Isolated Sublattices and their Application to Counting Closure Operators

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Motivation for Counting

- [BonzioPrabaldiValota2018] count bisemilattices
- [AlpayJipsen2020] count doubly idempotent semirings
- [QuinteroRamírezRuedaValencia2020] count join-endomorphisms
- [BerghammerBörmWinter2021] count topological spaces
- [AlpayJipsenSugimoto2021] count dℓ-structures

Motivation for Closures

- transitive closure of relations or graphs
- Kleene closure in language theory
- connected components in [Glück2017]
- most work deals with powerset lattices (Moore families)
- number of closure operators on $(\mathcal{P}(S), \subseteq)$ known only up to $|S| = 7$ [ColombIrlandeRaynaud2010]
- shown to be 14.087.648.235.707.352.472 [ColombIrlandeRaynaud2010]

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Closures

Definition

Given an ordered set S an endofunction c on S is called a *closure operator* if it fulfills the following properties for all $x, y \in S$:

- $x \leq c(x)$ (extensitivity)
- $x \leq y \Rightarrow c(x) \leq c(y)$ (isotony)
- $c(c(x)) = c(x)$ (idempotence)

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Given a lattice $(S, \leq)$ a subset $S' \subseteq S$ is called a *closure system* if it fulfills the following properties:

- $x, y \in S' \Rightarrow x \sqcap y \in S'$
- for every $s \in S$ there is a smallest $x \in S'$ such that $s \leq x$ holds.

The set of all closure systems of S is denoted by $\mathcal{C}(S)$.

Remark: The second condition implies the first one.

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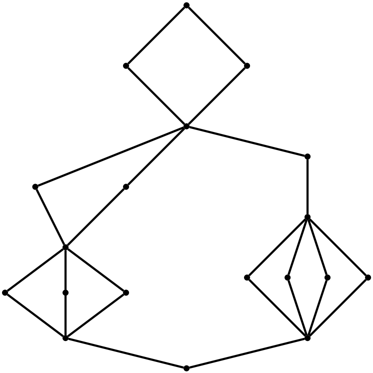
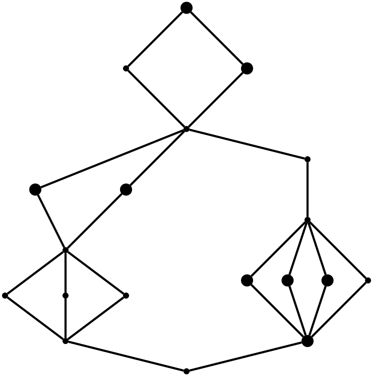
These two definitions are cryptomorphic.

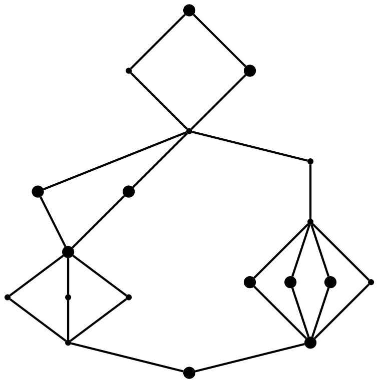
Example Lattice



No Closure Example

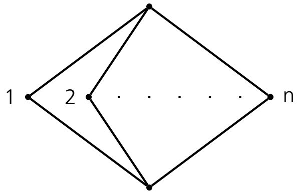


Closure Example

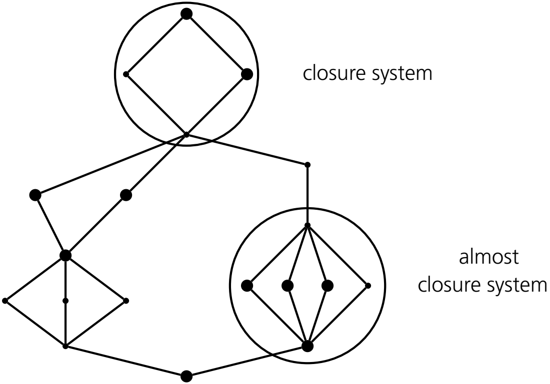


Counting Closures

- In general, counting closures is difficult:
- $|\mathcal{C}(\mathcal{P}(\{1,2,3,4,5,6,7\}),\subseteq)| = 14.087.648.235.707.352.472$ known since 2010.
- $\top$ (if exists) is element of every closure system
- Easy special cases:
- $|\mathcal{C}(\{1\ldots n\},\leq)| = 2^{n-1}$
- $|\mathcal{C}(\mathbf{diam}(n))| = 2 + 2n + (2^n - n - 1)$



Finding Substructures



Isolated Sublattices and Congruences

Definition

Let $(S, \leq)$ be a lattice. A subset $S' \subseteq S$ is called an *isolated sublattice* if it fulfills the following properties:

- S' is a sublattice with greatest element $\top_{S'}$ and least element $\bot_{S'}$.
- $\forall x \notin S' \forall y' \in S' : y' \leq x \Rightarrow \top_{S'} \leq x$
- $\forall x \notin S' \forall y' \in S' : x \leq y' \Rightarrow x \leq \bot_{S'}$

Remark: S and $\{s\}$ for $s \in S$ are isolated sublattices.

Lemma

Let S' be an isolated sublattice and define $\equiv_{S'}$ by $x \equiv_{S'} y \Leftrightarrow_{def} x = y \vee (x \in S' \wedge y \in S')$. Then $\equiv_{S'}$ is a congruence relation on $(S, \leq)$.

Reminder: An equivalence relation $\equiv$ is a congruence if the following holds:

- $x_0 \equiv y_0 \wedge x_1 \equiv y_1 \Rightarrow x_0 \sqcap x_1 \equiv y_0 \sqcap y_1$ and
- $x_0 \equiv y_0 \wedge x_1 \equiv y_1 \Rightarrow x_0 \sqcup x_1 \equiv y_0 \sqcup y_1$

More about Congruences

- A congruence $\equiv$ on a lattice $(S, \leq)$ induces a quotient lattice $(S/\equiv, \leq /\equiv)$

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More about Congruences

- A congruence $\equiv$ on a lattice $(S, \leq)$ induces a quotient lattice $(S/\equiv, \leq/\equiv)$
- $S/\equiv$ is a homomorphic image of S
- Are there relations between closure systems of S and closure systems of $S/\equiv_{S'}$?
- Yes, there are!

Various Kinds of Isolated Sublattices

Let $(S, \leq)$ be a lattice with greatest element $\top$.

Definition

An isolated sublattice S' of S is called a *summit isolated sublattice* if $\top_{S'} = \top$ holds.

Definition

An isolated sublattice S' of S is called an *isolated sublattice with bottleneck* if $\top'_S$ is meet-irreducible.

Definition

A subset $\hat{S} \subseteq S$ is called a *preclosure system* if $\hat{S} \cup \{\top\}$ is a closure system. The set of all preclosure systems is denoted by $\mathcal{PC}(S)$.

Remark: Note that $|\mathcal{PC}(S)| = 2 \cdot |\mathcal{C}(S)|$ holds.

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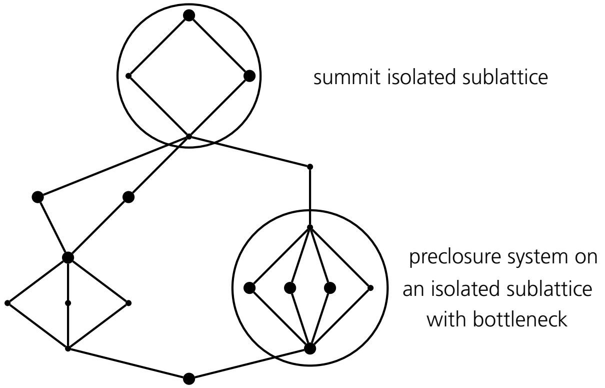
Examples


summit isolated sublattice
preclosure system on
an isolated sublattice
with bottleneck

Closure Systems and Isolated Sublattices

Notation: $C^{\{\}} =_{def} \{\{c\} \mid c \in C\}$

Lemma

Let $(S, \leq)$ be a lattice, S' an isolated sublattice of $(S, \leq)$ and consider a closure system C of $(S, \leq)$.

- *If $C \cap S' = \emptyset$ then $C^{\{\}}$ is a closure system of $S/{\equiv_{S'}}$.*
- *If $C \cap S' \neq \emptyset$ then $(C \backslash S')^{\{\}} \cup \{S'\}$ is a closure system of $S/{\equiv_{S'}}$.*

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Lemma

Let $(S, \leq)$ be a lattice and S' an isolated sublattice with bottleneck. Assume that $C_{S'}$ is a preclosure system of S' and let C' be a closure system of $S/{\equiv_{S'}}$ with $S' \in C'$. Then $C =_{def} \bigcup(C' \backslash \{S'\}) \cup C_{S'}$ is a closure system of $(S, \leq)$.

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Lemma

Let $(S, \leq)$ be a lattice and S' a summit isolated sublattice of S. Assume that $C_{S'}$ is a closure system of S' and let C' be a closure system of $S/{\equiv_{S'}}$. Then $C =_{def} \bigcup C' \backslash S' \cup C_{S'}$ is a closure system of $(S, \leq)$.

Remark: Note that $\top_{S/\equiv_{S'}} \in C'$ and $\top_S \in C_{S'}$ hold.

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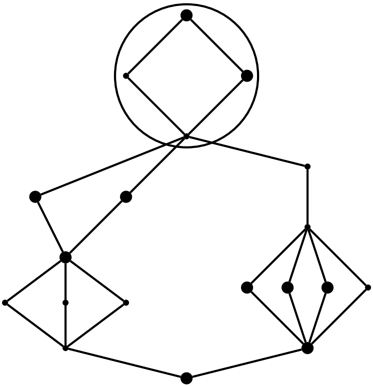
Illustration



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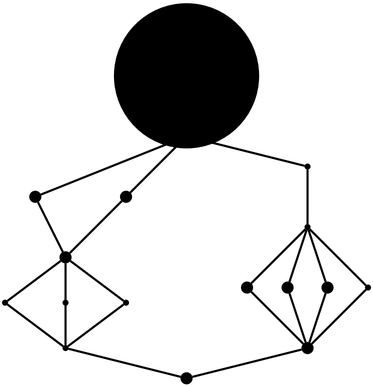

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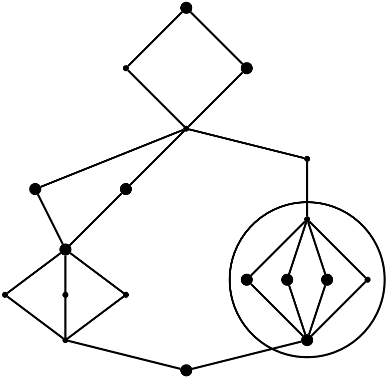

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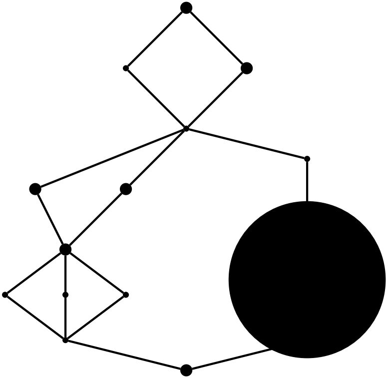
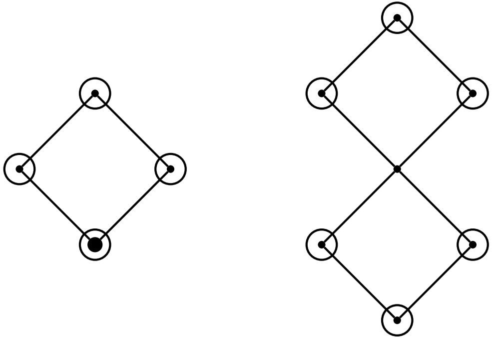


Necessity of Bottlenecks



Closure Systems and Isolated Sublattices

Lemma

Let S be a lattice and S' an isolated sublattice with bottleneck of S and assume that C' is a closure system on $S/{\equiv_{S'}}$ with $S' \notin C'$. Then $\bigcup C'$ is a closure system on S.

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Theorem

Let S' be an isolated sublattice with bottleneck of $(S, \leq)$, and consider a set $C \subseteq S$.

- *Assume that $C' =_{def} C \cap S' \neq \emptyset$ holds. Then C is a closure system of S iff C' is a nonempty preclosure system of S' and $(C \backslash S')^{\{\}} \cup \{S'\}$ is a closure system of $S/\equiv_{S'}$.*
- *Assume that $C \cap S' = \emptyset$ holds. Then C is a closure system of S iff $C^{\{\}}$ is a closure system of $S/\equiv_{S'}$.*

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- *Assume that $C \cap S' = \emptyset$ holds. Then C is a closure system of S iff $C^{\{\}}$ is a closure system of $S/\equiv_{S'}$.*

Theorem

Let S' be a summit isolated sublattice of a lattice $(S, \leq)$, and consider a set $C \subseteq S$. Then C is a closure system of S iff $C \cap S'$ is a closure system of S' and $(C \backslash S')^{\{\}} \cup \{S'\}$ is a closure system of $S/\equiv_{S'}$.

Towards a Recursive Counting Algorithm

Let $S', T \subseteq S$ be subsets of a lattice S and $x \in S$. Then we define:

- $\mathcal{C}(S)_T =_{def} \{C \in \mathcal{C}(S) \mid T \subseteq C\}$
- $\mathcal{C}(S)_{-x,T} =_{def} \{C \in \mathcal{C}(S)_T \mid x \notin C\}$
- $\mathcal{C}(S)_T^{S'} =_{def} \{C \in \mathcal{C}(S)_T \mid C \cap S' \neq \emptyset\}$
- $\mathcal{C}(S)_T^{-S'} =_{def} \{C \in \mathcal{C}(S)_T \mid C \cap S' = \emptyset\}$

This implies:

- $\mathcal{C}(S) = \mathcal{C}(S)_\emptyset$
- $\mathcal{C}(S)_T = \mathcal{C}(S)_{T \setminus \{\top_S\}} = \mathcal{C}(S)_{T \cup \{\top_S\}}$
- $\mathcal{C}(S)_T = \mathcal{C}(S)_{T \cup \{x\}} \dot{\cup} \mathcal{C}(S)_{-x,T}$, hence
- $|\mathcal{C}(S)_T| = |\mathcal{C}(S)_{T \cup \{x\}}| + |\mathcal{C}(S)_{-x,T}|$
- analogously $|\mathcal{C}(S)_T| = |\mathcal{C}(S)_T^{S'}| + |\mathcal{C}(S)_T^{-S'}|$

Formulae for the cardinalities in the case of chains or diamonds can easily be obtained.

Recursion for Isolated Sublattices with Bottleneck

Let S' be an isolated sublattice with bottleneck and consider an arbitrary set $T \subseteq S$ with $T \cap S' = \emptyset$. Then:

(1) $|\mathcal{C}(S)^{S'}_{T}| = |\mathcal{C}(S/\equiv_{S'})_{T\{\}\cup\{\{S'\}\}}| \cdot (|\mathcal{PC}(S')| - 1)$
(discard the empty preclosure)
(2) $|\mathcal{C}(S)^{-S'}_{T}| = |\mathcal{C}(S/\equiv_{S'})_{-\{S'\},T\{\}}|$
(3) $|\mathcal{C}(S/\equiv_{S'})_{T\{\}}| = |\mathcal{C}(S/\equiv_{S'})_{T\{\}\cup\{S'\}}| + |\mathcal{C}(S/\equiv_{S'})_{-\{S'\},T\{\}}|$
(4) $|\mathcal{PC}(S')| = 2 \cdot |\mathcal{C}(S')|$
(5) $|\mathcal{C}(S)_{T}| = |\mathcal{C}(S)^{S'}_{T}| + |\mathcal{C}(S)^{-S'}_{T}|$, hence:
(6) $|\mathcal{C}(S)_{T}| = |\mathcal{C}(S/\equiv_{S'})_{T\{\}\cup\{\{S'\}\}}| \cdot 2(|\mathcal{C}(S')| - 1) + |\mathcal{C}(S/\equiv_{S'})_{T\{\}}|$
(insert (1), (2), (3) and (4) into (5) and simplify)

Recursion for Summit Isolated Sublattices and Maximality

Analogously for a summit isolated sublattice with $T \cap S' = \emptyset$:

- $|\mathcal{C}(S)_T| = |\mathcal{C}(S/{\equiv_{S'}})_{T\{\}}| \cdot |\mathcal{C}(S')|$

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- This can be achieved by using inclusion-maximal nontrivial isolated sublattices.

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- Inclusion-maximal isolated sublattices are disjoint and can not contain quotients from earlier isolated sublattices (if only inclusion-maximal isolated sublattices are used).

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- Inclusion-maximal isolated sublattices are disjoint and can not contain quotients from earlier isolated sublattices (if only inclusion-maximal isolated sublattices are used).
- If possible, use a summit isolated sublattice.

Recursive Algorithm

function #CLOSURES(lattice S, set T)
  if S is a special case (chain, diamond) **then**
    return the respective number
  end if
  if S has a nontrivial useful summit isolated sublattice **then**
    $S' \leftarrow$ the inclusion maximal summit sublattice
    return #CLOSURES($S/\equiv_{S'}, T^{\{\}}$)·#CLOSURES($S', \emptyset$)
  end if
  if S has a useful isolated sublattice with bottleneck **then**
    $S' \leftarrow$ an inclusion maximal useful isolated sublattice with bottleneck
    return
#CLOSURES($S/\equiv_{S'}, T^{\{\}} \cup \{S'^{\{\}}\}$)·2(#CLOSURES($S', \emptyset$)-1)+#CLOSURES($S/\equiv_{S'}, T^{\{\}}$)
  end if
  compute and return $|\mathcal{C}(S)_T|$ by some brute force algorithm
end function

Running Time Considerations

- In general, there are exponenially many closure systems
- Brute force algorithms may take also exponential time
- Speed-up expectable if computation of isolated sublattices can be done in polynomial time
- See short talk on Friday

Future Work

Open issues:

- Identify other special lattices S with simple formulae for the cardinality of $\mathcal{C}(S)$
- More general structures than isolated sublattices with similar suitable properties?
- Implementation and evaluation
- Similar ideas for general orders?
- Applicable to counting monads on categories?