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Computational Complexity of Some Problems in Parametric Discrete Programming: I

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Abstract

We explore the computational complexity of certain issues regarding parametric linear and integer linear programming.

For example, we demonstrate that: (1) The equality of optimal value of two integer programs for all right-hand-sides (r.h.s.), is NP-complete either when the problem is stated in matrix or in functional form; (2) The equality of optimal value of two linear programs for all r.h.s. in matrix form is polynomial, but it becomes NP-complete when one desires equality for all r.h.s. in a polyhedral cone described by generators; (3) The equality of a general polyhedral function (allowing nested "maxes") to the value of a linear program in matrix form, or to another polyhedral function, is NP-complete; (4) The shortest expression, for the optimum to the subadditive dual of an integer program in matrix form, can require exponential space.

Key Words

1. Integer programming
2. Parametric programming
3. Computational complexity

COMPUTATIONAL COMPLEXITY OF SOME PROBLEMS IN PARAMETRIC DISCRETE PROGRAMMING: I

by C. E. Blair and R. G. Jeroslow[1]

This paper treats some problems of computational complexity in connection with the <u>mixed-integer program</u>:

$$(\mathrm{MIP}_b) \qquad \begin{aligned} &\inf\ cx + dy \\ &\text{subject to } Ax + By = b \\ &\qquad x, y \geq 0 \\ &\qquad x \text{ integer} \end{aligned}$$

The constraint matrices A, B and objective functions c, d will be assumed throughout to be fixed and rational. The right-hand-side (r.h.s.) vector b will vary.

The main focus of the paper is the two special cases of (MIP_b) in which either the integer variables x are entirely absent (A and c are empty), called the <u>linear program</u>, or the continuous variables y are entirely absent (B and d are empty), called the <u>(pure) integer program</u>. (Invariably, we drop the adjective "pure" for a pure integer program.) In Section 3, the <u>premultiplied</u> mixed-integer program is also of interest, which arises from $(\mathrm{MIP})_b$ when the r.h.s. "b" is replaced by "Cb" for a fixed rational matrix C (but again b varies). Pre-multiplied linear and integer programs arise in the same way, and also occur in the paper.

This paper requires some knowledge of both the variation of (MIP_b) with changes in the r.h.s., and computational complexity. In Section 2 we give relevant background for parametric programming. Our earlier papers [2], [3], [4], [5] are a primary source of results on this topic.

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[1]Work of the second author partially supported by NSF grant ECS 8001763.

[8] is our primary source for background on computational complexity.

To date, computational complexity measures have been applied to the study of various classes of programming problems. However, there are few results about parametric programming questions, such as ascertaining the validity of statements of the form:

(1.1) "For <u>every</u> r.h.s. b, the optimal values of the two linear programs $\min\{cx \mid Ax = b,\ x \geq 0\}$ and $\min\{\bar{c}\bar{x} \mid \bar{A}\bar{x} = b,\ \bar{x} \geq 0\}$ are equal"

The new logical quantifier "for every" appears a priori to complicate matters, generally raising complexity.

Perhaps the "surprise" of the results given here are some instances in which the complexity does not increase, as well as instances (like (1.1)) in which a parametric linear programming question is no less complex than its integer counterpart. In other instances, of course, complexity is strictly increased by the "for every" quantifier, at least if certain proposed hierarchies have distinct levels; we shall report on this phenomenon in a later paper.

The present paper is not intended as a complete study of all parametric questions, but rather it collects together several of the "lower complexity" results which can be obtained by methods developed in [4] and [5].

2. Background

In this section we give definitions and theorems which will be needed later. We will always assume that if $Ax + By = \vec{0}$ and $x, y \geq 0$ then $cx + dy \geq 0$. This implies (non-trivial, see [14] or [4, section 4]) that for any b either (MIP_b) has no feasible solutions or else there is an optimal solution. For given A, B, c, d the value function F(b) = value of optimal solution if (MIP_b) is feasible; $+\infty$ if (MIP_b) is not feasible.

Some of our work will deal with F(b) which are finite everywhere, an approach which is partly justified by the next result.

Theorem 2.1: [2, Theorem 4.6] Let F be the value function determined by A, B, c, d. There is a value function G defined by A', B', c', d' such that G(b) = F(b) where F(b) is finite, and G(b) is finite everywhere.

For pure integer programs we do not have the continuous variables y or the matrix B, but the idea of a value function is the same.

We will say that a given function $F: R^n \to R$ is a value function if a suitable A, B, c, d can be found.

For any natural number m we define the class of Gomory functions of m variables to be the smallest class of functions G^m such that

(i) If $\lambda \in Q^m$ then $F(b) = \lambda b$ is in G^m;

(ii) If $\alpha \geq 0$, α rational and $F \in G^m$ then $\alpha F \in G^m$;

(iii) If $F \in G^m$, $G(b) = \lceil F(b) \rceil$ = smallest integer $\geq F(b)$ is in G^m;

(iv) If $F, G \in G^m$ then $F + G \in G^m$;

(v) If $F, G \in G^m$ and $H(b) = \max\{F(b), G(b)\}$ then $H \in G^m$.

The Chvatal functions of m variables are the smallest class of functions C^m satisfying (i)-(iv). If we insist that all λ used in (i) have non-negative components we have the class of monotone Gomory or Chvatal functions MG^m or MC^m.

Each member of G^m or C^m can be defined by an expression using ceiling operations, plus signs, and so forth. We will occasionally refer to the length of a defining expression, which can be defined in a natural way.

Proposition 2.2: ([4]) Every member of G^m is the maximum of finitely many members of C^m. Moreover, the length of each defining expression for the members of C^m may be $\leq$ the length of the shortest defining expression for G^m.

It should be noted that the number of Chvatal functions needed to represent a Gomory function can grow exponentially as a function of the size of the expression of the Gomory function. This occurs when the Gomory function is the sum of many functions which are maxima.

The class of Gomory functions is identified with the class of value functions for pure integer programs by two results:

Theorem 2.3: ([4, Theorem 5.2]) For any A, c there is a Gomory function G, such that for all feasible b, G(b) is the objective function value of the optimal solution to the integer program with right-hand-side b.

Theorem 2.4: ([4, Theorem 3.13]) For any Gomory G there are A, c such that G(b) is the value of the optimal solution to (IP_b) for all integer vectors b.

If we fix a specific right-hand-side b, a Chvatal function suffices.

Corollary 2.5: For A, C, b given there is a Chvatal function F such that: (i) F(b) = value of optimal solution; (ii) if a^j = jth column of A then $F(a^j) \leq c_j$.

Proof: By Theorem 2.3 there is a Gomory function satisfying (i) and (ii). By Proposition 2.2 this function is a maximum of Chvatal functions. We choose the appropriate Chvatal function for the given b.

Q.E.D.

We will show in Section 6 that the size of the expression required in Corollary 2.5 may grow exponentially as a function of the size of A, c,b.

We will be concerned with placing problems related to value functions, Gomory functions, etc. within the polynomial-time hierarchy. We will use the theory as expounded in [8]. We have already referred to the size of matrices, of expressions defining functions, etc. The assumption is always made that the data are all rational. As usual, the precise definition of size is not crucial for NP-completeness results.

We will need to use the celebrated result of Khacian that the consistency of a linear program can be decided in polynomial time. The following result studies a parametric linear programming problem.

Theorem 2.6: ([10]) The following problem is NP-complete:

Instance: m·n matrix A, m·k matrix B, $b_0 \in Q^m$.

Question: Does there exist $c \in \{1,-1\}^m$ such that

$$\max \{cx \mid Ax + By = b_0,\ x,\ y \geq \vec{0}\} \neq \sum_1^m \max \{0, c_i\}?$$

Finally, we shall need the result of Borosh and Treybig placing integer program consistency in NP.

Theorem 2.7: ([6]) There is a polynomial q such that, given any system of linear inequalities of size S, the system has a solution in non-negative integer variables if and only if it has a non-negative integer solution of size $\leq q(S)$.

3. Results on Matrix Presentation of Parametric Problems

The results for a purely algebraic presentation of problems (i.e., utilizing only matrices and criterion vectors) are more accessible, as these do not require preliminary lemmas on function classes, etc. Consequently, we present these first, in this section.

The main development of this section is a parametric linear problem which is in P (Theorem 3.3) but becomes NP-complete in its pre-multiplied form (Theorem 3.11). Also of interest is a parametric integer problem which stays in NP (Theorem 3.5).

Lemma 3.1:

Let A and $\bar{A}$ be matrices with n respectively $\bar{n}$ columns $A = [a^{(j)}]$, $\bar{A} = [\bar{a}^{(j)}]$, having the same number of rows.

Then the assertion:

(3.1) "For all r.h.s. b, whenever there is $x \geq 0$ with $Ax = b$, then also there is $\bar{x} \geq 0$ with $\bar{A}\bar{x} = b$,"

is true if and only if the following assertion is true:

(3.2) "For all $j = 1, \ldots, n$ there is $\bar{x} \geq 0$ with $\bar{A}\bar{x} = a^j$."

This result remains valid when x and $\bar{x}$ are also required to be integer vectors in both (3.1) and (3.2).

Proof: The necessity of (3.2) is clear, since $Ax = a^j$ when x is the j-th unit vector. As to the sufficiency, if $\bar{x}^{(j)} \geq 0$ solves $\bar{A}\bar{x} = a^{(j)}$, and $x = (x_1, \ldots, x_n) \geq 0$ is such that $Ax = b$, then $\bar{A}(\sum_{n=1}^{n} x_j \bar{x}^{(j)}) = b$ and $\bar{x} = \sum_{j=1}^{n} x_j \bar{x}^{(j)}$ is a non-negative $\bar{n}$ vector. The same proof works when integrality is required of $\bar{x}^{(j)}$ and x, for then $\bar{x}$ is integer.

Q.E.D.

Theorem 3.2:

Let L_1 be the language of all quadruples $\langle A, c, \bar{A} \bar{c} \rangle$ of rational n respectively $\bar{n}$ vectors c resp. $\bar{c}$, and rational matrices A resp. $\bar{A}$ of n resp. $\bar{n}$ columns and the same number of rows, such that the following holds:

(3.3) "For all r.h.s. b, $\min\{cx \mid Ax = b, x \geq 0\}$ equals or exceeds $\min\{\overline{cx} \mid \overline{Ax} = b, \bar{x} \geq 0\}$."

Then L_1 is recognizable in polynomial time.

Remark: In assertions like (3.3) are included, by implication, minimum values of $+\infty$ (inconsistency) and $-\infty$.

Proof: Note that (3.3) holds exactly if the following holds:

(3.4) "For all z and b, whenever there is $x \geq 0$ with $Ax = b$, $cx \leq z$, then also there is $\bar{x} \geq 0$, $\overline{Ax} = b$, $\overline{cx} \leq z$."

With $A = [a^{(j)}]$ (cols.) and $c = (c_1, \ldots, c_n)$, by Lemma 3.1 we have (3.4) equivalent to:

(3.5) "For each $j = 1, \ldots, n$ there is an $\bar{x}^{(j)} \geq 0$ with $\overline{Ax}^{(j)} = a^{(j)}$, $\overline{cx}^{(j)} \leq c_j$."

By Khachian's result, (3.5) can be determined in polynomial time, since the number n of applications of Theorem 2.5 is less than linear in the length of $\langle A, c, \bar{A}, \bar{c} \rangle$. In fact, since sparse storage is ruled out, there is at least one word divider in the encoding of A for each column.

Q.E.D.

Theorem 3.3:

Theorem 3.2 still holds if the words "or exceeds" is dropped in (3.3). I.e., one can recognize, in polynomial time, whether two linear programs are equal in value for all r.h.s. b.

Proof: Two applications of Theorem 3.2, for both $z(b) \leq \bar{z}(b)$ and $\bar{z}(b) \leq z(b)$, require polynomial time. Here $z(b)$ and $\bar{z}(b)$ denote the respective value functions.

Q.E.D.

Theorem 3.4:

Let L_2 be the language of all quadruples $\langle A, c, \bar{A}, \bar{c} \rangle$ of rational n respectively $\bar{n}$ vectors c resp. $\bar{c}$, and rational matrices A resp. $\bar{A}$ of n resp. $\bar{n}$ columns and the same number of rows, such that the following holds:

(3.6) "For all r.h.s. b, $\min\{cx \mid Ax = b,\ x \geq 0,\ x \text{ integer}\}$ exceeds or equals $\min\{\overline{cx} \mid \overline{Ax} = b,\ \bar{x} \geq 0,\ \bar{x} \text{ integer}\}$."

Then L_2 is in NP.

Proof: We adapt the proof of Theorem 3.2.

Without loss of generality, and with a polynomial computation, we may assume that both c and $\bar{c}$ are integer. Then (3.6) holds exactly if:

(3.7) "For all b and integer z, whenever there is x_0, $x \geq 0$, both x_0 and x integer, with $Ax = b$, $xc + x_0 = z$, then also there is $\bar{x}_0$, $\bar{x} \geq 0$, both $\bar{x}_0$ and $\bar{x}$ integer, with $\overline{Ax} = b$, $\overline{cx} + \bar{x}_0 = z$."

By Lemma 3.1, we have (3.7) equivalent to:

(3.8) "For each $j = 1, \ldots, n$ there is an $\bar{x}^{(j)} \geq 0$, with $\bar{x}^{(j)}$ integer, and $\overline{Ax}^{(j)} = b$, $\overline{cx}^{(j)} \leq c_j$."

For each $j = 1, \ldots, n$ separately, there is a polynomial length of "guess" adequate to determine whether or not there is an $\bar{x}^{(j)} \geq 0$ integer with $\overline{Ax}^{(j)} = b$, $\overline{cx}^{(j)} \leq c_j$, by Theorem 2.6. The length n^2 vector $\tilde{x} = (\bar{x}^{(1)}, \ldots, \bar{x}^{(n)})$ long enough to store all possible guesses, need only be polynomial length. Hence the determination of (3.8) is in NP.

Q.E.D.

Theorem 3.5:

Define L_3 as in Theorem 3.3 with "exceeds or" deleted in (3.6). I.e., L_3 represents the problem of determining the equality of optimal value of two integer programs, for all r.h.s. b.

Then L_3 is NP-complete.

Proof: By Theorem 3.4, L_3 is the intersection of two NP sets; hence L_3 is in NP.

To prove that L_3 is NP-complete, it suffices to reduce any NP-complete set S to L_3 by a many-one function reduction. We chose for S the language of pairs A, b_0 defined in Theorem 2.7. I.e., $\langle A, b_0 \rangle \in S$ iff there is an integer $x \geq 0$ with $Ax = b_0$.

For an arbitrary matrix D, consider the question as to whether the value functions of the following two programs are equal:

$$\begin{aligned} &\min x_{n+1} + 2 \sum_{i=1}^{m} (z_i^+ + z_i^-) \\ (3.9) \qquad &\text{subject to } Dx + b_0 x_{n+1} + Iz^+ - Iz^- = v \\ &x, x_{n+1}, z^+, z^- \geq 0 \text{ and integer} \end{aligned}$$

and

$$\begin{aligned} &\min 2 \sum_{i=1}^{m} (z_i^+ + z_i^-) \\ (3.10) \qquad &\text{subject to } Dx + Iz^+ - Iz^- = v \\ &x, z^+, z^- \geq 0 \text{ and integer} \end{aligned}$$

In (3.9) and (3.10), D is m by n, $x = (x_1, \ldots, x_n)$, $z^+ = (z_1^+, \ldots, z_m^+)$, $z^- = (z_1^-, \ldots, z_m^-)$, $v = (v_1, \ldots, v_m)$, and I denotes an m by m identity matrix.

We claim that the optimal values of (3.9) and (3.10) are equal for all r.h.s. v, if and only if $\langle D, b_0 \rangle \in S$. Since the construction of the matrices and criterion vectors in (3.9) and (3.10) can be done in polynomial time, it defines a polynomial time function $f(\langle D, b_0 \rangle)$. If our claim is true, $\langle D, b_0 \rangle \in S$ iff $f(\langle D, b_0 \rangle) \in L_3$, and we will have established N -completeness of L_3.

If $\langle D, b_0 \rangle \in S$, let $x_0 \geq 0$ be an integer vector with $Dx_0 = b_0$. Then for any feasible solution to (3.9) with $x_{n+1} \geq 1$ we can, by changing x in the solution to $x + x_0 \geq 0$, obtain a new solution with x_{n+1} decreased by one unit. As this change strictly decreases criterion value (since z^+ and z^- are unchanged), in any optimal solution to (3.9) we have $x_{n+1} = 0$. Thus, if (3.9) has an optimal solution, it is also an optimal solution to (3.10). If (3.9) is inconsistent, so is (3.10). Note that (3.9) cannot be unbounded below in value (as the criterion value is always non-negative). Thus, in all cases, the value of (3.9) is that of (3.10); and our analysis is independent of v.

Suppose, on the other hand, that $\langle D, b_0 \rangle \notin S$. Then (3.9) has an optimal value of one for $v = b_0$, since $x = 0$, $x_{n+1} = 1$, $z^+ = z^- = 0$ is a solution, and no solution can have criterion value zero (for if $x_{n+1} = 0$, $z^+ = z^- = 0$, then $Dx = b_0$, contradicting $\langle D, b_0 \rangle \notin S$). Also, (3.10) has an optimal value at least two for $v = b_0$, as $z^+ = z^- = 0$ in a solution gives $Dx = b_0$, a contradiction. Thus the value function of (3.9) and (3.10) are unequal for $v = b_0$.

Q.E.D.

We next return to the problem of parametric linear programming, but in a pre-multiplied form. It is also significant that negation ("not") occurs in (3.11).

Lemma 3.6:

Let L_4 be the language of all quadruples $\langle A, c, \bar{A}, \bar{c}, \bar{D} \rangle$ of rational matrices A, $\bar{A}$, $\bar{D}$ and vectors c, $\bar{c}$ compatibly dimensioned, such that the following holds:

(3.11) "For some b, the value of $\min\{cx \mid Ax = b, x \geq 0\}$ does not equal that of $\min\{\overline{cx} \mid \overline{Ax} = \bar{D}b, x > 0\}$."

Then any set $S \in NP$ is polynomial time reducible to L_4 by a many-one function computation.

This result remains true if, in the definition of L_4, the matrix $\bar{D}$ is restricted to have only unit and zero rows. Furthermore, if one desires, A and c an be restricted to derive from the linear program:

$$\begin{aligned} &\min \sum_{j \in J} z_j^+ \\ (3.12) \qquad &\text{subject to } z_j^+ - z_j^- = b_j,\ j \in J, \\ &z^+, z^- \geq 0 \end{aligned}$$

where J is the set of indices for which $\bar{D}$ has a unit row. In addition, "some b" can (optionally) be replaced by "some b with coordinates ± 1." We can similarly require coordinates of 0, ± 1.

Proof: It suffices, for arbitrary $S \in NP$, to provide the necessary polynomial-time reduction to the sublanguage of L_4 which is defined by all the restrictions in connection with (3.12).

Note that the value function of (3.12) is $\sum_{j \in J} \max\{0, b_j\}$. Our result follows at once, by applying Theorem 2.6 in connection with the dual of the linear program cited there.

Q.E.D.

Lemma 3.7:

Let A, $\bar{A}$ and $\bar{D}$ be matrices, suitably dimensioned. Then the assertion:

(3.13) "For all r.h.s. b, whenever there is $x \geq 0$ with $Ax = b$, then also there is $\bar{x} \geq 0$ with $\overline{Ax} = \overline{Db}$"

is true if and only if the following assertion is true, where $A = [a^{(j)}]$ (cols.):

(3.14) "For all $j = 1, \ldots, n$ there is $\bar{x} \geq 0$ with $\overline{Ax} = \bar{D}a^{j}$."

This result remains valid when both x and $\bar{z}$ are required to be integer in (3.13) and (3.14).

Proof: Similar to that for Lemma 3.1.

Q.E.D.

By the following result, the direction "$\geq$" of the inequality cited in (3.11) is polynomial-time.

Theorem 3.8:

Let L_5 be the language of all quadruples $\langle A, c, \bar{A}, \bar{c}, \bar{D} \rangle$ of rational matrices A, $\bar{A}$, $\bar{D}$ and vectors c, $\bar{c}$ compatibly dimensioned, such that the following holds:

(3.15) "For all b, the value of $\min\{cx \mid Ax = b, x \geq 0\}$ equals or exceeds that of $\min\{\overline{cx} \mid \overline{Ax} = \bar{D}b, \bar{x} \geq 0\}$.

Then $L_5 \in P$.

Proof: Note, using Lemma 3.8, that (3.15) holds if and only if for $j = 1, \ldots, n$ there is a solution $\bar{x}^{(j)}$ to

$$\overline{cx} \leq c_j$$

(3.16) $$\overline{Ax} = \overline{Da}^{(j)}$$

where $A = [a^{(j)}]$ (cols.) and $c = (c_1, \ldots, c_n)$. By Khacian's result each instance of (3.16) is decidable in polynomial time, and only n instances are involved.

Q.E.D.

Our next result, which is needed in what follows, states that a certain specific kind of nonlinear consistency problem lies in NP.

<u>Lemma 3.9</u>: The language L_6 of all six-tuples $\langle P, Q, R, p, q, a_0 \rangle$ of rational matrices P, Q, R, rational vectors p, q and a rational scalar a_0, such that the following assertion holds:

(3.17) "There are Θ, w with

$$P\Theta \geq p$$

$$Qw \geq q$$

and $w^{tr}R\Theta > a_0$"

is in NP.

<u>Proof</u>: The polyhedron $P = \{\Theta | P\Theta \geq p\}$, if non-empty, has a finite basis [14]:

(3.18) $$P = \text{conv}\{\Theta^a | a \in A\} + \text{cone}\{\Theta^{-b} | b \in B\}$$

for non-empty finite index sets A, B. Assuming $P \neq \phi$, (3.17) holds if and only if there are scalars $\alpha_a \geq 0$, $a \in A$, and $\beta_b \geq 0$, $b \in B$, with $\sum_{a \in A} \alpha_a = 1$ and a vector w with:

(3.19) $$Qw \geq q$$

$$w^{tr}R\left(\sum_{a \in A} \alpha_a \Theta^a + \sum_{b \in B} \beta_b \Theta^{-b}\right) > a_0$$

One way for (3.19) to hold is for there to exist $\bar{\Theta}^b$, $b \in B$, and w with:

$$(3.20) \qquad \begin{aligned} &Qw \geq q \\ &w^{tr}R\bar{\Theta}^{b} > 0 \end{aligned}$$

Indeed, if (3.20) holds, for any $\Theta^0 \in P$ and $\sigma \geq 0$ large enough, $w^{tr}R(\Theta^0 + \sigma\Theta^{-b}) > a_0$ and $\Theta^0 + \sigma\bar{\Theta}^b \in P$. Otherwise, when (3.20) has no solution w for any $\bar{\Theta}^b$, in (3.19) we must actually have $w^{tr}R(\sum_{a A} \alpha_a\Theta^a) > a_0$ (since all $\beta_b \geq 0$). By $\sum_{a\in A} \alpha_a = 1$, there must exist $a \in A$ with $w^{tr}R\Theta^a > a_0$, i.e.,

$$(3.21) \qquad \begin{aligned} &Qw \geq q \\ &w^{tr}R\Theta^{a} > a_0 \end{aligned}$$

is consistent. Thus, if $P \neq \phi$, (3.17) holds if and only if (3.21) is consistent for some $a \in A$ or (3.20) is consistent for some $b \in B$.

The vectors Θ^a, $a \in A$, can be further specified as follows. These arise exactly as vectors of the form $\Theta^a = \Theta^1 - \Theta^2$ where (Θ^1, Θ^2) is a basic feasible solution to the linear system

$$(3.22) \qquad \begin{aligned} &P\Theta^1 - P\Theta^2 \geq p \\ &\Theta^1, \Theta^2 \geq 0 \end{aligned}$$

Similarly, the vectors $\bar{\Theta}^b = \Theta^1 - \Theta^2$ arise from basic feasible solutions of

$$(3.23) \qquad \begin{aligned} &\sum_i \Theta_i^1 + \sum_i \Theta_i^2 = 1 \\ &P\Theta^1 - P\Theta^2 \geq 0 \\ &\Theta^1, \Theta^2 \geq 0 \end{aligned}$$

These facts follow from the usual construction of a finite basis. While not all finite bases arise this way, at least one does.

We are now ready to give our proceedure for testing the validity of the assertion (3.17).

In polynomial time, we can test to see if $P \neq \phi$. If $P = \phi$, (3.17) is false, and no guesses are needed. If $P \neq \phi$, we proceed to make guesses.

Each guess consists of a pair of sets of columns, one for (3.22) and one for (3.23). Bases are identified by testing subsets of columns for linear independence, which is done in polynomial time (if a chosen subset fails this test, the overall proceedure fails). Each basis is then tested for feasibility; if the first is infeasible, the whole proceedure fails. If the second basis is infeasible, we put $\bar{\Theta}^b = 0$.

If we obtain two guesses Θ^a and $\bar{\Theta}^b$, we proceed to test both in (3.20) and (3.21) for in polynomial time. If either one of these systems is consistent, the proceedure is successful for these guesses; otherwise it fails.

Clearly, the proceedure has at least one success if at least one of the systems (3.20) or (3.21) are consistent. By our analysis, the latter event occurs exactly if (3.17) is true.

Q.E.D.

Lemma 3.10:

The language L_7 of all six tuples $\langle A, \bar{A}, D, \bar{D}, c, \bar{c} \rangle$ of rational matrices A, $\bar{A}$ D, $\bar{D}$ and vectors c, $\bar{c}$ such that the following assertion is true:

(3.24) "For some vector b, $\min\{\overline{cx} \mid \overline{Ax} = \bar{D}b,\ x \geq 0\}$ is strictly less than $\min\{cx \mid Ax = Db,\ x \geq 0\}$,"

is in NP.

Proof: Suppose that (3.24) is true.

When the second minimum is $+\infty$ (indicating inconsistency) for the vector b involved, the first cannot be $+\infty$, and so we must have a solution to the system:

$$\Theta A \leq c$$

$$\Theta Db > 0$$

$$A\bar{x} - \bar{D}b = 0$$

(3.25) $\quad \bar{x} \geq 0$

(The constraints $\Theta A \leq c$, $\Theta Db > 0$ are equivalent to the $+\infty$ value for consistency.) When the second minimum is finite for the vector b involved, this second minimum value is equal to $\max\{\Theta Db | \Theta A \leq c\}$. Hence there is then a solution to the system:

$$\Theta A \leq c$$

(3.26) $\quad \overline{cx} - \Theta Db < 0$

$$\overline{Ax} = \overline{Db}$$

$$\bar{x} \geq 0$$

Note that the second minimum cited in (3.24) cannot be $-\infty$ for the vector b involved.

Thus, the truth of (3.24) entails the consistency of either (3.25) or (3.26). The converse entailment is also easily verified. By Lemma 3.9, the condition (3.25) respectively (3.26) describe sets S_1 resp. S_2 which are in NP, are polynomial-time attainable from L_7. Thus, $S_1 \cup S_2$ is also in NP, and is equivalent to (3.24).

Q.E.D.

Theorem 3.11:

The language L_4 of Lemma 3.6 is NP complete.

<u>Proof</u>: By Lemma 3.6, we need only prove that $L_4 \in NP$. But $\langle A, c, \bar{A}, \bar{c}, \bar{D} \rangle \in L_4$ if and only if either $\langle A, \bar{A}, I, \bar{D}, c, \bar{c} \rangle \in L_7$ or $\langle \bar{A}, A, \bar{D}, I, \bar{c}, c \rangle \in L_7$. This fact gives a polynomial time function reduction of L_4 to the set $(L_7 \times \Sigma^*) \cup (\Sigma^* \times L_7)$, where Σ^* is the set of all words in the alphabet Σ of L_7. Since both $L_7 \times \Sigma^*$ and $\Sigma^* \times L_7$ are in NP by Lemma 3.10, so is L_4.

Q.E.D.

4. Some Complexity Results from Non-matrix Representation of Functions.

We begin with the problem of determining, given representations of two polyhedral functions F, G whether or not they define the same function. We shall see that the complexity of the problem depends on the type of representation allowed.

Theorem 4.1: Consider functions of the form $f_i(b) = w_i b + \sum_{j=1}^{n} |\lambda_{ij} b|$ where w_i, λ_{ij}, $b \varepsilon Q^m$. Let $F(b) = \max_{1 \leq i \leq L} f_i(b)$. The problem of determining whether $F(b_1, \ldots b_m) = \Sigma |b_j|$ for all b is NP-complete.

Proof: It is easy to show that the inequality is in NP. We can obtain this as a corollary of Theorem 5.2 in Section 5. To complete the proof we reduce our problem to a form of the NP-complete satisfiability problem:

Instance: functions $h_i(b_1, \ldots b_m) = \Sigma \alpha_{ij} b_j$ where $\alpha_{ij} = 0$, 1 or -1; $1 \leq i \leq L$, $1 \leq j \leq m$.

Question: Does there exist b such that $h_i(b) < \Sigma |\alpha_{ij} b_j|$ for $1 \leq i \leq L$?

Each b_j corresponds to a variable in the satisfiability problem-- $b_j < 0$ corresponds to value T, $b_j > 0$ to F. h_i corresponds to the ith clause with $\alpha_{ij} = 1$ if the jth variable appears $\alpha_{ij} = -1$ if the negation of the jth variable appears. The collection of clauses is satisfiable if and only if the answer to our question is yes.

Now we describe the reduction. Given h_i as in the instance define $f_i(b) = h_i(b) + \sum_{j=1}^{n} |\beta_{ij} b_j|$ where $\beta_{ij} = 1 - |\alpha_{ij}|$. Then $F(b) \neq \Sigma |b_j|$ iff, for some b, $F(b) < \Sigma |b_j|$ iff the answer to our question is yes.

Q.E.D.

Corollary 4.2: Let F be defined as in 4.1.

The problem of determining whether or not $F(b) = \min\{cx | Ax = b, x \geq 0\}$. for all b is NP-complete.

Proof: We choose c, A so that $\min\{cx | Ax = b\} = \Sigma |b_j|$ and apply Theorem 4.1 to establish NP-hardness. Again, showing that inequality is in NP is easy.

Q.E.D.

A different situation arises when we insist that the polyhedral functions be explicitly given as maxima of linear functions.

Theorem 4.3: The problem of deciding whether or not $F(b) = \max\{\lambda_1 b, \ldots \lambda_n b\} \leq G(b) = \max\{\lambda_1' b, \ldots \lambda_L' b\}$ is decidable in polynomial time.

Proof: From familiar results in linear inequality theory $F \leq G$ for all b if and only if $\lambda_i \in \mathcal{H}(\lambda_1', \ldots \lambda_L')$ for $1 \leq i \leq n$. For each i this can be determined in polynomial time by the Khacian algorithm.

Q.E.D.

Corollary 4.4: With F and G as in 4.3 we can determine whether or not F = G in polynomial time.

Next we present a partial result for the problem of deciding whether a maximum of linear forms and the value function of a linear program represent the same polyhedral function.

Theorem 4.5: Let $F(b) = \max\{\lambda_1 b, \ldots \lambda_N b\}$ and $G(b) = \min\{cx | Ax = b, x \geq 0\}$. Suppose that: (i) the LP in G is feasible for all b; (ii) $G(\vec{0}) \neq -\infty$ (hence $G(b) \neq -\infty$ for all b); (iii) no dual degeneracy is present, i.e., c is a linear combination of a subset of the rows of A only if all rows of A are linear combinations of that subset.

Then we can decide whether $F(b) = G(b)$ for all b in polynomial time.

Proof: We study the dual polyhedron $P = \{\lambda | \lambda A \leq c\}$. First we test whether each λ_1 is a member. Then we must determine whether $\lambda_1, \ldots \lambda_N$ includes all the extreme points of P. It suffices to compute, for each extreme λ_i, all the adjacent extreme points (property (iii) ensures we can do this) and verify they are also on the list. Also, by (i), (ii) P is a polytope.

Q.E.D.

We conjecture that Theorem 4.5 is true without assumption (iii). Possibly the device of perturbing c to eliminate dual degeneracy can be used.

The analogue of polyhedral functions for integer programming is the class of Gomory functions. Here we will establish that the problem of determining whether two Chvatal functions are the same is NP-hard. Since the class of Gomory functions includes the Chvatal functions this establishes NP-hardness for Gomory functions. In Section 5 we will show that inequality between Gomory functions is in NP. These two results imply the problem is NP-complete.

Theorem 4.6: Consider the NP-complete* problem

Instance: Non-negative integers a_{ij}, b_{ij}, c_i, d_i; $1 \leq i \leq m$, $1 \leq j \leq n$

Question: Are there $x_j = 0$ or 1 such that

$$c_i + \sum_{j=1}^{n} a_{ij}x_j \geq d_i + \sum_{j=1}^{n} b_{ij}x_j \text{ for } 1 \leq i \leq m?$$

---

*This problem is clearly equivalent to the problem of whether a system of linear inequalities has a solution in zero-one variables.

We can construct in polynomial time Chvatal functionf F, G such that the answer to the question is yes if and only if there is a z such that $F(z) \neq G(z)$.

Proof: The variables we will use in our functions will be $r_1, \ldots r_n$; $s_1, \ldots s_L$. Define $x_i(r_i) = \lceil r_i \rceil + \lceil -r_i \rceil$, $w_i(s_i) = \lceil s_i \rceil + \lceil -s_i \rceil$. x_i and w_i are zero or one depending on whether or not r_i and s_i are integers. We will define all our other functions in terms of x_i and w_i, which play the role of "zero-one variables." Define $u(x,w) = \lceil \frac{1}{M}(\Sigma x_i + \Sigma w_i) \rceil$ where M is chosen so that $u = 0$ if all x_i, w_i are zero, and 1 otherwise. The function u will enable us to use the constant 1 in our formulas.

Next we define, for $1 \leq i \leq m$, Chvatal functions $F_i(x_1, \ldots x_n; w_1, \ldots w_Q)$ and $G_i(x_1, \ldots x_n; w_1, \ldots w_Q)$ such that:

I. If all x and w are 0 or 1 and $u \neq 0$ (hence $u = 1$) then $F_i(x,w)$ and $G_i(x,w)$ are either 1 or 2. If $u = 0$, $F_i = G_i = 0$.

II. If $c_i + \sum_j a_{ij}x_j < d_i + \sum_j b_{ij}x_j$ then $F_i(x,w) \leq G_i(x,w)$ for all w.

III. If $c_i + \Sigma a_{ij}x_j \geq d_i + \Sigma b_{ij}x_j$ then, for some binary w, $F_i(x,w) = 2$ and $G_i(x,w) = 1$.

The formulas for F_i, G_i are

$$F_i = \left\lceil \frac{1}{M}\left((1+c_i)u + \Sigma a_{ij}x_j + w_1 + 2w_2 + \ldots + 2^Q w_Q\right) \right\rceil$$

$$G_i = \left\lceil \frac{1}{M}\left(d_i u + \Sigma b_{ij}x_j + w_1 + 2w_2 + \ldots + 2^Q w_Q\right) \right\rceil$$

where $M = 2^{Q+1} \geq 1 + c_i + \Sigma a_{ij} + \Sigma b_{ij}$. Verification of I-III is straightforward. For different i, different w-variables are used.

Finally we define

$$F = \left\lceil \frac{1}{2m-1} \sum_{1}^{m} F_i \right\rceil$$

$$G = \left\lceil \frac{1}{2m+1} \sum_{1}^{m} F_i + \frac{1}{m(2m+1)} \sum_{1}^{m} G_i \right\rceil$$

If $u = 0$, $F_i = G_i = F = G = 0$. If $u \neq 0$, $F = 2$ if all $F_i = 2$ and $F = 1$ otherwise. If $u \neq 0$, and at least one $F_i = 1$, then $G = 1$. If all $F_i = 2$, then $G = 1$ if all $G_i = 1$, and $G = 2$ otherwise.

Thus $F(x,w) = G(x,w)$ unless all $F_i = 2$ and all $G_i = 1$. By properties II and III this can only occur if x_i is a solution to our zero-one system of inequalities.

Q.E.D.

5. Systems of Inequalities Involving Gomory Functions are in NP

The main result of this section is

Theorem 5.1: There is a polynomial p such that for any rationals a_{ij}, d_i and Gomory functions G_j: $R^n \to R$ the system of inequalities

$$\sum_{j=1}^{k} a_{ij}G_j(x) \geq d_i, \quad i \in I_1$$

(5.1) $(I_1, I_2$ disjoint and finite)

$$\sum_{j=1}^{k} a_{ij}G_j(x) > d_i, \quad i \in I_2$$

has a solution only if it has a solution $x \in Q^n$ of size $\leq p(S)$ where S is the sum of the sizes of a_{ij}, d_j, and the expression defining G_j.

The proof requires several steps. We begin by reducing the problem to one dealing with Chvatal functions.

Lemma 5.2: If G is an expression of size S defining a Gomory function and $y \in R^n$ there are Chvatal functions D, C_i, C_i' such that: (i) For all x, if x satisfies the inequalities

(5.2) $\quad C_i(x) \geq C_i'(x) \qquad i \in I$

then G(x) = D(x); (ii) y satisfies (5.2); (iii) the size of the expression defining D is $\leq$ S; (iv) The sum of the sizes of C_i, C_i' is bounded by a polynomial in S.

Proof: We argue by induction on the expression G. If G is linear we take D = G and I empty. If $G = \lceil G_1 \rceil$ the induction hypothesis gives D, C_i, C_i' corresponding to G_1 and we simply replace D by $\lceil D \rceil$. The case $G = \alpha G_1$ is treated similarly. If $G = G_1 + G_2$ the induction hypothesis gives D_1 corresponding to G_1, D_2 to G_2. We take $D = D_1 + D_2$. The inequality system for G is the union of the systems for G_1, G_2.

The interesting case is $G = \max\{G_1,G_2\}$. Without loss of generality we assume $G(y) = G_1(y) \geq G_2(y)$. We take $D = D_1$. The inequality system for G combines the systems for G_1, G_2 with the additional inequality $D_1(x) \geq D_2(x)$.

Corollary 5.3: There is a polynomial q such that for any system (5.1) of size S, if (5.1) has a solution there is a system (5.1)' of size $\leq q(S)$ such that: (i) All the functions in (5.1)' are Chvatal functions; (ii) Any solution to (5.1)' is a solution to (5.1); (iii) (5.1)' has a solution.

Proof: Let y be a solution to (5.1). Apply Lemma 5.2 to each G_j. Form (5.1)' by replacing each G_j by the corresponding D and adding additional inequalities of the form (5.2) for each G_j.

Q.E.D.

From now on we assume that the G_j in (5.1) are Chvatal functions. The next step is to show that if (5.1) has a solution, it has a solution in which none of the denominators of its components is "too big."

Lemma 5.4: There is a polynomial q such that, given any system of linear inequalities

$$\text{(5.3)} \qquad \begin{array}{lll} \lambda_i x \geq s_i & i \in J_1 & \\ & & [\lambda_i \in Q^n] \\ \lambda_i x > s_i & i \in J_2 & \end{array}$$

let S = sum of sizes of components of all λ_i plus sizes of denominators of s_i. (5.3) has a solution only if it has a solution all of whose denominators are of size $\leq q(S)$.

Proof: Apply the finite basis theorem to $P = \{x | \lambda_i x \geq s_i, i \in J_1 \cup J_2\}$. If (5.3) has a solution, it has a solution which is a weighted average (all weights equal) of at most $|J_1 \cup J_2|$ extreme points of P plus an integer vector (corresponding to a sufficiently large multiple of the directions of infinity of P). Since each extreme point of P has a demoninator of bounded size (determinants) we are done.

Q.E.D.

Lemma 5.5: If a system (5.1) of size S has a solution there is a system (5.3) such that: (i) every solution to (5.3) is a solution to (5.1); (ii) (5.3) has a solution; (iii) the sum of the sizes of the denominators of (5.3) is bounded by a polynomial in S.

Proof: Let y be a solution to (5.1). Each Chvatal function G_i can be written as $G_i(x) = \lambda_i x + F_i(\lceil \lambda_{i1} x \rceil, \ldots \lceil \lambda_{iN} x \rceil)$ where the size of F(...) is $\leq$ the size of G_i. To construct the appropriate system (5.3) we have the inequalities $\lambda_{ij} x \leq \lceil \lambda_{ij} y \rceil$, $\lambda_{ij} x > \lceil \lambda_{ij} y \rceil - 1$ for all i,j together with inequalities of the form $(\Sigma a_{ij} \lambda_j) x \geq d_i - \Sigma a_{ij} F_j(y)$ $i \in I_1$ and similar inequalities for $i \in I_2$.

Q.E.D.

Corollary 5.6: There is a polynomial q such that, if (5.1) is of size S and has a solution, there is a solution with sum of sizes of denominators $\leq q(S)$.

Proof: We use Lemma 5.5 to construct the appropriate system (5.3) and invoke Lemma 5.4.

Q.E.D.

Next we show that any system (5.1) can be replaced by an integer program of polynomially bounded size. We require a preliminary result showing Chvatal functions have a periodicity property.

<u>Lemma 5.7</u>: Let G_j be a Chvatal function and e_i the ith unit vector. There are integers $N_{ij} > 0$, M_{ij} such that, for all x, $G_j(x + N_{ij}e_i) = G_j(x) + M_{ij}$. Moreover the sizes of N_{ij}, M_{ij} are polynomially bounded by the size of G_j.

<u>Proof</u>: We argue by induction on the expression defining G_j. If G_j is linear the result is immediate. If $G_j = \alpha G_j'$ the induction hypothesis gives N_{ij}', M_{ij}'. We take $N_{ij} = kN_{ij}'$, $M_{ij} = \alpha k M_{ij}'$ for suitable k. If $G_j = \lceil G_j' \rceil$, $N_{ij} = N_{ij}'$ and $M_{ij} = M_{ij}'$, if $G_j = G_j' + G_j''$ we take $N_{ij} = N_{ij}'N_{ij}''$, $M_{ij} = M_{ij}'N_{ij}'' + M_{ij}''N_{ij}'$.

Q.E.D.

We are now ready to complete the

<u>Proof of Theorem 5.1</u>: By corollary 5.3 we can find a system

$$(5.1)' \qquad \sum_{j=1}^{k} a_{ij}H_j(x) \geq d_i \qquad i \in J_1$$

$$\sum_{j=1}^{k} a_{ij}H_j(x) > d_i \qquad i \in J_2$$

in which all H_j are Chvatal functions, (5.1)' has a solution and x satisfies (5.1) if it satisfies (5.1)'. By Lemma 5.5 there is a solution y to (5.1)' with denominators bounded by a polynomial in the size of (5.1)' [hence, by a polynomial in the size of (5.1)]. Let N_{ij}, M_{ij} be as in Lemma 5.7 with $G_j = H_j$. Let $W_i = \prod_j N_{ij}$. Let z be such that: (i) the ith component of z is between zero and W_i; (ii) z - y is an integer linear combination of the vectors W_ie_i. The size of z is polynomial in the size of (5.1). For each $L \in J_1 \cup J_2$ there is a P_{Li} such that, for all x, $\sum_{j=1}^{k} a_{Lj}H_j(x + W_ie_i) = P_{Li} + \Sigma a_{Lj}H_j(x)$. Let $Q_L = \Sigma a_{Lj}H_j(z)$. The system of inequalities with integer variables t_i unconstrained in sign

$$\sum_{i=1}^{n} P_{Li} t_i \geq d_L - Q_L \qquad L \in J_1$$

(5.4)

$$\sum_{i=1}^{n} P_{Li} t_i > d_L - Q_L \qquad L \in J_2$$

is of polynomial size. Furthermore, if t_i is an integer solution to (5.4) then $x = z + \Sigma t_i (W_i e_i)$ is a solution to (5.1)', hence is a solution t (5.1). Since y coresponds to a solution to (5.4), (5.4) is consistent. By Theorem 2.7, (5.4) has a solution of polynomial size which gives a solution to (5.1) of polynomial size.

Q.E.D.

This result immediately implies that the problem of determining if $F(x) \neq G(x)$ for some x, where F and G are Gomory or Chvatal functions, is in NP. Theorem 4.6 then imples that these problems are NP-complete.

6. Subadditive Duality Requires Exponential Space

In this section, we provide an example of a subadditive dual, whose every Chvatal function solution requires exponential space to write down. We accomplish this by, essentially, examining a class of two-dimensional integer programs due to Bondy (and cited in [7], and the main content of our result is Theorem 6.3.

However, there are a fair number of technical results needed to convert this latter theorem to the form desired, which is Theorem 6.6. These technical results concern interrelations between certain proof systems and monotone Chvatal functions, as well as interrelations between monotone Chvatal functions (which are related to the inequality format $Ax \geq b$ for constraints) and Chvatal functions (which are related to equality format $Ax = b$).

Our proofs of the technical results are sketchy, since these are easy. Similarly, we discuss proof systems informally, to save space and reduce notation. More rigorous treatments of proof systems are available in [15], [17]. Proof systems were earlier used in optimization contexts, either implicitly or explicitly, in [1], [7], and [12].

The sentences of our proof system (our "logic") are numerical-valued linear inequalities in n indeterminates $x_1, \ldots, x_n$:

(6.1) $$a_1x_1 + a_2x_2 + \ldots + a_nx_n \geq b$$

(Actual rational numbers $a_1, \ldots a_n$, b written in binary occur in (6.1). In the system discussed in this paper, the sentences and the atomic sentences are the same (i.e., we do not allow logical connectives or quantifiers--this is a free variable system).

There are three "rules of deduction": 1) Nonnegative combinations; 2) Chvatal's rule; and 3) Weakening.

The rule of nonnegative combinations has any number t of <u>premises</u> $a_j x \geq b_j$, all of which are linear inequalities, and involves t nonnegative (rational) scalers $\lambda_1, \ldots, \lambda_t \geq 0$. Briefly put, it allows the "deduction" of $(\sum_i \lambda_i a_i)x \geq \sum_i \lambda_i b_i$ from the premises, and is symbolized:

$$(6.2) \qquad \frac{a_1 x \geq b_1,\ a_2 x \geq b_2,\ \ldots, a_t x \geq b_t}{(\sum_{1}^{t} \lambda_i a_i)x \geq \sum_{1}^{t} \lambda_i b_i}$$

The rule due to Chvatal has one premise, and it is symbolized:

$$(6.3) \qquad \frac{ax \geq b}{\lceil a \rceil x \geq \lceil b \rceil}$$

It allows the deduction of $3x_1 - 2x_2 \geq 5$ from $2.7x_1 - 2.7x_2 \geq 4.5$, for example.

The rule of weakening allows us to conclude less than we know, and serves some technical purposes. It is symbolized:

$$(6.4) \qquad \frac{ax \geq b}{a'x \geq b'}$$

It has one premise and requires that $a_j' \geq a_j$ for $j = 1, \ldots, n$ and $b' \leq b$.

As regards proofs, the only one-line proofs are sentences (6.1). All proofs of greater length are obtained inductively by use of the rules of deduction. As the rule (6.2) has several premises, proofs in this logic occur in "tree form," spread out at the top and coming to a last sentence at the bottom. The very topmost sentences are called the <u>axioms</u> of the proof; the last sentence is its <u>conclusion</u>. (Axioms occurring in multiple locations can be distinguished from each other or not, as desired.)

The intended interpretation of our proofs are: if the axioms are true for the quantities $x_1, \ldots, x_n$ whenever these quantities are nonnegative integers, the conclusion of the proof is also true. This interpretation can be established by induction on the number of rules of deduction used. Rule (6.2) is actually valid for all x, and rule (6.4) requires only $x \geq 0$. Chvatal's rule (6.3) is true since $ax \geq b$ implies $\lceil a \rceil x \geq b$ (by $x \geq 0$), and then, since $\lceil a \rceil x$ is integer for x integer, we conclude $\lceil a \rceil x \geq \lceil b \rceil$.

The logic L described is actually complete for consistent sets of axioms, i.e., it proves exactly the set of valid inequalities. This completeness property is the content of Shrijver's result [16]. We will not need completeness in this section.

Proofs in L yield monotone Chvatal functions (Proposition 6.1), and monotone Chvatal functions correspond to proof schema, i.e., a monotone Chvatal function together with a set of axioms yields a proof in L. Recall that a monotone Chvatal function is one in which all the "linear atoms" λv which occur are nonnegative ($\lambda \geq 0$). The detailed description of monotone Chvatal functions is in [4].

In what follows, the (Chvatal) degree of a proof Σ is the maximum number of occurences of the Chvatal rule (6.3) on any branch of Σ (viewing Σ as a tree). Also, the (nested) degree of a Chvatal function F(b) is the maximum length of a chain of occurences of the round-up operation $\lceil . \rceil$ in (an expression for) F(b), counting also the outermost occurrence of an operation. Note that the degree of F is a lower bound on the length of (an expression for) F.

Proposition 6.1:

To any proof Σ, there corresponds a monotone Chvatal function F of the same degree, such that the last line of Σ is a weakening of:

(6.5) $$\sum_{j=1}^{n} F(a_j)x_j \geq F(d)$$

In (6.5), the axioms of Σ are $Ax \geq d$, and $A = [a_j]$ (cols). (An arbitrary ordering of the axioms of Σ can be used to form the linear system $Ax \geq d$).

Proof: First, one establishes this lemma concerning the logic L: all occurrences of the weakening rule (6.4) can be moved to the very end of a proof, and be replaced by one occurrence, without changing the conclusions of Σ. Thus, it suffices to prove the proposition in the case that Σ has no occurrences of the weakening rule.

Now the proof proceeds by induction on the length of Σ.

For a proof of length one, put $A = a$, $d = b$, $F(v) = v$.

When the last line of Σ arises in a context (6.2) of nonnegative combinations, let F_k be the function associated with the subproof of Σ consisting of the line $a_k x \geq b_k$ and the subtree above (k=1, ..., t). Let the premise of the cited subproof be $A_k x \geq d_k$, $A_k = [a_{k_j}]$ (cols). If A_k has m_k vows, let $m = m_1 + \ldots + m_t$, and let $Ax \geq d$ denote the entire system $A_1 x \geq d_1, \ldots, A_t x \geq d_t$. Then define the monotone Chvatal function F on Q^m by:

$$F\begin{pmatrix} v_1 \\ \vdots \\ v_t \end{pmatrix} = \lambda_1 F_1(v_1) + \ldots \lambda_t F_t(v_t).$$

One easily verifies that this is the desired F. The degree of F is the maximum of the degree of the F_k, which is the degree of Σ.

When the last line of Σ arises in a context (6.3) of Chvatal's rule, and G is the function providing the preceeding line, let $F = \lceil G \rceil$. One easily verifies that the degree of F is that of Σ, as both have increased by one.

Q.E.D.

<u>Proposition 6.2</u>:

For any monotone Chvatal function $F = F(v)$ in m variables $v = (v \ldots, v_m)$, and any set of m axioms $Ax \geq d$ (where $A = [a_k]$ (rows) has m rows a_k), there is a proof Σ from axioms $Ax \geq d$ of (6.5), in which no weakening rule (6.4) is used. Moreover, the degree of Σ is that of F.

The proof of Proposition 6.2 is by induction on the degree of F, via ideas similar to those in the proof of Proposition 6.1. We give an example: to the monotone Chvatal function

$F(v_1, v_2) = 3\lceil 2v_1 + v_2\rceil + \lceil v_1 + \lceil v_2\rceil\rceil$ corresponds the proof schema:

$$\text{(6.6)} \qquad \begin{array}{cc} & v_2 \\ \cline{2-2} & v_1, \lceil v_2\rceil \\ v_1, v_2 & \\ \cline{1-1}\cline{2-2} 2v_1+v_2 & v_1+\lceil v_2\rceil \\ \cline{1-1}\cline{2-2} \lceil 2v_1+v_2\rceil & \lceil v_1+\lceil v_2\rceil\rceil \\ \hline \multicolumn{2}{c}{3\lceil 2v_1+v_2\rceil + \lceil v_1+\lceil v_1+\lceil v_2\rceil\rceil\rceil} \end{array}$$

Specifically, if $1.5x_1 - 2.5x_2 + .3x_3 \geq .7$ and $.2x_2 - 1.3x_3 \geq -1.4$ are taken as axioms, (7.6) yields the proof in Figure 1. In Figure 1 different occurences of the same axiom were viewed as instances of the one axiom. Next to a deduction line, we have indicated if (6.2) was used (L) or (6.3) was used (C).

<u>Theorem 6.3</u>:

Any proof of Σ with last line $-x_2 \geq 0$ from the axioms $2mx_1 - x_2 \geq 0$, $-2mx_1 - x_2 \geq -2m$, has degree at least m.

$$
\text{L}\;\dfrac{
\text{L}\;\dfrac{1.5x_1 - 2.5x_2 + .3x_2 \geq 7, \quad .2x_2 - 1.3x_3 \geq -1.4}{\text{C}\;\dfrac{3x_1 - 4.8x_2 - .7x_3 \geq 0}{3x_1 - 4x_2 \geq 0}}
\qquad
\text{L}\;\dfrac{\text{C}\;\dfrac{.2x_2 - 1.3x_3 \geq -1.4}{1.5x_1 - 2.5x_2 + .3x_3 \geq .7}, \quad x_2 - x_3 \geq 1}{\text{C}\;\dfrac{1.5x_1 - 1.5x_2 - .7x_3 \geq -.3}{2x_1 - x_2 \geq 0}}
}{11x_1 - 13x_2 \geq 0}
$$

Figure 1

Proof: The polytope P_m, consisting of all $(x_1,x_2) \geq 0$ which satisfy the axioms, has vertices (0,0), (1,0), and (1/2,m), and contains only the integer points (0,0) and (1,0).

For a (rational) polyhedron P, let P' denote the convex set defined by imposing, simultaneously, all possible deduced inequalities with proofs of degree zero or one. (From [13], P' is a rational polyhedron). Inductively set $P^{k+1} = (P^k)'$. Note also the monotonicity: $P \supseteq Q$ implies $P' \supseteq Q'$.

We show that:

(6.7) If $k < m$, P_m^k contains a point (x_1,x_2) with $x_2 > 0$.

Note that (6.7) establishes the theorem, since a proof of degree $k < m$ has a conclusion which is valid for all points in P_m^k.

By monotonicity, (6.7) follows from

(6.8) $P_m' \supseteq P_{m-1}$ for $m \geq 1$

E.g., if (6.8) is true, $P_m^2 \supseteq (P_{m-1})' \supseteq P_{m-2}$, etc., so that $P_m^k \supseteq P_{m-k}$ for $k \leq m$; yet $(1.2,\ 1) \in P_1$.

The proof of (6.8) is not hard.

Let $a_1x_1 + a_2x_2 \geq$ be any linear inequality with P_m to one side, such as can be derived without Chvatal operations (using (6.2) and (6.4) alone). Note that $\lceil a_1 \rceil x_1 + \lceil a_2 \rceil x_2 \geq b$ also has P_m to one side. Without loss of generality, b is as large as possible.

Since (0,0) (1,0) and (1/2,m) are the extreme points of P_m, we have

(6.9) $b = \min\{0,\ \lceil a_1 \rceil,\ 1/2\lceil a_1 \rceil + m\lceil a_2 \rceil\}$

Thus, if the result of a Chvatal operation (6.3) is nontrivial (i.e., if $\lceil b \rceil > b$), we have $b = q + 1/2$, where q is integer. Since (1/2, m-1) is interior to P_m, we have $1/2 \lceil a_1 \rceil + (m-1) \lceil a_2 \rceil \geq \lceil b \rceil = q + 1$. Thus, (1.2, m-1) satisfies the conclusion of any one use of Chvatal's rule Hence (0,0) (1,0), (1/2, m-1) $\epsilon P'_m$, and (6.8) is immediate.

Q.E.D.

Corollary 6.4:

Any monotone Chvatal function F which satisfies

$$F\binom{2m}{-2m} \leq 0$$

(6.10) $$F\binom{-1}{-1} \leq -1$$

$$F\binom{0}{-2m} > 0$$

has degree at least m.

Proof: This follows from Proposition 6.2 and Theorem 6.3

Q.E.D.

We note that our proof in Lemma 5.14 of [4] actually establishes the following result:

Lemma 6.5:

If F is a Chvatal function with $F(-e_j) \leq 0$ for $j = 1, \ldots, n$, there is a monotone Chvatal function F*, of the same degree as F, such that:

(6.11) $$F(v) = F^*(v) \text{ for all } v \; \epsilon \; z^m$$

We remark that, in (6.11) we can take $F^*(v) = \lambda v + F^{**}(v)$, where $\lambda \geq 0$, and all linear atoms θv of F** have $0 \leq \theta_j < 1$. However, we shall not need this sharper result below.

Corollary 6.6:

Any Chvatal function F which satisfies

$$F\binom{2m}{-2m} \le 0$$

$$F\binom{-1}{-1} \le -1$$

(6.12) $$F\binom{0}{-2m} \ge 0$$

$$F\binom{-1}{0} \le 0$$

$$F\binom{0}{-1} \le 0$$

has degree at least m.

Proof: This follows by Corollary 6.4 and Lemma 6.5.

Q.E.D.

We recall from [8], that the pure integer program in equality format

(6.13) $$\begin{aligned} &\min cx \\ &\text{subject to } Ax = b \qquad A = [a_j] \text{ (cols)} \\ &x \ge 0, \text{ integer} \end{aligned}$$

has, as its "subadditive dual"

(6.14) $$\begin{aligned} &\max F(b) \\ &\text{subject to } F(a^j) \le c_j \qquad j = 1, \ldots, n \\ &F \text{ Chvatal} \end{aligned}$$

We proved in [8] that, when (6.13) has a finite value, then its dual (6.19) has an equal finite value, which is attained (see Corollary 2.5).

Theorem 6.7:

Any optimal solution to the subadditive dual of

$$\begin{array}{llr} & \min \quad -x_2 & \\ (6.15) & \text{subject to } 2mx_1 - x_2 - x_3 & = 0 \\ & -2mx_1 - x_2 \qquad - x_4 & = -2m \\ & x_1, x_2, x_3, x_4, \geq 0 \text{ and integer} & \end{array}$$

has degree at least m.

Consequently, any such solution is of exponential length.

Proof: The integer program (6.15) is min $\{-x_2 \mid (x_1,x_2) \in P_m\}$, where P_m is as defined in the proof of Theorem 6.3. Since the only integer points in P_m are (0,0) and (1,0), the value of (6.15) is zero. Consequently, any optimal solution F to the dual of (6.15) satisfies (6.12), and Corollary 6.6 applies, showing that F has degree at least m.

As regards the "consequently," a degree m function F has at least m occurrences of the round-up operation $\lceil \cdot \rceil$, hence length at least m. For $m = 2^n$, the length of the program (6.15) is linear in n (as m is written in binary), yet the dual is of exponential length 2^n.

Q.E.D.

Corollary 6.8:

There are arbitrarily large integers $d \geq 1$, such that some Chvatal function of degree d is not equal (even for all integer vectors) to any Chvatal function of degree less than d.

Proof: If the results were false, there exists an integer d_0 such that any Chvatal function is equal (for all integer vectors) to some Chvatal function of degree d_0 or less. This contradicts Theorem 7.7.

Q.E.D.

In [13], the second author described an integer program, which is solvable using the Chvatal operation, but requires exponential time via branch-and-bound. In contrast, the program (6.15) requires exponential space to arrange its proof, hence at least exponential time for solution by algorithms based on Chvatal operations (such as Gomory's Method of Integer Forms), even though it is solved by branch-and-bound in one arbitration of a variable (set $x_2 = 0$ versus $x_2 = 1$). By juxtaposing these two examples, we see that there cannot be any purely theoretical result, for "general" integer programs, which shows the dominance of branch-and-bound over the special class of cutting-plane methods considered, or vice-versa. If theoretical results of dominance are established, they will require assumptions on the "structure" of the integer program (i.e., the types of quantities A, b, c) or the distribution of the data. Alternatively, such results are empirical (and hence even more likely to require structural assumptions, as experience has shown).

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