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## The Generalized Hierarchical Model: A New Approach to Resource Allocation Within Multilevel Organizations

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ABSTRACT

This paper describes a three-level organizational model or decomposition procedure named the Generalized Hierarchical Model (GHM). The GHM algorithm focuses upon a multiple criteria approach to hierarchical decision-making via mathematical programming decomposition theory. The GHM has been implemented on a CDC CYBER-175 computer and has been tested extensively. The results of this research tend to confirm that the GHM can offer a systematic approach to organizational design, multiperiod planning, and resource allocation in decentralized organizations.

# 1. INTRODUCTION

Since the development of the Dantzig-Wolfe [6,7] decomposition algorithm, a number of researchers have noted how closely the algorithm's solution procedure parallels the decision-making process found within a hierarchical organization. See [1,4,5,12] for a limited sample. Over the years, other decomposition algorithms have been developed for mathematical programming problems that possess a particular structure; works by Lasdon [19] and Geoffrion [16,17] have provided an excellent summary of these mathematical techniques. Because the solution procedures of these decomposition algorithms can mimick the information exchanges that occur within hierarchical organizations, they have been used to model organizational decision-making. Reviews of the economic and behavioral aspects of these applications are given in Ruefli [25] and Jennergren [18].

Recently, alternative formulations of decomposition models that accommodate multiple criteria optimization have evolved. Unlike their predecessors, these decomposition models were developed specifically to model the resource allocation process within a multilevel, decision-making hierarchy. The highest level of the hierarchy is assigned the task of generating goals for and/or allocating resources to the intermediate decision-makers. The decisions at the intermediate level are structured as goal programming problems that attempt to minimize deviations from the goals generated at the highest level. These deviations are minimized by selecting alternative proposals generated by subordinate units at the lowest level of the organization's hierarchy.

The initial effort in this area was Ruefli's Generalized Goal Decomposition (GGD) model [23,24]. He utilized the shadow prices (dual solutions) of the intermediate level decisions and the principles of generalized linear programming to coordinate the process of goal generation by the highest level decision-maker. Later Freeland [13] as well as Freeland and Baker [14] developed a model which also used shadow prices for coordinating decision-making in the hierarchy. However, the principles of Benders' partitioning procedure [3] were incorporated into the highest level decision in order to effect the process of goal generation. Freeland [15] later discussed the relationship of his model to the GGD model. As noted earlier the evolution of these algorithms represented a departure from the practice of defining an organizational model based upon a given decomposition procedure. That is, both the efforts of Freeland and Ruefli were explicitly dedicated to the construction of organizational models. Sweeney _et al_. [26] have characterized the models as "composition" approaches to organizational decision-making.

Although Ruefli [23, p. B510] noted the "insufficiency of [shadow] prices alone to coordinate the activities of an organization," neither the Freeland and Baker nor Ruefli algorithm utilized deviations from generated goals as a mechanism for coordinating decision making. Davis [8] and Davis and Talavage [9] realized the possible advantages of utilizing these deviations as a coordinative input. Their research led to the formulation of two new models: the Centralized Goal Decomposition (CGD) model, which relies upon goal deviations as a coordinative input to the highest level decision-maker, and the Hybrid Goal Decomposition (HGD) model which uses both deviations and simplex multipliers. Both the CGD and HGD models are formulated for a three-level organization. Further, an overall problem was defined for these algorithms, and this problem was identical to overall problem solved by Ruefli's GGD model. Finally, the convergence for both CGD and HGD models was shown.

Because the GGD, CGD, and HGD algorithms solve the same overall mathematical programming problem, their solutions' properties offer a vehicle for testing the efficacy of their respective coordinative mechanisms. After Davis and Talvage [9] applied these algorithms to several test examples, they concluded the following:

1) The simplex multiplier as a sole coordinative input to the highest level decision-maker was least effective. In fact, in all test examples, the GGD algorithm failed to converge without substantial modification.

2) The utilization of deviations as coordinative mechanims generated a solution which at convergence was nearly optimum.

3) The marginal advantage gained by using <u>both</u> simplex multipliers and deviations was small and did not guarantee optimality.

4) In most cases, the simplex multiplier was a detrimental coordinative input to the decisions at the lowest level. The computational properties of the goal programming decision associated with the intermediate level of the organization were shown to inhibit the efficiency of simplex multipliers as coordinative mechanisms.

Recently an article by Winkofsky *et al.* [31] presented a hierarchical decision process model (DPM) which utilizes a binary (0-1) goal programming model with preemptive priority factors (BGP) at each level of the organization. Further their coordinative mechanisms were either goal adjustment vectors (similar to Davis and Talavage's [9] deviations) or "broad changes in the current performance" [32, p. 274] levels for subordinate subsystems within the hierarchy. Their model is an extension of the work done by Ruefli [23], Freeland and Baker [14], and Davis and Talavage [9]. Although Winkofsky _et al_. focus upon the "composition" approach described in Sweeny, _et al_. [27], they do not demonstrate how their model's "derived organizational problem" is related to an "ideal organizational problem." (See [27] for an elaboration of these terms.) In other words, instead of viewing the DPM as a composition approach to an ideal problem, Winkofsky _et al_. provide an intriguing model designed to act as a simulation laboratory for testing the impact of alternative organizational policies and design issues on organizational decision-making and resource allocation.

The Generalized Hierarchical Model (GHM) presented in this study is an extension of these previous models, and it expands upon a two-level version of the model presented in [10]. It attempts to eliminate the negative properties of the previous algorithms while preserving their positive attributes. Although similar to the Winkofsky _et al_. BGP hierarchical model, the GHM retains the spirit of a true decomposition algorithm. The GHM has been tested on simulated as well as real-world administrative problems. The results of this testing indicate that the GHM can offer a systematic approach to the problems of organizational design, multiperiod planning, and resource allocation in decentralized organizations.

§ 2 specifies the structure of the GHM, and § 3 presents several theorems characterizing the solution and convergence properties of the model. § 4 relates the results of several computational tests of the GHM, and a final § provides concluding comments.

## 2. DEFINITION OF THE GENERALIZED HIERARCHICAL MODEL

This § presents the GHM, and for ease of exposition it will be divided into two subsections. §§ 2.1 will describe the organization and decision structure of the model. In this §§ an explicit statement for several key functions will be omitted; however, they will be detailed in §§ 2.2. By varying the functional form of these functions, several versions of the GHM (viz., linear, quadratic, etc.) are possible. §§ 2.2 will outline the linear version of the GHM, its simplest form.

### 2.1 Organizational and Decision Structure of the GHM

The decision-makers within the organization will be called subsystems. The model discussed in this paper employs three levels of hierarchical decision-making. However, the GHM can be modified to accommodate a two-level or n-level decision-making hierarchy. The subsystem at the highest level of the organization will be referred to as the supremal subsystem (see Figure 1). At the second level of the organization, there are M managing subsystems or managers. Finally, on the lowest level of the organization are N infimal subsystems or operating units. Using the assumption of M managing subsystems, there exists a series of integers, $r_0, r_1, \ldots, r_M$, such that infimal subsystems $r_{k-1} + 1$ through $r_k$ are subordinate to managing subsystem k. For consistency, assuming that there are N infimal subsystems, $r_0$ must equal zero while $r_M$ must equal N.

Each subsystem within the organization is assigned a specific decision. These decisions collectively describe the GHM and are given by

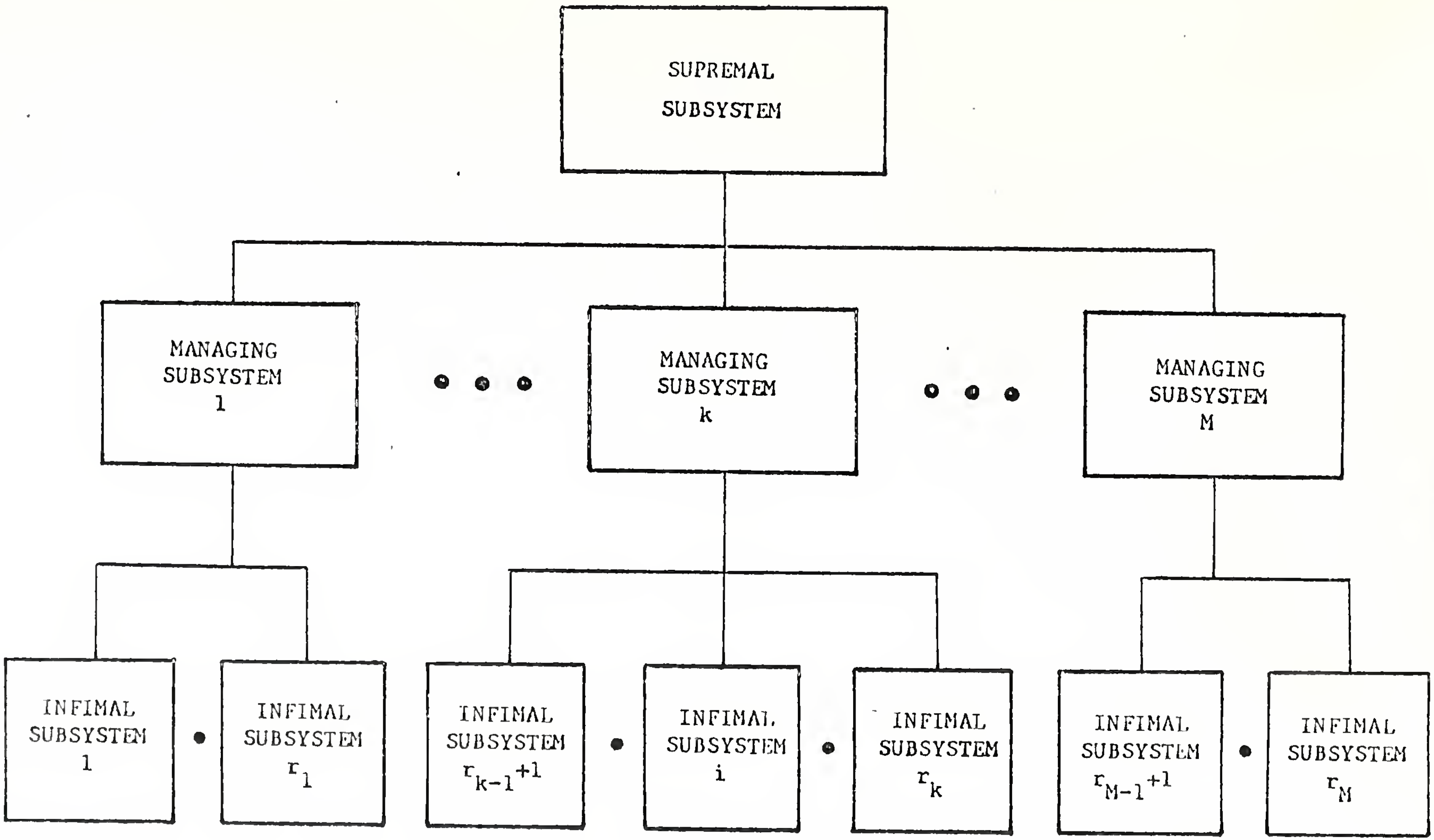


Figure 1. Three-level Organizational Structure

uations (1) through (14) in Table 1. Figure 2 details the assig f the decisions including the model's informational flows or coor ive echanisms. The individual decisions will now be detailed emphasi he nteraction between the subsystems on adjacent levels of the organ .

The first decision to be discussed is that of managing subsyst , defined by equations (5.k) through (11.i). Managing subsystem k wo sets of goals. The first goals are specified by a $m_k$-column ve $_k(t)$, which represents a set of external goals. These goals are ge rated by the supremal subsystem on interation t, and coordinate the ctions of the managing subsystems. The second set of goals, $g_k$, is $m'_k$-column vector of internal goals. These goals are constant through the optimization and can be regarded as a mechanism used by managing subsystem k to coordinate the efforts of each of its subordinate, infimal subsystems. The internal goals can also create a certain level autonomy for each managing subsystem.

Corresponding to the external goal vector, $G_k(t)$, are two $m_k$-column vectors, $Y_k^+(t)$ and $Y_k^-(t)$, representing positive and negative deviations from $G_k(t)$, respectively. Similarly, there are two $m'_k$-column vectors, $y_k^+(t)$ and $y_k^-(t)$, corresponding to positive and negative deviations from the internal goal vector, $g_k$. Managing subsystem k has at its disposal a set of $n_i$-column vector proposals, $X_i(1),\ldots,X_i(t)$, generated during the previous interations by the i-th infimal subsystem $(i=r_{k-1}+1,\ldots,r_k)$. In creating a composite proposal for each of its subordinate infimal subsystems, managing subsystem k can select any convex combination of the i-th infimal's operating proposals, $X_i(1),\ldots,X_i($ $(i=r_{k-1}+1,\ldots,r_k)$. To relate the infimals' proposals to the manager's goals, $G_k(t)$ and $g_k$, managing subsystem k has two sets of matrices,

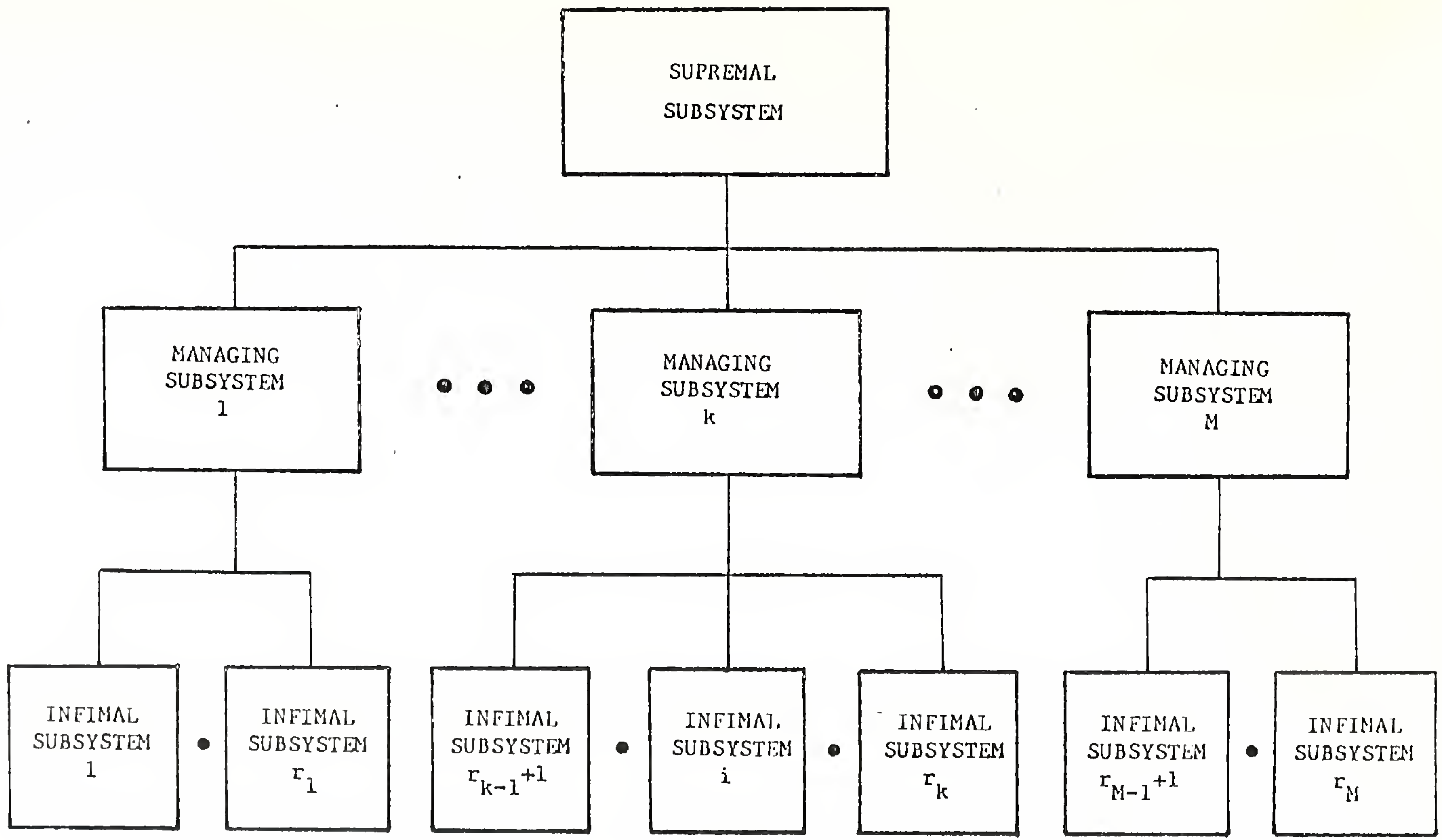


Figure 1. Three-level Organizational Structure

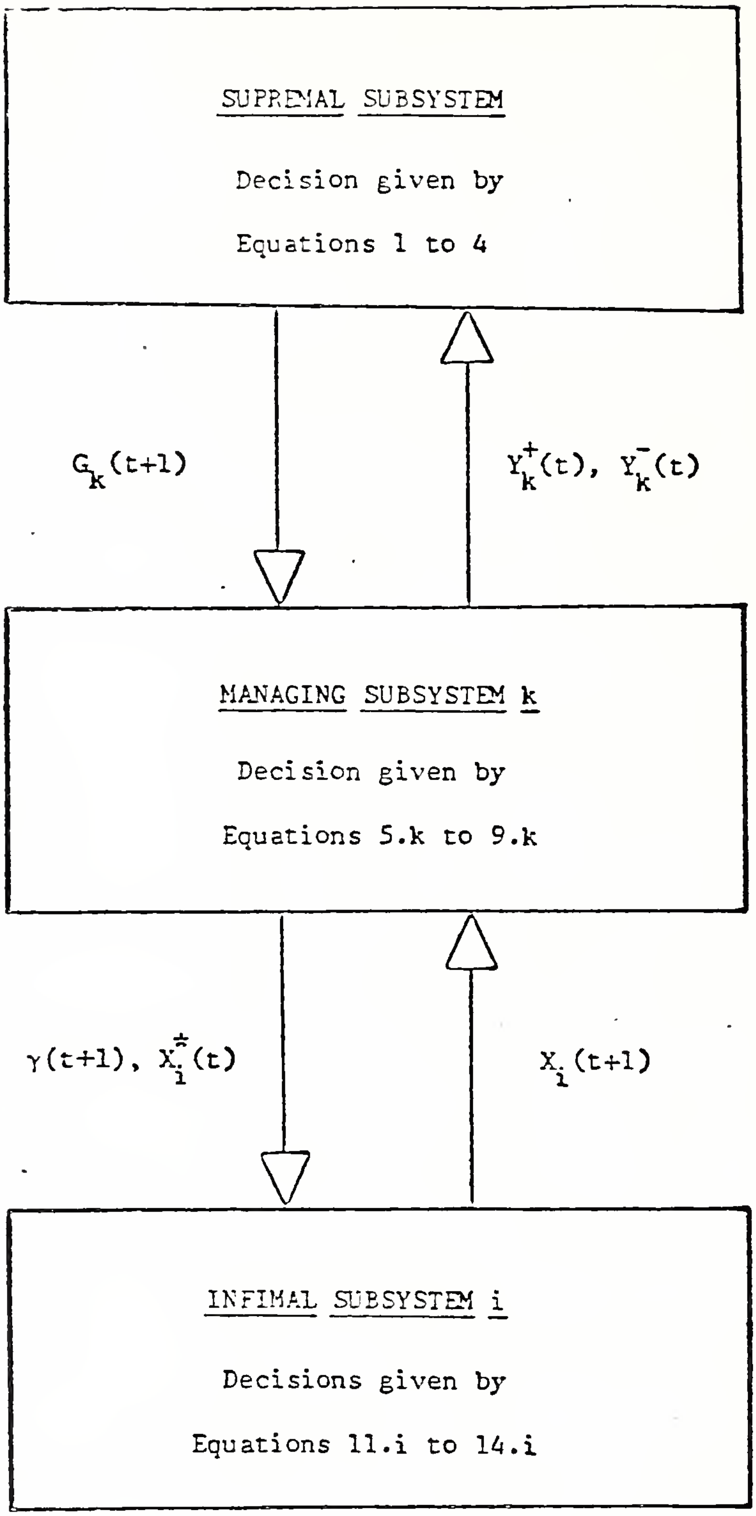


Figure 2. Detailed Structure for the Generalized Model Giving the Decision Assignment and Resulting Flows of Information

equations (1) through (14) in Table 1. Figure 2 details the assignment of the decisions including the model's informational flows or coordinative mechanisms. The individual decisions will now be detailed emphasizing the interaction between the subsystems on adjacent levels of the organization.

The first decision to be discussed is that of managing subsystem k, defined by equations (5.k) through (11.i). Managing subsystem k has two sets of goals. The first goals are specified by a $m_k$-column vector, $G_k(t)$, which represents a set of external goals. These goals are generated by the supremal subsystem on interation t, and coordinate the actions of the managing subsystems. The second set of goals, $g_k$, is a $m_k'$-column vector of internal goals. These goals are constant throughout the optimization and can be regarded as a mechanism used by managing subsystem k to coordinate the efforts of each of its subordinate, infimal subsystems. The internal goals can also create a certain level of autonomy for each managing subsystem.

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Table 1

Specification of the GHM

Supremal Subsystem's Problem

(1) Min $Z_0(t+1) = \Sigma_{k=1}^{M} \{c_{G_k} G_k(t+1) + W_k^+[S_k^+(t+1)] + W_k^-[S_k^-(t+1)]\}$
s.t.

(2.k) $I_{m_k} G_k(t+1) + I_{m_k} S_k^+(t+1) - I_{m_k} S_k^-(t+1) = G_k(t) + Y_k^+(t) - Y_k^-(t)$

for k=1,...,M where $I_{m_k}$ is the $(m_k \times m_k)$ - identity matrix

(3) $\Sigma_{k=1}^{M} P_k G_k(t+1) \lesseqgtr G_0$

(4) $G_k(t+1) \geq 0,\ S_k^+(t+1) \geq 0,\ S_k^-(t+1) \geq 0$

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Managing Subsystem k's Problem (k = 1,...,M)

(5.k) Min $Z_k(t) = \Sigma_{i=r_{k-1}+1}^{r_k} \Sigma_{j=1}^{t} c_i X_i(j)\ \lambda_i(j) + W_k^+[Y_k^+(t)] + W_k^-[Y_k^-(t)]$
s.t.

$+ w_k^+[y_k^+(t)] + w_k^-[y_k^-(t)]$

(6.k) $\Sigma_{i=r_{k-1}+1}^{r_k} \Sigma_{j=1}^{t} B_i X_i(j)\ \lambda_i(j) - I_{m_k} Y_k^+(t) + I_{m_k} Y_k^-(t) = G_k(t)$

(7.k) $\Sigma_{i=r_{k-1}+1}^{r_k} \Sigma_{j=1}^{t} B_i' X_i(j)\ \lambda_i(j) - I_{m_k'} y_k^+(t) + I_{m_k'} y_k^-(t) = g_k$

(8.i) $\Sigma_{j=1}^{t} \lambda_i(j) = 1,\ \lambda_i(j) \geq 0$ for $i = r_{k-1}+1,\ldots,r_k$

(9.k) $Y_k^+(t) \geq 0,\ Y_k^-(t) \geq 0$

The subsequent generation of the $n_i$-component goal vector for the infimal subsystem i is based upon equations (5.k) through (9.k) and is defined as

(10.i) $\gamma_i(t+1) = \Gamma_i[X_i^*(t),\ G_k(t),\ Y_k^+(t),\ Y_k^-(t),\ y_k^+(t),\ y_k^-(t)]$

for $i = r_{k-1}+1,\ldots,r_k$, where

(11.i) $X_i^*(t) = \Sigma_{j=1}^{t} X_i(j)\ \lambda_i(j)$

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Infimal Subsystem i's Problem (i = 1,...,N)

(12.i) Min $c_i X_i(t+1) + \omega_i^+[\sigma_i^+(t+1)] + \omega_i^-[\sigma_i^-(t+1)]$
s.t.

(13.i) $E_i[X_i(t+1)] - I_{n_i}\ \sigma_i^+(t+1) + I_{n_i}\ \sigma_i^-(t+1) = \gamma_i(t+1)$

(14.i) $D_i X_i(t+1) \lesseqgtr F_i$

(15.i) $X_i(t+1) \geq 0,\ \sigma_i^+(t+1) \geq 0,\ \sigma_i^-(t+1) \geq 0$

$B_i$ and $B'_i$, $(i=r_{k-1}+1,\ldots,r_k)$. $B_i$ is a $(m_k \times n_i)$-matrix which linearly relates each $X_i(j)$, $(j=1,\ldots,t)$, to $G_k(t)$. Similarly $B'_i$ is a $(m'_k \times n_i)$-matrix which linearly relates $X_i(j)$, $(j=1,\ldots,t)$, to $g_k$. These relationships are defined by equations (6.k) and (7.k), respectively.

Associated with each infimal subsystem's proposal is a $n_i$-row vector, $c_i$, such that the cost of a particular proposal vector, $X_i(j)$, is given by the inner product, $c_iX_i(j)$. Also associated with the deviation vectors, $Y_k^+(t)$ and $Y_k^-(t)$, there are two penalty cost functions $W_k^+[Y_k^+(t)]$ and $W_k^-[Y_k^-(t)]$, respectively. Similarly, associated with the vectors, $y_k^+(t)$ and $y_k^-(t)$, are the two penalty cost functions, $w_k^+[y_k^+(t)]$ and $w_k^-[y_k^-(t)]$. (A detailed specification of the functional form for these penalty functions will be given in §§ 2.2.) Each of these costs is included in equation (5.k).

Managing subsystem k's problem is one of generating a composite proposal, $X_i^*(t)$, for infimal subsystem i, $(i=r_{k-1}+1,\ldots,r_k)$. These composite proposals must satisfy equations (6.k) through (9.k) while the objective function given by equation (5.k) is minimized. As mentioned earlier, in generating each proposal, $X_i^*(t)$, the manager is allowed to consider only convex combinations of a subordinate infimal i's previous proposals, $X_i(j)$, $(j=1,\ldots,t)$. Thus, $X_i^*(t)$ is defined as

(11.i) $$X_i^*(t) = \Sigma_{j=1}^{t} X_i(j)\ \lambda_i(j)\quad (i=r_{k-1}+1,\ldots,r_k),$$

where $\Sigma_{j=1}^{t} \lambda_i(j) = 1$ and $\lambda_i(j) \geq 0$ for $j=1,\ldots,t$. Therefore, managing subsystem k is concerned with generating an optimum set of $\lambda_i(j)$'s.

Like previous organizational models, the GHM employs an iterative solution procedure. Once the k-th managing subsystem's problem has been solved, that is the optimum set of $\lambda_i(j)$'s and the deviations, $Y_k^+(t)$, $Y_k^-(t)$, $y_k^+(t)$ and $y_k^-(t)$, have been generated, each management unit has two alternatives to reduce its objective function value. First, each can ask the supremal subsystem for a new goal vector, $G_k(t+1)$, which will be more compatible with its subordinate infimals' proposals. Second, each manager can ask its subordinate infimal subsystems to generate new proposals which more closely conform to its current goals, $G_k(t)$ and $g_k$.

The second alternative will be discussed first. Having generated the optimal solution to its problem, the k-th managing subsystem generates a $\eta_i$-column vector of goals, $\gamma_i(t+1)$, for each of its infimals $(i=r_{k-1}+1,\ldots,r_k)$ using equation (10.i). (A detailed specification of $\Gamma_i$ is given in §§ 2.2.)

$$(10.i) \qquad \gamma_i(t+1) = \Gamma_i[X_i^*(t),\ G_k(t),\ Y_k^+(t),\ Y_k^-(t),\ y_k^+(t),\ y_k^-(t)]$$

The goal vector, $\gamma_i(t+1)$, is in turn used by infimal subsystem i in the selection of its next proposal vector, $X_i(t+1)$. This selection process is accomplished through the solution of the i-th infimal subsystem's goal programming problem given by equations (12.i) through (15.i). Associated with the goal vector, $\gamma_i(t+1)$, are two $\eta_i$-column vectors, $\sigma_i^+(t+1)$ and $\sigma_i^-(t+1)$, representing positive and negative deviations from $\gamma_i(t+1)$, respectively. To relate the proposal vector, $X_i(t+1)$, to the goal vector, $\gamma_i(t+1)$, each infimal subsystem has the linear function, $\beta_i[X_i(t+1)]$. (Again, the formal specification of this linear function will be detailed in §§ 2.2.) This relationship is defined by equation (13.i). The i-th infimal subsystem must also satisfy certain technological constraints given by the equation (14.i):

(14.i) $\quad D_i X_i(t+1) \lesseqgtr F_i,$

where $D_i$ is a $(\ell_i \times n_i)$-matrix of technological coefficients, and $F_i$ is a $\ell_i$-column vector stipulating a set of fixed, right-hand-side values.

In the solution process, each infimal subsystem must consider two costs. The first is the cost of the proposal, $X_i(t+1)$, given by the inner product, $c_i X_i(t+1)$. The second is the penalty cost associated with the deviations, $\sigma_i^+(t+1)$ and $\sigma_i^-(t+1)$, which are defined by the penalty functions, $\omega_i^+[\sigma_i^+(t+1)]$ and $\omega_i^-[\sigma_i^-(t+1)]$, respectively. (These functions will be defined in §§ 2.2.) The sum of these costs gives the objective function, equation (12.i). Once the infimal subsystem has determined its proposal vector, $X_i(t+1)$, these proposals are forwarded to its managing subsystem for consideration at the next iteration.

As indicated in Figure 2, $X_i^*(t)$ can also be passed to the i-th infimal subsystem $(i=r_{k-1}+1,\ldots,r_k)$ by manager k. Later it will be shown that $X_i^*(t)$ represents a feasible solution to the i-th infimal's problem on iteration t+1. Hence, $X_i^*(t)$ can be used as an initial feasible solution for the infimal's problem on iteration t+1. (The current GHM computer code does not pass $X_i^*$ to the i-th infimal. Only $\gamma_i(t+1)$ is used as the coordinative mechanism.)

While infimal subsystem i has been solving its problem, managing subsystem k has also sent its deviations, $Y_k^+(t)$ and $Y_k^-(t)$, from the current goal vector, $G_k(t)$, to the supremal subsystem. Because the supremal subsystem "knows" the goal values passed to managing subsystem k on the last iteration, the supermal can determine the goals that each managing subsystem could achieve with no deviations, given the k-th manager's current operating proposals. This goal vector is defined as

(16.k) $G_k^*(t) = G_k(t) + Y_k^+(t) - Y_k^-(t)$,

and corresponds to the right-hand-side of equation (2.k). The supremal subsystem is concerned with generating a new set of goals, $G_k(t+1)$ $(k=1,\ldots,M)$, which will minimize the deviations from the k-th managing subsystem's proposed operating levels, $G_k^*(t)$. These deviations are given as the $m_k$-column vectors, $S_k^+(t+1)$ and $S_k^-(t+1)$, for $k=1,\ldots,M$. This new set of goals must satisfy the constraints given by equation (3):

(3) $\Sigma_{k=1}^{M} P_k G_k(t+1) \lesseqgtr G_0$,

where $G_0$ is a $m_0$-column vector containing the organization's overall goals, and $P_k$ is a $(m_k \times m_0)$-matrix that linearly relates $G_k(t+1)$ to $G_0$. Having generated the optimal set of goals, $G_k(t+1)$, $(k=1,\ldots,M)$, the supremal subsystem sends the new goal vector, $G_k(t+1)$, to managing subsystem k. The managing subsystem then increments t to t+1, and the entire process is repeated.

Although several starting procedures are available, the simplest and perhaps the most efficient is as follows. On iteration one infimal subsystem i, $(i=1,\ldots,N)$, generates its minimum cost proposal considering only constraint (14.i). The penalty costs for deviations arising from equation (13.i) are neglected. Next, the k-th manager, $(k=1,\ldots,M)$, takes the proposals $X_i(1)$, $(i=r_{k-1}+1,\ldots,r_k)$, and generates the resulting deviations arising in equations (6.k) and (7.k) given that $G_k(1)$ equals zero, and $g_k$ is as assigned. The resulting deviations, $Y_k^+(1)$ and $Y_k^-(1)$ are then forwarded to the supremal subsystem for consideration on iteration two. Hence the supremal's problem is not solved on iteration one because $G_k(1)$, $(k=1,\ldots,M)$, was set to zero. It is interesting to note that this starting procedure closely resembles a zero-base budgeting system [21]. Also the GHM as well as other multiple criteria, decomposition algorithms [9,14,15,23] can be viewed as a mathematical model of a Planning, Programming, and Budgeting System (PPBS); see [24].

2.2. The Linear Version of the GHM

§§ 2.1 did not provide an explicit statement for several key functions. These include:

1. The penalty costs in equations (1), (5.k), and (12.i);
2. The function, $\Gamma_i$, in equation (10.i), which generates $\gamma_i(t+1)$ for each infimal i; and
3. The function, $\beta_i$, in equation (13.i), which relates $X_i(t+1)$ to $\gamma_i(t+1)$.

Through the explicit statement of these functions, several versions (viz., linear, quadratic, etc.) of the GHM can be formulated. It should be emphasized that the functional form chosen for these unspecified functions should be consistent throughout the hierarchy. That is, the penalty cost functions should be of the same polynomial order at each decision-making level. The penalty weights should also be consistent. The consistent weighting criterion placed upon the manager's and subordinate infimals' deviation vectors implies that the echelon behaves as a "cooperative" organization. That is, in the interaction between the managers and their respective subordinates, each infimal's objective function has been structured so that it will seek to minimize its manager's deviations and their associated costs. On the other hand, in the interaction between the supremal and the managers, the managers can retain some measure of autonomy through their internal goals. The supremal attempts to adjust the external goals so that the resulting costs of the the deviations are minimized over the ensemble of managing subsystems.

Because the deviation vectors, $\sigma_i^+(t+1)$ and $\sigma_i^-(t+1)$, represent positive and negative deviations from $\gamma_i(t+1)$, respectively, the penalty function in equation (12.i) also depends on the specific form of $\gamma_i(t+1)$. Finally, the specification of $\beta_i$ must also be consistent with the form of $\gamma_i(t+1)$, which is derived via $\Gamma_i$.

This paper will focus upon the linear version of the GHM, which is its simplest form. The functions for the linear version are given in Table 2 as equations (19.k) through (27.i).

The first step in defining the linear model is to specify the penalty cost functions, $W_k^+[Y_k^+(t)]$ and $W_k^-[Y_k^-(t)]$, as the inner products, $W_k^+Y_k^+(t)$ and $W_k^-Y_k^-(t)$, respectively. This corresponds to the utilization of an absolute error criterion for the penalty cost function, if the $m_k$-row vectors, $W_k^+$ and $W_k^-$, are such that $W_k^+$ equals $W_k^-$, for k=1,...,M. A similar formulation for $w_k^+[y_k^+(t)]$ and $w_k^-[y_k^-(t)]$ gives $w_k^+y_k^+(t)$ and $w_k^-y_k^-(t)$, respectively.

Next, the decision structure used by the managing subsystems to generate the $\gamma_i(t+1)$ goal vectors for their subordinates must be specified. In deriving these vectors for iteration t+1, managing subsystem k first assumes that $G_k(t)$ will be held constant. Next, manager k assumes that only infimal subsystem i will be able to generate a proposal, $X_i(t+1)$, which more closely satisfies the goal vectors, $G_k(t)$ and $g_k$. Using these assumptions, managing subsystem k subtracts the contribution of its other subordinate infimal subsystems from $G_k(t)$ and $g_k$ giving

$$(17.i)\qquad \gamma_i(t+1) = \begin{bmatrix} G_k(t) \\ \hline g_k \end{bmatrix} - \sum_{\substack{p=r_{k-1}+1 \\ p\neq i}}^{r_k} \begin{bmatrix} B_p \\ \hline B'_p \end{bmatrix} X_p^*(t), \quad (i=r_{k-1}+1,\ldots,r_k).$$

Table 2

Linear Specification of the GHM

(19.k) $W_k^+[Y_k^+(t)] = W_k^+ \, Y_k^+(t)$

where $W_k^+$ is a $m_k$-vector for $k = 1,\ldots,M$

(20.k) $W_k^+[Y_k^-(t)] = W_k^- \, Y_k^-(t)$

where $W_k^-$ is a $m_k$-vector for $k = 1,\ldots,M$

(21.k) $w_k^+[y_k^+(t)] = w_k^+ \, y_k^+(t)$

where $w_k^+$ is a $m_k'$-vector for $k = 1,\ldots,M$

(22.k) $w_k^-[y_k^-(t)] = w_k^- \, y_k^-(t)$

where $w_k^-$ is a $m_k'$-vector for $k = 1,\ldots,M$

(23.i) $$\gamma_i(t+1) = \begin{bmatrix} B_i \\ \hdashline B_i' \end{bmatrix} X_i^*(t) - \begin{bmatrix} Y_k^+(t) \\ \hdashline y_k^+(t) \end{bmatrix} + \begin{bmatrix} Y_k^-(t) \\ \hdashline y_k^-(t) \end{bmatrix}$$

for $i = 1,\ldots,N$

(24.i) $\beta_i[X_i(t)] = \begin{bmatrix} B_i \\ \hdashline B_i' \end{bmatrix} X_i(t)$ for $i = 1,\ldots,N$

(25.i) $\omega_i^+[\sigma_i^+(t)] = [W_k^+ \,\vdots\, w_k^+] \, \sigma_i^+(t)$ for $i = 1,\ldots,N$

$\omega_i^-[\sigma_i^-(t)] = [W_k^- \,\vdots\, w_k^-] \, \sigma_i^-(t)$ for $i = 1,\ldots N$

(27.i) $n_i = m_k + m_k'$ for $i = 1,\ldots,N$

Using equations (6.k) and (17.i), the definition of $\gamma_i(t+1)$ by equation (23.i) follows. Thus given the goal, $\gamma_i(t+1)$, each infimal subsystem attempts to generate $X_i(t+1)$ such that the following conditions are fulfilled:

$$(18.i) \qquad \beta_i[X_i(t+1)] = \begin{bmatrix} B_i \\ \text{---} \\ B_i' \end{bmatrix} X_i(t+1) = \gamma_i(t+1)$$

This definition of $\beta_i$ is given in equation (24.i). In general, however, infimal subsystem i will not be able to generate a proposal, $X_i(t+1)$, which satisfies this equality. Accordingly, the $\eta_i$-deviation vectors, $\sigma_i^+(t+1)$ and $\sigma_i^-(t+1)$, are introduced. The linear versions of the penalty cost functions $\omega_i^+[\sigma_i^+(t+1)]$ and $\omega_i^-[\sigma_i^-(t+1)]$ are defined by equations (25.i) and (26.i). For consistency, the penalty weights associated with deviations for infimal subsystem i are identical to those for corresponding deviations at the managing subsystem.

The strategy for generating $\gamma_i(t+1)$ presented in this paper places stringent demands upon the infimal subsystems. It requires that each infimal single-handedly attempts to eliminate the deviations from its manager's current solution. Alternative approaches which prorate the manager's deviations among the subordinate infimal subsystems are possible and have been investigated. Prorating schemes could be based upon proportional resource utilization and/or goal contribution as well as the marginal cost associated with a given infimal's elimination of a unit of a goal deviation. Each of these alternatives has been computationally tested. Although it is perhaps managerially unenlightened, the strategy described in equation (23.i) has proven to be the most efficient at squeezing out what Williamson [31] as well as Schiff and Lewin [26] have referred to as "organizational" slack.

As noted earlier, the GHM can accommodate n levels of hierarchical decision-making. For example, if one fixes the value of $G_k(t)$ to a constant vector, $G_k$, then the interaction between managing subsystem k and its subordinate infimal units is similar to that of a standard two-level decomposition procedure. Also, a four-level version is possible if one replaces a given infimal subsystem with a submanager and an associated ensemble of infimal subsystems. Of course this substitution process could be repeated to generate n levels of decision-making.

In contrast with previous organizational models the technique of merging penalty functions with operating proposal costs in order to effect the solution by the column generators is novel. This approach allows the GHM to act as a pure decomposition procedure. This concept is discussed in detail in Davis [10]. In addition, one should note the ommission of <u>all</u> cost vectors in the three level model creates a simplified goal programming structure, similar to the Ruefli [23,24] and Freeland [13, 15] models. This approach requires an <u>a priori</u> specification of penalty weights that will subsequently remain constant throughout optimization Thus the linear version of the GHM utilizes a linearly additive utility formulation.

## 3. ANALYTICAL PROPERTIES OF THE GHM

The GHM has been designed to mimick the decision-making process within a hierarchical organization, and thus it can be classified as an organizational model. The fact that the GHM is concerned with decision-making in an organization necessarily implies the existence of a global or real-world problem for that organization. This issue raises two concerns to which modelers must address themselves. The first is the compatability of the overall (i.e. undecomposed) problem solved by a specific organizational model with the global problem faced by the organization. The second is the efficacy of an organizational model in solving its version of the overall problem.

In either concern, the specification of the model's overall problem is of primal importance. Unless a modeler has a reasonable understanding of this overall problem, a model's applicability to the global or real-world problem cannot be evaluated. Unfortunately, a discussion of the difficulties and problems encountered in defining and enumerating global problems are beyond the scope of this paper. Nevertheless, the need for a global problem specification remains. This section will focus upon the definition of the overall problem solved by the GHM and will access the ability of the GHM to solve this problem.

The overall mathematical programming problem for the GHM is given in equations (28) through (34) in Table 3. Theorem 1 states that the GHM generates a feasible solution to this overall problem at each iteration.

THEOREM 1: The Generalized Hierarchical Model given by equation (1) through equation (15) generates a feasible solution to the overall problem, equation (28) through equation (34), at every iteration. Proof of Theorem 1 is given in Appendix 1.

The efficacy of the GHM in solving the overall problem will now be discussed. Proof of Theorem 2 demonstrates convergence of the GHM.

THEOREM 2: If a finite optimal solution exists to the overall problem given by equations (28) through (34), then the linear version of the

Table 3

The Overall Problem Solved by the GHM

(28) $$\text{Min } \Sigma_{k=1}^{M} \Sigma_{i=r_{k-1}+1}^{r_k} c_i X_i + \Sigma_{k=1}^{M} \{W_k^+ [Y_k^+] + w_k^+ [y_k^+] + W_k^- [Y_k^-] + w_k^- [y_k^-]\} + \Sigma_{k=1}^{M} c_{G_k} G_k$$

subject to:

(29.1) $$\Sigma_{i=1}^{r_1} B_i X_i \quad - Y_1^+ \quad + Y_1^- \quad - G_1 \quad = 0$$

$\vdots$

(29.M) $$\Sigma_{i=r_{M-1}+1}^{r_M} B_i X_i \quad - Y_M^+ \quad + Y_M^- \quad - G_M \quad = 0$$

(30.1) $$\Sigma_{i=1}^{r_1} B_i' X_i \quad - y_1^+ \quad + y_1^- \quad = g_1$$

$\vdots$

(30.M) $$\Sigma_{i=r_{M-1}+1}^{r_M} B_i' X_i \quad - y_M^+ \quad + y_M^- \quad = g_M$$

(31.1) $$D_1 X_1 \quad \lesseqgtr F_1$$

$\vdots$

(31.N) $$D_N X_N \quad \lesseqgtr F_N$$

(32) $$\Sigma_{k=1}^{M} P_k G_k \quad \lesseqgtr G_0$$

(33) $$X_i \geq 0 \quad \text{for } i = 1,\ldots,N$$

(34) $$Y_k^+ \geq 0,\ Y_k^- \geq 0,\ y_k^+ \geq 0,\ y_k^- \geq 0,\ G_k \geq 0 \quad \text{for } k=1,\ldots,M$$

<u>GHM, given by equations (1) through (15) and (19) through (27), converges to a finite limit.</u>

Proof of the Theorem is given in Appendix 2. The proof is accomplished by defining a composite objective function for the model. At each iteration, this composite objective function is given by equation (35).

$$(35) \qquad Z(t) = \Sigma_{i=1}^{N} c_i X_i^*(t) + \Sigma_{k=1}^{M} [c_{G_k} G_k(t) + W_k^+ Y_k^+(t) + W_k^- Y_k^-(t) + w_k^+ y_k^+(t) + w_k^- y_k^-(t)]$$

Note that this equation is obtained by inserting the iteration t solution values generated by the GHM into equation (28), the objective function for the overall problem.

Theorem 1 has shown that the solution given by the GHM on each iteration is a feasible solution to the overall problem. If we assume that a finite optimum solution exists for the overall problem, say $Z_0$, then it can be concluded that:

$$(36) \qquad Z(t) \geq Z_0 \text{ for } t = 1,2,\ldots$$

Proof of Theorem 2 shows that the sequence, Z(1), Z(2),..., is monotonically decreasing. That is:

$$(37) \qquad Z(t+1) \leq Z(t) \text{ for } t = 1,2,\ldots$$

Because each Z(t) is bounded from below by equation (36), a Cauchy sequence results. It can thus be concluded that a cluster point exists, or equivalently, for any $\varepsilon > 0$, there exists t* such that:

(38) $|Z(t^*) - Z(t^*+k)| < \varepsilon$ for $k=1,2,...$

In other words, the GHM converges.

Equations (37) and (38) are crucial for the proof of the GHM's convergence. In the proof, several assumptions have been made in order to demonstrate that the iterative objective function sequence is such that:

(39) $Z(1) > Z(2) > ... > Z(t^*-1) > Z(t^*) = Z(t^*+k)$ for $k = 1,2,...$

These assumptions were made so that the proof of Theorem 2 would analytically display the same convergence characteristics demonstrated in computational testing of the GHM. In computational practice, the linear version of the GHM demonstrates the convergence pattern seen in equation (39), where t* is usually less than ten and is often less than five. Thus, in application, the assymptotic convergence implied by equation (38) is being satisfied in a more stringent respect. That is:

(40) $|Z(t^*) - Z(t^*+k)| = 0$ for $t^*$ finite and $k = 1,2,...$

Further in the proof of Theorem 2, it is shown that Corollary 1 follows.

COROLLARY 1. The following characteristics will exist at convergence of the model on iteration t*: $X_i(t^*+1) = X_i^*(t^*+1)$ for $i=1,...,N$. That is, the optimal proposal generated by managing subsystem k, $(k=1,...,M)$, for infimal subsystem i, $(i=r_{k-1}+1,...,r_k)$, at iteration $t^*+1$ is identical to the optimal proposal generated by infimal subsystem i.

Recall infimal subsystem i's constraints:

(13.i) $$\begin{bmatrix} B_i \\ \hline B'_i \end{bmatrix} X_i(t) - I_{n_i}\sigma_i^+(t) + I_{n_i}\sigma_i^-(t) = \gamma_i(t) \text{ and}$$

(14.i) $$D_i X_i(t) \lesseqgtr F_i,$$

where $\begin{bmatrix} B_i \\ \hline B'_i \end{bmatrix} X_i(t) = \beta_i[X_i(t)]$ (see equation 24.i).

Constraint (13.i) places no restriction on $X_i(t)$. Any $X_i(t)$ which satisfies equation (14.i) also satisfies equation (13.i). The purpose of equation (13.i) is to direct infimal subsystem i toward the selection of $X_i(t)$, by minimizing deviations from the goal vector $\gamma_i(t)$. Most previous decomposition algorithms considered only equation (14.i). This implies that use of a linear objective function by infimal subsystem i requires that its $X_i(t)$ necessarily be an extreme point of the polytope for equation (14.i). This estabished the necessity that managing subsystem k consider convex combinations of previous proposals (extreme points),

(11.i) $$X_i^*(t) = \Sigma_{j=1}^t X_i(j)\ \lambda_i(j)$$

(8.i) $$\Sigma_{j=1}^t \lambda_i(j) = 1,\ \lambda_i(j) \geq 0$$

in order to generate interior points of the polytope for equation (14.i).

By using constraint (13.i), infimal subsystem i is free to generate any point in its feasible polytope as an optimum decision. Each infimal is no longer totally dependent upon its managing subsystem to generate an interior point for an optimum proposal. Furthermore, Corollary 1 states that at convergence the optimal proposal generated by infimal subsystem i corresponds to the one generated by its managing subsystem.

Corollary 1 shows that the GHM possesses a desired behavioral property often sought in organizational modeling (see [21]). In the GHM each decision-maker generates its own limiting solution. Guidance in selecting solutions flows from above. In contrast, each of the multiple criteria organizational models [9,13,23,24] discussed earlier requires each managing subsystem to generate the limiting solution for its subordinate infimal systems.

The remainder of this section will discuss the solution properties of the model. As stated earlier, Theorem 2 guarantees at least assymptotic convergence of the model to a limiting solution for the overall organizational problem given in equations (28) through (34). Further, this limiting solution is feasible and bounded. Computational testing has indicated that the GHM converges to a limiting solution faster than the previous models discussed in the introduction. In addition, no hueristic starting procedures are required for this convergence.

A possible explanation for the improved convergence characteristics of the model follows. All of the previous multiple criteria models use simplex multipliers associated with managing subsystem k's problem as a mechanism for coordinating the decisions at one or more levels of the organization. Use of simplex multipliers can create convergence problems for organizational models that utilize a goal programming format.

Recall that the manager's problem attempts to choose least costly proposals and to force the goal deviation vectors to zero. The associated goal programming formulation of this decision creates computational difficulties. First, the number of proposals, $X_i(j)$ $(j=1,\ldots t)$, that managing subsystem k uses to generate the composite vectors $X_i^*(t)$

for $i=r_{k-1}+1,\ldots,r_k$, at iteration t is usually less than the total number of constraints. For example, the constraints (excluding non-negativity conditions) in the k-th manager's problem originate from three sources: external and internal goals plus a convex combination proposal constraint for each subordinate infimal. In total there should be $m_k + m_k' + r_k - r_{k-1}$ constraints. Computational testing has indicated that the number of $X_i(j)$'s $(j=1,\ldots,t)$ actually selected in the convex combination search seldom exceeds three for each infimal. Given this operational maximum for all subordinate infimals, there should be no more than $3x[r_k - r_{k-1}]$ $\lambda_i(j)$'s $(i=r_{k-1}+1,\ldots,r_k$ and $j=1,\ldots,t)$ in solution as basic variables. If the total number of constraints exceeds $3x[r_k - r_{k-1}]$ by a considerable margin, problems can occur. Given these conditions, the process of forcing goal deviation vectors to zero often causes the primal problem of the managing subsystem to become degenerate. As a result, the associated dual to this problem has multiple optimum solutions. Recall from the theory of linear programming that when the primal problem achieves optimality, the simplex multiplier and the dual solution are essentially the same (they can differ in sign). Thus, if multiple dual solutions occur, then any decomposition procedure or organizational model using simplex multipliers as coordinating mechanisms can experience a breakdown in coordination. This results from the fact that the managing subsystem does not have a unique specification for its simplex multipliers.

Even if one could disregard these difficulties, use of the simplex multiplier can create additional problems. For example, the value of a simplex multiplier in a linear programming problem is dependent upon the cost associated with a basic variable and not the basic variable's solution value. Thus, if a particular component of the deviation vector is a basic variable, its influence on the simplex multiplier is independent of its value in the current basic solution. Instead, it is dependent upon the penalty cost assigned to it. Thus, a simplex multiplier can have the same value if a particular deviation were $10^{-3}$ or $10^{3}$. Therefore, usage of the simplex multiplier only establishes that a particular deviation is a basic variable, but it offers no guidance on how far goals must be adjusted. On the other hand, the utilization of deviations conveys considerably more information on the success that managing subsystem k has had in meeting goals, and indicates how far certain goals should be adjusted. This additional information tends to result in faster convergence.

Although the GHM appears to be in some ways better than the previous multiple criteria, organizational models, the optimality of its limiting solution cannot be guaranteed. It is possible that the linear version of the GHM can converge to a near but nonoptimal solution to the overall problem given in equations (28) through (34). However, this nonoptimality is not unique to the GHM. The potential for nonoptimal solutions is brought about by several factors. First, Ruefli's three level GGD model [23] and Freeland's correction to it [15] may not generate feasible solutions to the overall problem. Neither applies a "convex combination" proposal constraint in the manager's problem. That is, there is no equivalent $\lambda_i(j)$ in their models. This omission implies that each manager can consider the "origin" or null $X_i$ vectors. Thus the Ruefli and Freeland models have difficulties if constraint (14.i) or its overall problem counterpart (31.i) place nonzero minimum proposal levels on a given infimal. This difficulty can be easily overcome by inserting the type of convex combination constraints imposed by equations (6.k) through (8.i).

Unfortunately more serious problems exist in these multiple criteria organizational models. The previous discussion on degeneracy indicated possible sources of ambiguity imbedded in the shadow price coordinative mechanisms. This is further complicated by the fact that unlike the Dantzig and Wolfe [6] model, no true master program exists for the decomposition of the overall problem by these three level multiple criteria algorithms. Instead, each uses a group of partial master programs, which are coordinated through the supremal subsystem. In addition, no individual decision-maker is concerned with the organization's overall objective function.

Our computational testing has shown that nonoptimal solutions are possible in the linear versions of all three level multiple criteria decomposition algorithms. We should stress that a quadratic version of the GHM [2,11] overcomes this nonoptimality problem. However, this improvement is achieved at the expense of significantly higher computational difficulty and cost.

## 4. EXPERIENCE WITH COMPUTATIONAL TESTING

Although the first multiple criteria, organizational model was introduced over ten years ago, implementations (successful or otherwise) of the algorithms are rare. Ruefli has noted:

> Perhaps the most discouraging aspect of the subject [analytic models of resource allocation]

> we are discussing is the persistent lack of applications. Modeling efforts usually take place at the theoretical level and little has been done to link these efforts to actual problem situations and actual (or even strongle representative) data. [25, p. 361]

He continues:

> Indeed, one of the very good reasons why these models have not seen wider use is probably because they are not very useful. However, until selected efforts are made to apply these models and the problems and successes are reported, we must continue to operate in substantial ignorance of real-world problems vis-à-vis the models. [25, p. 362]

This § briefly describes the results of computational testing of the GHM. The results of these applications tend to confirm that organizational models based upon mathematical programming techniques can offer a systematic and viable approach to systems and organizational design, multiperiod planning, and resource allocation in real-world, decentralized organizations.

In recent months four separate problem formulations have been tested utilizing the GHM. They include Ruefli's Department of Defense (DOD) military plan [24], the Davis and Talvage Transhipment problem [9], the Davis design structure formulation [10], and the Whitford university planning model [29,30]. The size of these problems varies. For example, a statement of the overall problem [see equations (28) through (34)] of the Ruefli DOD formulation contained 42 variables and 38 constraints. In contrast the Whitford university model's overall problem has over 7,200 variables and 2,800 constraints.

In order to eliminate data manipulation errors and to reduce the man-hours required to formulate or solve a particular problem using the GHM, a FORTRAN computer code was written for a Control Data Corporation CYBER-175 computer. This program utilized a separate set of sparse matrix optimization subroutines [21].

This code offered the benefits of speed and computational ease. For example, data preparation for the DOD problem required .75 hours while the university problem required eighty hours. Compilation of the GHM code required 3.5 CPU seconds. Convergence of the DOD problem was attained in 3.4 CPU seconds, while the university model required 97.7 seconds. Both formulations converged in four iterations. Further no starting hueristic procedures were necessary. As indicated earlier use of goals and deviations from goals as coordinating mechanisms avoids potential difficulties in organizational coordination caused by simplex multipliers. Thus it appears that avoidance of simplex multipliers obviates the need for good starting solutions cited in [5]. In contrast to previous studies by Davis [8], Davis and Talavage [9], Christensen and Obel [5], and those cited by Dirickx and Jennergren [12, pp. 86-97], the GHM's results are promising. Indeed if one excludes the first iteration (required for model initialization), the number of planning and programming information exchanges or iterations is similar to those actually experienced in most decentralized organizations. Unfortunately, a detailed description of these applications is beyond the scope of this paper. However, the interested reader is referred to Whitford [28,29] and Whitford and Davis [30].

## 5. SUMMARY AND CONCLUSIONS

The purpose of this paper has been to describe the Generalized Hierarchical Model (GHM), a multilevel, multiple criteria decomposition procedure. The GHM is an outgrowth of several decomposition algorithms developed during the last twenty years. However, its principal coordinative mechanisms are not the traditional shadow prices but performance targets or operating goals, and the resulting deviations from those goals. These goal deviations provide guidance on how far goals must be adjusted to achieve organizational objectives and priorities.

Although the GHM cannot guarantee that it will generate the optimum solution to an organization's overall problem, its limiting solution is nearly optimal. A FORTRAN computer code has been written to implement the GHM. Testing of the algorithm has demonstrated rapid computational speed and convergence. Not only do the informational exchanges within the algorithm closely resemble those found in actual hierarchical organizations, the number of iterations or information exchanges required for convergence seldom exceed the typical number of planning and/or budgeting reviews found in most organizations.

Although the GHM offers an interesting approach to organizational modeling, much work remains. Currently research is in progress which utilizes the GHM to investigate the model's applicability in several organizational, decision-making settings. However, based upon our initial results, the GHM appears to provide a systematic approach to investigating issues related to organizational design, multiperiod planning, and resource allocation in decentralized organizations and systems.

## Appendix 1

## Proof of Theorem 1

THEOREM 1: The Generalized Hierarchical Model given by equation (1) through equation (15) generates a feasible solution to the overall problem, equation (28) through equation (34), at every iteration.

Proof: The proof begins by observing that constraints (6.k) and (7.k) for managing subsystem k are identical to constraints (29.k) and (30.k) of the overall problem if one defines

$$(41.i)\quad \begin{cases} X_i = X_i^*(t) = \Sigma_{j=1}^{t} X_i(j)\,\lambda_i(j) & \text{for } i = r_{k-1}+1,\ldots,r_k, \\ \Sigma_{j=1}^{t} \lambda_i(j) = 1 & \text{for } i = r_{k-1}+1,\ldots,r_k, \\ \lambda_i(j) \geq 0 & \text{for } i = r_{k-1}+1,\ldots,r_k, \text{ and} \\ & \text{for } j = 1,\ldots,t. \end{cases}$$

In the overall problem, each $X_i$ must satisfy the following constraint:

$$(42.i)\qquad D_i X_i \lesseqgtr F_i.$$

Within the GHM's solution process the i-th infimal subsystem on iteration j must consider a similar constraint in generating $X_i(j)$:

$$(14.i)\qquad D_i\, X_i(j) \lesseqgtr F_i \quad \text{for } j = 1,\ldots,t.$$

Because $X_i^*(t)$ is a convex combination of $X_i(1)$ through $X_i(t)$, $X_i^*(t)$ must necessarily satisfy equation (31.i) of the overall problem.

In addition, managing subsystem k has the external goal vector, $G_k(t)$. In the overall problem, the vector $[G_1^T, G_2^T, \ldots, G_M^T]^T$ (T indicates transpose) must satisfy the constraint:

$$(32) \qquad \Sigma_{k=1}^{M} P_k G_k \lesseqgtr G_0.$$

In generating the goals, $G_k(t)$, (k = 1,...,M), the supremal subsystem must also satisfy the constraint:

$$(3) \qquad \Sigma_{k=1}^{M} P_k G_k(t) \lesseqgtr G_0.$$

Because the vectors, $Y_k^+(t)$, $Y_k^-(t)$, $y_k^+(t)$ and $y_k^-(t)$, are generated via constraints (6.k) and (7.k), which are identical to constraints (29.k) and (30.k), it is possible to conclude that values for

$$(42) \quad \begin{cases} X_i^*(t) & \text{for } i = 1,\ldots,N \text{ and} \\ G_k(t) & \\ Y_k^+(t),\ Y_k^-(t) & \text{for } k = 1,\ldots,M \\ y_k^+(t),\ y_k^-(t) & \end{cases}$$

derived through the decomposition algorithm provide a feasible solution to the overall problem. Q.E.D.

Appendix 2

Proof of Theorem 2

THEOREM 2: If a finite optimal solution exists to the overall problem given by equations (28) through (34), then the linear version of the GHM, given by (1) through (15) and (19) through (27), converges to a finite limit.

Proof. To facilitate the proof of this theorem, several assumptions are made. First, it will be assumed that a finite optimal solution exists to the overall problem. Also, the following set of behavioral assumptions effecting decision-making at each level of the organization are operative.

1) If at iteration t+1, $G_k(t)$ is an alternate optimal solution to the supremal's decision, then $G_k(t)$ will be returned to the managing subsystem k as $G_k(t+1)$.
2) If at iteration t+1, $X_i^*(t)$ is an alternate optimal solution to infimal i's decision, then $X_i^*(t)$ will be returned to managing subsystem k as $X_i(t+1)$.
3) If at iteration t, the set of proposals, $X_i^*(t-1)$, $(i = r_{k-1}+1,\ldots,r_k)$, represents an alternative optimum solution to managing subsystem k's problem, then $X_i^*(t) = X_i^*(t-1)$, $(i = r_{k-1}+1,\ldots,r_k)$.

In every case, these behavioral assumptions are utilized only if alternate optimal solutions exist to the subsystem's current decision. Computational experience has shown that these assumptions hasten convergence of the model. Furthermore, they serve as a mechanism to preventing cycling. The convergence of the model will now be shown.

The proof initially focuses upon the supremal subsystem's decision at iteration t+1. From each managing subsystem k, the supremal subsystem has the feedback information, $Y_k^+(t)$ and $Y_k^-(t)$, (k=1,...,M). The supremal's problem is:

(1) Min $\Sigma_{k=1}^{M} [c_{G_k} G_k(t+1) + W_k^+ S_k^+(t+1) + W_k^- S_k^-(t+1)]$

s.t.

(2) $G_k(t+1) + S_k^+(t+1) - S_k^-(t+1) = G_k(t) + Y_k^+(t) - Y_k^-(t)$ for k=1,...,M,

(3) $\Sigma_{k=1}^{M} P_k G_k(t+1) \lesseqgtr G_0$, and

(4.k) $G_k(t+1) \geq 0, \; S_k^+(t+1) \geq 0, \; S_k^-(t+1) \geq 0$ for k=1,...,M.

The solution

(43) $$\left\{ \begin{array}{l} G_k(t+1) = G_k(t) \\ S_k^+(t+1) = Y_k^+(t) \\ S_k^-(t+1) = Y_k^-(t) \end{array} \right\} \quad \text{for } k=1,\dots,M$$

is feasible for this problem, and therefore, the following condition may be assumed:

(44) $$\Sigma_{k=1}^{M} [c_{G_k} G_k(t+1) + W_k^+ S_k^+(t+1) + W_k^- S_k^-(t+1)] \leq \Sigma_{k=1}^{M} [c_{G_k} G_k(t) + W_k^+ Y_k^+(t) + W_k^- Y_k^-(t)].$$

Unless equation (44) holds as a strict inequality, then by assumption 1, $G_k(t+1) = G_k(t)$, and equation (43) gives the supremal's solution. If $G_k(t+1)$ does not equal $G_k(t)$, then equation (44) guarantees that the overall performance or composite objective of the organization defined in equation (35)

$$(35) \qquad Z(t) = \Sigma_{j=1}^{t} \Sigma_{i=1}^{N} c_i X_i(j)\, \lambda_i(j) + \Sigma_{k=1}^{M} [c_{G_k}\, G_k(t) + W_k^+\, Y_k^+(t) + W_k^-\, Y_k^-(t) + w_k^+\, y_k^+(t) + w_k^-\, y_k^-(t)]$$

will be improved on iteration t+1. To see this, assume that on iteration t+1, managing subsystem k's problem is:

$$(45.k) \qquad \text{Min} \quad \Sigma_{i=r_{k-1}+1}^{r_k} \Sigma_{j=1}^{t+1} c_i X_i(j)\lambda_i(j) + W_k^+\, Y_k^+(t+1) + W_k^-\, Y_k^-(t+1) + w_k^+\, y_k^+(t+1) + w_k^-\, y_k^-(t+1)$$

$$(46.k) \qquad \text{s.t.} \quad \Sigma_{i=r_{k-1}+1}^{r_k} \Sigma_{j=1}^{t+1} B_i X_i(j)\lambda_i(j) - Y_k^+(t+1) + Y_k^-(t+1) = G_k(t+1),$$

$$(47.k) \qquad \Sigma_{i=r_{k-1}+1}^{r_k} \Sigma_{j=1}^{t+1} B_i' X_i(j)\lambda_i(j) - y_k^+(t+1) + y_k^-(t+1) = g_k, \text{ and}$$

$$(48.i) \qquad \Sigma_{j=1}^{t+1} \lambda_i(j) = 1, \ \lambda_i(j) \geq 0$$

for $i=r_{k-1}+1,\ldots,r_k$.

Let $\lambda_i^*(j)$, $(i=r_{k-1}+1,\ldots,r_k)$ and $(j=1,\ldots,t)$, be the optimal set of $\lambda_i(j)$'s on the last iteration. The solution

$$(49.k)\quad \begin{cases} \lambda_i(j) = \lambda_i^*(j) \text{ for } j=1,\ldots,t \text{ and } i=r_{k-1}+1,\ldots,r_k \\ \lambda_i(t+1) = 0 \quad \text{for } i=r_{k-1}+1,\ldots,r_k \\ Y_k^+(t+1) = S_k^+(t+1) \\ Y_k^-(t+1) = S_k^-(t+1) \\ y_k^+(t+1) = y_k^+(t) \\ y_k^-(t+1) = y_k^-(t) \end{cases}$$

is a feasible solution to managing subsystem k's problem for iteration t+1. Using the feasibility of $S_k^+(t+1)$ and $S_k^-(t+1)$ in manager k's solution, equation (49.k), and the inequality of equation (35), equation (50) follows.

$$\Sigma_{k=1}^M\{[\Sigma_{j=1}^{t+1} \Sigma_{i=r_{k-1}+1}^{r_k} c_i X_i(j)\lambda_i(j)] + c_{G_k} G_k(t+1) + W_k^+ Y_k^+(t+1) + W_k^- Y_k^-(t+1) + w_k^+ y_k^+(t+1) + w_k^- y_k^-(t+1)\} \leq$$

$$(50)\quad \Sigma_{k=1}^M\{[\Sigma_{j=1}^{t} \Sigma_{i=r_{k-1}+1}^{r_k} c_i X_i(j)\lambda_i^*(j)] + c_{G_k} G_k(t+1) + W_k^+ S_k^+(t+1) + W_k^- S_k^-(t+1) + w_k^+ y_k^+(t) + w_k^- y_k^-(t)\} \leq$$

$$\Sigma_{k=1}^M\{[\Sigma_{j=1}^{t} \Sigma_{i=r_{k-1}+1}^{r_k} c_i X_i(j)\lambda_i^*(j)] + c_{G_k} G_k(t) + W_k^+ Y_k^+(t) + W_k^- Y_k^-(t) + w_k^+ y_k^+(t) + w_k^- y_k^-(t)\}.$$

Using equations (35) and (50), it can be shown that

(51) $Z(t+1) \leq Z(t)$.

Furthermore, the solution for iteration t+1 is distinct from that for iteration t if and only if

(52) $Z(t+1) < Z(t)$.

Next, the interaction between managing subsystem k and its infimal subsystems i, $(i=r_{k-1}+1,\ldots,r_k)$, will be discussed. For iteration t, managing subsystem k has the optimal solution:

$$(53)\quad \begin{cases} X_i^*(t), \ (i=r_{k-1}+1,\ldots,r_k) \\ Y_k^+(t),\ Y_k^-(t),\ y_k^+(t),\ y_k^-(t). \end{cases}$$

In defining the GHM's <u>behavioral</u> interactions of manager k and its i-th infimal, manager k "assumes" that the supremal subsystem will return the goal $G_k(t)$ as $G_k(t+1)$. On this basis, manager k "asks" each of its subordinate infimal subsystems to generate the proposal vector that will best meet the goal vector, $G_k(t)$, assuming that the other infimal subsystems' proposals will be held constant. Mathematically, the manager subtracts the contribution of the other infimal subsystems from the right hand side of constraints (6.k) and (7.k) giving:

$$(54.i)\quad \begin{bmatrix} B_i \\ \hline B_i' \end{bmatrix} X_i^*(t) - Y_k^+(t) + Y_k^-(t) = \begin{bmatrix} G_k(t) \\ \hline g_k \end{bmatrix} - \sum_{\substack{j=r_{k-1}+1 \\ j\neq i}}^{r_k} \begin{bmatrix} B_j \\ \hline B_j' \end{bmatrix} X_j^*(t) \qquad (i=r_{k-1}+1,\ldots,r_k).$$

By definition, the left hand side of equation (54.i) is $\gamma_i(t+1)$. The manager then passes the goal, $\gamma_i(t+1)$, along with its proposed action for infimal i, $X_i^*(t)$, to the i-th infimal subsystem.

Infimal subsystem i then has the following problem:

(55.i) $$\text{Min}\quad c_i X_i(t+1) + [W_k^+ \vdots w_k^+]\,\sigma_i^+(t+1) + [W_k^- \vdots w_k^-]\,\sigma_i^-(t+1),$$

(56.i) $$\text{s.t.}\begin{bmatrix} B_i \\ \hline B_i' \end{bmatrix} X_i(t+1) - \sigma_i^+(t+1) + \sigma_i^-(t+1) = \begin{bmatrix} B_i \\ \hline B_i' \end{bmatrix} X_i^*(t) - \begin{bmatrix} Y_k^+(t) \\ \hline y_k^+(t) \end{bmatrix} + \begin{bmatrix} Y_k^-(t) \\ \hline y_k^-(t) \end{bmatrix}$$

(57.i) $$D_i X_i(t+1) \lesseqgtr F_i, \text{ and}$$

(58.i) $$X_i(t+1) \geq 0,\ \sigma_i^+(t+1) \geq 0,\ \sigma_i^-(t+1) \geq 0.$$

The solution

(59.i) $$X_i(t+1) = X_i^*(t)$$
$$\sigma_i^+(t+1) = \begin{bmatrix} Y_k^+(t) \\ \hline y_k^+(t) \end{bmatrix}$$
$$\sigma^-(t+1) = \begin{bmatrix} Y_k^-(t) \\ \hline y_k^-(t) \end{bmatrix}$$

is a feasible solution to infimal subsystem i's problem. Therefore, it follows that:

(60.i) $$c_i X_i(t+1) + [W_k^+ \vdots w_k^+]\sigma_i^+(t+1) + [W_k^- \vdots w_k^-]\sigma_i^-(t+1) \leq$$
$$c_i X_i^*(t) + [W_k^+ \vdots w_k^+] \begin{bmatrix} Y_k^+(t) \\ \hline y_k^+(t) \end{bmatrix} + [W_k^- \vdots w_k^-] \begin{bmatrix} Y_k^-(t) \\ \hline y_k^-(t) \end{bmatrix} .$$

If equation (60.i) holds as an equality, then (59.i) gives at least an alternate optimal solution to infimal subsystem i's decision. Thus by assumption 2, (61.i) $$X_i(t+1) = X_i^*(t).$$

Using equations (35), (45), and (59.i), it follows that if $X_i(t+1) \neq X_i^*(t)$, then $Z(t+1) < Z(t)$. Furthermore if $Z(t) = Z(t+1)$, then the assumptions guarantee that:

(62) $$\begin{cases} X_i(t+1) = X_i^*(t) & \text{for } i = 1,\ldots,N \quad \text{and} \\ G_k(t+1) = G_k(t) & \text{for } k = 1,\ldots,M. \end{cases}$$

The optimal solution for managing subsystem k, is thus given by

(63) $$\begin{cases} X_i^*(t+1) = X_i^*(t) & \text{for } i = r_{k-1}+1,\ldots,r_k \quad \text{and} \\ Y_k^+(t+1) = Y_k^+(t) & \\ Y_k^-(t+1) = Y_k^-(t) & \\ y_k^+(t+1) = y_k^+(t) & \text{for } k = 1,\ldots,M \\ y_k^-(t+1) = y_k^-(t) & . \end{cases}$$

This implies that the supremal subsystem's and the infimal subsystems' decisions on iteration t+2 are identical to those of iteration t+1. Therefore, the algorithm has converged.

The previous discussion indicates that the <u>seratium</u> solutions will take on the following characteristics:

(64) $$Z(1) > Z(2) > \ldots. > Z(t^*) = Z(t^*+1) = \ldots = Z(t^*+\tau) = \ldots \text{ (for } \tau \geq 0).$$

The only step that remains is to show the existence of $t^*$. Recall that a finite optimum solution to the overall problem, $Z_0$, is assumed to exist. Further by Theorem 1, the solution of the model at every iteration is a feasible solution to the overall problem. Therefore, it follows that:

(65) $Z(t) \geq Z_0$ for all t.

Since the monotonically decreasing series seen in equation (64) is bounded from below by equation (65), the existence of $t^*$ follows such that for any $\varepsilon > 0$,

(66) $| Z(t^*+\tau) - Z(t^*) | < \varepsilon$ for $\tau \geq 0$.

Q.E.D.

The following corollary was derived in the proof of Theorem 2 and will be given without further proof.

COROLLARY 1. The following characteristics will exist at convergence of the model on iteration t*: $X_i(t^*+1) = X_i^*(t^*+1)$ for $i = 1,\ldots,N$. That is, the optimal proposal generated by managing subsystem k (k=1,...,M) the infimal subsystem i ($i=r_{k-1}+1,\ldots,r_k$) at iteration t*+1 is identical to the optimal proposal generated by infimal subsystem i.

Equations (59.i) and (63) verify Corollary 1. Further, equation (63) demonstrates that at convergence solutions within the hierarchy of the GHM will be identical for iterations $(t^*+\tau)$ $(\tau \geq 0)$.

## References


1. Baumol, W. J. and Fabian, T., "Decomposition, Pricing for Decentral ization and External Economies," Management Science, Vol. 11, No. 1 (September 1964), pp. 1-31.

2. Ben Afia, K., "Computational Testing and Improvement of a Multilevel Decomposition Model," Unpublished Ph.D. Dissertation, University of Illinois at Urbana-Champaign, Urbana, Illinois (1980).

3. Benders, J. F., "Partitioning Procedures for Solving Mixed-Variables Programming Problems," Numerische Mathematik, Vol. 4, No. 3 (January-February 1960), pp. 238-252.

4. Burton, R. M. and Obel, B., "The Efficiency of the Price, Budget, and Mixed Approaches under Varying a priori Information Levels for Decentralized Planning," Management Science, Vol. 26, No. 4 (April 1980), pp. 401-417.

5. Christensen, J. and Obel, B., "Simulation of Decentralized Planning in Two Danish Organizations Using Linear Programming Decomposition," Management Science, Vol. 24, No. 15 (November 1978), pp. 1658-1667.

6. Dantzig, G. B. and Wolfe, P., "Decomposition Principles for Linear Programs," Operations Research, Vol. 8, No. 1 (February 1960), pp. 101-111.

7. Dantzig, G. B., Linear Programming and Extensions, Princeton University Press, New Jersey, 1963, pp. 448-469.

8. Davis, W. J., "Three Level Echelon Models for Organizational Coordination," Unpublished Ph.D. Dissertation, Purdue University, West Lafayette, Indiana (1975).

9. Davis, W. J. and Talavage, J. J., "Three Level Models for Hierarchical Organization," OMEGA, Vol. 5, No. 6 (December 1977), pp. 709-720.

10. Davis, W. J., "A Generalized Decomposition Procedure and Its Application to Engineering Design," Journal of Mechanical Design, Vol. 100, No. 4 (October 1978), pp. 739-746.

11. Davis, W. J. and Ben Afia, K., "Two Level Decomposition Models via Linear and Quadratic Penalty Functions and Sequential Goal Generation," Engineering Working Paper, 1980, University of Illinois at Urbana-Champaign.

12. Dirickx, Y. M. I. and Jennergren, L. P., Systems Analysis by Multi-Level Methods, John-Wiley, New York, 1979.

13. Freeland, J. R., "Conceptual Models for the Resource Allocation Decision Process in Hierarchical Decentralization Organizations," Unpublished Ph.D. Dissertation, Georgia Institute of Technology, Atlanta, Georgia, 1973.

14. Freeland, J. R. and Baker, N. R., "Goal Partitioning in a Hierarchical Organization," OMEGA, Vol. 3, No. 6 (December 1975), pp. 673-688.

15. Freeland, J. R., "A Note on Goal Decomposition in a Decentralized Organization," Management Science, Vol. 23, No. 1 (September 1976), pp. 100-102.

16. Geoffrion, A. M., "Large-Scale Linear and Nonlinear Programming," in Optimization Methods for Large-Scale Systems, Wesmer, D. A., Ed., McGraw-Hill, New York, 1971, pp. 47-74.

17. Geoffrion, A. M., "Elements of Large-Scale Mathematical Programming," Parts I and II, Management Science, Vol. 16, No. 11 (July 1970), pp. 652-691.

18. Jennergren, L. P., "Studies in the Mathematical Theory of Decentralized Resource-Allocation," Unpublished Ph.D. Dissertation, Stanford University, Stanford, California, (1971).

19. Lasdon, L. S., Optimization Theory for Large Systems, Macmillan, New York, 1970.

20. Marsten, R. E., "XMP: A Structural Library of Subroutines for Experimental Mathematical Programming," Technical Report #351, 1978, Department of MIS, University of Arizona.

21. Mesarović, M. D., Macko, D., and Takahara, Y., Theory of Hierarchical Multilevel Systems, Academic Press, New York, 1970.

22. Phyrr, P. A., "Zero-Base Budgeting," Harvard Business Review, Vol. 48, No. 6 (November-December 1970), pp. 111-121.

23. Ruefli, T. W., "A Generalized Goal Decomposition Model," Management Science, Vol. 17, No. 8 (April 1971), pp. 505-518.

24. Ruefli, T. W., "PPBS--An Analytic Approach," in Budgeting of Interrelated Activities, Vol. II, Byrne, F. R., et. al., Eds., North Holland, Amsterdam, 1971, pp. 161-209.

25. Ruefli, T. W., "Analytical Models of Resource-Allocation in Hierarchical Multilevel Systems," Socio-Economic Planning Sciences, Vol. 8, No. 6 (December 1974), pp. 353-363.

26. Schiff, M. and Lewin, A. Y., "Where Traditional Budgeting Fails," Financial Executive, Vol. 36, No. 5 (May 1968), pp. 57-62.

27. Sweeny, D. J., Winkofsky, E. P., Roy, P., and Baker, N. R., "Composition vs. Decomposition: Two Approaches to Modelling Organizational Decision Processes," Management Science, Vol. 24, No. 14 (October 1978), pp. 1491-1499.

28. Whitford, D. T., "Resource Allocation in a University Setting: A Generalized Hierarchical Goal Decomposition Model," Unpublished Ph.D. Dissertation, Georgia State University, Atlanta, Georgia, 1980.

29. Whitford, D. T., "A Generalized Hierarchical Goal Decomposition Model of Resource Allocation within a University," College of Commerce Working Paper #705, 1980, University of Illinois at Urbana-Champaign.

30. Whitford, D. T. and Davis, W. J., "Resource Allocation in a University Environment: A Test of the Ruefli, Freeland, and Davis Goal Programming Decomposition Algorithms," College of Commerce Working Paper #735, 1980, University of Illinois at Urbana-Champaign.

31. Williamson, O. E., The Economics of Discretionary Behavior: Managerial Objectives in a Theory of the Firm, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

32. Winkofsky, E. P., Baker, N. R., and Sweeny, D. J., "A Decision Process Model of R&D Resource Allocation in Hierarchical Organizations," Management Science, Vol. 27, No. 3 (March 1981), pp. 268-283.

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