal{N}$.

[6] Our analysis can be extended to the generalized framework where bidders have general valuation functions $v_{i,\mathcal{A}}$ for the entire assignment $\mathcal{A} \in \mathbf{A}$, with the restriction that $v_{i,\mathcal{A}} = 0$ if $\mathcal{A}_i = \emptyset$, i.e. allowing for allocative externalities between purchasers but no externalities on non-traders as in Lamy [28]. With respect to the present analysis, the price vector would have to be enriched to allow for contingent prices that do not depend solely on $H \subseteq G$ but on the whole assignment $\mathcal{A} \in \mathbf{A}$.

Definition 3 We say that *bidders are substitutes* if the characteristic function w is submodular, i.e. if for all $M \subseteq M' \subseteq \mathcal{N}$ and all $j \in \mathcal{N}$ we have $w(M \cup \{j\}) - w(M) \geq w(M' \cup \{j\}) - w(M')$.

A key payoff vector is the Vickrey payoff vector, denoted by $\gamma^V := (\gamma_i^V)_{i \in \mathcal{N} \cup \{0\}}$, such that bidder i's payoff γ_i^V equals $w(\mathcal{N}) - w(\mathcal{N} \setminus \{i\})$ and the seller receives the revenue $\gamma_0^V = w(\mathcal{N}) - \sum_{l \in \mathcal{N}} \gamma_l^V$. Then we define the set of Core payoffs, denoted by $Core(\mathcal{N}, v)$, related to this characteristic function w:

$$Core(\mathcal{N}, v) = \left\{ (\gamma_i)_{i \in \mathcal{N} \cup \{0\}} \in \mathbb{R}_+^{N+1} \mid (a): \sum_{i \in \mathcal{N} \cup \{0\}} \gamma_i = w(\mathcal{N});\ (b): \ (\forall S \subseteq \mathcal{N})\ w(S) \leq \sum_{i \in S \cup \{0\}} \gamma_i \right\}.$$

(a) is the feasibility condition meaning that a Core payoff vector implements an efficient assignment, i.e. an assignment $\mathcal{A} \in Arg\max_{\mathcal{A} \in \mathrm{A}} \sum_{i \in \mathcal{N}} v_{i, \mathcal{A}_i}$, whereas the inequalities (b) mean that the payoffs are not blocked by any coalition S.[7]

Remark 2.1 The payoff vector resulting from a positive transfer of payoffs from a given bidder i to the seller remains in the Core if the initial payoff vector is in the Core and provided that γ_i remains nonnegative. This comes from the fact that the inequalities (b) are not altered if $i \in S$ and that the inequalities (b) are only strengthened if $i \notin S$. In particular, the payoff vector such that $\gamma_0 = w(\mathcal{N})$ and $\gamma_i = 0$ for all $i \in \mathcal{N}$ belongs to the Core which is thus non empty. As a corollary of the following proposition we obtain that the set of CE payoffs is non-empty and thus also the set $BOCE(\mathcal{N}, v)$.

Proposition 2.1 (Bikhchandani and Ostroy [5])

- $Core(\mathcal{N}, v) = CEP(\mathcal{N}, v)$
- *The Vickrey payoff vector is a competitive equilibrium payoff vector if and only if the bidder-optimal frontier is a singleton. In such a case they coincide:* $BOCE(\mathcal{N}, v) = \{\gamma^V\}$.
- *If bidders are substitutes, then for any set of bidders* $M \subseteq \mathcal{N}$ *we have* $w(\mathcal{N}) - w(\mathcal{N} \setminus M) \geq \sum_{j \in M} [w(\mathcal{N}) - w(\mathcal{N} \setminus \{j\})]$. *This latter condition is equivalent to the Vickrey payoff vector being a competitive equilibrium payoff vector.*

[7] We assume implicitly that the seller is present in any coalition.

From proposition 2.1, the set $BOCE(\mathcal{N}, v)$ is also called the bidder-optimal frontier of the Core. A crucial condition in the analysis of ascending auctions is whether the Vickrey payoff vector is a CE payoff vector. This is thus equivalent to the fact that the bidder-optimal frontier a singleton and closely related to the slightly stronger 'bidders are substitutes' condition.

The bulk of our analysis consists in implementing exactly either a payoff vector that belongs to the set $BOCE(\mathcal{N}, v)$ or the Vickrey payoff vector. In some circumstances, we are considering an approximate implementation perspective instead of an exact one. To this aim, we consider the following definition.

Definition 4 A vector $x \in \mathbb{R}^m$ is said to ϵ-approximate a set $K \subset \mathbb{R}^m$ if there is a vector $y \in K$ such that $|x_i - y_i| \leq \epsilon$ for any $i = 1, \ldots, m$.

2.2 Ascending auctions

For our impossibility results we mainly consider the class of standard ascending auctions introduced by de Vries et al. [14]. First this class imposes that price adjustments are driven solely through demand revelation. Second it imposes a full linkage between final prices and final monetary transfers on the contrary to Gul and Stacchetti [22]'s analysis: prices in the auction are thus not artificial constructs. Furthermore, we also introduce two additional ingredients with respect to their model: price increments and the possibility of a price discount stage after the auction dynamic ends which relax the linkage between final prices and final monetary transfers.

Definition 5 A *price path* is a function $P : [0, 1] \rightarrow \mathbb{R}_+^{2^G \times N}$. For each bundle of goods $H \subseteq G$, interpret $P_{i,H}(t)$ to be the price seen by bidder i for bundle H, at "time" t. A price path is *ascending* if for any $i \in \mathcal{N}$ and $H \subseteq G$ the function $P_{i,H}(t)$ is nondecreasing in t. A price path involves ϵ*-increments* (with $\epsilon \geq 0$) if for any $t, t' \in [0, 1]$ either $P_{i,H}(t) = P_{i,H}(t')$ or $|P_{i,H}(t) - P_{i,H}(t')| \geq \epsilon$. Let Π_ϵ denote the set of all ascending price paths with ϵ-increments.

We say that a price path (or abusively a pricing system or an auction) is anonymous if $P_{i,H}(t) = P_{j,H}(t)$ for any $i, j \in \mathcal{N}$ and $H \subseteq G$, linear if $P_{i,H1}(t) + P_{i,H2}(t) = P_{i,H1 \cup H2}(t)$ for any $i \in \mathcal{N}$ and $H1, H2 \subseteq G$ with $H1 \cap H2 = \emptyset$ and combinatorial otherwise. Note that the English button auction is anonymous. However, a typical

English auction with increments fails to be anonymous according to our terminology: the provisional winning bidder is not facing the same price as the provisional losing bidders who have to bid the winning price plus a positive increment in order to stay active.

Definition 6 A *standard ascending auction* [respectively an *ascending auction with price discounts*] with ϵ-increments is a pair of functions $\pi : \mathbb{R}_+^{2^G \times N} \to \Pi_\epsilon$ and $\xi : \mathbb{R}_+^{2^G \times N} \to \mathbb{R}_+^N \times \mathbf{A}$ such that:
(i) for all valuation profiles $v, v' \in \mathbb{R}_+^{2^G \times N}$, if $D_i(\pi(v)[t]; v) = D_i(\pi(v')[t]; v')$ for any $t \in [0, t^*]$ and $i \in \mathcal{N}$, then $\pi(v) = \pi(v')$ on $[0, t^*]$.
(ii) the final assignment $\mathcal{A} = \xi_2(v)$ satisfies demand according to the final prices $\pi(v)[1]$, i.e. $\mathcal{A}_i \in D_i(\pi(v)[1]; v)$ for any $v \in \mathbb{R}_+^{2^G \times N}$, and the price $[\xi_1(v)]_i = [\pi(v)[1]]_{i,\mathcal{A}_i}$ is charged to bidder i [respectively (ii) there exists a discount function $\delta : \mathbb{R}_+^{2^G \times N} \to \mathbb{R}_+^{2^G \times N}$ such that for all valuation profiles $v, v' \in \mathbb{R}_+^{2^G \times N}$, if $D_i(\pi(v)[t]; v) = D_i(\pi(v')[t]; v')$ for any $t \in [0, 1]$ and $i \in \mathcal{N}$, then $\delta(v) = \delta(v')$ and $[\delta(v)]_{i,H} \leq [\pi(v)[1]]_{i,H}$ for any $H \subseteq G$ and $i \in \mathcal{N}$, the final assignment $\mathcal{A} = \xi_2(v)$ satisfies demand according to the final discounted prices, i.e. $\mathcal{A}_i \in D_i([\delta(v)]_i; v)$ for any $v \in \mathbb{R}_+^{2^G \times N}$ and the price $[\delta(v)]_{i,\mathcal{A}_i}$ is charged to bidder i].

The property (i) guarantees that the price vector is rising and so information is revealed only through demand revelation in the auction. In the same way, the discount function in an ascending auction with price discounts is defined such that it depends solely on the demand revelation history and the ascending price path that is associated to the auction. For our positive results, the discount function we use is actually much simpler than what our definition is allowing since it uses only the final vector of prices $\pi(v)[1]$ and the corresponding demand sets $(D_i(\pi(v)[1]; v))_{i \in \mathcal{N}}$.[8]

The class of ascending auctions with price discounts is a superset of the class of standard ascending auctions and is then mentioned briefly as ascending auctions. The idea of a price discount stage has been first introduced by Mishra and Parkes [34].[9] Nevertheless, to implement the Vickrey payoff vector, an additional element

[8] If we have in mind that bidders may refine their valuations in the course of the auction process, this last property seems desirable. Otherwise, it would allow the price discounts to depend on out-of-date information about bidders' demand.

[9] The 'clinching rule' in Ausubel [2, 3] is implicitly a price discount stage. In those papers the emphasis is on 'simple' dynamic auction mechanisms rather than on the ascending status of those auctions which belong to our class of ascending auctions. The 'clinching rule' corresponds to a

of departure with respect to the usual ascending auctions proposed in the literature to generalize the English auction is also implicitly used: once a competitive equilibrium has been found the auction dynamic does not stop. Mishra and Parkes [34]'s analysis relies crucially on the research of a stronger equilibrium concept, universal competitive equilibrium, that will be briefly discussed at the end of section 4. As emphasized in the introduction, the idea of price discounts per se does not necessarily stand in conflict with the desirable features that sustain our interest in the development of 'ascending auction' formats. However, without additional restrictions, it opens the door to solutions that almost mimic -from a pure private value perspective- the sealed-bid generalized Vickrey auction as the one where bidders are asked to report their full demand through ascending price adjustments that are not linked to the entire demand revelation from all bidders. The following *minimality* property formalizes the need to strengthen the linkage between bidders' demand revelation and price adjustments. As a preliminary, we define for a given price vector $p \in \mathbb{R}_+^{2^G \times N}$:

$$L_S^*(p) := \text{Arg} \max_{\mathcal{A} \in \mathbf{A} | \mathcal{A}_i \in \{\emptyset, D_i(p;v)\}, i \in S} \sum_{i \in S} p_{i,\mathcal{A}_i}$$

to denote the set of the revenue maximizing assignments for the seller in economy $E(S)$, $S \subseteq \mathcal{N}$, among those that assign to every bidder either a bundle from his demand set or the $\emptyset$ bundle at price vector p.

The central requirement of the minimality property is that the vector of prices for a given bidder i is raised at some price vector p only if there is an assignment $\mathcal{A} \in L_{\mathcal{N}}^*(p)$ such that bidder i's demand is not satisfied. A bidder such that $\mathcal{A}_i \in D_i(p; v)$ for any $\mathcal{A} \in L_{\mathcal{N}}^*(p)$ is called a provisionally winning bidder. In other words, we impose a mild linkage between price shifts and bidders' demand: the price vector of a provisionally winning bidder can not be pushed up. In particular, it means that bidder i's price vector is frozen once the empty set belongs to his demand set.

Definition 7 An ascending auction (π, ξ) is *minimal* if for any $v \in \mathbb{R}_+^{2^G \times N}$, the price path $P = \pi(v)$ satisfies:

discount function that depends not solely on the final price vector, as in Mishra and Parkes [34] and in the auctions we will consider next, but on the whole price path. See also Bikhchandani and Ostroy [6] for an interpretation of Ausubel [2]'s ascending auction for multiple units of a homogeneous object in term of a primal-dual algorithm with respect to Bikhchandani and Ostroy [5]'s linear programming formulation.

- $P_{i,\emptyset}(t) = 0$ for any $i \in \mathcal{N}$ and $t \in [0,1]$,[10]
- if $P_{i,H}(t) \neq P_{i,H}(t')$ for some $H \subseteq G$, $i \in \mathcal{N}$ and $t' > t$ then there exists $t^* \in [t,t')$ and $\mathcal{A} \in L^*_{\mathcal{N}}(P(t^*))$ such that $\mathcal{A}_i \notin D_i(P(t^*);v)$,
- if $H \subseteq H'$, then $P_{i,H}(t) \leq P_{i,H'}(t)$ for any $t \in [0,1]$ and $i \in \mathcal{N}$.

When there is a unique good for sale, the price path of a minimal ascending auction reduces to a current price for each bidder that is raising over time. The minimality property requires that if a given bidder demands the good at time t at a price that is strictly the highest among the current vector of prices, then this price is not rising at time t. This property is satisfied in all practical versions of the English auction to the best of our knowledge.

Ascending auction in the literature

First, the ascending auctions we consider seem -at first glance- to be limited to what is called a 'clock auction' in the literature, i.e. auctions where it is not the bidders but the auctioneer that raises the prices according to bidders' reported demand sets. Nevertheless, auctions where it is on the contrary the bidders that update their bids as in the simultaneous ascending auction in Milgrom [32], in the package auction in Ausubel and Milgrom [4] or in the hierarchical package bidding in Goeree and Holt [21] can be reframed to fit in our class of ascending auctions when we limit ourselves to appropriate 'straightforward strategies'. As an example, let us consider Milgrom [32]'s simultaneous ascending auction under his straightforward bidding rule. It corresponds to the following algorithm: start from null prices and then apply the following price dynamic until it stops and where winning bidders pay then their final personalized prices: for any bidder $i \in \mathcal{N}$, pick $D_i^*(t) \in D_i(P(t), v)$, consider a vector of winning prices $(p_1(t), \ldots, p_{|G|}(t))$ where for each good $g \in G$, we have $p_g(t) := \max_{i \in \mathcal{N} | g \in D_i^*(t)} P_{i,g}(t)$, select for each good g a provisional winning bidder, denoted by $w_g(t)$, that belongs to $Arg\max_{i \in \mathcal{N} | g \in D_i^*(t)} P_{i,g}(t)$ and then raise the price in the next bidding round (at time $t + \Delta t$) in the following way $P_{i,g}(t + \Delta t) := P_{i,g}(t)$ if $i = w_g(t)$, otherwise let $P_{i,g}(t + \Delta t) := p_g(t) + \epsilon$.[11] Thus Milgrom [32]'s simultaneous ascending auction can be viewed as a standard ascending auction

[10] We allow that $P_{i,H}(0) > 0$ if $H \neq \emptyset$, i.e. we allow minimal opening bids.

[11] Milgrom [32] considers multiplicative instead of additive bidding increments. This detail is innocuous.

according to our terminology. In a similar way, Ausubel and Milgrom [4] and Goeree and Holt [21]'s combinatorial auctions can be viewed as ascending auctions with price discounts: contrary to the linear auction in Milgrom [32], the personalized price for a given bundle $H \subseteq G$ and for a given bidder i may rise such that it does not belong anymore to bidder i's demand set while the seller would maximize her revenue by assigning the bundle H to bidder i at its current price minus the increment, i.e. such that it belongs to bidder i's demand set after a price discount of the bidding increment on the given bundle H. The need for ϵ-discounts, where ϵ is the size of the increments between each bids in the auction dynamic, or equivalently the need to take into account all previously submitted bids is developed in section 6.

On the whole, we emphasize the strength of our impossibility results since almost all auctions that appeared in the literature under the terminology 'ascending auctions' belong to our general class of ascending auctions. We are aware of only two exceptions: Ausubel [3]'s dynamic auction for heterogenous goods with multiple parallel auctions in order to implement the Vickrey payoff vector under substitutable preferences while maintaining linear pricing or Perry and Reny [39]'s dynamic auction for homogenous goods under interdependent valuations in order to implement an efficient assignment.[12]

Second, the only ascending auctions in the literature that fail to be minimal, to the best of our knowledge, are the ones proposed by Mishra and Parkes [34]. At first glance, the anonymous and linear auctions in Demange et al. [15] and Gul and Stacchetti [22] fail to be minimal since a provisionally winning bidder can face a price increase: this is purely a framing effect, those auctions can be defined alternatively without modifying the final outcome under truthful reporting and such that they remain minimal (but such that their pricing system is no longer anonymous). We do not incorporate the minimality property in the definition of an ascending auction for clarification purposes. However, it is precisely our aim to analyze what can be implemented under (standard) ascending auctions that are minimal. In this perspective, we emphasize that the formats we use in our possibility results, i.e. the

[12] In the same way as price discounts per se do not stand in conflict with what should be viewed as an ascending auction, we should not dispose a priori of the idea of multiple price pathes. However, it may stand in conflict with what we have captured under the minimality property defined for ascending auctions with a single price path: if a bidder is provisionally a winning bidder then we should not ask him further information on his preferences. See also footnote 5. Such developments are left for further research.

QCE-invariant ascending auctions defined below and their variants introduced later, are all minimal.

Definition 8 Price vector p and assignment $\mathcal{A}$ are a *quasi-CE* of economy $E(S)$, for some $S \subseteq \mathcal{N}$, if $\mathcal{A}_i \in D_i(p; v)$ for every bidder $i \in S$ and $\mathcal{A} \in L^*_S(p)$. Price p is called a quasi-CE price vector of economy $E(S)$.

Contrary to CE price vectors, an assignment that corresponds to a quasi-CE price vector is not necessarily efficient. For any quasi-CE price vector p, let $\overline{\gamma}(p)$ denote the corresponding payoff vector that is defined analogously to the corresponding definition for a CE: $\overline{\gamma}_0(p) = \max_{\mathcal{A} \in \mathbf{A} | \mathcal{A}_i \in \{\emptyset, D_i(p;v)\}, i \in S} \sum_{i \in S} p_{i,\mathcal{A}_i}$ and $\overline{\gamma}_i(p) = \gamma_i(p)$ for any $i \in \mathcal{N}$.

For our various existence results of an appropriate ascending auction we consider variations from the following class of QCE-invariant ascending auctions with ϵ-increments: at each stage, bidders are asked to report their demand sets for the current price vector, the seller chooses an assignment that maximizes her revenue according to those demand sets, then for bidders that do not obtain an assignment in their demand set the corresponding prices are increased by the increment ϵ, if all bidders receive an assignment in their demand set then the auction stops and the current price vector is used for pricing the goods.

Definition 9 A QCE-invariant ascending auction with ϵ-increments ($\epsilon > 0$) is defined as follows:
(S0) The auction starts at the zero price vector.
(S1) In round t of the auction, with price vector p^t:

(S1.1) Collect the demand sets of the bidders at price vector p^t

(S1.2) If p^t is a quasi-CE price vector with respect to reported demand sets, then go to Step S2 with $T := t$.

(S1.3) Else, select a temporary winning assignment $\mathcal{A}^t \in L^*_{\mathcal{N}}(p^t)$ and a (non-empty) set of temporary losers $\mathcal{L}^t \subset \mathcal{N}$ such that $\mathcal{A}^t_i \notin D_i(p^t; v)$ for any $i \in \mathcal{L}^t$ who will see a price increase.[13]

(S1.4) If $i \in \mathcal{L}^t$ and $H \in D_i(p^t; v)$, then $p^{t+1}_{i,H} := p^t_{i,H} + \epsilon$. Else, $p^{t+1}_{i,H} := p^t_{i,H}$. Repeat from Step (S1.1).

[13]This set is not empty. Otherwise, p^t would be a quasi-CE price vector and the algorithm would have stopped in the previous stage (S1.2).

(S2) The auction ends with the final assignment of the auction being any $\mathcal{A}^T \in L^*_{\mathcal{N}}(p^T)$ and the final payment of every bidder $i \in \mathcal{N}$ being $p^T_{i,\mathcal{A}^T_i}$, where p^T is the final price vector of the auction.

The construction guarantees that any QCE-invariant ascending auction is a standard ascending auction.[14] Let k_i^t denote the optimal profit that bidder i can expect at round t, i.e. $k_i^t := \max_{H \subseteq G} \{v_{i,H} - p^t_{i,H}\}$. The ascending nature of the auction guarantees that k_i^t is nonincreasing in t.

If valuations are integers, then QCE-invariant ascending auctions with unit increments belong to the class of "uQCE-invariant(0) auction for the main economy" considered by Mishra and Parkes [34]. However, the other inclusion fails. First, the set of bidders that may face a price increase is reduced in our definition to those bidders that may be temporary losers, while Mishra and Parkes [34] are allowing any price increase for a given bidder provided that the empty set is not in his demand set. This is precisely the reason why our auctions are minimal contrary to those in Mishra and Parkes [34]. Second, Step (S1.2) in Mishra and Parkes [34] is more restrictive since it requires that p^t is a CE price vector with respect to reported demand sets instead of the weaker notion of quasi-CE. If valuations are integers and with unit increments -more generally if the valuations' and increments' grids fit- as it will be considered in sections 4 and 5, the two equilibrium notions would coincide in such auctions as a corollary of our subsequent lemma 2.3. However, in general it is not the case. The reason why we use the quasi-CE notion instead of the CE notion in our definition is that there is no guarantee with the latter notion that the algorithm in definition 9 would end. On the contrary, the algorithm in definition 9 and its variants end in a finite number of rounds since prices are strictly increasing from one round to the next such that the demand set of all bidders would be reduced to the empty bundle after a finite number of rounds if the algorithm does not stop. This would raise a contradiction since the algorithm would then stop immediately.

If the empty bundle is in one given bidder's demand set then we will call such a bidder an inactive bidder. In a QCE-invariant auction, his prices remain fixed and his demand set is thus unchanged such that his inactive status is fixed until the end of the auction's dynamic. On the contrary, a bidder whose demand set contains

[14] Up to some additional notation, the structure with countable "rounds" can equivalently be reframed in the framework with a price path on the interval $[0, 1]$.

solely non-empty bundles is called an active bidder.

Lemma 2.1 *Consider a QCE-invariant ascending auction with ϵ-increments at a given round t. There are two kinds of assignments for a given bidder $i \in \mathcal{N}$:*

- *Type 1: $p^t_{i,\mathcal{A}_i} = 0$ and $v_{i,\mathcal{A}_i} - p^t_{i,\mathcal{A}_i} \leq k^t_i$.*
- *Type 2: $p^t_{i,\mathcal{A}_i} > 0$ and $k^t_i - \epsilon < v_{i,\mathcal{A}_i} - p^t_{i,\mathcal{A}_i} \leq k^t_i$.*

Definition 10 A price vector p is semi-truthful if $p_{i,H} > 0 \Rightarrow H \in D_i(p; v)$, for any $i \in N$ and $H \subseteq G$.

An alternative equivalent definition is given in the following lemma.

Lemma 2.2 *A price vector p is semi-truthful if and only if for any $i \in \mathcal{N}$ there exists a unique $k_i \in [0, v_{i,G}]$ such that p is characterized by $p_{i,H} = \max\{v_{i,H} - k_i, 0\}$. Furthermore $k_i = \gamma_i(p)$.*

For any $i \in \mathcal{N}$ and $H \in D_i(p; v)$, any price vector satisfies $p_{i,H} = v_{i,H} - \gamma_i(p)$. If p is semi-truthful, then for any $H \notin D_i(p; v)$, we have $p_{i,H} = 0$, which shows the necessary part. The sufficient part is straightforward.

Under integer valuations and with unit increments, we obtain from lemma 2.1 the equalities $k^t_i = v_{i,\mathcal{A}_i} - p^t_{i,\mathcal{A}_i}$ for any assignment with $p^t_{i,\mathcal{A}_i} > 0$, i.e. the price vector is semi-truthful along the price path, and then the following lemma guarantees that QCE-invariant auctions are implementing CE payoffs under truthful reporting.

Lemma 2.3 *If a price vector p is a quasi-CE and is semi-truthful, then p is a CE. In other words, we have*

$$\max_{\mathcal{A} \in \mathbf{A} | \mathcal{A}_i \in \{\emptyset, D_i(p;v)\}} \sum_{i \in \mathcal{N}} p_{i,\mathcal{A}_i} = \max_{\mathcal{A} \in \mathbf{A}} \sum_{i \in \mathcal{N}} p_{i,\mathcal{A}_i}. \tag{1}$$

Remark 2.2 In order to prove the inclusion $Core(\mathcal{N}, v) \subseteq CEP(\mathcal{N}, v)$ (proposition 2.1), Bikhchandani and Ostroy [5] have shown that any point γ in the Core can be priced by a semi-truthful CE. More precisely, they consider the semi-truthful price vector defined in the following way: $p_{i,H} := \max\{v_{i,H} - \gamma_i, 0\}$ for any $i \in \mathcal{N}$ and $H \subseteq G$. In the following, this vector is denoted by $\mathcal{P}(\gamma)$. for any $\gamma \in \mathbb{R}^N_+$.

Definition 11 An ascending auction is *incentive compatible* on a domain V if truthful reporting (with respect to demand sets) at each "time" $t \in [0, 1]$ by all bidders is a (complete information) equilibrium for any realization $v \in V$.

Incentive compatibility on a domain V means also that if all opponents are bidding truthfully then it is a dominant strategy to bid truthfully.

Apart from section 6, our analysis does not impose the use of strictly positive increments. For simplification purposes, this part of the analysis is then limited to environments with integer valuations. For our possibility results the aim is to avoid the technicalities related to introduction of ascending auctions without increments. With a single good for sale, the simplest minimal ascending auction without increments is the English button auction where bidders can exit the auction at any time while the price rises continuously and stops at the time where at least all bidders except one have exited the auction. With multiple goods, the continuous time versions of the auction we propose are straightforward but tedious to define. For our following impossibility results, they hold when bidders are restricted to integer valuations and hold thus a fortiori with general (continuous) valuations.

3 An illustrative example

In one simple example with two identical goods and four bidders, we illustrate some problematic features of standard ascending auctions. We consider more specifically the QCE-invariant ascending auction with unit increments where, in case of ties at some time t, the temporary winning assignment $\mathcal{A}^t$ is the assignment which assigns the greatest number of units to bidders with the highest indices while the maximal set of temporary losers is chosen. Table 1 details the progress of the auction. The bundles which have prices in (.) are in the demand set of the respective bidders. We put emphasis on the auction dynamic since it is instructive with regards to the intuition of our impossibility results in propositions 4.1 and 6.1: at the earliest stages of the auction, prices may be pushed in an inappropriate direction.

We consider that bidder 1 is valuing 10 the first item and 0 an additional item, bidder 2 and 3 are identical and are valuing any additional item 4. For the moment, preferences are satisfying the gross substitutes condition. Let us introduce an additional bidder 4 who has complement preferences: he values the bundle of the two items 7, but values 0 a single item. Note also that bidder 4 could be labeled as a 'dummy bidder': he does not modify the structure of the set of CE payoffs which is given by

$$CEP(\mathcal{N}, v) = \left\{(\widehat{\gamma}_i)_{i \in \mathcal{N} \cup \{0\}} \in \mathbb{R}^{N+1} \mid \quad \widehat{\gamma}_0 = 14 - \widehat{\gamma}_1; \quad \widehat{\gamma}_1 \in [0, 6]\ ;\ \widehat{\gamma}_2 = \widehat{\gamma}_3 = \widehat{\gamma}_4 = 0\right\}.$$

The Vickrey payoff vector equals $(8, 6, 0, 0, 0)$ which belongs to $CEP(\mathcal{N}, v)$. Bidder 4 is thus neutral from both a Vickrey or a bidder-optimal CE implementation point of view: he does not change the final outcome which consists in assigning the items either to the couple $\{1, 2\}$ or to $\{1, 3\}$ and to make pay the amount 4 to the purchasers. If bidder 4 were absent, then we could apply Ausubel and Milgrom [4]'s results (see also Mishra and Parkes [34] in a more general class of ascending auctions) since all bidders would have substitutes preferences which guarantees that the bidders are substitutes condition holds and so that the auction dynamic would implement the Vickrey payoffs. Nevertheless, the mere presence of bidder 4 disturbs the dynamic of the auction. The final payoff vector is no longer the Vickrey payoff vector as shown in Table 1. The reason for this is that in early rounds the auction forces bidder 1 to compete against bidder 4 while the preferences of bidder 2 and 3 that are revealed later in the auction dynamic imply that such high prices were not needed for bidder 1 to block the coalition composed of bidder 4. More specifically overbidding occurs twice in the auction: between rounds 6 and 7 and then between rounds 10 and 11. At the end, bidder 1 has two pay two monetary units more than according to his Vickrey payoff. Somehow clumsily, he bids above 4 because he should internalize the externality imposed only on his opponents $\{4\}$, an externality which is stronger than the one he imposes on the bigger set of opponents $\{2, 3, 4\}$. This is exactly those situations that the 'bidders are substitutes' condition avoids by guaranteeing that the externality terms $w(S) - w(S \setminus \{i\})$ are nondecreasing in S.

This undesirable feature suggests to add a stage to the QCE-invariant ascending auction where the auctioneer reduces incrementally the bids of some bidders such that the price vector still remains a CE. This stage will be called the final discount stage. Given the reported demand sets at the last round in Table 1, when a CE price vector has been reached, the seller knows that if bidder 1's prices are reduced such that the price for one or two items for bidder 1 equals 4 then the final price vector

Table 1: Progress of the QCE-invariant ascending auction with unit increments

	bidder 1		bidder 2		bidder 3		bidder 4	
	One	Two	One	Two	One	Two	One	Two
Values →	10	10	4	8	4	8	0	7
↓ Rounds								
1	(0)	(0)	0	(0)	0	(0)	0	(0)
	$\mathcal{A}^1$: bidder 4 receives two items.							
	The revenue is 0. $\mathcal{L}^1 = \{1, 2, 3\}$.							
2	(1)	(1)	0	(1)	0	(1)	0	(0)
	$\mathcal{A}^2$: bidder 3 receives two items.							
	The revenue is 1. $\mathcal{L}^2 = \{1, 2, 4\}$.							
3	(2)	(2)	0	(2)	0	(1)	0	(1)
	$\mathcal{A}^3$: bidder 2 receives two items.							
	The revenue is 2. $\mathcal{L}^3 = \{1, 3, 4\}$.							
4	(3)	(3)	0	(2)	0	(2)	0	(2)
	$\mathcal{A}^4$: bidder 1 receives two items.							
	The revenue is 3. $\mathcal{L}^4 = \{2, 3, 4\}$.							
5	(3)	(3)	0	(3)	0	(3)	0	(3)
	$\mathcal{A}^5$: bidder 4 receives two items.							
	The revenue is 3. $\mathcal{L}^5 = \{1, 2, 3\}$.							
6	(4)	(4)	(0)	(4)	(0)	(4)	0	(3)
	$\mathcal{A}^6$: bidder 3 receives two items.							
	The revenue is 4. $\mathcal{L}^6 = \{1, 2, 4\}$.							
7	(5)	(5)	(1)	(5)	(0)	(4)	0	(4)
	$\mathcal{A}^7$: bidders 1 and 2 receive one item.							
	The revenue is 6. $\mathcal{L}^7 = \{3, 4\}$.							
8	(5)	(5)	(1)	(5)	(1)	(5)	0	(5)
	$\mathcal{A}^8$: bidders 1 and 3 receive one item.							
	The revenue is 6. $\mathcal{L}^8 = \{2, 4\}$.							
9	(5)	(5)	(2)	(6)	(1)	(5)	0	(6)
	$\mathcal{A}^9$: bidders 1 and 2 receive one item.							
	The revenue is 7. $\mathcal{L}^9 = \{3, 4\}$.							
10	(5)	(5)	(2)	(6)	(2)	(6)	(0)	(7)
	$\mathcal{A}^{10}$: bidder 4 receives two items.							
	The revenue is 7. $\mathcal{L}^{10} = \{1, 2, 3\}$.							
11	(6)	(6)	(3)	(7)	(3)	(7)	(0)	(7)
	$\mathcal{A}^{11}$: bidders 1 and 3 receive one item.							
	The revenue is 9. $\mathcal{L}^{11} = \{2\}$ is unsatisfied.							
12	(6)	(6)	(4)	(8)	(3)	(7)	(0)	(7)
	$\mathcal{A}^{12}$: bidders 1 and 2 receive one item.							
	The revenue is 10. $\mathcal{L}^{12} = \{3\}$ is unsatisfied.							
13	(6)	(6)	(4)	(8)	(4)	(8)	(0)	(7)
	$\mathcal{A}^{13}$: bidders 1 and 3 receive one item.							
	The revenue is 10. A CE price vector is reached.							
Final payoffs: $(10, 4, 0, 0, 0)$; Vickrey payoffs: $(8, 6, 0, 0, 0)$								
Final payoffs with infinitesimal increments: $(8.75, 6.25, 0, 0, 0)$								

remains a CE.[15] More precisely, she knows that if bidders have bid truthfully, then such a price vector belongs to $BOCE(\mathcal{N}, v)$, as it will be shown in section 4. Since the Vickrey payoff vector is a CE payoff vector in our example, then it means that such a price discount leads to the Vickrey payoff vector and truthful reporting is thus an equilibrium. On the contrary, truthful reporting is not an equilibrium in the 'original' auction without price discounts: bidder 1 would profitably deviate from truthful reporting by reporting that he values only slightly above 4 one or two units.

On the whole, this example illustrates that, in those QCE-invariant auctions, the final payoff may not lie in the set of bidder-optimal CE payoffs, even if the Vickrey payoff vector is a CE payoff vector, while adding a final discount stage restores the implementation of the Vickrey payoffs and thus incentive compatibility. Thanks to the price discount stage, we show more generally in section 4 that truthful reporting is an equilibrium which leads to the single-valued bidder-optimal frontier of the Core, hence the Vickrey payoffs, not solely under the condition that bidders are substitutes but under the weaker condition that the Vickrey payoff vector belongs to $BOCE(\mathcal{N}, v)$.

Technical remark Even if valuations are integer-valued, increments that are smaller than 1 have an influence on the final payoffs in this example. We emphasize that the overbidding problem in the earliest rounds of the auction is not an artifact resulting from positive increments. In appendix C, we summarize the dynamic of the auction in the continuous version of the auction where the increment becomes infinitesimal. Bidder 1's overbidding is persistent.

4 Bidder-optimal CE selecting auctions

Proposition 4.1 *Under general (integer) valuations and even if the Vickrey payoff vector is guaranteed to be a competitive equilibrium payoff vector, there is no standard ascending auction that yields a bidder-optimal CE under truthful bidding.*

We build an example with 11 different goods and 5 bidders such that:[16] in the neighborhood of the null prices, the demand sets, which are singletons, are known ex

[15] Note that any strict price discount for the winner among 2 and 3 would drive the final payoff vector out of the set of CE payoffs.

[16] The nice feature of the example is to make the proof simple from its intrinsic symmetry between all bidders. Similarly to Gul and Stacchetti [22]'s impossibility result, the exact number of bidders and goods required to face the impossibility is an open question.

ante ; however the efficient assignment is not known ex ante such that the auctioneer has to raise strictly the prices on some bidders' most preferred assignments to learn about bidders' preferences and thus to be able to implement an efficient assignment and a fortiori a point in $BOCE(\mathcal{N}, v)$. Nevertheless, our construction guarantees also that each bidder may obtain his most preferred assignment at a null price for any point in $BOCE(\mathcal{N}, v)$ for some realization of the joint preferences of his opponents. With such an example, no standard ascending auction can implement a payoff in $BOCE(\mathcal{N}, v)$: if it were, then the price of some bidders should be strictly higher than 0 for their respective most preferred assignments in order to learn about bidders' preferences, but then the realization of bidders' valuations could be such that those preferences imply that the price of a given bidder who has suffered from an initial price's increase (on his most preferred assignments) occurs to be null, which raises a contradiction. Such an example is built in the proof which is relegated to appendix D.

As it is well-known, the impossibility result is circumvented when bidders are substitutes (Ausubel and Milgrom [4], de Vries et al. [14], Mishra and Parkes [34]). Within the class of QCE-invariant ascending auctions we also show that for general valuations the final payoff vector corresponds 'roughly' to a weak bidder-optimal CE payoff, an insight that has not been noted in those previous papers.

Proposition 4.2 *Assume integer valuations and truthful bidding. For any QCE-invariant ascending auction with unit increments, the final assignment is efficient and the final payoff vector* $(k_i^T)_{i\in\mathcal{N}}$ *1-approximates the set of weak bidder-optimal CE payoffs* $wBOCE(\mathcal{N}, v)$*, i.e. there exists* $\gamma \in wBOCE(\mathcal{N}, v)$ *such that* $|\gamma_i - k_i^T| \leq 1$ *for any* $i \in \mathcal{N}$*. Furthermore, if bidders are substitutes, then the final payoff vector equals the Vickrey payoff vector.*

We now move to the possibility result with respect to the implementation of a bidder-optimal CE payoff under general valuations with an ascending auction by means of an appropriate price discount function.

First we introduce some additional notation. For any vector $e \in \mathbb{R}_+^N$, let $\beta(e; p) = p'$ denote the price vector such that $p'_{i,H} = \max\{p_{i,H} - e_i, 0\}$ for any $i \in \mathcal{N}$ and $H \subseteq G$. Note that if p is a semi-truthful price vector, then $\beta(e; p)$ is a semi-truthful price vector for any $e \in \mathbb{R}_+^N$.[17]

[17] Let $k \in \mathbb{R}_+^N$ such that $p = \mathcal{P}(k)$, i.e. $k_i = \gamma_i(p)$ for any $i \in \mathcal{N}$. For any $e \in \mathbb{R}_+^N$, we have

Definition 12 *A vector $e \in \mathbb{R}_+^N$ is called an admissible discount with respect to a quasi-CE price vector p if there exists $\mathcal{A} \in \mathbf{A}$ such that $e_i \leq p_{i,\mathcal{A}_i}$, $(p, \mathcal{A})$ is a quasi-CE of the main economy and $\mathcal{A} \in Arg\max_{\mathcal{A}\in\mathbf{A}|\mathcal{A}_i\in\{\emptyset,D_i(p;v)\}} \sum_{i\in\mathcal{N}} [\beta(e;p)]_{i,\mathcal{A}_i}$.*

For any price vector p and its corresponding demand sets $\mathcal{D} = (D_i(p))_{i\in\mathcal{N}}$ such that p is a quasi-CE, let:

$$H(p, \mathcal{D}) := \{p' \in \mathbb{R}_+^{2^G\times N} \mid \exists e \in \mathbb{R}_+^N \text{ an admissible discount with respect to p } : p' = \beta(e;p)\}.$$

We emphasize that the set $H(p, \mathcal{D})$ relies on bidders' valuations only through the demand sets $\mathcal{D}$. Let $\Gamma(p, \mathcal{D}, v)$ be the corresponding set of payoff vector for a given valuation profile $v \in \mathbb{R}_+^{2^G\times N}$, i.e. $\Gamma(p, \mathcal{D}; v) := \{h \in \mathbb{R}_+^N : \exists p' \in H(p, \mathcal{D})$ such that $h = \gamma(p')\}$. Hereafter, $H(p, \mathcal{D})$ and $\Gamma(p, \mathcal{D}; v)$ are respectively denoted by $H(p)$ and $\Gamma(p)$ to alleviate notation.

Lemma 4.1 *If p is a semi-truthful quasi-CE, then $\Gamma(p) = [\gamma(p)]^+ \cap CEP(\mathcal{N}, v)$.*[18]

Lemma 4.1 means also that the information imbedded in a given semi-truthful quasi-CE price vector p allows to implement all CE payoffs that are bigger (according to bidders' payoffs) than the payoff vector corresponding to this original CE price vector: this is done by using all admissible discounts with respect to p. In particular, we can implement a bidder-optimal CE payoff as it is done below with maximal discount rules.

Definition 13 A *maximal discount rule* δ is a function that assigns to any quasi-CE price vector p and its corresponding demand sets $\mathcal{D}$ the price vector $\delta(p, \mathcal{D})$ such that:

- $\delta(p, \mathcal{D}) \in H(p)$: there exists then $e \in \mathbb{R}_+^N$ such that $\delta(p, \mathcal{D}) = \beta(e;p) \in H(p)$.
- There is no $e^* \in \mathbb{R}_+^N$ such that $\beta(e^*;p) \in H(p)$ and $e_i^* \geq e_i$ for any $i \in \mathcal{N}$ and $e_i^* > e_i$ for some i.

A maximal discount rule is thus one that selects a payoff vector that is a strict bidder Pareto-optimum in the set $\Gamma(p)$. From lemma 4.1, we obtain thus that it selects equivalently a bidder-optimal competitive payoff.

$\beta(e;p) = \mathcal{P}(k + e)$ since $\max\{\max\{v_{i,H} - k_i, 0\} - e_i, 0\} = \max\{v_{i,H} - (k_i + e_i), 0\}$.

[18] For some payoff vector $\gamma = (\gamma_i)_{i=0,\ldots,N} \in \mathbb{R}^{N+1}$, let $[\gamma]^+ := \{\gamma' : \gamma_i' \geq \gamma_i \text{ for } i = 1, \ldots, N\}$.

Corollary 4.3 *For any maximal discount rule δ, if p is a quasi-CE semi-truthful price vector and $\mathcal{D}$ its corresponding demand sets, then $\gamma(\delta(p, \mathcal{D})) \in BOCE(\mathcal{N}, v)$.*

If $[\gamma(p)]^+ \cap BOCE(\mathcal{N}, v)$ is a singleton, then there is a unique candidate for $\delta(p, \mathcal{D})$. Otherwise, there are various candidates to be a solution. According to the desired properties on the selection rule (e.g. monotonicity), one can pick specific solutions as investigated similarly by the literature on transferable utility cooperative games (e.g. Arin and Inarra [1] and Dutta [17]).

We now establish the links between maximal discount rules that is the key practical innovation of the paper and the discount function that has been proposed by Mishra and Parkes [34]. For any quasi-CE price vector p, its corresponding demand sets $\mathcal{D}$ and for any $i \in \mathcal{N}$, let $\overline{e}_i(p, \mathcal{D}) := \max_{p' \in H(p), \mathcal{A} \in \mathbf{A}} p_{i,\mathcal{A}} - p'_{i,\mathcal{A}}$. In a nutshell, $\overline{e}_i(p, \mathcal{D})$ corresponds to the greatest payoff increase that bidder i may expect in a maximal discount rule from the quasi-CE p. Let $\overline{e}(p) = (\overline{e}_i(p))_{i=1,\ldots,N}$. From any semi-truthful CE price vector p and its corresponding demand sets $\mathcal{D}$, proposition 4.4 establishes that $\overline{e}(p)$ coincides with the discount function proposed by Mishra and Parkes [34]. Note that our construction gives a more interpretable definition of their discount function as the largest discount for a bidder such that the price vector remains a quasi-CE. Henceforth it is called the MP discount rule.

Proposition 4.4 *Consider a quasi-CE semi-truthful price vector p.*

The discount function δ_{MP} such that $\delta_{MP}(p, \mathcal{D}) = \beta(\overline{e}(p, \mathcal{D}); p)$ corresponds to the one proposed by Mishra and Parkes [34].

As a corollary, the price discounts in a maximal discount rule are smaller than in the MP discount rule. Furthermore, bidders' payoffs after the MP discount rule are smaller than the Vickrey payoffs: $\gamma_i(\delta_{MP}(p, \mathcal{D})) \leq \gamma_i^V$ for any $i \in \mathcal{N}$.

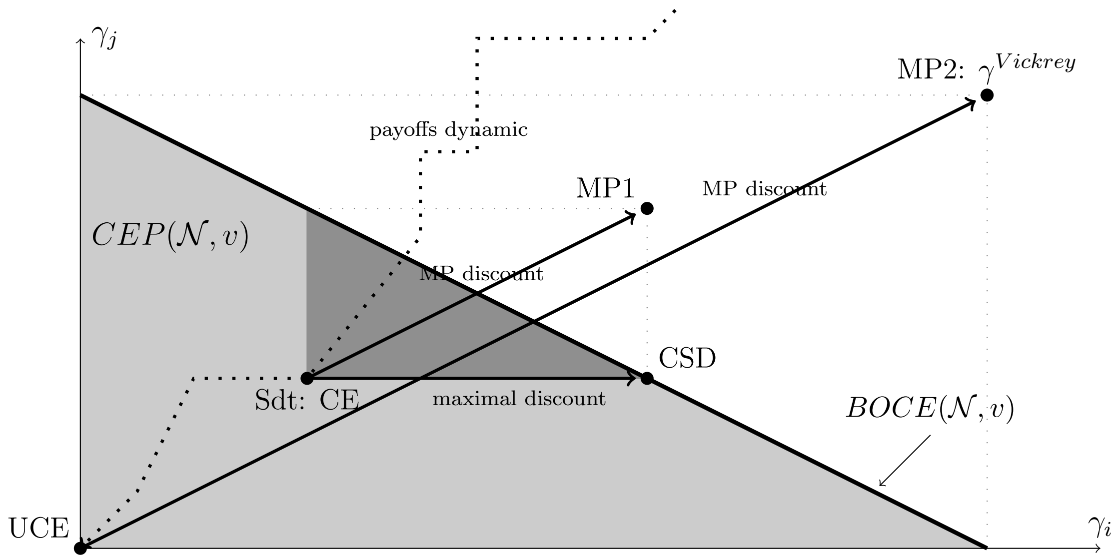

Figure 1

We extend the class of QCE-invariant auctions to allow a price discount stage after the auction dynamic stops.

Definition 14 A QCE-invariant ascending auction with ϵ-increments and with a maximal price discount rule [MP discount rule] δ is a QCE-invariant ascending auction with ϵ-increments with Step (S2) replaced by (S2(δ)) "The auction ends with the final assignment of the auction being any $\mathcal{A} \in L^*_{\mathcal{N}}(\delta(p^T))$ and the final payment of every bidder $i \in \mathcal{N}$ being $[\delta(p^T)]_{i,\mathcal{A}_i}$, where p^T is the final price vector of the auction and δ is a maximal price discount rule [MP discount rule]".

From corollary 4.3, the use of a maximal discount rule after a QCE-invariant auction yields thus a bidder-optimal CE payoff.

Proposition 4.5 *Assume integer valuations and truthful bidding. Any QCE-invariant ascending auction with unit increments and with a maximal price discount rule implements a bidder-optimal CE payoff vector.*

On the contrary, this result does not hold in general with the MP discount rule. A QCE-invariant ascending auction with unit increments and with a MP discount rule implements a payoff vector that is bigger than the one implemented in the auction where the MP discount rule has been replaced by a maximal price discount rule and smaller than the Vickrey payoff vector (Proposition 4.4). See Figure 1 for an illustration where the auctions' acronyms are those introduced in Table 2.

auction price discounts acronym	QCE no Sdt	QCE MP MP1	QCE max CSD	uQCE MP MP2
Minimal	YES			NO
Incentive compatible under general preferences	NO			YES
Incentive compatible if Vickrey is a CE payoff	NO	YES		
Incentive compatible if bidders are substitutes	YES			
Implements a CE payoff under general preferences	YES	NO	YES	NO

Table 2: Properties of the various ascending auctions

Previous ascending auctions in the literature that use combinatorial pricing systems (see Parkes [37], Ausubel and Milgrom [4] and de Vries et al. [14]) are QCE-invariant auctions that involve no discounts and guarantee incentive compatibility only on the domain where bidders are substitutes. On the contrary, the minimal ascending auction with discounts presented in proposition 4.5 guarantees incentive compatibility under the weaker condition that the Vickrey payoff vector is a competitive equilibrium payoff vector. According to such a perspective the format we propose is a strict improvement with respect to the literature while keeping the minimality property that captures the ascending flavor as summarized in Table 2. The last column corresponds to the auction proposed by Mishra and Parkes [34] to implement the Vickrey payoff vector under general valuations. It belongs thus to a larger class of auctions than QCE-invariant auctions (with possibly price discounts): it resorts to a non-minimal ascending auction. The rest of this section is devoted to a discussion on such auctions to implement some specific bidder-optimal payoff vectors.

4.1 Non-minimal ascending auctions

The notion of competitive equilibrium guarantees that an efficient assignment is known. To be able to compute the Vickrey payoff point, a more stringent condition is needed as developed in Mishra and Parkes [34] and Lahaie and Parkes [27]: the notion of universal competitive equilibrium (UCE) price which requires that the price vector is a CE price not only for the main economy but also for all marginal economies.

Definition 15 A price vector p is called a *universal competitive equilibrium* (UCE) [*universal quasi competitive equilibrium* (quasi-UCE)] price vector if p is a CE [quasi-CE] price vector of economy $E(S)$ for every $S \subseteq \mathcal{N}$ with $|S| \geq N - 1$.

As formalized in proposition 5.2 in next section, the computation of the Vickrey payoff vector with an ascending auction would require the violation of the minimality property. As a corollary, an ascending auction that ends in a UCE price vector for general valuations would violate the minimality property.

At this stage, we have tried solely to implement a bidder-optimal CE payoff and not a specific payoff vector in $BOCE(\mathcal{N}, v)$ according to some specific selection rule.[19] It brings us to move to the problem of determining the entire set of bidder-optimal CE payoffs from a given CE price p. In the same way as the price vector that implements the Vickrey payoff vector can be computed from a UCE price vector, we could conjecture that the set of bidder-optimal semi-truthful CE price vectors can be computed from a quasi-UCE semi-truthful price vector. This conjecture is not true under general valuations as shown by the following example.

Example Consider four bidders and three goods **a**,**b** and **c**. Consider that bidder 1 (resp. 2 and 3) values $V \geq 4$ any bundle containing the good **a** (resp. **b** and **c**) and 0 any other bundle. Consider that bidder 4 values 8 the bundle **abc** and 0 any other bundle. The efficient assignment is the one that gives the goods **a**, **b** and **c** respectively to bidders 1, 2 and 3. Consider then the semi-truthful price vector p such that $\gamma(p) = (12, V-4, V-4, V-4, 0)$, i.e. p is the semi-truthful price vector characterized by $p_{1,\mathbf{a}} = p_{2,\mathbf{b}} = p_{3,\mathbf{c}} = 4$ and $p_{4,\mathbf{abc}} = 8$. p is a UCE price vector for any $V \geq 4$. Nevertheless, the set of bidder-optimal semi-truthful CE payoff vectors depends strictly on V for $V \geq 4$. If $V \geq 8$, then the price vector p^* such that $p^*_{i,H} = 0$ for $i = 1, 2$ and any H, $p^*_{3,H} = 8$ if $\mathbf{c} \subseteq H$ and 0 otherwise and $p^*_{4,H} = p_{4,H}$ for any H is a bidder-optimal semi-truthful CE price vector. On the contrary, if $V < 8$, p^* is not a CE price vector.

Since the knowledge of the full set of bidder-optimal semi-truthful CE price vectors implies the knowledge of the semi-truthful price vector that implements the Vickrey payoff vector,[20] then it means that we need to explore bidders' preferences strictly more than with a UCE price vector. In other words, the computation of

[19] See Erdil and Klemperer [18] for a new proposal on this topic.

[20] For a given bidder, the Vickrey payoff corresponds to the upper bound of his payoffs among the set $BOCE(\mathcal{N}, v)$ (Bikhchandani and Ostroy [5]).

the entire set of bidder-optimal semi-truthful CE price vectors requires a greater violation of the minimality property than in Mishra and Parkes [34].

5 Ascending Vickrey auctions

As a corollary of propositions 2.1 and 4.5, we obtain a positive result for the implementation of the Vickrey payoff vector with some minimal ascending auctions under preferences such that the Vickrey payoff vector is a competitive equilibrium payoff vector.

Proposition 5.1 *Assume integer valuations and truthful bidding. In the class of joint preferences such that the Vickrey payoff vector is a competitive equilibrium payoff vector, any QCE-invariant ascending auction with unit increments and with a maximal price discount rule yields the Vickrey payoff vector.*

When the Vickrey payoff vector is a competitive equilibrium payoff vector, then the MP discount rule also yields the Vickrey payoffs (Proposition 4.4). Consequently, proposition 5.1 also holds for any QCE-invariant ascending auction with unit increments and with the MP discount rule. Nevertheless, as discussed in section 7, there are then robustness criteria that would make the use of a maximal price discount rule strictly profitable to the MP discount rule although they coincide on the (truthful) equilibrium path.

Mishra and Parkes [34]'s resolution of the impossibility to implement the Vickrey payoffs with a standard ascending auction relies on an ascending auction with discounts that fails to be minimal. Next result shows that this departure from one essential feature that makes ascending auctions desirable with respect to their sealed-bid counterpart can not be avoided.

Proposition 5.2 *Under general (integer) valuations, there is no minimal ascending auction that yields the Vickrey payoff vector under truthful bidding.*

We build an example with two goods and three bidders where only one bidder fails the gross substitutes condition. Contrary to proposition 4.1, the impossibility result naturally does not hold if the Vickrey payoff vector is guaranteed to be a competitive equilibrium where proposition 5.1 would apply.

6 The traps of increments

Though real-life auctions typically involve price increments that do not fit with the valuation grid, most of the literature on ascending auctions consider the restriction of integer valuations coupled with unit increments such that bidders' indifference curves can be fully recovered. In the environment with unit-demand, the ascending auction in Demange et al. [15] is shown to be robust to price increments: the approximate auction leads to an approximation of the Vickrey payoffs. In the same vein and for the simultaneous ascending auction under substitutable preferences, Milgrom [32] establishes a bound on the efficiency loss that depends linearly on the increment.[21] The starting point of this section is an impossibility result, proposition 6.1, that challenges the usual perspective that increments are adding just a noise that vanishes when they are chosen sufficiently small: with general valuations and for any standard ascending auction with ϵ-increments with $\epsilon > 0$, the efficiency loss does not vanish when the increment goes to zero. More precisely, this kind of discontinuity occurs when at least one bidder values the goods as complements.

Proposition 6.1 *Suppose that there are two goods $G = \{a, b\}$ and $N \geq 3$. Suppose one bidder's valuation function, say v_1, fails the gross substitutes condition. Then there exists a class of gross substitutes valuation functions for the other bidders, $(\mathrm{V}_j)_{2\leq j\leq N}$, such that under truthful bidding:*

there exists $\alpha > 0$ such that no standard ascending auction with ϵ-increments with $\epsilon > 0$ yields an assignment whose welfare α-approximates the welfare from an efficient assignment for each profile from $v_1 \times \mathrm{V}_2 \times \cdots \times \mathrm{V}_N$.

The proof which is relegated in appendix I relies on a very simple intuition. In order to outbid some bidders that have complementary preferences, the auctioneer has to raise the prices of several bidders. Some of those challenging bidders may prefer to quit the auction though the dynamic of the auction may reveal latter that such a bidder makes a strictly positive contribution to the welfare. This is exactly the same intuition as the one that drives the example in section 3: the auctioneer has pushed the prices too high in an inappropriate way. At first glance, this impossibility result could be viewed as an artifact of the present definition of an ascending auction

[21] See also Crawford and Knoer [11] and Kelso and Crawford [25] for continuity results with respect to the existence of a competitive equilibrium in respectively the unit demand case and general valuations satisfying the gross substitute condition.

where it is the auctioneer instead of the bidders themselves that raises the bids: if it were the bidders that raise their bids, then bidders would never quit the auction. However, as discussed at the end of this section, this problem gives some insights for the practical implementation of multi-object ascending auctions where bidders that do not bid actively have to exit the auction according to so-called activity rules that are used in real-life multi-object auctions.

The impossibility to approximate the welfare from an efficient assignment in proposition 6.1 crucially relies on the failure of substitutability. If bidders value the goods as substitutes, the simultaneous ascending auction considered by Milgrom [32] is a standard ascending auction with strictly positive increments that implements approximately an efficient assignment without any need of a combinatorial pricing system: see Theorem 2 in Milgrom [32] for a proper formalization. Nevertheless, this positive result under substitutable preferences does not mean that *any* 'natural' standard ascending auction becomes efficient when the increments vanish. The following example shows that QCE-invariant ascending auctions with ϵ-increments may fail to be efficient when the increment goes to zero even if bidders are substitutes.

Example Consider two identical bidders with additive preferences and two goods $\mathbf{a}, \mathbf{b}$ such that bidder 1 values the good $\mathbf{a}$ 10 and good $\mathbf{b}$ 5, while bidder 2 values the good $\mathbf{a}$ 5 and good $\mathbf{b}$ 10. In a QCE-invariant ascending auctions with ϵ-increments, the price dynamic takes the following form under truthful bidding: in a first step, only the prices on the bundle $\mathbf{ab}$ are rising. Bidders 1 and 2 alternate to be the winning bidder. In a second step, bidders are starting to bid on the individual goods. However, at each step the winning assignment remains one where both units are assigned to the same bidder, alternatively 1 and 2, such that when the losing bidder demand one good alone, then the opponent bidder demands only the bundle $\mathbf{ab}$ such that any assignment $L^*_{\mathcal{N}}(p^t)$ allocates both goods to one bidder. Finally, when the auction stops, the final assignment is never the efficient one that gives good $\mathbf{a}$ to bidder 1 and good $\mathbf{b}$ to bidder 2. Note also that bidders' payoff goes to 0 when ϵ goes to 0 while the corresponding Vickrey payoff equals 5.

In the above example, it is clear that truthful bidding is not an appealing strategy profile.[22]

Definition 16 A price vector p is called a *pseudo-CE* price vector (of the main

[22]See Börgers and Dustmann [7] for an empirical failure of truthful bidding.

economy) if there is an assignment $\mathcal{A} \in \mathbf{A}$ such that $\mathcal{A}_i \notin D_i(p;v)$ implies that $p_{i,\mathcal{A}_i} > 0$ for every bidder $i \in \mathcal{N}$ and such that $\mathcal{A} \in L_{\mathcal{N}}(p)$.

Remark 6.1 CE price vectors are obviously pseudo-CE. If a price vector is semi-truthful, then the converse holds since $p_{i,\mathcal{A}_i} > 0$ implies that $\mathcal{A}_i \in D_i(p;v)$.

Definition 17 A QCE-invariant ascending auction with ϵ-increments and with ϵ-discounts is a QCE-invariant ascending auction with ϵ-increments with Step (S1.2) replaced by "If p^t is a pseudo-CE price vector with respect to reported demand sets, then go to Step S2 with $T := t$", (S1.3) replaced by "Else, select a temporary winning assignment $\mathcal{A}^t \in L_{\mathcal{N}}(p^t)$ and a (non-empty)[23] set of temporary losers $\mathcal{L}^t \subset \mathcal{N}$ such that $\mathcal{A}_i^t \notin D_i(p^t;v)$ and $p^t_{i,\mathcal{A}_i^t}$ for any $i \in \mathcal{L}^t$ who will see a price increase" and (S2) replaced by "The auction ends with the final assignment of the auction being any $\mathcal{A}^T \in L_{\mathcal{N}}(p^T)$ and the final payment of every bidder $i \in \mathcal{N}$ being $p^T_{i,\mathcal{A}_i^T}$ if $\mathcal{A}_i^T \in D_i(p;v)$, $p^T_{i,\mathcal{A}_i^T} - \epsilon$ if $\mathcal{A}_i^T \notin D_i(p;v)$ and $p^T_{i,\mathcal{A}_i^T} > 0$ and 0 otherwise, where p^T is the final price vector of the auction".

Auctions with ϵ-discounts can be viewed 'roughly' as an auction where, at each step, the seller maximizes her revenue not solely according to the current set of bids but taking into account all previous submitted bids. While the notion of a quasi-CE requires that $\mathcal{A}_i^T \in D_i(p^T;v)$ at the last round T, the notion of a pseudo-CE requires the weaker condition that $\mathcal{A}_i^T \in D_i(p^t;v)$ for some round $t \leq T$, i.e. the bundle $\mathcal{A}_i^T$ has been demanded by bidder i in the auction history. A key point, that results from lemma 2.1, is that in such a case, if from the final price p^T the price for the bundle $\mathcal{A}_i^T$ is discounted by an increment ϵ, then the bundle $\mathcal{A}_i^T$ belongs to the demand set with respect to the final discounted price vector.

Remark 6.2 Under integer valuations and with unit increments, the price p^t is semi-truthful at each round t such that any QCE-invariant ascending auction with unit increments and with ϵ-discounts ends at a semi-truthful pseudo-CE price vector. From remark 6.1, it ends thus at a CE price vector such that the discount stage vanishes and the auction coincides thus with a QCE-invariant ascending auction with unit increments.

[23]This set is not empty. Otherwise, p^t would be a pseudo-CE price vector and the algorithm would have stopped in the previous stage (S1.2).

The following proposition shows how proposition 4.2 is robust to ϵ-increments if we allow ϵ-discounts. The other possibility results in sections 4 and 5 can be handled in the same way. Let $\overline{k}^T = (\overline{k}_i^T)_{i\in\mathcal{N}}$ denote the final bidder-payoff vector (after the ϵ-discounts). Note that $\overline{k}_i^T \in [k_i^T, k_i^T + \epsilon]$ for any $i \in \mathcal{N}$.

Proposition 6.2 *For any QCE-invariant ascending auction with ϵ-increments and with ϵ-discounts under truthful bidding:*

- *The final bidder-payoff vector $\overline{k}^T$ $[(N+1)\cdot\epsilon]$-approximates the set of weak bidder-optimal competitive CE payoffs $wBOCE(\mathcal{N}, v)$, i.e. there exists $\gamma \in wBOCE(\mathcal{N}, v)$ such that $|\gamma_i - \overline{k}_i^T| \leq (N+1)\cdot\epsilon$ for any $i \in \mathcal{N}$.*
- *The welfare at the final assignment $[N\cdot\epsilon]$-approximates the welfare from an efficient assignment.*
  As a corollary, for any vector of valuations $v \in \mathbb{R}_+^{2^G\times N}$, there exists ϵ^* such that any QCE-invariant ascending auction with ϵ-increments implements an efficient assignment if $\epsilon \leq \epsilon^*$.
- *If bidders are substitutes, then the final bidder-payoff vector $\overline{k}^T$ $[(N+1)\cdot\epsilon]$-approximates the Vickrey payoff vector.*

Activity Rules in combinatorial auctions[24] QCE-invariant ascending auctions with ϵ-increments are closely related to the combinatorial auction formats that have been proposed for some licenses by the FCC as discussed by the Public Notices DA 00-1075 [19] and DA 07-3415 [20]. The seemly differences between our class of ascending auctions and simultaneous ascending auctions as developed by the FCC are misleading: the former are clock auctions where bidders are reporting demand set while bidders are submitting bids in the latter. At first glance, it seems thus that the exit of a bidder that contributes strictly to the welfare cannot occur with sufficiently low increments. However, in those latter ascending formats, the need for activity rules as emphasized by Milgrom [32] could restore the issue: the typical

[24]For a rigorous formalization of activity rules, see Harsha et al. [23]. In particular, they introduce 'strong activity rules' which allow only reports that are compatible with some class of preferences (possibly larger than ours, e.g. allowing budget constraints). In our framework and with integer valuations and unit increments, it corresponds to impose to the bidders that their demand set should increase over time. We emphasize that such a restriction would be inappropriate for practical design in a perspective where valuations are not fixed but where bidders are refining their knowledge about their valuations in the process of the auction as in Compte and Jehiel [8, 9] or similarly if valuations were interdependent.

activity rule is to require that bidders that do not obtain any good in the temporary winning assignment have to submit an active bid in order to remain eligible to *stay* in the auction, i.e. such that all his previously submitted bids can be used by the auctioneer to maximize her revenue. Without any proper formalization, the F.C.C. report [19] emphasizes that the activity rules should take into account the intricacies of package bidding design: "Retained bids include the provisional winning bids, plus bids that have the potential to become provisional bids because of changes in other bids in subsequent rounds. Assuming that bids in the auction may only rise, bids that could never be winning bids are not retained". Indeed, such a rule does not define precisely the way to retain a bid and thus a bidder as active: it is not clear how a given bid cannot appear as a potential winning bid in subsequent rounds and thus that such an activity rule is really binding to give proper incentive to bid actively. An alternative interpretation of proposition 6.2 is a theoretical foundation for the following activity rule: bidders that do not obtain any good in the temporary winning assignment have to submit an active bid in order to remain eligible to *bid* in the auction but all bids that have been submitted by an inactive bidder, who is thus not eligible to submit additional bids in the auction, are remaining as active bids when the auctioneer seeks a provisional winning assignment. This is precisely the rule retained by the FCC [20] for the first combinatorial spectrum auction for the Block C of the 700 MHz licenses bands run in 2008 where losing eligibility does not mean the exit of the auction but corresponds to the inability to place additional bids. The FCC has followed the combinatorial design proposed and investigated experimentally by Goeree and Holt [21] where the set of possible packages is limited and tailored to some hierarchical structure that kills the computational issues that are traditionally associated with combinatorial auctions.

7 Conclusion

The main methodological contribution of the paper is to refine the definition of an ascending auction pioneered by Gul and Stacchetti [22] and then de Vries et al. [14] who introduced the idea of a linkage between final prices and bidders' monetary transfers. On the one hand, we relax this latter linkage by allowing a final price discount stage as in Mishra and Parkes [34]. On the other hand, we introduce the minimality property that establishes a linkage between demand revelation and how prices are rising. It allows us to revisit the issue of the implementation of the Vickrey payoff vector through an ascending auction. We have also considered the issue of the implementation of a bidder-optimal competitive equilibrium. In the case where the Vickrey payoff vector is a competitive equilibrium payoff, a case that has received a careful attention here because the discount stage we propose brings then a clear benefit with respect to the literature, the two implementation problems coincide. Contrary to Vickrey implementation that has a theoretical foundation in term of incentive compatibility, the foundation for the implementation of a bidder-optimal competitive equilibrium is less clear. Nevertheless, Day and Milgrom [13] argue that auctions that implement bidder-optimal competitive equilibrium payoffs are maximizing bidders' incentives for truthful reporting among those that implement competitive equilibrium payoffs and are thus robust to shill bidding contrary to Vickrey auctions as shown by Yokoo et al. [43].[25] We also emphasize the surprising insight that in the case where the Vickrey payoff vector is a competitive equilibrium payoff, then an auction that implements a bidder-optimal competitive equilibrium payoff may be strictly preferable to an auction that implements the Vickrey payoff vector though they both yield the same outcome under truthful bidding: the point is that those auctions differ outside the (truthful) equilibrium path and, e.g., the former are robust to shill bidding or to losing bidders' joint deviations while the latter are not.[26]

Coming back to our preliminary motivations that sustain our interest in developing ascending multi-object formats, we emphasize that the arguments in favor of ascending formats have been formalized mainly for the single-unit environment. A better understanding of how those arguments can be extended and possibly strengthened to multi-object environments is left for further research.

[25] Day and Milgrom [13]'s argument with regards to revenue monotonicity is not correct except when there are only two goods for sale as investigated by Lamy [29].

[26] See Lamy [28] for illustrations of such an insight under preferences involving allocative externalities.

References

[1] J. Arin and E. Inarra. Egalitarian solutions in the core. *International Journal of Game Theory*, 30:187–193, 2001.

[2] L. Ausubel. An efficient ascending-bid auction for multiple objects. *Amer. Econ. Rev.*, 94(5):1452–1475, 2004.

[3] L. Ausubel. An efficient dynamic auction for heterogenous commodities. *Amer. Econ. Rev.*, 96(3):602–629, 2006.

[4] L. Ausubel and P. Milgrom. Ascending auctions with package bidding. *Frontiers of Theoretical Economics*, 1(1), 2002.

[5] S. Bikhchandani and J. Ostroy. The package assignment model. *Journal of Economic Theory*, 107(2):377–406, 2002.

[6] S. Bikhchandani and J. Ostroy. Ascending price vickrey auctions. *Games and Economic Behavior*, 55:215–241, 2006.

[7] T. Börgers and C. Dustmann. Strange bids: Bidding behaviour in the united kingdom's third generation spectrum auction. *Economic Journal*, 115(505):551–578, 2005.

[8] O. Compte and P. Jehiel. The wait-and-see option in ascending price auctions. *Journal of the European Economic Association*, 2:494–503, 2004.

[9] O. Compte and P. Jehiel. Auctions and information acquisition: Sealed-bid or dynamic formats? *RAND J. Econ.*, 38(2):355–372, 2007.

[10] P. Cramton, Y. Shoham, and R. Steinberg. *Combinatorial Auctions*. MIT Press, 2006.

[11] V. Crawford and E. Knoer. Job matching with heterogeneous firms and workers. *Econometrica*, 49(2):437–50, 1981.

[12] G. Das Varma. Standard auctions with identity-dependent externalities. *RAND J. Econ.*, 33(4):689–708, 2002.

[13] R. Day and P. Milgrom. Core-selecting package auctions. *International Journal of Game Theory*, 36(3):393–407, 2008.

[14] S. de Vries, J. Schummer, and R. Vohra. On ascending vickrey auctions for heterogenous objects. *Journal of Economic Theory*, 132:95–118, 2007.

[15] G. Demange, D. Gale, and M. Sotomayor. Multi-item auctions. *Journal of Political Economy*, 94(4):863–872, August 1986.

[16] J. Dubra, F. Echenique, and A. Manelli. English auctions and the stolper-samuelson theorem. *Journal of Economic Theory*, 144(2):825–849, 2009.

[17] B. Dutta. The egalitarian solution and reduced game properties in convex games. *International Journal of Game Theory*, 19:153–69, 1990.

[18] A. Erdil and P. Klemperer. A new payment rule for core-selecting package auctions. September 2009.

[19] F.C.C. (federal communications commission) comment sought on modifying the simultaneous multiple round auction design to allow combinatorial (package) bidding. Public Notice, May 2000.

[20] F.C.C. (federal communications commission) comment sought on competitive bidding procedures for auctiion 73. Public Notice, August 2007.

[21] J. K. Goeree and C. A. Holt. Hierarchical package bidding: A paper & pencil combinatorial auction. *Games and Economic Behavior*, In Press, Corrected Proof, 2009.

[22] F. Gul and E. Stacchetti. The english auction with differentiated commodities. *Journal of Economic Theory*, 92:66–95, 2000.

[23] P. Harsha, C. Barnhart, D. Parkes, and H. Zhang. Strong activity rules for iterative combinatorial auctions. *M.I.T. Operations Research Center Working Paper*, 2009.

[24] P. Jehiel and B. Moldovanu. *Advances in economics and econometrics*, chapter 3 Allocative and informational externalities in auctions and related mechanisms, pages 102–135. Cambridge University Press, 2006.

[25] A. Kelso and V. Crawford. Job matching, coalition formation, and gross substitutes. *Econometrica*, 50(6):1483–1504, 1982.

[26] V. Krishna. Asymmetric english auctions. *Journal of Economic Theory*, 112(2):261–288, 2003.

[27] S. Lahaie and D. C. Parkes. On the communication requirements of verifying the vcg outcome. *Proceedings ACM Conference on Electronic Commerce*, 2008.

[28] L. Lamy. Buyer substitutability with allocative externalities.

[29] L. Lamy. Core-selecting package auctions: a comment on revenue-monotonicity. *International Journal of Game Theory*, forthcoming, 2009.

[30] H. Leonard. Elicitation of honest preferences for the assignment of individuals to positions. *Journal of Political Economy*, 91(3):461–79, 1983.

[31] C. Mezzetti. Mechanism design with interdependent valuations: Efficiency. *Econometrica*, 72(5):1617–1626, 2004.

[32] P. Milgrom. Putting auction theory to work: The simultaneous ascending auction. *Journal of Political Economy*, 108(2):245–272, 2000.

[33] P. Milgrom and R. Weber. A theory of auctions and competitive bidding. *Econometrica*, 50:1089–1122, 1982.

[34] D. Mishra and D. C. Parkes. Ascending price vickrey auctions for general valuations. *Journal of Economic Theory*, 132:335–366, 2007.

[35] D. Mishra and D. C. Parkes. Multi-item vickrey-dutch auctions. *Games and Economic Behavior*, 66(1):326–347, 2009.

[36] B. Moldovanu and M. Tietzel. Goethe's second-price auction. *Journal of Political Economy*, 106(4):854–859, 1998.

[37] D. C. Parkes. ibundle: An efficient ascending price bundle auction. *Proceedings ACM Conference on Electronic Commerce*, pages 148–157, 1999.

[38] M. Perry and P. Reny. An efficient auction. *Econometrica*, 70(3):1199–1212, 2002.

[39] M. Perry and P. J. Reny. An efficient multi-unit ascending auction. *Review of Economic Studies*, 72:567–592, 2005.

[40] T. Rockafellar. *Convex Analysis*. Princeton University Press, 1970.

[41] M. Rothkopf, T. Teisberg, and E. Kahn. Why are vickrey auctions rare? *Journal of Political Economy*, 98(1):94–109, 1990.

[42] L. Ye. Indicative bidding and a theory of two-stage auctions. *Games and Economic Behavior*, 58(1):181–207, 2007.

[43] M. Yokoo, Y. Sakurai, and S. Matsubara. The effect of false-name bids in combinatorial auctions: new fraud in internet auctions. *Games and Economic Behavior*, 46:174–188, 2004.

Appendix

A Proof of lemma 2.1

The inequality $v_{i,\mathcal{A}_i} - p^t_{i,\mathcal{A}_i} \leq k^t_i$ comes from the definition of the demand set of bidder i. It remains to show that $p^t_i(\mathcal{A}_i) > 0$ implies that $k^t_i - \epsilon < v_{i,\mathcal{A}_i} - p^t_{i,\mathcal{A}_i}$. Suppose by contradiction that $p^t_i(\mathcal{A}_i) > 0$ and $p^t_{i,\mathcal{A}_i} \geq v_{i,\mathcal{A}_i} - k^t_i + \epsilon$. From time t's perspective, consider the last round t^* where the price on assignment $\mathcal{A}_i$ has been raised for bidder i (such a round exists since $p^t_i(\mathcal{A}_i) > 0$): we have then $p^{t^*}_{i,\mathcal{A}_i} = p^t_{i,\mathcal{A}_i} - \epsilon \geq v_{i,\mathcal{A}_i} - k^t_i > v_{i,\mathcal{A}_i} - k^{t^*}_i$ where the equality comes from the definition of t^* and the last inequality comes from Step (1.4) where all bundles that belong to the demand set at t^* see a price increase such that optimal profit at t are strictly lower than at t^*. Finally, we obtain that $\mathcal{A}_i \notin D_i(p^{t^*})$ which raises a contradiction with Step (S1.4) where only prices for assignments in the demand set are raised.

B Proof of lemma 2.3

Consider p a price vector that is semi-truthful and a quasi-CE. We show that $L^*_{\mathcal{N}}(p) \subseteq L_{\mathcal{N}}(p)$ such that p is thus a CE. Pick $\mathcal{A} \notin L_{\mathcal{N}}(p)$. There exists thus $\overline{\mathcal{A}} \in L_{\mathcal{N}}(p)$ such that $\sum_{i\in\mathcal{N}} p_{i,\overline{\mathcal{A}}_i} > \sum_{i\in\mathcal{N}} p_{i,\mathcal{A}_i}$. We then build $\mathcal{A}^* \in \mathbf{A}$ such that, for any $i \in \mathcal{N}$, $\mathcal{A}^*_i := \emptyset$ if $p_{i,\overline{\mathcal{A}}_i} = 0$ and $\mathcal{A}^*_i := \overline{\mathcal{A}}_i$ otherwise. Since p is semi-truthful we have $\mathcal{A}^*_i \in \{\emptyset, D_i(p;v)\}$ for all $i \in \mathcal{N}$. Furthermore $\sum_{i\in\mathcal{N}} p_{i,\mathcal{A}^*_i} = \sum_{i\in\mathcal{N}} p_{i,\overline{\mathcal{A}}_i}$. Finally we obtain $\sum_{i\in\mathcal{N}} p_{i,\mathcal{A}^*_i} > \sum_{i\in\mathcal{N}} p_{i,\mathcal{A}_i}$ and thus $\mathcal{A} \notin L^*_{\mathcal{N}}(p)$. $L^*_{\mathcal{N}}(p) \subseteq L_{\mathcal{N}}(p)$ then implies the equality (1).

C Price path under infinitesimal increments

D Proof of proposition 4.1

Let A, $\{B_k\}_{k=1,\cdots,5}$, $\{C_k\}_{k=1,\cdots,5}$ denote the goods. Each bidder $i = 1,\cdots,5$ values $\overline{V} > 0$ any bundle that contains the goods A, B_i and C_i, while any alternative bundle is valued either $\underline{V}$ or 0 with $\overline{V} < 2\cdot\underline{V}$. In the neighborhood of the null prices, the demand set of each bidder i is thus known ex ante: it corresponds to

Table 3: Progress of the QCE-invariant ascending auction with infinitesimal increments

	bidder 1		bidder 2		bidder 3		bidder 4	
	One	Two	One	Two	One	Two	One	Two
Values →	10	10	4	8	4	8	0	7
↓ Phases								
1	(0)	(0)	0	(0)	0	(0)	0	(0)
	Agents are bidding on the most preferred assignments 3/4 of the time while their most preferred assignments are selected 1/4 of the time							
2	(4)	(4)	(0)	(4)	(0)	(4)	0	(4)
	Agent 1 is bidding on the most preferred assignment 1/5 of the time Agents 2 and 3 are bidding on the most preferred assignment 3/5 of the time Agent 4 is bidding on the most preferred assignment 4/5 of the time 1/5 of the time: two items are assigned to agent 4 2/5 of the time: one item is assigned to agents 1 and 2 2/5 of the time: one item is assigned to agents 1 and 3							
3	(4.75)	(4.75)	(2.25)	(6.25)	(2.25)	(6.25)	(0)	(7)
	Agents 2 and 3 are bidding on the most preferred assignment 1/2 of the time 1/2 of the time: one item is assigned to agents 1 and 2 1/2 of the time: one item is assigned to agents 1 and 3							
4	(4.75)	(4.75)	(4)	(8)	(4)	(8)	(0)	(7)
	Selected assignment: bidders 1 and 3' receive one item. The revenue is 8.75. A CE price vector is reached.							
Final payoffs: $(8.75, 5.25, 0, 0, 0)$; Vickrey payoff vector: $(8, 6, 0, 0, 0)$								

the subset of the bundles with the smallest prices among the set of the bundles that contain the goods A, B_i and C_i. Furthermore, from the symmetry between the bidders, it is sufficient to argue that raising strictly the minimum price from the bundles that contain the goods A, B_1 and C_1 for bidder 1 may prevent the implementation of a payoff in $BOCE(\mathcal{N}, v)$ since the preferences of the remaining bidders may imply that the monetary transfer of bidder 1 should be null for any payoff in $BOCE(\mathcal{N}, v)$. More precisely, we exhibit a realization of the preferences such that the set $BOCE(\mathcal{N}, v)$ is a singleton and then corresponds to the Vickrey payoff vector. Among the remaining bundles that do not contain the goods A, B_1 and C_1, bidder 1 is valuing all bundles 0. Among the remaining bundles that do not contain the goods A, B_2 and C_2, suppose that bidder 2 values $\underline{V}$ the bundles containing the bundle $G_2 = \{B_3, B_4, B_5, C_4\}$. Similarly, bidder 3 values $\underline{V}$ the bundles containing the bundle $G_3 = \{B_2, B_4, B_5, C_5\}$, bidder 4 values $\underline{V}$ the bundles containing the bundle $G_4 = \{C_2, C_3, C_5, B_2\}$ and finally bidder 5 values $\underline{V}$ the bundles containing the bundle $G_5 = \{C_2, C_3, C_4, B_3\}$. Below we assume that i, j, k and l belong to $\{2, \cdots, 5\}$ and are distinct. Note that $G_2 \cap G_4 = \emptyset$, $G_3 \cap G_5 = \emptyset$ and that, any other couples i, j, we have $G_i \cap G_j \neq \emptyset$. We have also $G_i \cap \{A, B_j, C_j\} \neq \emptyset$.

The characteristic function corresponding to such preferences is given by: $w(\{1\}) = w(\{i\}) = \overline{V}$; $w(\{1, i\}) = \overline{V} + \underline{V}$ $w(\{i, j\}) = 2\underline{V}$ if $\{i, j\} = \{2, 4\}$ or $\{i, j\} = \{3, 5\}$ otherwise $w(\{i, j\}) = \overline{V}$; $w(\mathcal{N} \setminus \{1, i\}) = 2\underline{V}$, $w(\mathcal{N} \setminus \{i, j\}) = \overline{V} + 2\underline{V}$ if $\{i, j\} = \{2, 4\}$ or $\{i, j\} = \{3, 5\}$ otherwise $w(\mathcal{N} \setminus \{i, j\}) = \overline{V} + \underline{V}$; $w(\mathcal{N} \setminus \{1\}) = 2\underline{V}$ and $w(\mathcal{N} \setminus \{i\}) = \overline{V} + 2\underline{V}$ and finally $w(\mathcal{N}) = \overline{V} + 2\underline{V}$. The Vickrey payoff vector is $(2\underline{V}, \overline{V}, 0, 0, 0, 0)$: bidder 1 pays a null price for the bundle $\{A, B_1, C_1\}$. From proposition 2.1 it remains to check that this payoff is in the Core by checking the coalitional constraints from the expression of w.

E Proof of proposition 4.2

First note that the auction dynamic does not stop if a CE has not been reached since from lemma 2.3 it would imply that we have not reached a quasi-CE because prices are semi-truthful along the price path. Second, any CE is a quasi-CE such that the auction dynamic stops once a CE outcome has been reached.

So we are sure to end in the set of CE payoffs. Let p the final CE where the auction stops. Suppose that the final payoff $(\gamma_i(p))_{i \in \mathcal{N}}$ does not $1-$approximates $wBOCE(\mathcal{N}, v)$. We now show that it would imply that the payoff vector γ' where $\gamma'_0 = \gamma_0(p) - N$ and $\gamma'_i = \gamma_i(p) + 1$ for any $i \in \mathcal{N}$ belongs to $CEP(\mathcal{N}, v)$ or equivalently belongs to the Core. Suppose that it is not the case. From standard convex analysis (see in Rockafellar [40]), the Core is a polyhedral convex set and there exists thus a hyperplane separating the Core and the point γ'. Thus there is a point in the interval[27] $[\gamma(p), \gamma']$ which belongs to the weak bidder-optimal frontier raising thus a contradiction with $(\gamma_i(p))_{i \in \mathcal{N}}$ not $1-$approximating the set $wBOCE(\mathcal{N}, v)$. Finally we have proved that the outcome γ' is in the Core. Then as pointed by remark 2.1, the whole cube $C = \left\{z \in \mathbb{R}^{N+1} | \gamma_i(p) \leq z_i \leq \gamma'_i\right\}$ is included in the Core or following remark 2.2, equivalently, the set of semi-truthful prices $C_{\mathcal{P}} = \{p \in \mathbb{R}_+^{2^G \times N} \mid \exists \gamma \in C : p = \mathcal{P}(\gamma)\}$ is included in $CEP(\mathcal{N}, v)$. In the previous round of any QCE-invariant ascending auction with 1-increments, the state of the price vector in the algorithm

[27]For $x, y \in \mathbb{R}^{N+1}$ with $x \leq y$, let $[x, y]$ denote the set $\{z \in \mathbb{R}^{N+1} \mid \exists \lambda \in [0, 1] : z = \lambda \cdot x + (1-\lambda) \cdot y\}$.

was necessary in $C_{\mathcal{P}}$, which raises a contradiction with the aforementioned point that the auction dynamic stops once a CE has been reached.

Remark Note that if we choose the sets of temporary losers in the steps (S1.3) such that they are singletons[28], then the final payoff vector is exactly a weak bidder-optimal CE payoff vector.

The result when bidders are substitutes can be obtained by following the proof of de Vries et al. [14]'s Theorem 4. Note that this is exactly what we will do in Appendix J but under an additional ingredient: general increments.

F Proof of lemma 4.1

Consider a semi-truthful price p. Let $\gamma \in \mathbb{R}_+^N$ such that $p = \mathcal{P}(\gamma)$, i.e. $\gamma_i = \gamma_i(p)$ for any $i \in \mathcal{N}$. As a preliminary, note that we have $D_i(\mathcal{P}(\gamma); v) = \{H \subseteq G \mid v_{i,H} \geq \min\{\gamma_i, v_{i,G}\}\}$. As a corollary, under semi-truthful prices, when prices shrink then the demand set can only shrink. In particular, if $\emptyset \notin D_i(p; v)$ then $\emptyset \notin D_i(\beta(e; p); v)$ for any $e \in \mathbb{R}_+^N$.

We first show the inclusion $\Gamma(p) \subseteq [\gamma(p)]^+ \cap CEP(\mathcal{N}, v)$. First $\Gamma(p) \subseteq [\gamma(p)]^+$ since bidder payoffs increase after prices discounts. To obtain the inclusion $\Gamma(p) \subseteq CEP(\mathcal{N}, v)$, we show that if p a semi-truthful quasi-CE, then $[p' \in H(p) \Rightarrow p' \in CEP(\mathcal{N}, v)]$. Consider $p' \in H(p)$. So there exists $e \in \mathbb{R}_+^N$ and an assignment $\mathcal{A}^*$ such that $p' = \beta(e; p)$, $e_i \leq p_{i,\mathcal{A}_i^*}$, $\mathcal{A}^* \in \operatorname{Arg\,max}_{\mathcal{A} \in \mathbf{A} | \mathcal{A}_i \in \{\emptyset, D_i(p)\}} \sum_{i \in \mathcal{N}} p'_{i,\mathcal{A}_i}$ and $(p, \mathcal{A}^*)$ is a quasi-CE. We have $\mathcal{A}_i^* \in D_i(p)$ and $e_i \leq p_{i,\mathcal{A}_i^*}$ which guarantees thus that $\mathcal{A}_i^* \in D_i(\beta(e; p))$ since p is semi-truthful. $D_i(p'; v) \subseteq D_i(p; v)$ implies that $\max_{\mathcal{A} \in \mathbf{A} | \mathcal{A}_i \in \{\emptyset, D_i(p)\}} \sum_{i \in \mathcal{N}} p'_{i,\mathcal{A}_i} \geq \max_{\mathcal{A} \in \mathbf{A} | \mathcal{A}_i \in \{\emptyset, D_i(p')\}} \sum_{i \in \mathcal{N}} p'_{i,\mathcal{A}_i}$ and thus that $\mathcal{A}^* \in \operatorname{Arg\,max}_{\mathcal{A} \in \mathbf{A} | \mathcal{A}_i \in \{\emptyset, D_i(p')\}} \sum_{i \in \mathcal{N}} p'_{i,\mathcal{A}_i}$, i.e. $\mathcal{A}^* \in L_{\mathcal{N}}^*(p')$. Finally, we have shown that $(p', \mathcal{A}^*)$ is a quasi-CE and thus with lemma 2.3 we conclude that p is a CE price vector.

Then it remains to show the inclusion $[\gamma(p)]^+ \cap CEP(\mathcal{N}, v) \subseteq \Gamma(p)$ holds for any semi-truthful vector p. Consider $\gamma^* \in [\gamma(p)]^+ \cap CEP(\mathcal{N}, v)$ or equivalently $\gamma^* \in [\gamma(p)]^+ \cap Core(\mathcal{N}, v)$. It is then sufficient to show that $\mathcal{P}(\gamma^*) \in H(p)$ in order to obtain $\gamma^* \in \Gamma(p)$. By means of standard calculations, we have for any $\gamma \in \mathbb{R}_+^N$

[28] From a practical perspective where the pace of the auction is an important issue as in spectrum auctions, it would slow the auction dynamic.

with $\gamma_i \le v_{i,G}$ for any $i \in \mathcal{N}$:

$$\begin{aligned}\max_{\mathcal{A}\in\mathbf{A}} \sum_{j\in\mathcal{N}} [\mathcal{P}(\gamma)]_{j,\mathcal{A}_j} &= \max_{\mathcal{A}\in\mathbf{A}} \sum_{j\in\mathcal{N}} \max\{0, v_{j,\mathcal{A}_j} - \gamma_j\} \\ &= \max_{S\subseteq\mathcal{N}} \max_{\mathcal{A}\in\mathbf{A}} [\sum_{j\in S} v_{j,\mathcal{A}_j} - \gamma_j] \\ &= \max_{S\subseteq\mathcal{N}} [w(S) - \sum_{j\in S} \gamma_j].\end{aligned}$$

Furthermore, if $\gamma^* \in Core(\mathcal{N}, v)$, then $S = \mathcal{N}$ is a solution of the maximization program $\max_{S\subseteq\mathcal{N}} [w(S) - \sum_{j\in S} \gamma_j^*]$ or equivalently from the calculation above there exists $\mathcal{A}^*$ with $\mathcal{A}_i^* \in D_i(\mathcal{P}(\gamma^*); v)$ for any $i \in \mathcal{N}$ which is a solution of the maximization program $\max_{\mathcal{A}\in\mathbf{A}} \sum_{j\in\mathcal{N}} [\mathcal{P}(\gamma^*)]_{j,\mathcal{A}_j}$. Consider the vector $e = \gamma^* - \gamma(p) \in \mathbb{R}_+^N$ (since $\gamma^* \in [\gamma(p)]^+$) and the assignment $\mathcal{A}^*$. First we have $\mathcal{P}(\gamma^*) = \beta(e, p)$ from the definition of the function $\mathcal{P}(.)$. Second, $\mathcal{A}_i^* \in D_i(p; v)$ since $D_i(\mathcal{P}(\gamma^*); v) \subseteq D_i(\mathcal{P}(\gamma(p)); v) = D_i(p; v)$. Third, $\mathcal{A}_i^* \in D_i(p; v) \cap D_i(\mathcal{P}(\gamma^*); v)$ implies that $v_{i,\mathcal{A}_i^*} = p_{i,\mathcal{A}_i^*} + [\gamma(p)]_{i,\mathcal{A}_i^*} = [\mathcal{P}(\gamma^*)]_{i,\mathcal{A}_i^*} + [\gamma^*]_{i,\mathcal{A}_i^*}$ and thus $e_i = p_{i,\mathcal{A}_i^*} - [\mathcal{P}(\gamma^*)]_{i,\mathcal{A}_i^*} \le p_{i,\mathcal{A}_i^*}$. Fourth $\mathcal{A}^* \in \max_{\mathcal{A}\in\mathbf{A}} \sum_{i\in\mathcal{N}} [\mathcal{P}(\gamma^*)]_{i,\mathcal{A}_i} = \max_{\mathcal{A}\in\mathbf{A}|\mathcal{A}_i\in\{\emptyset, D_i(p;v)\}} \sum_{i\in\mathcal{N}} [\mathcal{P}(\gamma^*)]_{i,\mathcal{A}_i}$ where the last equality is satisfied because $\mathcal{A}_i^* \in D_i(p; v)$ for any $i \in \mathcal{N}$. Last, for any $\mathcal{A} \in \mathbf{A}$ we have $\sum_{i\in\mathcal{N}} [\mathcal{P}(\gamma(p))]_{i,\mathcal{A}_i} \le \sum_{i\in\mathcal{N}} [\mathcal{P}(\gamma^*)]_{i,\mathcal{A}_i} + \sum_{i\in\mathcal{N}} e_i$ and thus $\max_{\mathcal{A}\in\mathbf{A}} \sum_{i\in\mathcal{N}} [\mathcal{P}(\gamma(p))]_{i,\mathcal{A}_i} \le \max_{\mathcal{A}\in\mathbf{A}} \sum_{i\in\mathcal{N}} [\mathcal{P}(\gamma^*)]_{i,\mathcal{A}_i} + \sum_{i\in\mathcal{N}} e_i$ while we have $\sum_{i\in\mathcal{N}} [\mathcal{P}(\gamma(p))]_{i,\mathcal{A}_i^*} = \sum_{i\in\mathcal{N}} [\mathcal{P}(\gamma^*)]_{i,\mathcal{A}_i^*} + \sum_{i\in\mathcal{N}} e_i = \max_{\mathcal{A}\in\mathbf{A}} \sum_{i\in\mathcal{N}} \mathcal{P}(\gamma^*) + \sum_{i\in\mathcal{N}} e_i$. Finally we obtain that $\mathcal{A}^* \in L_{\mathcal{N}}^*(p)$. Gathering those last five points we have shown precisely that $\mathcal{P}(\gamma^*) \in H(p)$ which completes the proof.

G Proof of proposition 4.4

The price discount rule defined by Mishra and Parkes [34] (p. 148) for bidder i at a quasi-CE price vector p is given by:

$$\overline{e}_i^{MP}(p) := \max_{\mathcal{A}\in\mathbf{A}} \sum_{j\in\mathcal{N}} p_{j,\mathcal{A}_j} - \max_{\mathcal{A}\in\mathbf{A}} \sum_{j\in\mathcal{N}\setminus\{i\}} p_{j,\mathcal{A}_j}.$$

We now show that $\overline{e}_i^{MP}(p) = \overline{e}_i(p)$ for any $i \in \mathcal{N}$ if p is a semi-truthful CE price vector. Let e^i denote the vector in $\mathbb{R}_+^N$ such that $e_j^i = 0$ if $j \ne i$ and $e_i^i = 1$. In a similar calculation as one lead in appendix F, for any scalar $\lambda \ge 0$ we have by means of equation (1) and standard calculations:

$$\begin{aligned}\max_{\mathcal{A}\in\mathbf{A}|\mathcal{A}_j\in\{\emptyset,D_j(p;v)\}}\sum_{j\in\mathcal{N}}[\beta(\lambda e^i;p)]_{j,\mathcal{A}_j} = \max_{\mathcal{A}\in\mathbf{A}}\sum_{j\in\mathcal{N}}[\beta(\lambda e^i;p)]_{j,\mathcal{A}_j} &= \gamma_0(\beta(\lambda e^i;p)) \\ &= \max_{S\subseteq\mathcal{N}}[w(S)-\sum_{j\in S}\gamma_j(\beta(\lambda e^i;p))] \\ = \max\{\max_{S\subseteq\mathcal{N},i\notin S}[w(S)-\sum_{j\in S}\gamma_j(p)],\max_{S\subseteq\mathcal{N},i\in S}[w(S)-\sum_{j\in S}\gamma_j(p)]-\lambda\} \\ = \max\{\max_{\mathcal{A}\in\mathbf{A}}\sum_{j\in\mathcal{N}\setminus\{i\}}p_{j,\mathcal{A}_j},(\max_{\mathcal{A}\in\mathbf{A}}\sum_{j\in\mathcal{N}}p_{j,\mathcal{A}_j})-\lambda\}\end{aligned}$$

The positive scalar $\overline{e}_i(p)$ is precisely defined as the greatest scalar $\lambda \geq 0$ such that there is $\mathcal{A} \in \mathbf{A}$ with $\mathcal{A} \in Arg\max_{\mathcal{A}\in\mathbf{A}|\mathcal{A}_j\in\{\emptyset,D_j(p;v)\}}\sum_{j\in\mathcal{N}}[\beta(\lambda e^i;p)]_{j,\mathcal{A}_j}$, $\mathcal{A} \in Arg\max_{\mathcal{A}\in\mathbf{A}}\sum_{j\in\mathcal{N}}p_{j,\mathcal{A}_j}$ and with $\mathcal{A}_j \in D_j(p;v)$ for any j. Thus we have for any $i\in\mathcal{N}$:

$$\max_{\mathcal{A}\in\mathbf{A}|\mathcal{A}_j\in\{\emptyset,D_j(p;v)\}}\sum_{j\in\mathcal{N}}[\beta(\overline{e}_i(p)e^i;p)]_{j,\mathcal{A}_j} = \max_{\mathcal{A}\in\mathbf{A}}\sum_{j\in\mathcal{N}}p_{j,\mathcal{A}_j}-\overline{e}_i(p).$$

Finally we obtain from the above calculation that $\overline{e}_i(p) = \overline{e}_i^{MP}(p)$. From the definition of $\overline{e}_i(p)$, the price discounts in a maximal discount rule are smaller than in the MP discount rule.

It remains to show that bidders' payoffs after the MP discount rule are smaller than Vickrey payoffs: $\gamma_i(\delta_{MP}(p,\mathcal{D})) \leq \gamma_i^V$ for any $i\in\mathcal{N}$. Let $\mathcal{A}^*$ be an assignment such that $(p,\mathcal{A}^*)$ is a quasi-CE and thus a CE since p is semi-truthful. The inequality above is thus equivalent to:

$$\overbrace{v_{i,\mathcal{A}_i^*}-p_{i,\mathcal{A}_i^*}+\sum_{j\in\mathcal{N}}p_{j,\mathcal{A}_j^*}-\max_{\mathcal{A}\in\mathbf{A}}\{\sum_{j\in\mathcal{N}\setminus\{i\}}p_{j,\mathcal{A}_j}\}}^{=\gamma_i(p)+\overline{e}_i^{MP}(p)} \leq \overbrace{\sum_{j\in\mathcal{N}}v_{j,\mathcal{A}_j^*}-\max_{\mathcal{A}\in\mathbf{A}}\{\sum_{j\in\mathcal{N}\setminus\{i\}}v_{j,\mathcal{A}_j}\}}^{=\gamma_i^V}.$$

Since $\mathcal{A}_i^* \in D_i(p;v)$ for any $i\in\mathcal{N}$ this is also equivalent to

$$\sum_{j\in\mathcal{N}\setminus\{i\}}[\gamma(p)]_i+\max_{\mathcal{A}\in\mathbf{A}}\{\sum_{j\in\mathcal{N}\setminus\{i\}}p_{j,\mathcal{A}_j}\} \geq \max_{\mathcal{A}\in\mathbf{A}}\{\sum_{j\in\mathcal{N}\setminus\{i\}}v_{j,\mathcal{A}_j}\}.$$

This last inequality holds since $[\gamma(p)]_i + p_{j,\mathcal{A}_j} \geq v_{j,\mathcal{A}_j}$ for any $j\in\mathcal{N}$ and $\mathcal{A}\in\mathbf{A}$.

H Proof of proposition 5.2

Consider two heterogeneous goods a and b. Let V_1 denote the set of preferences such that $v_{1,a} = v_{1,b} = 0$ and $v_{1,ab} = x_1 \in \mathbb{N}$ such that bidder 1's valuation function fails the gross substitutes condition if $x_1 > 0$. Let V_2 [resp. V_3] denote the set of preferences such that $v_{2,ab} = v_{2,a} = x_2 \in \mathbb{N}$ and $v_{2,b} = 0$ [resp. $v_{3,ab} = v_{3,a} = x_3 \in \mathbb{N}$ and $v_{3,b} = 0$]. Bidders' valuations are reduced to the three integers x_1, x_2 and x_3. We show that there is no minimal ascending auction that yields the Vickrey payoffs on the domain $V_1 \times V_2 \times V_3$. Consider a moment in time t where the demand sets of bidders 1, 2 and 3 do not contain the empty set. Since the price path is ascending while the price for the empty bundle remains zero in a minimal auction, then the demand sets of bidders 1, 2 and 3 do not contain the empty set at any time $t' \leq t$. At such a moment in time, the unique information we have with regards to bidders' valuation functions is: $x_1 > P_{1,ab}(t)$, $x_2 > P_{2,a}(t)$ and $x_3 > P_{3,b}(t)$. In a minimal auction, the provisional revenue raised by an assignment in $L^*_{\mathcal{N}}(P(t))$ is either $P_{1,ab}(t)$ if the assignment that allocates both goods to bidder 1 belongs to $L^*_{\mathcal{N}}(P(t))$ or $P_{2,a}(t) + P_{3,b}(t)$ if the assignment that allocates a to bidder 2 and b to bidder 3 belongs to $L^*_{\mathcal{N}}(P(t))$. We now show that in any minimal ascending auction that yields the Vickrey payoffs the inequalities $P_{1,ab}(t) - 2 \leq P_{2,a}(t) + P_{3,b}(t) \leq P_{1,ab}(t) + 4$ should be satisfied at any time t such that the demand sets of bidders 1, 2 and 3 do not contain the empty set for any time $t' < t$. Suppose on the contrary that one of those inequalities fails to hold.

First, suppose that $P_{1,ab}(t) > P_{2,a}(t) + P_{3,b}(t) + 2$. If bidder 1's prices have never been raised, then we have necessarily $P_{1,ab}(t) = P_{1,ab}(0) > 2$ such that the auction can never make a distinction between the cases $x_1 = 0, 1$ or 2 which raises a contradiction with the efficiency property since the final efficient assignment would strictly depend on the exact value of $x_1 \in \{0, 1, 2\}$ if $x_2 = 1$ and $x_3 = 0$. Thus there exists a point in time where bidder 1's prices have been raised and from the minimality property there exists $t' < t$ such that $P_{1,ab}(t') \leq P_{2,a}(t') + P_{3,b}(t') \leq P_{2,a}(t) + P_{3,b}(t)$. However, it means that if bidder 1 reports a null demand set after such a jump, then bidder 1 can have at least two distinct valuations: the interval $(P_{1,ab}(t'), P_{1,ab}(t))$ contains at least two two integers, denote s and $s+1$ the two lowest integers in this set and that are strictly above $P_{2,a}(t) + P_{3,b}(t)$. Since prices are increasing, then there is no way to learn which one of those valuations is the right one. However, such an uncertainty could matter in term of Vickrey pricing since it could prevent the computation of the Vickrey payoffs. Let x_2 and x_3 be equal to the lowest integers that are respectively strictly above $P_{2,a}(t)$ and $P_{3,b}(t)$. Then we have either $x_2 < s+1$ or $x_3 < s+1$ (otherwise we have $2+s \geq x_2+x_3 \geq 2(s+1)$ which raises a contradiction since $s>0$). Suppose that bidders' valuations are such that the efficient assignment is to give item a to 2 and b to 3, i.e. $x_2+x_3 > s+1$ (e.g. if the inequality $x_2 < s+1$ holds, then choose x_3 high enough). Then the Vickrey payoffs differ whether bidder 1's valuation x_1 is set to s or $s+1$: the payment will differ for bidder 2 if $x_3 < s+1$ or for bidder 3 if $x_2 < s+1$.

Second, suppose that $P_{1,ab}(t) < P_{2,a}(t) + P_{3,b}(t) - 4$. In the same way, it would mean that a jump has occurred for one of the prices $P_{2,a}$ and $P_{3,b}$. Otherwise, $P_{2,a}(t) = P_{2,a}(0)$ and $P_{3,b}(t) = P_{3,b}(0)$, which implies that $P_{2,a}(t) + P_{3,b}(t) > 4$ and then that either $P_{2,a}(0) > 2$ or $P_{3,b}(0) > 2$ which would prevent the implementation of the efficient assignment on the domain $V_1 \times V_2 \times V_3$. More precisely, a jump of an extend greater than 2 has occurred such that the precise valuation of of those bidders can not be learned anymore since prices are increasing. However, if bidder 1's valuation were high enough such that the efficient assignment is to give him the bundle ab with the price x_2+x_3 and a contradiction is raised with the implementation of the Vickrey payoffs.[29]

In a nutshell, we have shown that the prices $P_{1,ab}(t)$, $P_{2,a}(t)$ and $P_{3,b}(t)$ can never be shifted by strictly more than 2. Otherwise there would be an irreversible lack of information to implement the Vickrey payoffs since prices are increasing.

Consider the subset of $V_1 \times V_2 \times V_3$ such that $x_1 = 20$, $x_2, x_3 \in \{17, 18, 19, 20\}$. The auction dynamic should be such that at some time t^* we have $20 \leq P_{1,ab}(t^*) < 22$. Otherwise, if $P_{1,ab}(t) < 20$ for any $t \in [0,1]$, there would be no way to learn that bidder 1's valuation equals 20 or a greater valuation such that the efficient assignment would be to assign the bundle ab to bidder 1. At this time t^*, bidder 1's prices are then frozen. Our previous analysis has shown that $P_{1,ab}(t^*) \geq P_{2,a}(t^*) + P_{3,b}(t^*) - 4$ such that we have either $P_{2,a}(t) < 13$ or $P_{3,a}(t) < 13$. Without loss of generality, say that $P_{2,a}(t) < 13$. Then there is no way to learn precisely bidder 2's valuation in the set $\{17, 18, 19, 20\}$ since after $P_{2,a}(t) \geq 17$, then any further price increase is

[29] Under general continuous valuations we would obtain the equality $P_{2,a}(t) + P_{3,b}(t) = P_{1,ab}(t)$ up to any time where the demand sets of bidders 1, 2 and 3 do not contain the empty set in any minimal ascending auction that yields the Vickrey payoffs.

frozen. Otherwise it would violate the minimality assumption. By setting only one price above 17, it is not possible to distinguish surely between four valuations.[30] On the whole it is not possible to compute the Vickrey payoff of the bidder $j \in \{2, 3\}$ with $j \neq i$.

Finally, we have shown that there is no minimal ascending auction that yields the Vickrey payoffs on the domain $V_1 \times V_2 \times V_3$.[31]

I Proof of proposition 6.1

Under an ascending auction with ϵ-increments with $\epsilon > 0$ then each price can take only a finite number of values. Each time the price vector of a given bidder is shifted in a way that his demand set is uncertain then we will speak of a price inquiry to this bidder.

Suppose without loss of generality that bidder 1's valuation function fails the gross substitutes condition: $v_1(ab) > v_1(a) + v_1(b) \geq 0$. Let $z = v_1(ab) - v_1(a) - v_1(b)$. Let V_2 [resp. V_3] denote the set of preferences such that $v_2(ab) = v_2(a) \in (v_1(a), v_1(a) + v_1(b) + z]$ and $v_2(b) = 0$ [resp. $v_3(ab) = v_3(a) \in (v_1(b), v_1(a) + v_1(b) + z]$ and $v_3(b) = 0$]. Let $\alpha = \min\{v_1(a), v_1(b), z/2\}$. We show that no ascending auction with ϵ-increments with $\epsilon > 0$ yields an assignment whose welfare α-approximates the welfare from an efficient assignment for each profile from $v_1 \times \mathrm{V}_2 \times \cdots \times \mathrm{V}_n$. The price dynamic can be viewed as a list of inquiries to refine our knowledge on bidders 2 and 3's preferences. Without additional knowledge, $v_2(a) + v_3(b)$ may lie anywhere in the interval $(v_1(ab) - z, v_1(ab) + z]$ which would not guarantee better than an z-approximation of the welfare by choosing to assign the bundle ab to bidder 1. Thus with inquiries in the auction, then the final assignment does not α-approximate the welfare from an efficient assignment for each profile from $v_1 \times \mathrm{V}_2 \times \cdots \times \mathrm{V}_n$. Inquiries

[30]With 3 valuations, it is possible by setting the price at the middle valuation since bidders are reporting the entire demand set.

[31]Note that the proof would not work if bidder 1's valuation function were known ex ante. Otherwise, we could build an auction where the price of bidder 1 for the bundle ab is set initially very high which does not violate minimality since at the null price bidder 1 may receive no items at the supply set of the seller then the complete preferences of the remaining bidders can be revealed by continuously rising the prices up to the valuation without violating minimality since the supply set is reduced to the assignment where the bundle ab is assigned to bidder 1. More generally, a formulation of the impossibility result in proposition 5.2 as under proposition 6.1 or the impossibility result of de Vries et al [14] where the valuation function of the bidder who fails to satisfy the gross substitutes condition is known ex ante does not hold.

on bidders 2's preferences thus take the form of a shift on the prices $p^t_{2,a}$ and $p^t_{2,ab}$, more precisely on the minimum of those two prices since it is this minimum that matters for reporting either the null assignment as ones's demand set or a non-null demand set. We show that the very first inquiry (which exists) may create an inefficiency that is strictly greater than $z/2$ which will prove the result. Without loss of generality consider that this is a inquiry for bidder 2 (our argument works exactly in the same way for bidder 3 with respect to inquiries on $\min\{p^t_{3,a}, p^t_{3,ab}\}$ which completes the proof). The first inquiry takes the form of a price vector such that $\min\{p^t_{2,a}, p^t_{2,ab}\} \in (v_1(a), v_1(a) + v_1(b) + z)$ (otherwise the demand set is known). For some valuation realizations, we have $v_2(ab) = v_2(a) < \min\{p^t_{2,a}, p^t_{2,ab}\}$ and $v_3(ab) = v_3(b) = v_1(a)+v_1(b)+z$ such that the demand set is null for bidder 2 and would be null forever given the ascending nature of the auction and thus in the final assignment of a standard ascending auction bidder 2 receives no item such that the welfare difference between the final assignment and the efficient one is greater than α.

J Proof of proposition 6.2

Before entering the proof itself, we introduce additional notation and preliminary results.

For any round t, consider the modified valuation profile $\overline{v}^t$ defined in the following way: if $\emptyset \in D_i(p^t; v)$, then $\overline{v}^t_{i,H} = p^t_{i,H}$ for any $H \subseteq G$; otherwise $\overline{v}^t_{i,H} = p^t_{i,H} + k^t_i$ for H such that $p^t_{i,H} > 0$ and $\overline{v}^t_{i,H} = v_{i,H}$ for H such that $p^t_{i,H} = 0$. Note that for any $H \in D_i(p^t; v)$, we have $\overline{v}^t_{i,H} = v_{i,H}$. First the definition of $\overline{v}^t$ guarantees that p^t is semi-truthful according to the valuation profile $\overline{v}^t$. Second, from lemma 2.1, we have $\overline{v}^t_{i,H} \in [v_{i,H}, v_{i,H} + \epsilon)$ for any $i \in \mathcal{N}$ and $H \subseteq G$ with $p^t_{i,H} > 0$. In any other case, we have $\overline{v}^t_{i,H} = v_{i,H}$.

Then we define the set of bidder-Core payoffs, denoted by $bCore(\mathcal{N}, v)$:

$$bCore(\mathcal{N}, v) = \left\{ (\widehat{\gamma}_i)_{i\in\mathcal{N}} \geq 0 \mid \ (\forall S \subseteq \mathcal{N}) \sum_{i\in\mathcal{N}\setminus S} \widehat{\gamma}^t_i \leq w(\mathcal{N}) - w(S) \right\}.$$

We define also $bCore(\mathcal{N}, v)+\lambda = \left\{ (\widehat{\gamma}_i)_{i\in\mathcal{N}} \geq 0 \mid \ \forall S \subseteq \mathcal{N}, \ \sum_{i\in\mathcal{N}\setminus S} \widehat{\gamma}^t_i \leq w(\mathcal{N}) - w(S) + \lambda \right\}$ for any scalar λ.

In the same way as for the Core, we consider also the bidder-optimal frontier and the weak bidder-optimal frontier of the bidder-Core denoted respectively by $bBOCE(\mathcal{N}, v)$ and $bwBOCE(\mathcal{N}, v)$.

Lemma J.1 *For any QCE-invariant auction with ϵ-increments and with ϵ-discounts, at any round t, we have:*

$$(bCore(\mathcal{N}, v) - N\epsilon) \subseteq bCore(\mathcal{N}, \overline{v}^t) \subseteq (bCore(\mathcal{N}, v) + N\epsilon). \qquad (2)$$

Furthermore, any $\gamma \in (bCore(\mathcal{N}, v) + \lambda) \setminus (bCore(\mathcal{N}, v) - \lambda)$ λ-approximates the set $bwCore(\mathcal{N}, v)$ for any scalar $\lambda \geq 0$.

Proof For any round t, any $i \in \mathcal{N}$ and $H \subseteq G$, we have $\overline{v}^t_{i,H} \in [v_{i,H}, v_{i,H} + \epsilon)$. We obtain thus the inequalities:

$$w(\mathcal{N}; v) - w(S; v) - N\epsilon \leq w(\mathcal{N}; \overline{v}^t) - w(S; \overline{v}^t) \leq w(\mathcal{N}; v) - w(S; v) + N\epsilon. \qquad (3)$$

Consider $\gamma \geq 0$ such that $\gamma \notin bCore(\mathcal{N}, \overline{v}^t)$. Then there exists $S \subseteq \mathcal{N}$ such that $\sum_{i \in \mathcal{N} \setminus S} \gamma_i > w(\mathcal{N}; \overline{v}^t) - w(S; \overline{v}^t)$ and then we obtain from (3) that $\sum_{i \in \mathcal{N} \setminus S} \gamma_i > w(\mathcal{N}; v) - w(S; v) - N\epsilon$ and finally that $\gamma \notin (bCore(\mathcal{N}, v) - N\epsilon)$. Consider $\gamma \geq 0$ such that $\gamma \in bCore(\mathcal{N}, \overline{v}^t)$. Then we have $\sum_{i \in \mathcal{N} \setminus S} \gamma_i \leq w(\mathcal{N}; \overline{v}^t) - w(S; \overline{v}^t)$ for any $S \subseteq \mathcal{N}$ and then we obtain from (3) that $\sum_{i \in \mathcal{N} \setminus S} \gamma_i \leq w(\mathcal{N}; v) - w(S; v) + N\epsilon$ and finally that $\gamma \in (bCore(\mathcal{N}, v) + N\epsilon)$. On the whole we have shown the inclusions in (2). We now move to the second part of the lemma.

Take $\gamma \in (bCore(\mathcal{N}, v)) \setminus (bCore(\mathcal{N}, v) - \lambda)$. Let $\delta = \min_{S \subsetneq \mathcal{N}} \{\frac{w(\mathcal{N}) - w(S) - \sum_{i \in \mathcal{N} \setminus S} \gamma_i}{N - |S|}\}$. Since $\gamma \in bCore(\mathcal{N}, v)$, we have $\delta \geq 0$. Furthermore, since $\gamma \notin (bCore(\mathcal{N}, v) - \lambda)$, there exists a set $S \subseteq \mathcal{N}$ such that $\sum_{i \in \mathcal{N} \setminus S} \gamma_i > w(\mathcal{N}) - w(S) - \lambda$. We obtain thus that $\delta \leq \frac{\lambda}{N - |S|} \leq \lambda$. Consider then the bidder-payoff vector $\gamma^* \geq 0$ such that $\gamma^*_i = \gamma_i + \delta$ for any $i \in \mathcal{N}$. The construction guarantees that $\sum_{i \in \mathcal{N} \setminus S} \gamma^*_i \leq w(\mathcal{N}) - w(S)$ for any $S \subseteq \mathcal{N}$ while the inequality stands as an equality for any $S \in Arg \min_{S \subsetneq \mathcal{N}} \{\frac{w(\mathcal{N}) - w(S) - \sum_{i \in \mathcal{N} \setminus S} \gamma_i}{N - |S|}\}$. Thus γ^* belongs to the set $bwCore(\mathcal{N}, v)$. Since $0 \leq \delta \leq \lambda$, we have $|\gamma^*_i - \gamma_i| \leq \lambda$ and we have thus shown that the bidder-payoff vector γ λ-approximates the set $bwCore(\mathcal{N}, v)$.

A similar construction shows that any bidder-payoff $\gamma \in (bCore(\mathcal{N}, v) + \lambda) \setminus (bCore(\mathcal{N}, v))$ λ-approximates the set $bwCore(\mathcal{N}, v)$.

We conclude the proof after noting that $(bCore(\mathcal{N}, v) + \lambda) \setminus (bCore(\mathcal{N}, v) - \lambda) =$

$(bCore(\mathcal{N}, v)) \setminus (bCore(\mathcal{N}, v) - \lambda) \cup (bCore(\mathcal{N}, v) + \lambda) \setminus (bCore(\mathcal{N}, v))$ which holds since $(bCore(\mathcal{N}, v) - \lambda) \subseteq bCore(\mathcal{N}, v) \subseteq (bCore(\mathcal{N}, v) + \lambda)$. **CQFD**

We now enter into the proof itself.

Let p^T the pseudo-CE where the auction stops before the discount stage and $\mathcal{A}^T \in L_{\mathcal{N}}(p^T)$. From remark 6.1, we obtain that p^T is a CE with respect to the valuation profile $\overline{v}^T$ since prices are remaining semi-truthful at any round t according to the modified valuation profile $\overline{v}^t$. CE assignments are efficient and we have so $\mathcal{A}^T \in \operatorname{Arg\,max}_{\mathcal{A} \in \mathbf{A}} \sum_{i \in \mathcal{N}} \overline{v}^T_{i,\mathcal{A}_i}$. As noted above, we have also that $\overline{v}^T_{i,H} \in [v_{i,H}, v_{i,H}+\epsilon]$ for any $i \in \mathcal{N}$ and $H \subseteq G$. Let $\mathcal{A}^* \in \operatorname{Arg\,max}_{\mathcal{A} \in \mathbf{A}} \sum_{i \in \mathcal{N}} v_{i,\mathcal{A}_i}$. We have thus:

$$\sum_{i \in \mathcal{N}} v_{i,\mathcal{A}^*_i} \geq \sum_{i \in \mathcal{N}} \overline{v}_{i,\mathcal{A}^T_i} - N \cdot \epsilon \geq \sum_{i \in \mathcal{N}} v_{i,\mathcal{A}^*_i} - N \cdot \epsilon.$$

Remember that for any $H \in D_i(p^T; v)$, we have $\overline{v}^T_{i,H} = v_{i,H}$. We have thus $\overline{v}^T_{i,\mathcal{A}^T_i} = v_{i,\mathcal{A}^T_i}$ and we conclude finally that the final assignment $\mathcal{A}^T$ $[N \cdot \epsilon]$-approximates the welfare from an efficient assignment as $\mathcal{A}^*$.

In the same way as in the proof of proposition 4.2 in appendix E, note that the auction dynamic stops if and only if p^t is a CE with regards to the valuation profile $\overline{v}^t$.

On the one hand, the bidder-payoff dynamic can not stop at round T at a vector $\gamma \geq 0$ such that $\gamma \notin (bCore(\mathcal{N}, v) + N\epsilon)$. Otherwise we obtain from lemma J.1 $\gamma \notin bCore(\mathcal{N}, \overline{v}^T)$ which raises a contradiction. In particular, this means that, at round T, we have:

$$k^T_i \leq (w(\mathcal{N}) - w(\mathcal{N} \setminus \{i\}) + N \cdot \epsilon. \qquad (4)$$

On the other hand, with a similar argument as in appendix E, the bidder-payoff dynamic can not stop at round T at a vector $\gamma \in (bCore(\mathcal{N}, v) - (N+1)\epsilon)$. Otherwise it would mean that the algorithm has not stopped at a vector $\gamma' \in (bCore(\mathcal{N}, v) - N\epsilon)$ at some round t. From lemma J.1, this latter condition implies that $\gamma' \notin bCore(\mathcal{N}, \overline{v}^t)$ which raises a contradiction.

On the whole we obtain that the final bidder-payoff vector $(\overline{k}^T_i)_{i \in \mathcal{N}} \in (bCore(\mathcal{N}, v) + (N+1)\epsilon) \setminus (bCore(\mathcal{N}, v) - (N+1)\epsilon)$ and thus from lemma J.1, it $(N+1)\epsilon$-approximates the set $bwCore(\mathcal{N}, v)$.

We now prove the last part of the proposition. Suppose now that bidders are substitutes.

Suppose by contradiction that by monotonicity of the price adjustment process, there exists a round t and a bidder $l \in \mathcal{N}$ such that $k_i^{t-1} \geq w(N) - w(N \setminus \{i\}) - N\epsilon$ and $k_i^t < w(N) - w(N \setminus \{i\}) - N\epsilon$. Since bidder i sees a price increase at period $t-1$, there exists $\mathcal{A} \in L_{\mathcal{N}}(p^{t-1})$ such that $p^t_{i,\mathcal{A}_i} = 0$. There exists thus $\overline{\mathcal{A}} \in L_{\mathcal{N}}(p^{t-1})$ such that $\overline{\mathcal{A}}_i = \emptyset$. Let $M = \{j \in \mathcal{N} : \overline{\mathcal{A}}_j \neq \emptyset\}$. Let $\widehat{\mathcal{A}}$ be an assignment yielding value $w(M \cup \{i\})$. $\widehat{\mathcal{A}}$ is chosen such that $\widehat{\mathcal{A}}_j = \emptyset$ if $j \notin M \cup \{i\}$. Let $M' = \{j \in \mathcal{N} : \widehat{\mathcal{A}}_j \neq \emptyset\} \subseteq M \cup \{i\}$.

$$
\begin{aligned}
\sum_{j \in M} p^t_{j,\overline{\mathcal{A}}_j} &= \sum_{j \in M} v^t_{j,\overline{\mathcal{A}}_j} - k^t_j \\
&\leq \sum_{j \in M} (v_{j,\overline{\mathcal{A}}_j} - k^t_j + \epsilon) \\
&\leq w(M) - \sum_{j \in M} k^t_j + |M|\epsilon \\
&\leq w(M) - \sum_{j \in M} k^t_j + N\epsilon && [M \subseteq \mathcal{N} \setminus \{i\}] \\
&< w(M) - \sum_{j \in M} k^t_j + (w(N) - w(N \setminus \{i\}) - k^t_i) && [\text{induction assumption}] \\
&\leq w(M) - \sum_{j \in M \cup \{i\}} k^t_j + (w(M \cup \{i\}) - w(M)) && [\text{bidders are substitutes}] \\
&= \sum_{j \in M'} v_{j,\widehat{\mathcal{A}}_j} - \sum_{j \in M \cup \{i\}} k^t_j \\
&\leq \sum_{j \in M'} v_{j,\widehat{\mathcal{A}}_j} - \sum_{j \in M'} k^t_j && [k^t_j \geq 0 \text{ and } M' \subseteq M \cup \{i\}] \\
&\leq \sum_{j \in M'} p^t_{j,\widehat{\mathcal{A}}_j} && [\text{lemma 2.1}].
\end{aligned}
$$

We have established that $\sum_{j \in M} p^t_{j,\overline{\mathcal{A}}_j} < \sum_{j \in M'} p^t_{j,\widehat{\mathcal{A}}_j}$ and thus raised a contradiction with $\overline{\mathcal{A}} \in L_{\mathcal{N}}(p^{t-1})$ and thus proved that k^T, the final payoff outcome once the algorithm stops, satisfies $k^T_i \geq \gamma^V_i - N \cdot \epsilon$ for any $i \in \mathcal{N}$. From (4), we have $k^T_i \leq \gamma^V_i + N \cdot \epsilon$. On the whole we obtain that k^T $[N \cdot \epsilon]$-approximates the Vickrey payoff γ^V. Finally after the ϵ discounts, the final bidder-payoff vector $\overline{k}^T$ $[(N+1) \cdot \epsilon]$-approximates the Vickrey payoff γ^V.