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ON LINEAR OPTIMAL CONTROL PROBLEMS WITH MULTIPLE QUADRATIC CRITERIA

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by

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Abstract

Linear optimal control problems with multiple quadratic criteria and their efficient points are studied using penalty function methods. Procedures for approximating efficient points and solutions to bicriterion control problems are given for cases with and without magnitude restraints. Omitted proofs and details can be found in [6].

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ON LINEAR OPTIMAL CONTROL PROBLEMS WITH MULTIPLE QUADRATIC CRITERIA

1. Introduction. Consider the system

(1.1) $\dot{x} = A(t) + B(t)u \quad (t_o \leq t \leq T_o)$

with initial condition

(1.2) $x(t_o) = x_o$

and n criterion functions

(1.3)
$$f_i(u) = \langle x(T_o) - \xi_i, W_i[x(T_o) - \xi_i] \rangle$$
$$+ \int_{t_o}^{T_o} \langle \tilde{x}_i(t) - C_i(t)x(t), Q_i(t)[\tilde{x}(t) - C_i(t)x(t)] \rangle \, dt$$
$$- \int_{t_o}^{T_o} \langle u(t), R_i(t)u(t) \rangle \, dt, \quad (1 \leq i \leq n),$$

where

the matrices $A(t)$ and $B(t)$ are mxm and sxm respectively, continuous on $[t_o, T_o]$,

the control u: $[t_o, T_o] \to R^s$ is integrable,

the vectors $\xi_i \in R^m$ and the symmetric matrices $W_i \in R^{mxm}$ are constant,

the functions $\tilde{x}_i: [t_o, T_1] \to R^m$ are continuous for some $T_1 \geq T_o$,

the matrices $C_i(t)$ are mxm and continuous on $[t_o, T_1]$,

the matrices $Q_i(t)$ and $R_i(t)$ are continuous and symmetric, and

the $R_i(t)$ are positive definite on $[t_o, T_1]$.

In this paper we study optimal control problems, stated in terms of (1.1) - (1.3).

2. Statement of the problems. An efficient point u^o over a class of controls $\mathcal{C}$ is a control $u^o \in \mathcal{C}$ such that for no ther $u \in \mathcal{C}$

$$f_i(u) \geq f_i(u^o) \qquad 1 \leq i \leq n$$

with at least one strict inequality.

The class of controls $\mathcal{C}$ in the above definition will be taken here as one of the following two classes.

$L^{2,s}(t_oT_o)$ - the space of controls u with $\int_{t_o}^{T_o} |u(t)|^2 dt < \infty$, where $|\ |$ denotes the Euclidean norm.

$\mathcal{U}$ - the class of measurable controls with $|u(t)| \leq 1$ a.e. on $[t_o, T_o]$.

The following two efficient point problems will be studied here:

E_1 Find all efficient points over $L^{2,s}(t_o, T_o)$.

E_2 Find all efficient points over $\mathcal{U}$.

For the bicriterion case $n = 2$ we study two other optimal control problems

M_1 Maximize $\min\{f_1(u), f_2(u)\}$ subject to $u \in L^{2,s}(t_o, T_o)$

M_2 Maximize $h[f_1(u), f_2(u)]$ subject to $u \in \mathcal{U}$,

where $h\colon R^2 \to R$ is continuous and non-decreasing in each of its arguments on the non-negative orthant R^2_+, and quasiconcave over the interior of R^2_+. We further assume that

$|u(t)| \leq 1$ a.e. on $[t_o, T_o] \Rightarrow f_i(u) > 0$, $i = 1,2$, which is guaranteed, for example, if W_i are positive definite and $|x_o|$ is sufficiently large.

We observe that the objective function of M_1 is a special case of the objective function $h(x,y)$ of M_2. Other examples are

$$h(x,y) = x^\beta y \quad ; \qquad \beta > 0$$

or

$$h(x,y) = c_1x^{\beta_1} + c_2y^{\beta_2} \; ; \quad c_1, c_2, \beta_1, \beta_2 \geq 0.$$

(For other examples see e.g.[2] p. 40)

3. Solution of E_1. In what follows let the vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfy

(3.1) $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_i \geq 0 \quad 1 \leq i \leq n.$

For each such α we define the following single criterion analog of E_1:

P_1^α Maximize $\sum_{i=1}^{n} \alpha_i f_i(u)$ subject to $u \in L^{2,s}(t_o, T_o)$

First we require:

Lemma 3.1. There exists a constant $N > 0$ such that if $T_o - t_o \leq N$ then each f_i is strictly concave and bounded above over $L^{2,s}(t_o, T_o)$.

Proof. Concavity follows from negativity of the second Gateaux derivatives of the f_i, verified as in [1], Section 8.4, by using the Hölder inequality. Boundedness is similarly proved.

The following Theorem establishes existence and uniqueness for solutions of P_1^α.

Theorem 3.1. There exists a constant $N^1 > 0$ such that if $T_o - t_o \leq N^1$, then for each α satisfying (3.1) the problem P_1^α has a unique solution.

Proof. Let $f^\alpha = \sum_{i=1}^{n} \alpha_i f_i$. By Lemma 3.1, if $T_o - t_o \leq N$ then the condition

(3.2) $\frac{d}{d\epsilon} f^\alpha(u^\alpha + \epsilon v) = 0$ at $\epsilon = 0$ for any $v \in L^{2,s}(t_o, T_o)$

is necessary and sufficient for u^α to be the unique solution of P_1^α. We recall that for each u a unique solution to (1.1) - (1.2) is determined by

(3.3) $x(t) = S(t,t_o)x_o + \int_{t_o}^{t} S(t,\sigma)\, B(\sigma)\, u(\sigma)\, d\sigma$

where $S(t,\sigma)$ is the fundamental solution of $\dot{x} = A(t)x$. Using (3.3) it follows that (3.2) can be written in the form

(3.4) $u^\alpha = A^\alpha u^\alpha$

where A^α is an operator mapping $C^{0,s}(t_0,T_0)$, the space of continuous R^s- valued functions on (t_0,T_0) with the sup norm, into itself. It can then be shown that there exists a constant $N^1 > 0$ such that $T_0 - t_0 \le N^1$ guarantees that each possible A^α in (3.4) is a contraction.

The problems E_1 and P_1^α are related by following Theorem.

Theorem 3.2. Let $T_0 - t_0 \le \min\{N, N^1\}$.

Then

(i) If the vector α satisfying (3.1) is positive (i.e. if $\alpha_i > 0$, $1 \le i \le n$), then the solution u^α of P_1^α is an efficient point over $L^{2,s}(t_0,T_0)$.

(ii) If u^0 is an efficient point over $L^{2,s}(t_0,T_0)$ then for some α^0 satisfying 3.1, u^0 is the solution of $P_1^{\alpha^0}$.

Proof. (i) is obvious. (ii) is proved as in the finite dimensional case, see e.g. [3], Section 7.4.

A method for approximating a solution of P_1^α is suggested by Theorem 3.1-namely, u^α in (3.4) is the uniform limit of a sequence of successive approximations. Another approach for solving P_1^α, the "synthesis" or "feedback" approach (see e.g. [1] Section 8.6 or [4] Section 5.2) uses a Riccati matrix differential equation whose solution is then used to express u^α.

4. Solution of E_2. For α satisfying (3.1) we define the following single criterion analog of E_2:

P_2^α Maximize $\sum_{i=1}^{n} \alpha_i f_i(u)$ subject to $u \in \mathcal{U}$

Although $\mathcal{U}$ is not a compact subset of $L^{2,s}(t_o,T_o)$, the consistency of $\{P_2^\alpha\}$ is guaranteed by the following Theorem.

Theorem 4.1. Let $T_o - t_o \leq N$ where N is as in Lemma 3.1. Then each problem P_2^α has a unique solution.

The relations between E_1 and P_1^α, studied in Theorem 3.2, hold also for E_2 and P_2^α:

Theorem 4.2. Let $T_o - t_o \leq N$ where N is as in Lemma 3.1. Then:

(i) If the vector α satisfying (3.1) is positive, then the solution of P_2^α is an efficient point over $\mathcal{U}$.

(ii) If u^o is an efficient point over $\mathcal{U}$ then u^o is the solution of P_2^α for some α_o satisfying (3.1).

Using the penalty function approach of [5], a computational scheme for approximating the solutions of P_2^α will now be outlined.

Let $\rho(u)$ denote the Euclidean distance between a vector $u \in R^s$ and the unit ball $\{u: |u| \leq 1\}$ in R^s. For any $0 < \epsilon \leqq 1$ and α satisfying (3.1) define the function

$$f_\epsilon^\alpha(u) = f^\alpha(u) - \frac{1}{\epsilon}\int_{t_o}^{T_o} \rho^2(u(t))\, dt$$

where

$$f^\alpha(u) = \sum_{i=1}^{n} \alpha_i f_i(u).$$

Let

$$c^\alpha = \sup_{u \in \mathcal{U}} f^\alpha(u)$$

and

$$c_\epsilon^\alpha = \sup_{u \in L^{2,s}(t_o,T_o)} f_\epsilon^\alpha(u)$$

The following Theorem can then be proved using Fillipov's Lemma and Hölder's inequality.

Theorem 4.3. Let $T_o - t_o \le N$.

Then $\bar{c}^{\alpha}_{\epsilon}$ decreases to c^{α} as $\epsilon \to 0$.

For any positive integer k we define the following function in $L^{2,s}(t_o,T_o)$

$$f^{\alpha}_k(u) = f^{\alpha}(u) - \int_{t_o}^{T_o} |u(t)|^{2k} dt$$

and

$$c^{\alpha}_k = \sup_{u \in L^{2,s}(t_o,T_o)} f^{\alpha}_k(u).$$

Theorem 4.4. c^{α}_k converges to c^{α} as $k \to \infty$

Finally, under the simplifying assumption $R_i(t) = I$, $1 \le i \le n$, we can prove:

Theorem 4.5. If $T_o - t_o \le \min\{N, N^1\}$ then for each α and k as above there is a unique $u^{\alpha}_k \in L^{2,s}(t_o,T_o)$ at which the supremum of f^{α}_k is attained. Furthermore

$$f_i(u^{\alpha}_k) \longrightarrow f_i(u^{\alpha}), \quad 1 \le i \le n,$$

where u^{α} is the unique element at which the supremum of f^{α} over $\mathcal{U}$ is attained.

The proof of Theorem 4.5 exhibits each u^{α}_k as a solution of

(4.1) $M_k(u^{\alpha}_k) = T^{\alpha}_k[M_k(u^{\alpha}_k)]$

where the map M_k: $R^s \longrightarrow R^s$ is given by

$$M_k(u) = 2(1 + k|u|^{2k-2})u$$

and T^{α}_k is an operator mapping $C^{o,s}(t_o,T_o)$ into itself, determined by the stationarity condition

$$\frac{d}{d\epsilon} f_k(u^{\alpha} + \epsilon v) = 0 \text{ at } \epsilon = 0 \text{ for any } v \in L^{2,s}(t_o,T_o).$$

Each T_k^{α} is a contraction if $T_o - t_0 \leq N^1$. A successive approximation procedure for solving equations like (4.1) is found in [5], Section 5.

5. Solution of M_1 and M_2. The following results are proved similarly to the corresponding results in [2].

Lemma 5.1. If $T_o - t_o \leq N$, then M_1 and M_2 have optimal solutions, and at least one solution of each is efficient.

Theorem 5.1. Let $T_o - t_o \leq N$.
Then the following function of α

$$h[f_1(u^{\alpha}),\ f_2(u^{\alpha})]$$

is unimodal on [0,1].

Using the customary search techniques for finding the supremum of a unimodal function, Theorems 3.1, 3.2, Lemma 5.1 and Theorem 5.1 constitute a procedure for approximately solving M_1, while Theorems 4.2, 4.5, Lemma 5.1 and Theorem 5.2 similarly constitute a procedure for approximately solving M_2.

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