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Constructive Characterizations of the Value Function of a Mixed-Integer Program

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Abstract

We study value functions for mixed-integer programs and also functions for deciding when an MIP is feasible. Examples are given to show that the inductive structure in pur integer programs is not preserved. However, a generalized class of value functions for pre-multiplied MIPs does have an inductive structure.

Constructive Characterizations of the Value Function of a Mixed-Integer Program

by C. E. Blair and R. G. Jeroslow[1]

The mixed-integer program is the optimization program with linear constraints, and integer and continuous variables:

$$(\mathrm{MIP}_b) \quad \begin{array}{ll} & \inf\ cx + dy \\ \text{subject to} & Ax + By = b \\ & x,y \geq \vec{0} \quad x \text{ integer} \end{array}$$

The constraint matrices A,B and objective functions c,d will be assumed throughout to be fixed and rational. Our primary concern is the value function z(b), defined to be the objective function value of the optimal solution to (MIP_b), as b varies over all rational right-hand-side vectors.* If (MIP_b) is inconsistent we set $z(b) = +\infty$. Except where otherwise stated we assume that $z(\vec{0}) = 0$. This is equivalent to assuming that whenever (MIP_b) is feasible, it is bounded in value.

The focus in our present paper has been on characterizations of value functions, i.e., necessary and sufficient properties of value functions. Our earlier work ([4], [14]) provided necessary properties only. This focus has, in turn, led us to study inductively-defined classes of function.

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[1] The work of the second author has been partially supported by NSF grant ECS 8001763.

*Most of our results hold in the case when b,c,d are arbitrary reals. However the rationality of A,B is crucial to our work.

In [2] we studied the value function of a pure integer program, i.e., one in which there is no y, B, or d. We defined the class of Gomory functions inductively as the smallest class containing linear functions and closed under rational sums, non-negative rational mulipli-cation, taking maxima, and ceiling (next-higher integer) operations. It was established that for each A,c there is a Gomory function F such that $F(b) = z(b)$ for all feasible b. A converse showing that for every Gomory function F there is a corresponding integer program was also established. (Formal definitions and precise statements will be given in the next section.)

These results are analogues of the simpler situation in parametric linear programming. If we remove the ceiling operations from the defin-ition of Gomory functions, the resulting class is the set of polyhedral functions. This is precisely the class of value functions of linear programs. The ceiling operations provide the additional tool needed for the requirement that all vbls are integer.

For pure integer programs, Gomory functions also provide "consis-tency testers." For any matrix A, there is a Gomory function F such that $F(b) \leq 0$ if and only if the integer program with right-hand-side b is feasible. Again, there is a converse which constructs a pure in-teger program for every Gomory function.

Our present paper examines the interrelationships between Gomory functions and MIPs.

Section 2 contains preliminary definitions and statements of theorems from earlier papers.

In section 3, we show that every MIP has a "consistency tester" which is a Gomory function. This follows from the results for pure integer programs in a straightforward manner. Several easy consequences of this result are estabished.

Section 4 shows that the other results for pure integer programs do not generalize to MIPs. Examples are given of a Gomory function that is not a consistency tester and of an MIP whose value function is not a Gomory function. The fundamental pathology underlying these examples is that the inductive structure breaks down, e.g., the sum of two value functions is not necessarily a value function. This theme is explored in depth in section 6.

Section 5 identifies precisely which Gomory functions are consistency testers. A constructive procedure is given by which, given a closed-form expression for a Gomory function, it can be determined whether or ot it is the consistency tester for some integer program.

Section 6 addresses the problem of value functions. The class of value functions is imbedded in a class of functions corresponding to mixed-integer programs in which the right-hand-side is pre-multiplied by a matrix. This larger class of problems preserves the inductive structure and is of independent practical interest. The subclass of the pre-multiplied value functions corresponding to value functions in the original sense is identified algorithmically. We show that the closure of the Gomory functions under infimal convolution provides exactly the value functions for the pre-multiplied constraint sets. A closed-form expression for value functions (both ordinary and pre-multiplied) as the minimum of finitely many Gomory functions is also given.

2. Definitions and Preliminary Results

In this section we present the material which will be needed later in approximately the order in which it is needed. Motivation will be supressed in the interests of brevity.

A. Value functions. As described in section 1, if A,B are $m \times r$, $m \times s$ rational matrices $c \in Q^r$, $d \in Q^s$ [Q = the rationals], then A,B,c,d determine a value function $z\colon\ Q^m \to Q \cup \{+\infty\}$. $z(b)$ is the objective function value of the optimal solution to (MIP_b).* $z(b) = +\infty$ means (MIP_b) is inconsistent, i.e., has no feasible solutions. Any value function which is not finite everywhere can be extended to one which is, as shown by the result:

Theorem 2.1 [4, thm 4.6]: Let z be the value function determined by A,B,c,d. There are A',B',c',d' such that $z'(b) < +\infty$ for all b and $z'(b) = z(b)$ if $z(b) < +\infty$.

B. Chvatal and Gomory Functions: In [2] we defined several inductive classes of functions.

The Gomory functions are the smallest class $\mathcal{G}$ of functions such that

(5.1) If $\lambda \in Q^m$ then $F \in \mathcal{G}$ where $F(b) = \lambda b$

(5.2) If $F \in \mathcal{G}$ then $G \in \mathcal{G}$ where $G(b) = \lceil F(b) \rceil$ = smallest integer $\geq F(b)$

(5.3) If $F,G \in \mathcal{G}$ then $H \in \mathcal{G}$ where $H(b) = F(b) + G(b)$

(5.4) If $F \in G$, $\alpha \in Q$, $\alpha \geq 0$ then $H \in G$ where $H(b) = \alpha F(b)$

(5.5) If $F,G \in \mathcal{G}$ then $H \in \mathcal{G}$ where $H(b) = \max\{F(b),G(b)\}$

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*The existence of an optimal solution to (MIP_b) is a non-trivial result due to Meyer [15, see also 2, thm 4.6].

The Chvatal functions are the smallest class of functions satisfying (5.1) - (5.4).

<u>Proposition 2.2</u> [2, prop. 2.18]: If F is a Gomory function, there are Chvatal functions, C_1, C_2, ... C_N such that $F(b) = \max(C_1(b), \ldots C_N(b))$.

The monotone Gomory functions are a subclass of the Gomory functions obtained by restricting the λ in (5.1) to have all components non-negative. The unrestricted Gomory functions correspond to using (5.1) as given and allowing all rational α (positive or negative) in (5.4).

C. <u>Integer Programs and Gomory Functions</u>. As in the introduction, if A is an mxr rational matrix and $c \in Q^r$ we have, for each $b \in Q^m$ the integer program

$$(IP_b) \qquad \inf cx$$
$$Ax = b$$
$$x \geq 0 \qquad x \text{ integer.}$$

Here z(b) is the value of the optimal solution to (IP_b).

<u>Theorem 2.3</u> [2, theorems 5.1, 5.2]: For any A,c if $z(\vec{0}) = 0$ there are Gomory functions F,G such that: (i) (IP_b) is consistent iff $G(b) \leq 0$; (ii) If $G(b) \leq 0$ $F(b) = z(b)$.

A function satisfying (i) is called a consistency tester for (IP_b). The proof of theorem 2.3 is constructive.

<u>Theorem 2.4</u> [2, theorem 3.13]: Let F,G be Gomory functions and suppose $G(b) > 0$ if b is not an integer vector. Then there are A,c such that (i) and (ii) hold.*

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*We will discuss the constructive aspects of 2.4 in a forthcoming paper.

Monotone Gomory functions are the appropriate class for the study of integer programs in inequality form:

$$\begin{array}{lll} & \inf cx & \\ (IIP_b) & Ax \leq b & \\ & x \geq 0, & x \text{ integer} \end{array}$$

Theorem 2.5 [2, thm. 5.15, 5.16]: For any A,c if $z(\vec{0}) = 0$ there are monotone Gomory functions F,G such that (i) and (ii) hold.

D. Monoids and Subadditive Functions. A rational monoid is a set $M \subset Q^m$ which contains the zero vector and is closed under addition. M is a finitely generated monoid iff there is a finite $F \subset M$ such that if $F \subset M'$ and M' is a monoid then $M \subset M'$. Equivalently, a finitely generated monoid is the set of feasible right-hand sides of some integer program (the columns of A are the members of F). M is an integer monoid if every element of M is an integer vector.

Theorem 2.6 [13]: The intersection of two finitely generated integer monoids is a finitely generated integer monoid.

A function $F\colon M \to R \cup \{+\infty\}$ is subadditive iff $f(x) + f(y) \geq f(x + y)$ for all $x,y \in M$.

Proposition 2.7 [12]: Gomory functions, value functions of integer programs, and value functions of mixed-integer programs are defined on monoids and are all subadditive in their domains.

3. Consistency Testers are Gomory Functions

We shall say that a function G is a consistency tester for (MIP_b) if, for all right-hand-sides $b \in Q^m$,

$$(3.1) \qquad G(b) \leq 0 \quad \leftrightarrow \quad (MIP_b) \text{ is consistent}$$

The Gomory functions were developed in connection with pure integer programs (the case s=0 in (MIP_b) of no continuous variables). Nevertheless, this class of functions is adequate to account for the consistency testers for all mixed-integer programs (MIP_b), as we show in this section. This will contrast with the fact that there are (MIP_b) value functions which are not Gomory functions, as we shall show in Section 4. We also establish some related results on the adequacy of Gomory functions to represent certain relations and value functions, but postpone to Section 6 a fuller treatment of these matters, after the negative results in Section 4 serve to motivate our interest in the "pre-multiplied mixed integer programs" of Section 6.

Lemma 3.1: Let C be a rational matrix. There exist finitely many rational vectors $\theta^1,\ldots\theta^t$ such that, for all v,

$$(3.2) \qquad Cu \geq v \text{ is consistent} \leftrightarrow \theta^i v \leq 0 \text{ for } i=1,\ldots,t$$

Proof: This result is well-known (see e.g. [18, Chapter 1] where the issue of the field in which the θ^i lie is treated explicitly). In fact, the θ^i can be taken to be the extreme rays of the rational polyhedral cone $\{\theta \geq 0 \mid \theta C = 0\}$.

Q.E.D.

Our next result is related to Benders decomposition [1].

Theorem 3.2: The consistency tester for (MIP_b) is a Gomory function. Moreover, if d=0 in the criterion function cx+dy of (MIP_b), the value function of (MIP_b) is a Gomory function on its domain of definition.

Proof: Using Lemma 3.1, there are rational vectors $\theta^1,\dots,\theta^t$ such that

(3.3) $\quad By = v,\ y \geq 0$ is consistent $\leftrightarrow \theta^i v \leq 0$ for $i=1,\dots,t$

By (3.3), (MIP_b) is consistent if and only if there is some integer vector $x \geq 0$ with

(3.4) $\quad D(b - Ax) \leq 0$

where $D = [\theta^i]$ (rows).

Let A' be the rational matrix whose rows are $\theta^i A$ for $i=1,\dots t$. By Theorem 2.5 there is a monotone Gomory function F with, for all $w \in R^t$,

(3.5) $\quad F(w) \leq 0 \leftrightarrow A'x \geq w,\ x \geq 0$ and integer, has a solution

Let G(b) be the function obtained from F by substituting, for the variable w_i of F, the linear form $\theta^i b$. Thus, if $F(w) = F(w_1,\dots,W_t)$, we have

(3.6) $\quad G(b) = F(\theta^1 b,\dots,\theta^t b)$

Since F is a Gomory function, G is a Gomory function.

We have

$$(3.7)\qquad G(b) \le 0 \leftrightarrow F(\theta^1 b,\ldots,\theta^t b) \le 0$$

$$\leftrightarrow A'x \ge Db,\ x \ge 0 \text{ integer, has a solution}$$

$$\leftrightarrow D(b - Ax) \le 0,\ x \ge 0 \text{ integer, has a solution}$$

$$\leftrightarrow Ax + By = b;\ x,y \ge 0;\ x \text{ integer, has a solution}$$

Thus, G is a consistency tester for (MIP_b).

Now suppose that $d=0$ in the criterion function $cx+dy$ of (MIP_b). By Theorem 2.5 there is a monotone Gomory function $H(w)$ which provides the value of the pure integer program $\inf\{cx|A'x \ge w,\ x \ge 0 \text{ and integer}\}$ wherever that integer program is consistent. As we have seen, for w of the form $w=Db$, the latter integer program is consistent precisely if (MIP_b) is consistent. Moreover, for any solution x to this pure integer program with $w=Db$, the criterion value $cx=cx+0y$ is also a possible criterion value for (MIP_b), and vice-versa. Hence the optimal value of (MIP_b) and $\inf\{cx|A'x \ge Db,\ x \ge 0 \text{ and integer}\}$ is the same. As H is a monotone Gomory function, $K(b) = H(Db)$ is a Gomory function which is also the value function of (MIP_b).

Q.E.D.

We next show that the epigraph of a value function can be identified by a Gomory function.

<u>Corollary 3.3</u>: Let $z(b)$ be the value function of (MIP_b). Then there is a Gomory function $G(z,b)$ such that

$$(3.8)\qquad G(z,b) \le 0 \leftrightarrow z \ge z(b)$$

In particular, $z(b)$ is the smallest value of z satisfying $G(z,b) \leq 0$.

Proof: $z \geq z(b)$ if and only if the following mixed integer constraint set is consistent.

$$cx+dy \leq z, \quad Ax + By = b; \quad x,y \geq 0; \quad x \text{ integer} \tag{3.9}$$

The result is then an immediate consequence of Theorem 3.2. The "in particular" also follows at once.

Q.E.D.

4. Some Negative Results

In this section we show that some of the results relating to pure integer programs do not extend to mixed integer programs.

Theorem 3.2 showed that each MIP has a consistency tester which is a Gomory function. We begin with some examples of Gomory functions that are not consistency testers for any MIP, which shows that theorem 3.2 does not have a converse.

Example 4.1: Let $g(\alpha) = \lceil \alpha \rceil + \lceil -2\alpha \rceil$. $g(\alpha) > 0$ for α close to zero. Hence, if g were a consistency tester it would have to be for an $(\mathrm{MIP})_b$ with no continuous variables. Since $g(\alpha) \leq 0$ for $\alpha \geq 1$, we would have to have all $\alpha \geq 1$ feasible as a right-hand-side, which is impossible.

Note that $\lceil \alpha \rceil$ and $\lceil -2\alpha \rceil$ are consistency testers, so the consistency testers are not closed under addition. Our next example shows they are not closed uner maxima either.

Example 4.2: Let $h_1(\alpha,\beta) = \lceil \alpha \rceil + \lceil -\alpha \rceil$, $h_2(\alpha,\beta) = \max\{-\beta,\beta-\alpha\}$. h_1 is the consistency tester for an MIP with integer columns (1,0); (-1,0) and continuous columns (0,1); (0,-1). h_2 is the consistency tester for an MIP with continuous columns (1,0); (1,1). The function $g = \max\{h_1,h_2\}$ has $g(\alpha,\beta) > 0$ for all (α,β) close to the origin. Hence, as in example 4.1, g cannot be the consistency tester for an MIP with continuous columns. But $g(1,\beta) \leq 0$ for $0 \leq \beta \leq 1$, so at least one continuous column is necessary.

Next we analyze the situation with respect to value functions for constraint sets (MIP_b). In view of theorem 2.1, we may restrict ourselves to those situations in which (MIP_b) is consistent for all b. Neither theorem 2.3 nor 2.4 extends to constraint sets (MIP_b).

Lemma 4.3: If a Chvatal function is continuous at the origin it is linear.

Proof: We argue by induction on the formation of f. If f is linear we are done. If $f = \alpha g$ ($\alpha > 0$) f is continuous at the origin only if g is. By induction hypothesis, g continuous at origin implies g linear, hence f is linear.

It is easy to show that Chvatal functions are zero at the origin and lower semicontinuous. If $f = \lceil g \rceil$ is continuous at the origin g(b) must be non-positive for all b close to the origin. This implies g must be continuous at the origin. By induction, this means g is linear. If $\lceil g \rceil$ is continuous and g is linear, g must be the constant mapping, hence f is also.

If $f = g_1 + g_2$ is continuous at the origin, the semicontinuity of g_1, g_2 implies they are both continuous at the origin. By induction g_1 and g_2 are linear, hence f is linear.

Q.E.D.

Corollary 4.4. If a Gomory function is continuous at the origin, it is a maximum of finitely many linear functions.

Proof: By proposition 2.2, a Gomory function is the maximum of finitely many Chvatal functions. Since the Chvatal functions are all lower semi-continuous and have value zero at the origin, they must be continuous, hence linear. Q.E.D.

Example 4.5: The MIP with one constraint

$$\begin{array}{ll} \min & w + z \\ \text{subject to} & x - y + w - z = \alpha \\ & x,y \text{ integer } x,y,w,z \geq 0 \end{array}$$

has value function $\delta(\alpha) = \min \{\lceil \alpha \rceil - \alpha, \lceil -\alpha \rceil + \alpha\}$

δ is continuous at the origin, but is not a maximum of linear functions. By corollary 4.4, δ is not a Gomory function.

Example 4.6: $\lceil \alpha \rceil + \lceil -\alpha \rceil$ is not a value function. Since we want (MIP_b) to be feasible for all b, B must have at least one continuous column. But if b is such a column then the value of the optimal solution to $(MIP_{\varepsilon b})$ would have to be less than one for small positive ε. Since $\lceil \alpha \rceil + \lceil -\alpha \rceil = 1$ for α close to zero, it cannot be a value function.

Note that $\lceil \alpha \rceil$ and $\lceil -\alpha \rceil$ are value functions.

Example 4.7: Consider the constraint set

$$x + y_1 + y_2 = b_1$$

$$-x + y_1 + y_3 = b_2$$

$$x, y_i \geq 0 \text{ x integer.}$$

For any α_1, $\alpha_2 > 0$ b_1, b_2 can be chosen so that the line segment connecting $(x=0, y_1=\alpha_1)$ and $(x=1, y_1=\alpha_2)$ is a facet of the convex hull of the feasible region. Hence the feasible region may contain facets of arbitrary slope. In pure integer programs only finitely many slopes may occur as b varies [2, theorem 6.2].

5. Which Gomory Functions are Consistency Testers?

Theorem 3.2 and examples 4.1, 4.2 imply that the set of consistency testers is a proper subset of the set of Gomory functions. The main purpose of this section is to characterize this subset. A constructive procedure is described such that, given a closed-form expression for a Gomory function G we can determine whether or not matrices A, B exist such that for all b, $G(b) \leq 0$ iff (MIP_b) has feasible solutions.

Lemma 5.1: Given a Chvatal function $C: R^k \to R$. There are $\lambda^o, \lambda^1, \ldots \lambda^N \in Q^k$, a natural number D, and a monotone Chvatal function $E: R^N \to R$ such that: (i) For all $v \in R^k$ $C(v) = \lambda^o v + E(\lceil \lambda^1 v \rceil, \ldots \lceil \lambda^N v \rceil)$; (ii) $E(e_i) > 0$, $1 \leq i \leq N$ where e_i is the ith unit vector; (iii) For all v, $E(\lceil \lambda^1 v \rceil, \ldots \lceil \lambda^N v \rceil)$ may be written as a rational number with denominator D.

Proof: Our argument is by induction on the formation of C. If C is linear we may take N=D=1, E to be the identity, $\lambda_1 = \vec{0}$, and $\lambda_o v = C(v)$. If $C = \frac{K}{M} C_1$, where K,M are natural numbers and C_1 is a Chvatal function, the induction hypothesis implies that there are λ_1^i, D_1, E_1 such that (i) - (iii) hold. We take $D = MD_1$; $\lambda^o = \frac{K}{M}\lambda_1^o$; and $E = \frac{K}{M}E_1$.

Similarly, if $C = C_1 + C_2$ we take $N = N_1 + N_2$, $E(\alpha_1, \ldots \alpha_N) = E_1(\alpha_1, \ldots \alpha_{N_1}) + E_2(\alpha_{N_1+1}, \ldots \alpha_N)$; $D = D_1 D_2$; $\lambda^o = \lambda_1^o + \lambda_2^o$; $\lambda^i = \lambda_1^i$, $1 \leq i \leq N_1$; $\lambda^i = \lambda_1^{i-N_1}$, $N_1 + 1 \leq i \leq N$.

If $C = \lceil C_1 \rceil$; $\lambda^o = \vec{0}$, $D = 1$; $N = N_1 + 1$; $\lambda^i = \lambda_1^i$, $1 \leq i \leq N_1$; $\lambda^N = D_1 \lambda_1^o$; and $E(\alpha_1, \ldots \alpha_N) = \lceil E_1(\alpha_1, \ldots \alpha_{N-1}) + \frac{1}{D_1}\alpha_N \rceil$. This completes the induction.

Q.E.D.

Lemma 5.2: Let $H: R^k \to R$ be a Gomory function. There are $\lambda^1, \ldots \lambda^N \in Q^k$ and a monotone Gomory function $F: R^N \to R$ such that

(5.1) $G(v) = F(\lceil \lambda^1 v \rceil, \ldots \lceil \lambda^N v \rceil) \leq 0$ iff $H(v) \leq 0$

(5.2) $F(e_i) > 0$, $1 \leq i \leq N$.

Proof: Recall from proposition 2.2 that there are Chvatal function $C_1, \ldots C_M$ such that $H(v) = \max_{1 \leq j \leq M} C_j(v)$. Next apply Lemma 5.1 to each C_j to obtain λ_j^i, $1 \leq i \leq N_j$ and E_j such that $C_j(v) = \lambda_j^o v + E_j(\lceil \lambda_j^1 v \rceil, \ldots \lceil \lambda_j^N v \rceil)$. With D_j as in (iii) above note that $B_j(v) = \frac{1}{D_j} D_j \lambda_j^o v + E_j(\lceil \lambda_j^i v \rceil, 1 \leq i \leq N_j) \leq 0$ iff $C_j(v) \leq 0$. It is simple to choose F, λ^i so that $G(v) = \max_{1 \leq j \leq M} B_j(v)$. (5.2) follows from (ii) above.

Q.E.D.

Lemma 5.3: Let G be given by (5.1). Let $T = \{v \mid \lambda^i v = 0, 1 \leq i \leq N\}$. Assume T is non-trivial. Let $\theta^1, \ldots \theta^L$ be a basis for T. Then G is a consistency tester if and only if

$J(v) = \max\{G(v); \lceil \theta^i v \rceil, 1 \leq i \leq L; \lceil (-\Sigma\theta^i) v \rceil\}$ is a consistency tester.

Proof: Recall from linear algebra that for every v there is a unique representation $v = v_1 + v_2$ where $v_1 \in T$ and $\theta^i v_2 = 0$, $1 \leq i \leq L$.

Suppose that J is a consistency tester, i.e., that there are A, B such that $J(b) \leq 0$ iff (MIP_b) has feasible solutions. Form a new MIP by adding columns $\theta^1, \ldots \theta^L$, $(-\Sigma\theta^i)$ to B. If b is feasible for the new MIP there is a b' such that $b - b' \in T$ and $J(b') \leq 0$, hence $G(b) = G(b') \leq J(b') \leq 0$. If $G(b) \leq 0$ then $b = b_1 + b_2$ as above. $b_1 \in T$ implies $G(b_2) = G(b)$, hence $J(b_2) \leq 0$, b_2 is feasible for the original MIP, hence b is feasible for the new MIP. Thus, G is a consistency tester for the new MIP.

Conversely, suppose that G is a consistency tester and that A, B define the appropriate MIP for G. To form the appropriate MIP for J take each column v of A or B, decompose $v = v_1 + v_2$ as above and replace v by v_2. $J(b) \leq 0$ only if $G(b) \leq 0$ and $\theta^i b = 0$, $1 \leq i \leq L$. If $b = b_1 + b_2$ is feasible for the MIP for G then b_2 is feasible for the MIP for J, because the map from b to b_2 is linear. $\theta^i b = 0$, $1 \leq i \leq L$ implies $b_1 = \vec{0}$, hence $b = b_2$. If b is feasible for the MIP for J then there is a b' feasible to the MIP for G with $b - b' \in T$. Since all columns v of the MIP for J satisfy $\theta^i v = 0$, b feasible implies
$J(b) = \max\{G(b),0\} = \max\{G(b'),0\} = 0$.

Q.E.D.

From this point onwards, we confine our attention to those Gomory functions G satisfying (5.1), (5.2), and

(5.3) $\lambda^i v = 0$, $1 \leq i \leq N$ implies $v = \vec{0}$.

If we are given a formula for an arbitrary Gomory function H, Lemma 5.2 enables us to construct a Gomory function G satisfying (5.1) and (5.2) which is a consistency tester iff H is. If G does not satisfy (5.3), then the function J constructed in lemma 5.3 does, and is a consistency tester if and only if G is. Thus a procedure for determining whether a Gomory function satisfying (5.1) - (5.3) is a consistency tester suffices to solve the original problem.

<u>Theorem 5.4</u>: Let G satisfy (5.1) - (5.3). If G is a consistency tester then G is a consistency tester for an (MIP) in which the columns of B are the extreme rays of the cone $C = \{v | \lambda^i v \leq 0,\ 1 \leq i \leq N\}$.

Proof: Note that (5.3) implies C is a pointed cone and standard results of convexity theory imply that every member of C is a non-negative linear combination of the extreme rays of C.

If A, B is an MIP corresponding to G and w is a column of B then we claim $w \in C$. If $w \notin C$ then for some $\alpha > 0$ $\lceil \lambda^i(\alpha w) \rceil \geq 0$, $1 \leq i \leq N$ and $\lceil \lambda^j(\alpha w) \rceil = 1$ for at least one j. $G(\alpha w) \geq F(e_j)$ by the monotonicity of F and $F(e_j) > 0$ by (5.2). If w is a column of B we must have $G(\alpha w) \leq 0$ for all positive α, so our claim is established. Further, if $w \in C$ $G(w) \leq F(\vec{0}) = 0$ so all members of C must be feasible for the MIP. From these two results we conclude that the MIP obtained by replacing B by the matrix of extreme rays of C has the same feasible set, hence also has G as a consistency tester.

Q.E.D.

An example illustrates a subtle way in which a function may fail to be a consistency tester. It motivates our subsequent analysis.

Example 5.5: Let $G(\alpha_1,\alpha_2) = \lceil \alpha_1 + \alpha_2 \rceil + \lceil \alpha_1 - \alpha_2 \rceil + \lceil \alpha_1 - 2\alpha_2 \rceil$. G satisfies (5.1) - (5.3) hence we may apply theorem 5.4. The extreme rays of C are $w_1 = (-1,1)$, $w_2 = (-2,-1)$. For $1 < \alpha_1 < 1.5$ $G(\alpha_1, 1 + \alpha_1) = 0$. For any $\epsilon > 0$ $G((\alpha_1, 1 + \alpha_1) - \epsilon w_i) > 0$, $i = 1,2$. If G were a consistency tester for an MIP in which the columns of B are w_1, w_2 then $(\alpha_1, 1 + \alpha_1)$ must be a feasible right-hand side. Moreover, the MIP would have to have a feasible solution with $y_1 = y_2 = 0$, because otherwise $(\alpha_1, 1 + \alpha_1) - \epsilon w_i$ would be feasible for some $\epsilon > 0$. However, the integrality requirement on the x vbls implies there must be infinitely many α_1 such that no suitable feasible solution exists. Hence, G cannot be a consistency tester.

We shall show that example 5.5 represents the typical way in which a function fails to be a consistency tester. Theorem 5.4 enables us to explicitly identify the columns of the B-matrix. The set of points which are integer combinations of the A-matrix columns is discrete. If a line segment (in this example the open segment with end points (1,2) and (1.5,2.5)) can be identified such that each point of the segment would have to be an integer combination of A-columns, then the function cannot be a consistency tester.

<u>Definition 5.6</u>: Let G be a specified Gomory function in the form (5.1). v is defined to be a maximal vector if and only if: (i) $G(v) \leq 0$ and (ii) whenever $G(w) \leq 0$ and $\lambda^i w \geq \lambda^i v$ for $1 \leq i \leq N$, then $w = v$.

<u>Lemma 5.7</u>: Let G satisfy (5.1) - (5.3). If G is the consistency tester for (MIP) and b is a maximal vector, then for some integer $x \geq 0$ $Ax = b$.

<u>Proof</u>: Since $G(b) \leq 0$ there are $x,y \geq \vec{0}$, x integer with $Ax + By = b$. By theorem 5.4 we may assume that if w is a column of B, $\lambda^i w \leq 0$, $1 \leq i \leq N$. If $y \neq \vec{0}$, then there is $\vec{0} \leq y' \leq y$ such that $b - By' \neq b$. Since $Ax + B(y - y') = b - By'$, $G(b - By') \leq 0$. Since $y' \geq 0$ implies $\lambda_i(b - By') \geq \lambda_i b$, this contradicts property (ii) of maximal vectors.

Q.E.D.

<u>Definition 5.8</u>: For G of the form (5.1) let

$$(5.4)\quad J_v = \{w \mid \lceil \lambda^i w \rceil = \lceil \lambda^i v \rceil,\ \lceil -\lambda^i w \rceil = \lceil -\lambda^i v \rceil,\ 1 \leq i \leq N\}$$

<u>Proposition 5.9</u>: If $w \in J_v$, $G(v) = G(w)$.

<u>Lemma 5.10</u>: If v is maximal and $w \in J_v$, then w is maximal.

<u>Proof</u>: Property (i) holds for w by 5.9. If property (ii) fails, we have $w' \neq w$ with $\lambda^i w' \geq \lambda^i w$ and $G(w') \leq 0$. For some $\alpha > 0$ we have $v' = v + \alpha(w' - w)$ such that $\lceil \lambda^i v' \rceil \leq \lceil \lambda^i w' \rceil$, $1 \leq i \leq N$. Monotonicity of F implies $G(v') \leq G(w') \leq 0$, which contradicts the maximality of v.

Q.E.D.

Lemma 5.11: If G satisfying (5.1) - (5.3) is a consistency tester, then for every maximal vector v, $J_v = \{v\}$.

Proof: If $w \in J_v$ and $w \neq v$ then every vector $\alpha w + (1 - \alpha)v$, $0 \leq \alpha \leq 1$ is in J_v, hence is a maximal vector by 5.10. If G were a consistency tester, then 5.7 would imply that every convex combination of w and v would be an integer linear combination of the columns of A, which is impossible.

Q.E.D.

We shall establish the converse to 5.11 later. First, we examine the conditions under which $J_v = \{v\}$.

Lemma 5.12: Let G satisfy (5.1) - (5.3). $J_v = \{v\}$ if and only if $\lambda^i v$ is integer for a linearly independent subset of size k from $\{\lambda^1, \ldots \lambda^N\}$.

Proof: Recall that the $\lambda^i \in Q^k$ and that (5.3) implies the set of all λ^i has dimension k. The "if" direction follows immediately from the definition of J_v since the condition given implies that any member $w \in J_v$ would have to solve a non-singular system of k equations in k unknowns.

Conversely if $J_v = \{v\}$ let $S = \{i | \lambda^i v \text{ is integer}\}$. It must be the case that $\lambda^i w = 0$ for all $i \in S$ implies $w = \vec{0}$ (otherwise $v + \alpha w \in J_v$ for small α). The conclusion follows by linear algebra.

Q.E.D.

For technical reasons we need to establish the ways in which denominators of members of J_v depend on the λ^i.

Lemma 5.13: For every $v \in R^k$, J_v contains a rational member.

Proof: For $1 \leq i \leq N$, choose α_i, β_i as follows: if $\lambda^i v$ is integer $\alpha_i = \beta_i = \lambda^i v$; otherwise $-\lceil -\lambda^i v \rceil < \beta_i < \lambda^i v < \alpha_i < \lceil \lambda^i v \rceil$. Let $Q = \{w | \beta_i \leq \lambda^i v \leq \alpha_i, 1 \leq i \leq N\}$. $Q \subset J_v$ is a non-empty ($v \in Q$) polytope with rational defining inequalities, hence it has a rational member.

Q.E.D.

Proposition 5.14: Let M be an integer such that $M\lambda^i$ is an integer vector, $1 \leq i \leq N$. Then for all v, $J_{v+Me_j} = J_v + Me_j$, $1 \leq j \leq k$, (recall e_j = jth unit vector).

Proof: $J_{v+Me_j} = \{w | \lceil \lambda^i w \rceil = \lceil \lambda^i v \rceil + M\lambda^i e_j; \lceil -\lambda^i w \rceil = \lceil -\lambda^i v \rceil - M\lambda^i e_j, 1 \leq i \leq N\} = \{w | w - Me_j \in J_v\}$.

Q.E.D.

Lemma 5.15: There is a finite $F \subset Q^k$ such that, for any w, there is a $v \in F$ for which $J_w = J_v + u$ for some integer vector u.

Proof: By using 5.14, we can find u, w such that $J_w = J_v + u$ where $\vec{0} \leq v \leq \vec{M}$ and u is an integer vector. For v in a bounded region of space, only finitely many sets J_v occur. By 5.13 each of these sets has a rational member, which we may choose for F.

Corollary 5.16: There are natural numbers D_1, D_2 such that (i) if $J_v = \{v\}$ then every component of v is a rational with denominator D_1. (ii) if $J_v \neq \{v\}$ then there is a rational vector w all of whose components are rationals with denominator D_2 such that at least one component cannot be written as a rational with denominator D_1.

Proof: (i) follows immediately from 5.15. To establish (ii) note that if J_v is not a singleton, then the rational polytope Q defined in 5.13 is not a singleton, hence it has a rational member whose components do not all have denominator D_1. By 5.15, we may take D_2 to be the least common multiple of the denominators arising from those J_v, $v \in F$ that are not singletons.

Q.E.D.

Lemma 5.17: Let $F:R^N \to R$ be a Gomory function, $w \in R^N$. If $F(w) + F(-w) > 0$ then for every $u \in R^N$ and L there is a natural number M such that $F(u + Mw) + MF(-w) \geq F(u) + L$.

Proof: We argue by induction on F. If F is linear, $F(w) + F(-w) = 0$ and the conclusion holds vacuously. If $F = \alpha F_1$, $\alpha > 0$, then by induction hypothesis there is an M such that $F_1(u + Mw) + MF_1(-w) \geq F_1(u) + \frac{L}{\alpha}$ and the conclusion follows.

If $F = F_1 + F_2$ we may assume $F_1(w) + F_1(-w) > 0$. By induction hypothesis there is an M such that $F_1(u + Mw) + MF_1(-w) \geq F_1(u) + L$. Since Gomory functions are subadditive $F_2(u + Mw) + MF_2(-w) \geq F_2(u)$, and we obtain the conclusion by adding the two inequalities.

If $F = \lceil F_1 \rceil$ and $F_1(w) + F_1(-w) \leq 0$ then subadditivity implies $F_1(u + Mw) = F_1(u) + MF_1(w)$ for all integer M. Since $F(w) + F(-w) > 0$, $F_1(w) = -F_1(-w)$ is not integer. Choose M so that $MF_1(w) + M\lceil -F_1(w) \rceil \geq L + 1$. Then $F(u + Mw) = \lceil F_1(u) + MF_1(w) \rceil \geq \lceil F_1(u) + L + 1 - M\lceil F_1(-w) \rceil \rceil = \lceil F_1(u) + L + 1 \rceil - MF(-w) \geq F(u) + L - MF(-w)$. If $F_1(w) + F_1(-w) > 0$ then we may apply the induction hypothesis to F_1 and obtain M such that $F_1(u + Mw) + MF_1(-w) \geq F_1(u) + L + 1$ and the conclusion follows.

If $F = \max\{F_1, F_2\}$ and $F_1(w) + F_1(-w) > 0$, then the induction hypothesis implies there is an M such that $F_1(u + Mw) + MF_1(-w) \geq F(u) + L$. Since $F \geq F_1$, the conclusion follows. The remaining case is

$F_1(w) + F_1(-w) = F_2(w) + F_2(-w) = 0$. In this case $F_i(u + Mw) = F_i(u) + MF_i(w)$, $i=1,2$. We may assume $F(w) = F_1(w)$ and $F(-w) = F_2(-w)$. Choose M so that $M(F(w) + F(-w)) \geq L + F(u) - F_1(u)$ and the conclusion follows.

Q.E.D.

Lemma 5.18: Let G satisfy (5.1) - (5.3), $G(v) \leq 0$. Let $T_v = \{w | \lambda^i w \geq \lambda^i v,\ 1 \leq i \leq N\}$. There is a $w \in T_v$ which is a maximal vector (definition 5.6).

Proof: Let $z = (\lambda^1 v, \ldots \lambda^N v)$. For $1 \leq j \leq N$ either $F(e_j) + F(-e_j) = 0$ or $F(e_j) + F(-e_j) > 0$ by subadditivity. In the first case $F(z + Ne_j) = F(z) + NF(e_j)$. In the second case, we may apply lemma 5.17. In either case, there is a natural number M_j such that $F(z + M_j e_j) > 0$. Let $M = \max M_j$. Using the monotonicity of F, we conclude that if $w \in T_v$ and $G(w) \leq 0$ then $\lambda^i w \leq \lambda^i v + M$. By (5.3) $\{w | \lambda^i v \leq \lambda^i w \leq \lambda^i v + M\}$ is compact. Hence there is a $w^1 \in T_v$ such that (i) $G(w^1) \leq 0$ (ii) if $w \in T_v$ and $G(w) \leq 0$ then $\lambda^1 w \leq \lambda^1 w^1$.

For $2 \leq j \leq N$ we obtain $w^j \in T_v$ such that (i) $G(w^j) \leq 0$ (ii) $\lambda^i w^j = \lambda^i w^{j-1}$, $1 \leq i \leq j-1$ (iii) if $w \in T_v$ satisfies (i) and (ii) then $\lambda^j w \leq \lambda^j w^j$. w^N is the desired maximal vector.

Q.E.D.

Theorem 5.19: Let G satisfy (5.1) - (5.3). Then G is a consistency tester if and only if $J_v = \{v\}$ for every maximal vector v.

Proof: We established the "only if" part of the result in 5.11. If $J_v = \{v\}$ for every maximal v we construct matrices A,B to form the appropriate (MIP). The columns of B consist of the extreme rays of the

cone $\{w | \lambda^i w \leq 0,\ 1 \leq i \leq N\}$. To construct A we consider the function $G': R^k \to R$ defined by $G'(b_1, \ldots b_k) =$
$\max\{\lceil b_i \rceil - b_i,\ 1 \leq i \leq k;\ G(\frac{b_1}{D_1}, \ldots \frac{b_k}{D_1})\}$, where D_1 is given by 5.16. $G'(b) \leq 0$ iff b is an integer vector and $G(\frac{b}{D_1}) \leq 0$. By 2.4 a matrix A' can be constructed such that $G'(b) \leq 0$ iff b is a non-negative integer combination of columns of A'. The columns of A consist of the columns of A' divided by D_1. If $G(b) \leq 0$ and all components of b have denominator D_1 then b is a non-negative integer combination of columns of A. In particular, 5.16 implies that every maximal vector is such a combination. For any b, if $G(b) \leq 0$, 5.18 implies there is a maximal vector v such that $\lambda^i(b - v) \leq 0$. Since $b - v$ is a non-negative linear combination of columns of B, we have a feasible solution to (MIP_b).

Q.E.D.

<u>Lemma 5.20</u>: If v is a maximal vector and $v = \Sigma\alpha_i w_i$ where α_i are natural numbers and $G(w_i) \leq 0$ then the w_i are maximal vectors.

<u>Proof</u>: If some w_j is not maximal then there is $w_j' \neq w_j$ with $\lambda^i w_j' \geq \lambda^i w_j$ and $G(w_j') \leq 0$. But subadditivity would imply $G(v - \alpha_j w_j + \alpha_j w_j') \leq 0$, which would contradict the maximality of v.

Q.E.D.

<u>Theorem 5.21</u>: Let G satisfy (5.1) - (5.3). Suppose that $w_1, \ldots w_L$ are such that for every b whose components are rationals with denominator D_2, $G(b) \leq 0$ iff b is a non-negative integer combination of the w_i. Then G is a consistency tester iff every w_i which is a maximal vector satisfies $J_{w_i} = \{w_i\}$.

<u>Proof</u>: If G is a consistency tester, we may apply 5.11. If G is not a consistency tester, then 5.19 implies there is a maximal v for

which $J_v \neq \{v\}$. By 5.16 and 5.10 we may assume every component of v is a rational with denominator D_2, and that at least one component is not a rational with denominator D_1. By hypothesis, v is a non-negative integer combination of w_i, which are maximal by 5.19. At least one maximal w_j does not have all its components with denominator D_1. By 5.16, $J_{w_j} \neq \{w_j\}$.

Q.E.D.

Theorem 5.19 gives us our procedure for testing whether or not a given G is a consistency tester. As described previously, G is modified so that (5.1) - (5.3) are satisfied. Then D_2 is calculated. Using theorem 2.4 we construct the appropriate set of w_i. For each w_i, it can be determined if it is a maximal vector and if $J_{w_i} = \{w_i\}$.

It must be admitted that this description of which Gomory functions are consistency testers is cumbersome, but we do not think there is a really neat characterization. In particular, we conjecture that the problem of deciding whether or not a given formula for a Gomory function is a consistency tester is NP - complete.

6. Pre-multiplied Mixed Integer Constraint Sets

We now first summarize the information provided by this paper up to the present section, and give a perspective on it.

The Gomory functions were developed to account for pure integer programs [2]. There is a priori no reaon to expect this class of functions to account for mixed integer programs (MIP_b).

The barriers to a treatment of (MIP_b) by Gomory functions were highlighted in Section 4.

First of all, the Gomory functions are, by construction, a class of functions inductively closed under the operations of maximum, sum, and round-up; neither the class of consistency testers or value functions of (MIP_b) are so closed. Since the Gomory functions do provide all consistency testers for (MIP_b) by Theorem 3.2, that class must also contain functions which are not consistency testers for (MIP_b). Our efforts in Section 5 were directed at obtaining some algorithmic procedure for identifying the consistency testers for (MIP_b) within the much larger class of Gomory functions.

The second barrier to a treatment of (MIP_b) by Gomory functions, is that some value functions of problems (MIP_b) are not Gomory functions. To this fact, we add the complication, as before, that some value functions for (MIP_b) are Gomory functions, even though their sum or maximum or round-up is not a value function for a problem (MIP_b). Thus, on the one hand, there will be Gomory functions which are not value functions, and on the other hand, value functions which are not Gomory functions. The situation for value functions is thus even more complex than for consistency testers.

Despite these barriers, matters are simpler than might appear. We proceed to account for the discrepancies by, first, addressing the easier issue of consistency testers, and asking the question: is there a class of constraints for which the Gomory functions are exactly the class of consistency testers?

Observe that, if $H(b)$ is a Gomory function and C is a rational matrix, then $G(v) = H(Cv)$ is a Gomory function. If H happens also to be the consistency tester for (MIP_b), then G is the consistency tester for this "premultiplied" mixed integer constraint set:

$$(PMIP_v) \qquad Ax + By = Cv;\ x,y \geq 0;\ x \text{ integer}$$

This motivates a consideration of such constraint sets. They would arise in practice, for example, when certain right-hand-sides of a mixed integer program depend linearly on others. By Theorem 3.2, consistency testers for ($PMIP_v$) are all Gomory functions, so such constraint sets are the smallest class of constraints which may possibly allow us to answer the question of the previous paragraph.

In the following, if S is the set of all r.h.s. v such that ($PMIP_v$) holds (for certain rational A, B, and C), we call S a <u>finitely generated mixed monoid</u>.

<u>Theorem 6.1</u>: The Gomory functions provide exactly a class of consistency testers for the constraint sets ($PMIP_v$).

Moreover, the sets S for which there is a rational matrix Λ and a finitely generated integer monoid M with

$$(6.1) \qquad S = \bigcup_{m \in M} \{v \mid \Lambda v \leq m\}$$

are exactly the finitely-generated mixed monoids.

<u>Proof</u>: It suffices to show, in order, these implications: if S is a finitely generated mixed monoid, its consistency tester is a Gomory function; if G is a Gomory function, then there is a rational matrix Λ and a finitely generated integer monoid M such that (6.1) holds, for $S = \{v \mid G(v) \leq 0\}$; if (6.1) holds, then S is a finitely generated mixed monoid.

Our previous discussion established the first implication, using Theorem 3.2.

As to the second implication, let G be a Gomory function. Since the second implication concerns G only through the set $S = \{v \mid G(v) \leq 0\}$, by Lemma 5.2, we may assume that G has the form

$$(6.2) \qquad G(v) = F(\lceil \lambda^1 v \rceil, \ldots, \lceil \lambda^t v \rceil)$$

where $F(w) = F(w_1, \ldots, w_t)$ is a monotone Gomory function.

We define $H\colon \; R^t \to R$ by

$$(6.3) \qquad H(w) = \max \{F(w), \lceil w_i \rceil - w_i, \; i = 1, \ldots t\}$$

$H(w) \leq 0$ iff w is an integer vector and $F(w) \leq 0$. We apply Theorem 2.4 with $F = 0$, $G = H$ to obtain an integer program such that $H(w) \leq 0$ iff (IP_w) is consistent. We let M be the monoid generated by the columns of A and let Λ be the matrix with ith row λ_i, $1 \leq i \leq t$ and (6.1) holds.

As to the third implication, suppose that (6.1) holds. Then if M is generated by m^j for $j = 1,.., s$ we have:

(6.4) $v \in S \leftrightarrow$ for some $m \in M$, $\Lambda v \leq m$

$\leftrightarrow$ there exist $x,y \geq 0$ with x integer and

$$\Lambda v = \sum_{j=1}^{s} m^j x_j = I_y$$

where I is the identity matrix. Thus, in (PMIP_v) we may take $C = \Lambda$, $B = -I$, and $A = [m^j]$ (cols).

Q.E.D.

The finitely generated mixed monoids have certain interesting closure properties, some of which follow directly from closure properties of Gomory functions, and some of which imply new closure properties of Gomory functions. We provide some examples in the immediately following discussion and the next two results.

Let two finitely generated mixed monoids be defined by:

(6.5a) $S_1 = \{v | A_1 x + B_1 y = C_1 v;\ x,y \geq 0;\ x \text{ integer}\}$

(6.5b) $S_2 = \{v | A_2 x + B_2 y = C_2 v;\ x,y \geq 0;\ x \text{ integer}\}$

for rational matrices A_i, B_i, C_i and $i = 1,2$. One can prove directly that their intersection $S_1 \cap S_2$ is a mixed monoid; and one easily notes that this corresponds to the fact that the maximum of two Gomory functions is a Gomory function. It is also easy to prove directly that, if α_1 and α_2 are rational scalars, then $\alpha_1 S_1 + \alpha_2 S_2$ is a finitely generated mixed monoid.

Proposition 6.2: If S is a finitely generated mixed monoid, and $v \varepsilon S$ is partitioned $v = (u,w)$, then the following is also a finitely generated mixed monoid:

$$S' = \{u \mid \text{ for some } w, \text{ we have } (u,w) \varepsilon S\}. \tag{6.6}$$

Proof: Let $S = \{v \mid Ax + By = Cv \text{ for some } x,y \geq 0 \text{ with } x \text{ integer}\}$ where A, B and C are rational matrices. Partition $C = [D \vdots E]$ conformally with $v = (u,w)$. Then we have:

$$S' = \{u \mid Ax + By - Ew_1 + Ew_2 = Du \text{ for some } x,y,w_1,w_2 \geq 0 \text{ with } x \text{ integer}\} \tag{6.7}$$

From (6.7) S' is a finitely generated mixed monoid.

Q.E.D.

Corollary 6.3: If G(u,w) is a Gomory function, there is a Gomory function H(u) such that:

$$H(u) \leq 0 \leftrightarrow \min_{w} G(u,w) \leq 0. \tag{6.8}$$

Proof: This follows directly from Theorem 6.1 and Proposition 6.2.

Q.E.D.

We next turn our attention to the value function of ($PMIP_v$) with respect to a chosen criterion function $cx + dy$, i.e., functions of the form $z(v) = \inf \{cx + dy \mid Ax + By = Cv \text{ for some } x,y \geq 0 \text{ with } x \text{ integer}\}$, where A, B and C are rational matrices. As usual, we assume throughout our discussion that $z(0) > -\infty$. (If $z(0) = -\infty$, then $z(v) = -\infty$ whenever ($PMIP_v$) is consistent, and this case is therefore trivial).

If $G(z,v)$ is a Gomory function of the scalar z and vector v, we introduce the symbolism

$$(6.9) \qquad F(v) = \mu z(G(z,v) \leq 0)$$

to define the function $F(v)$ = the least value of z such that $G(z,v) \leq 0$. If no z exists with $G(z,v) \leq 0$, we define $F(v) = +\infty$ in (6.9).

Theorem 6.4: The value functions of constraints (PMIP_v) are exactly the functions F defined by (6.9), where $G(z,v)$ is a Gomory function. The function F is a value function of mixed integer program exactly if $G(z,v)$ is a consistency tester for such a program.

Proof: First, we show that the value function of (PMIP_v) has the desired form (6.9). By Corollary 3.3, there is a Gomory function H with

$$(6.10) \qquad \mu z(H(z,b) \leq 0) = \inf \{cx + dy \mid Ax + By = b; x,y > 0;\ x \text{ integer}\}$$

Hence $\mu z(H(z,Cv) \leq 0)$ is the value function of (PMIP_v), and as C is a rational linear transformation, $G(z,v) = H(z,Cv)$ is a Gomory function.

Now suppose that $G(z,v)$ is a Gomory function. By Theorem 6.1, there exists rational matrices A, B, C and a rational column vector C_0 with

$$(6.11) \qquad G(z,v) \leq 0 \leftrightarrow Ax + By = C_0 z + Cv \text{ for some } x,y, \geq 0 \text{ with } x \text{ integer}$$

From (6.11) we see at once that

$$(6.12)\qquad F(v) = \inf \{z_1 - z_2 | Ax + By - C_0 z_1 + C_0 z_2 = Cv \text{ with } x,y,z_1,z_2 \geq 0 \text{ and } x \text{ integer}\}$$

where F is defined by (6.9).

In the case of a mixed-integer program (i.e., C = I = the identity matrix, in ($PMIP_v$)), we found that (6.9) held for the function G given as the consistency tester of the mixed-integer constraints (3.9). Conversely, in the case that G(z,v) is a consistency tester for a mixed-integer constraint set, we can take $C_0 = e_1$ = the first unit vector in (6.14), and

$$(6.13)\qquad C = \left[\begin{array}{c} 0 \\ \hline I \end{array}\right]$$

in (6.11), where 0 denotes a zero row and I an identity matrix. We can conformally partition A and B thus:

$$(6.14)\qquad A = \left[\begin{array}{c} c \\ \hline A' \end{array}\right] \quad , \qquad B = \left[\begin{array}{c} d \\ \hline B' \end{array}\right]$$

where c,d are rational row vectors and A', B' are rational matrices. Again, (6.12) holds, hence

$$(6.15)\qquad F(v) = \inf \{z | z = cx + dy,\ A'x + B'y = v; x,y \geq 0;\ x \text{ integer}\}$$

$$= \inf \{cx + dy | A'x + B'y = v; x,y \geq 0;\ x \text{ integer}\}.$$

Thus, F is the value function of a mixed integer program.

Q.E.D.

Our next result provides another expression for pre-multiplied value functions.

Theorem 6.5: The value function $z(v)$ of a pre-multiplied constraint set ($PMIP_v$) is equal to the minimum of finitely many Gomory functions where $z(v)$ is defined, provided that $z(\vec{0}) > -\infty$. Furthermore, the minimum of Gomory functions will be subadditive on R^m.

Proof: By Theorem 6.4 we may assume that

$$(6.16) \qquad z(v) = \mu z(G(z,v) \leq 0)$$

where G is a Gomory function. By Lemma 5.2, we may assume that

$$(6.17) \qquad G(z,v) = F(\lceil \alpha_1 z+\lambda^1 v \rceil, \ldots, \lceil \alpha_t z+\lambda^t v \rceil)$$

where $F\colon R^t \to R$ is a monotone Gomory function. By Theorem 2.1 we may assume $z(v)$ is finite for all v.

Define $N = \{i | \alpha_i < 0$. N must be non-empty if $z(\vec{0}) < -\infty$. For all v the integers $n_i(v) = \lfloor \alpha_i z(v) + \lambda_i v \rfloor$, $i \varepsilon N$ are well defined. ($\lfloor x \rfloor$ = largest integer $\leq x$).

For $i \varepsilon N$, $-n_i(b)$ is also the optimal value of this pre-multiplied constraint set, which has an integer objective function value:

$$\min - (n_i^+ - n_i^-)$$

$$(6.18) \qquad \text{subject to} \qquad \begin{aligned} &(n_i^+ - n_i^-) - \alpha_i(z_1 - z_2) \leq \lambda^i v \\ &Ax + By - C_0(z_1 - z_2) = Cv \\ &x,\ y,\ z_1,\ z_2,\ n_i^+,\ n_i^- \geq 0 \\ &x \text{ integer};\ n_i^+,\ n_i^- \text{ integer} \end{aligned}$$

where A, B, C are rational matrices and C_0 is a rational column vector, chosen so that

$$(6.19) \qquad G(z,v) \leq 0 \leftrightarrow Ax + By = C_0 z + Cv$$

By Theorem 6.1, (6.19) holds for suitable A, B, C, and C_0.

By Theorem 3.2 the value function of (6.18) equals a Gomory function pre-multiplied by a matrix, where defined. Hence there is a Gomory function $F_i(v)$ such that $-n_i(v) = F_i(v)$ for all v.

We claim that if for all $v \in Q^m$ there is an $i\varepsilon N$ such that $n_i(v) = \alpha_i z(v) + \lambda_i v$. If this were not the case there would be a $z' < z(v)$ such that $\lceil \alpha_i z' + \lambda^i v \rceil \leq \lceil \alpha_i z(v) + \lambda^i v \rceil$, $1 \leq i \leq t$. The monotonicity of F would imply that $G(z',v) \leq 0$, which would contradict (6.16).

Since the definition of $n_i(v)$ implies $n_i(v) \leq \alpha_i z(v) + \lambda_i v$ for all $i\varepsilon N$, it follows that

$$(6.20) \qquad z(v) = \min_{i\varepsilon N} \left(-\frac{1}{\alpha_i}\right)(\lambda_i v - n_i(v))$$

wherever z(v) is well defined. Since the $\alpha_i < 0$ and $-n_i(v) = F_i(v)$ we have the desired representation.

From prop 2.7, z is subadditive, hence by (6.20) the minimum on the right is subadditive for all v.

Q.E.D.

Theorem 6.4 provides one characterization for value functions of both premultiplied and ordinary mixed integer programs, via the algorithmic procedure of Section 5 for identifying consistency tests for mixed-integer programs. In the remainder of this section, we seek other inductive characterizations of pre-multiplied value functions and derive consequences of these characterizations.

The <u>infimal convolution</u> [16] of a finite set $f_1, \ldots, f_t$ of functions, denoted $\text{infcon}\{f_1, \ldots, f_t\}$, is the function f defined by

$$f(v) = \inf\{f_1(v^1) + \ldots + f_t(v^t) \mid v^1 + \ldots + v^t = v\} \tag{6.21}$$

It is possible to have $f(v) = -\infty$ even if all f_i are finite valued. One easily proves that only the infimal convolution of functions two-at-a-time need be defined. Indeed, if we set

$$g_1 = f_1 \tag{6.22a}$$

$$g_{i+1} = \text{infcon}\{f_{i+1}, g_i\} \text{ for } 1 \leq i \leq t-1 \tag{6.22b}$$

we have

$$g_t = \text{infcon}\{f_1, \ldots, f_t\} \tag{6.23}$$

To handle occurrences of $-\infty$ which can arise due to infimal convolution, we shall adhere to these conventions, whenever $r \in R \cup \{-\infty\}$:

$$(-\infty)\cdot r = r\cdot(-\infty) = -\infty \text{ if } r > 0; \; (-\infty)\cdot 0 = 0\cdot(-\infty) = 0; \tag{6.24}$$

$$r + (-\infty) = (-\infty) + r = -\infty.$$

The <u>mixed Gomory functions</u> is the class of functions inductively defined by these clauses:

(6.25a) All linear functions λv with $\lambda \in Q^m$ are mixed Gomory;

(6.25b) If F_1 and F_2 are mixed Gomory and α and β are non-negative rationals, then $\alpha F_1 + \beta F_2$ is mixed Gomory;

(6.25c) If F_1 and F_2 are mixed Gomory, then $\max\{F_1,F_2\}$ is mixed Gomory;

(6.25d) If F is mixed Gomory, then $\lceil F \rceil$ is mixed Gomory;

(6.25e) If F_1 and F_2 are mixed Gomory, then $\operatorname{infcon}\{F_1,F_2\}$ is mixed Gomory.

Note that the mixed Gomory functions take values in $R \cup \{-\infty\}$. This class of functions can alternatively be defined, as inductively obtained by closing the class of Gomory functions under infimal convolution.

We next state an alternative characterization of the value functions of pre-multiplied constraint sets.

Theorem 6.6: The class of value functions of pre-multiplied constraint sets, including the value function which is identically $-\infty$ where defined, are exactly the class of mixed Gomory functions restricted to the domain of definition of the value function. Moreover, for any mixed Gomory function F with $F(0) > -\infty$, there is a premultiplied constraint set defined for all r.h.s. v, for which F is the value function.

Our proof of Theorem 6.6 will proceed by several lemmata, which also have some independent interest.

Lemma 6.7: Suppose that $z_1(b)$ and $z_2(b)$ are pre-multiplied value functions (where one or both functions may be identically $-\infty$ where defined).

Then the following functions are pre-multiplied value functions (including possibly the $-\infty$ value function):

a) $\alpha z_1 + \beta z_2$ when $\alpha,\beta \geq 0$ are rational scalar;

b) $\max\{z_1,z_2\}$;

c) $\lceil z_1 \rceil$

d) $\text{infcon}\{z_1,z_2\}$

Proof: For notational purposes, set:

(6.26) $$z_i(b) = \inf\{c^i x + d^i y \mid A^i x + B^i y = C^i b;\ x,y \geq 0;\ x \text{ integer}\};\ i=1,2;$$

where A^i, B^i, C^i are rational matrices and c^i and d^i are rational vectors.

All the above results a) to d) are proven by first writing these constraints:

(6.27) $$A^1x + B^1y - C^1\beta^1 + C^1\gamma^1 = 0;$$

$$A^2u + B^2v - C^2\beta^2 + C^2\gamma^2 = 0;$$

$$x,u,y,v,\beta^1,\beta^2,\gamma^1,\gamma^2 \geq 0;\ x,u \text{ integer}$$

and then appending some further constraints and an objective function z.

For (a), we append the constraints $z - \alpha c^1x - \alpha d^1y - \beta c^2u - \beta d^2v = 0$; $\beta^1 - \gamma^1 = b$, and $\beta^2 - \gamma^2 = b$.

For (b), we append the two constraints: $z - c^1x - d^1y \geq 0$, $z - c^2x - d^2y \geq 0$, $\beta^1 - \gamma^1 = b$, and $\beta^2 - \gamma^2 = b$.

For c), we append the constraints: $\beta^2 = \gamma^2 = 0$ and $z \geq c^1x + d^1y$; and we also require that z be an integer variable.

For d), we append the constraints $z - c^1x - d^1y - c^2u - d^2v = 0$ and $\beta^1 - \gamma^1 + \beta^2 - \gamma^2 = b$.

Since all right-hand-sides are, in all cases a) to d), either zero or a component of b, we do have a pre-multiplied constraint set.

We leave it to the reader to check that the constraint sets in a) to d) define the value functions desired.

Q.E.D.

<u>Lemma 6.8</u>: Suppose that the function

(6.28) $$g(v) = \min_{i=1,\ldots,t} f_i(v)$$

is subadditive. Then

(6.29) $$g = \text{infcon}\{f_1, \ldots, f_t\}.$$

<u>Proof</u>: Let $i = 1, \ldots, t$. Using (6.21) with $v^i = v$ and $v^j = 0$ for $j \neq i$, we find that $f(v) \leq f_i(v)$. Since $i = 1, \ldots, t$ is arbitrary, we have $f(v) \leq g(v)$ for all v, where $f = \text{infcon}\{f_1, \ldots, f_t\}$.

Now suppose that g is subadditive. Let $v^1, \ldots, v^t$ be chosen arbitrarily so that $v^1 + \ldots + v^t = v$. We have

(6.30) $$g(v) \leq g(v^1) + \ldots + g(v^t) \leq f_1(v^1) + \ldots + f_t(v^t).$$

Taking the infimum on the right in (6.28), we obtain $g(v) \leq f(v)$ where $f = \text{infcon}\{f_1, \ldots, f_t\}$, using (6.21). Hence $g(v) = f(v)$ for all v.

Q.E.D.

<u>Proof of Theorem 6.6</u>:

Let $z(v)$ be a value function of a pre-multipled constraint set. If $z(0) = -\infty$, then $z(v) = -\infty$ whenever $z(v)$ is defined. Thus, on its domain of definition, $z(v)$ equals the mixed Gomory function $\text{infcon}\{F_1(v^1) + F_2(v^2) | v^1 + v^2 = v\}$ where $F_1(v^1) = \sum_j v_j^1$ and $F_2(v^2) = -2 \sum_j v_j^2$ (as the latter infimal convolution is identically $-\infty$)

If $z(0) > -\infty$, Theorem 6.5 and Lemma 6.8 provide the desired result, since by (6.25e) the infimal convolution of Gomory functions is a mixed Gomory function, and value functions are subadditive.

For the converse, suppose that F is a mixed Gomory function. If $F(v) = \lambda v$, F is the value function of the linear program

$$\text{(6.31)} \qquad \begin{aligned} &\inf \lambda(y^1 - y^2) \\ \text{subject to } \; & y^1 - y^2 = v \\ & y^1, y^2 \geq 0 \end{aligned}$$

For the cases (6.25b) through (6.25e) in the definition of a mixed Gomory function, Lemma 6.7 applies. Moreover, if $F(0) > -\infty$, F is the value function of a mixed integer constraint set with $z(0) > -\infty$. Hence $z(v) > -\infty$ for all v where $z(v)$ is defined. However, F is everywhere defined, since $\infty > F(v) = z(v) > -\infty$. The first strict inequality obtains since infimal convolution cannot give a $+\infty$ value when applied to functions with values in $R \cup \{-\infty\}$.

Q.E.D.

Theorem 6.9: The class of mixed Gomory functions F with $F(0) > -\infty$ is identical with the class of minima of a finite set of Gomory functions, such that the minimum is subadditive.

Proof: If F is a mixed Gomory function with $F(0) > -\infty$, by Theorem 6.6 it is the value function of an everywhere consistent pre-multiplied constraint set. Then Theorem 6.5 shows that F is a minimum of finitely many Gomory functions. Moreover, F is subadditive, since $F(v) = G(Cv)$ for a rational maxtrix C, where G is the value function of a mixed integer program, and G is subadditive by proposition 2.7.

For the converse, let a subadditive minimum of Gomory functions be given. By Lemma 6.8, this minimum is also the infimal convolution of the same Gomory functions. By (6.25e), it is also mixed Gomory function.

Q.E.D.

Although mixed Gomory functions is a class of functions which properly contains the Gomory functions, this class does not extend the capability to test for consistency, as our next result shows (compare with Theorem 6.1).

Theorem 6.10: The Gomory functions provide exactly a class of consistency tests for constraint sets ($PMIP_v$).

Proof: By Theorem 6.1, every constraint set ($PMIP_v$) has a consistency tester which is a Gomory, and hence a mixed Gomory, function.

For the converse, let F be a mixed Gomory function. By Theorem 6.5, these are rational matrices A, B, C and vectors c and d such that

(6.32) $$F(v) = \inf\{cx + dy \mid Ax + By = Cv;\ x,y \geq 0;\ x \text{ integer}\}$$

Since the infimal value of a mixed-integer program in rationals is attained [15], we have:

(6.33) $$F(v) \leq 0 \leftrightarrow \text{there are } x,y \geq 0 \text{ with } x \text{ integer and } Ax + By = Cv,\quad cx + dy \leq 0$$

The constraint set on the right in (6.33) is a pre-multiplied constraint set of the form ($PMIP_v$).

Q.E.D.

We are now in a position to sharpen Theorem 3.2 for the case $d = 0$.

Theorem 6.11: The class of value functions $z(v)$ of pre-multiplied constraint sets with $z(0) > -\infty$ and with c integer and $d = 0$ in the criterion function, are exactly the class of Gomory functions F, such that $F(v)$ is integer for all v, when the latter functions are restricted to the domain of definition of $z(v)$. Moreover, for any Gomory function F, such that $F(v)$ is integer for all v, there is a premultiplied constraint set, having criterion $cx + dy$ with c integer and $d = 0$, which is consistent for all r.h.s. v, and for which F is the value function.

Proof: For a pre-multiplied constraint set of the type described, let its r.h.s. configuration Cv be replaced by a vector b. By Theorem 3.2 for the case $d = 0$, there is a Gomory function $G(b)$ which provides the optimal value of the resulting mixed integer program, and note that $z(v) = G(Cv)$ on its domain of definition. Moreover, as c is an integer vector, $G(b)$ is integral for all consistent r.h.s. v. The desired Gomory function is therefore $F(v) = G(Cv)$.

For the converse, suppose that F is a Gomory function such that $F(v)$ is integer for all v. Then by Theorem 6.6, we have (6.32). By the integrality property of F:

$$F(v) = \inf\{z_1 - z_2 | Ax + By = Cv;\ z_1 - z_2 + cx - dy = 0;\ x,y,z_1,z_2 \geq 0;\ x,z_1,z_2 \text{ integer}\} \tag{6.34}$$

The criterion function (6.34) has integral coefficients for the integer variables, and zero coefficients for the remaining variables.

Q.E.D.

Corollary 6.12: For any $\varepsilon > 0$ and any mixed Gomory function G with $G(0) > \infty$, there is a Gomory function F which approximates G uniformly to within ε, from above:

$$0 \leq F(v) - G(v) < \varepsilon \quad \text{for all } v \tag{6.35}$$

Proof: Let $D \geq 1$ be an integer such that $\varepsilon > \frac{1}{D}$. We have that $H(v) = \lceil DG(v) \rceil$ is integer valued for all v, and it is a mixed Gomory function with $H(0) > -\infty$. By Theorem 6.6, there is a pre-multiplied constraint set, defined for all r.h.s. v, for which H is the value function. By Theorem 6.11, H is actually a Gomory function. Hence $F(v) = H(v)/D$ is a Gomory function, and we have (6.35).

Q.E.D.

We next provide a result which may be interpreted as a converse to Theorem 6.6.

Theorem 6.13: Let F and G be mixed Gomory functions. Then there is a pre-multiplied constraint set ($PMIP_v$) for which G is a consistency tester, and an objective function $cx + dy$ such that $F(v)$ is the optimal value of the pre-multipled program whenever $G(v) \leq 0$.

Proof: By Theorems 6.6, and 6.10, there are rational matrices A_i, B_i, C_i (i=1,2) and rational c and d such that

$$F(v) = \inf\{cx + dy | A^1x + B^1y = C^1v;\ x,y \geq 0;\ x \text{ integer}\} \tag{6.36}$$

$$G(v) \leq 0 \leftrightarrow \text{there are } x,y \geq 0 \text{ with } x \text{ integer and } A^2x + B^2y = C^2v \tag{6.37}$$

The desired pre-multipled optimization problem is then

(6.38) $$\inf cx + dy$$
$$\text{subject to } A^1x + B^1y = C^1v$$
$$A^2u + B^2w = C^2v$$
$$x,y,u,v \geq 0$$
$$x,u \text{ integer}$$

Q.E.D.

We conclude with a result about pre-multiplied constraint sets in the form (6.1), which provides an alternate (but not directly computational) perspective on when a pre-multiplied constraint set is actually that of a mixed-integer program.

Theorem 6.14: A set S is exactly the set of feasible r.h.s. b of a mixed integer program (MIP_b), if and only if there exists a finitely generated integer monoid M and a matrix of rationals Λ which, both satisfy (6.1), and are such that the equality system

(6.39) $$\Lambda v = m$$

has a solution for every $m \in M$.

Proof: Suppose that A,B are rational matrices with

(6.40) $$v \in S \leftrightarrow v = Ax + By \text{ for some } x,y \geq 0 \text{ with } x \text{ integer}$$

Let $\lambda^1, \ldots, \lambda^t$ be a finite set of generators for the cone $\{\lambda \geq 0 | \lambda B \leq 0\}$. Without loss of generality the λ^i are integer vectors such that $\lambda^i A$ is integer for $i = 1, \ldots, t$. As in Theorem 3.2, (6.41) $v \quad S \leftrightarrow$ for some $x \geq 0$ integer,

$$\lambda^i Ax \geq \lambda^i v \text{ for } i = 1, \ldots, t$$

$\leftrightarrow$ for some $m \in M$,

$$m \geq \Lambda v$$

where M is the integral monoid generated by the $\lambda^i A$ and where $\Lambda = [\lambda^i]$ (rows).

For the converse, suppose that (6.1) holds and that (6.39) is solvable for all $m \in M$. Observe that there is a rational matrix Γ with this ("pseudo-inverse") property, which can easily be constructed from Smith Normal Form:

(6.42) $\Lambda v = d$ is solvable $\rightarrow \Lambda(\Gamma d) = d$.

Let $m^1, \ldots, m^u$ be a finite basis for M and let $v^1, \ldots, v^z$ be a finite rational basis for the cone $\{v | \Lambda v \leq 0\}$. Then

(6.43) $v \in S \leftrightarrow$ for some $m \in M$, $\Lambda v \leq m$

$\leftrightarrow$ for some $m \in M$, $\Lambda v \leq \Lambda\Gamma m$

$\leftrightarrow$ for some $m \in M$, $\Lambda(v - \Gamma m) \leq 0$

$\leftrightarrow$ for some $y_1, \ldots, y_j \geq 0$, and some $m \in M$,

$$v - \Gamma m = \sum_{j=1}^{s} y_j v_j$$

$\leftrightarrow$ for some integer $x_1, \ldots, x_u \geq 0$ and scalars $y_1, \ldots, y_j \geq 0$,

$$v = \sum_{k=1}^{u} (\Gamma m^k) x_k + \sum_{j=1}^{s} v^j y_j.$$

We have thus constructed a mixed integer program for which S is the set of feasible r.h.s.

Q.E.D.

References

[1] J. F. Benders, "Partitioning Procedures for Solving Mixed-Variables Programming Problems," Numerische Mathematik 4 (1962), pp. 238-252.

[2] C. E. Blair and R. G. Jeroslow, "The Value Function of an Integer Program," to appear in Mathematical Programming (available as paper MS-80-12, College of Management, Georga Institute of Technology).

[3] C. E. Blair and R. G. Jeroslow, "The Computational Complexity of Sensitivity Analysis in Discrete Programming," in preparation.

[4] C. E. Blair and R. G. Jeroslow, "The Value Function of a Mixed-Integer Program: I" and ":II," Discrete Mathematics 19 (1977), pp. 121-138, and Discrete Mathematics 25 (1979), pp. 7-19.

[5] V. Chvátal, "Edmonds Polytopes and a Hierarchy of Combinatorial Problems," Discrete Mathematics 4 (1973), pp. 305-337.

[6] J. Edmonds and R. Giles, "A min-max Relation for Submodular Functions on Graphs," Annals of Discrete Mathematics 1 (1977), pp. 185-204.

[7] R. E. Gomory, "An Algorithm for Integer Solutions to Linear Programs," in Graves and Wolfe, eds., Recent Advances in Mathematical Programming, 1963, pp. 269-302.

[8] R. E. Gomory, "Some Polyhedra Related to Combinational Problems," Journal of Linear Algebra and its Applications 2 (1969), pp. 451-558.

[9] R. E. Gomory and E. L. Johnson, "Some Continuous Functions Related to Corner Polyhedra: I" and ":II," Mathematical Programming 3 (1972), pp. 23-85 and pp. 359-389.

[10] J. E. Graver, "On the Foundations of Linear and Integer Linear Programming I," Mathematical Programming 9 (1975), pp. 207-226.

[11] D. Hilbert, "Uber die Theorie der Algebraischen Formen," Mathematische Annalen 36 (1890), pp. 475-534.

[12] R. Jeroslow, "Cutting-plane Theory: Algebraic Methods," Discrete Mathematics 23 (1978), pp. 121-150.

[13] R. Jeroslow, "Some Basis Theorems for Integral Monoids," Mathematics of Operations Research 3 (1978), pp. 145-154.

[14] R. Jeroslow, "Minimal Inequalities" Mathematical Programming 17 (1979), pp. 1-15.

[15] R. R. Meyer, "On the Existence of Optimal Solutions to Integer and Mixed-Integer Programming Problems," Mathematical Programming 7 (1974), pp. 223-235.

[16] R. T. Rockafeller, Convex Analysis, Princeton University Press, Princeton, NJ, 1970.

[17] A. Schrijver, "On Cutting Planes," Department of Mathematics, Eindhoven University, 1979.

[18] J. Stoer and C. Witzgall, Convexity and Optimization in Finite Dimensions I, Springer-Verlag, New York, 1970.

[19] L. Wolsey, "Integer Programming Duality: Price Functions and Sensitivity Analysis," London School of Economics and CORE, November 1978.

[20] L. Wolsey, "The b-hull of an Integer Program," London School of Economics and CORE, February 1979.

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