Embedding and Weight Distribution for Quantum Annealing

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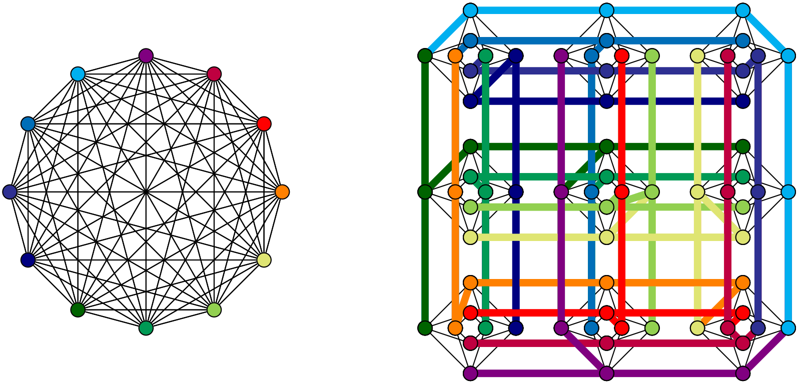
Outline

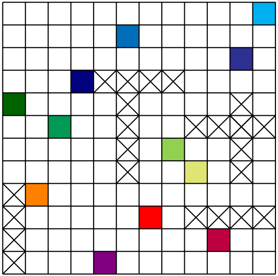
1. Embedding

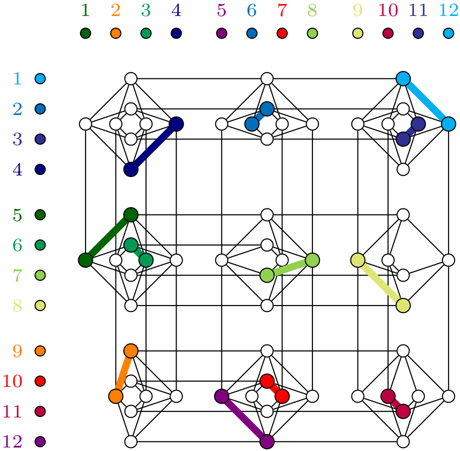
2. Weight Distribution

Complete Graph Embedding



Complete Graph Embedding in Broken Chimera





Embedding ILP

- Symmetric Chimera graph of size $n \in \mathbb{N}$, $N = \{1, ..., n\}$
- Binary variables $x \in \{0, 1\}^{N \times N}$ with

$$x_{rc} = \begin{cases} 1, & \text{if crossroad from row } r \text{ to column } c \text{ is used} \\ 0, & \text{otherwise} \end{cases}$$

  $x_{rc} = 0$ if one of the qubits is broken

- At maximum one row is matched to each column and vice versa

$$\sum_{r \in N} x_{rc} \leq 1 \quad \forall c \in N$$

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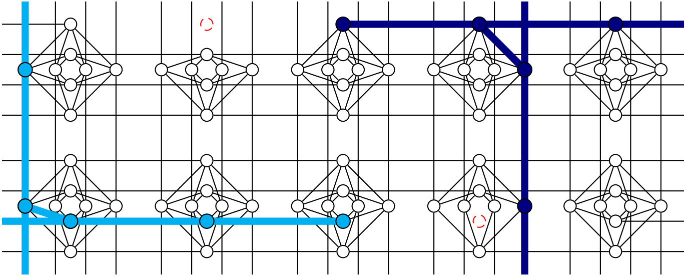
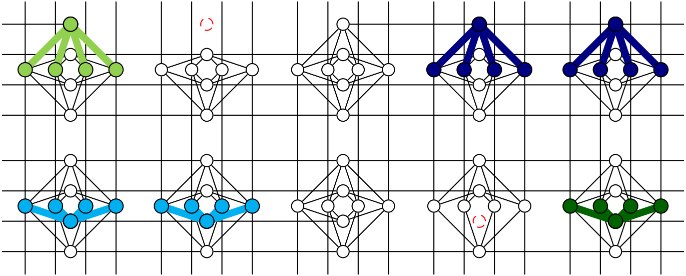
- Maximizing matched rows and columns $\sum_{rc \in N^2} x_{rc}$

Embedding ILP – Forbidden Combinations



additional constraints $x_{rc} + x_{\tilde{r}\tilde{c}} \leq 1$

Embedding ILP – Forbidden Combinations



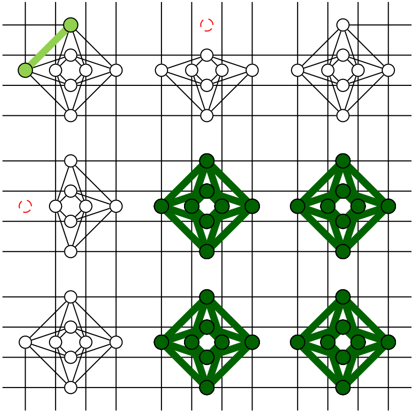
additional constraints

$$\sum_{rc \in I_{\text{left}}^{\text{below}}} x_{rc} + \sum_{rc \in I_{\text{right}}^{\text{above}}} x_{rc} \leq 1$$

simplified because of matching constraints

$$\sum_{rc \in I_{\text{left}}^{\text{above}}} x_{rc} + \sum_{rc \in I_{\text{right}}^{\text{below}}} x_{rc} \leq 1$$

Embedding ILP – Forbidden Combinations



$$x_{\tilde{r}\tilde{c}} + \sum_{r \in I_{\text{rows}}} x_{rc} \leq 1 \quad \forall c \in I_{\text{cols}}$$

or

$$x_{\tilde{r}\tilde{c}} + \sum_{c \in I_{\text{cols}}} x_{rc} \leq 1 \quad \forall r \in I_{\text{rows}}$$

depending on interval sizes

Analysis

- Restricted bipartite matching problems are NP-hard
  - even for cardinality 1 constraints
  - reusable once computed
- Specific structure of constraints allows splitting into subproblems
  - for each constraint 2 options to choose
  - resulting constraint is weaker than matching constraint
  - simple bipartite matching problems with no additional constraints $\mathcal{O}(n^3)$
- More broken qubits
  - more pairs introducing constraints
  - fixed-parameter tractable in number of broken qubits $\mathcal{O}(n^3\,2^{|B|^2})$

Results – Exact Optimization

ratio \ size	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34
0.005	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.91	0.89	0.82	0.85
0.02	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.98	0.97	0.75	0.74	0.71	0.61		
0.03	1.00	1.00	1.00	1.00	1.00	1.00	0.95	0.68	0.58						
0.04	1.00	1.00	1.00	1.00	1.00	0.89	0.60								
0.05	1.00	1.00	1.00	1.00	0.93	0.66									
0.1	0.93	0.86	0.82	0.67	0.51										
0.2	0.61	0.53	0.49												

solved in 1h: 10/10 | 1/10-9/10 | 0/10

Average ratio of maximum possible complete graph size ($4 \cdot$ size)

- Implemented in SCIP
- Still fast for current hardware graphs
- D-Wave 2000Q: size 16, ratio ≈ 0.008 → 64 nodes embeddable

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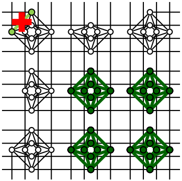
Results – Heuristic Approach

ratio \ size	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34
0.005	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.02	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.98	0.92
0.03	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.98	0.93	0.89	0.84	0.80	0.80	0.75	
0.04	1.00	1.00	1.00	1.00	1.00	0.96	0.91	0.83	0.79	0.75	0.68	0.62	0.58	0.52	
0.05	1.00	1.00	1.00	0.98	0.90	0.83	0.79	0.70	0.63	0.57	0.50	0.46	0.40		
0.1	0.90	0.77	0.68	0.51	0.44	0.38	0.29	0.27	0.19						
0.2	0.44	0.29	0.24	0.12	0.10	0.06	0.02								

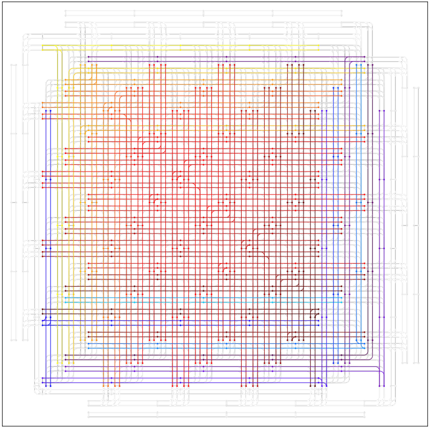
Average ratio of maximum possible complete graph size ($4 \cdot$ size)

- Removed unlikely crossroads before optimization
- Reduces amount of constraints significantly
- Much faster, still close to maximum possible



Next Steps

- Testing how solution quality and time changes when
  - not all crossroads of different broken qubits are removed
  - different heuristics are applied
- Investigate whether the construction can be transferred to Pegasus



Embedding of K_{62} in P_5

(*Next-Generation Topology of D-Wave Quantum Processors*, D-Wave, technical report, figure 5)

Embedded Ising Function

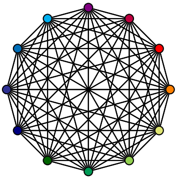
From $f(t) = \sum_{v \in V_G} H_v t_v + \sum_{\{v,w\} \in E_G} J_{vw} t_v t_w$ with $t \in \{-1, 1\}^{V_G}$

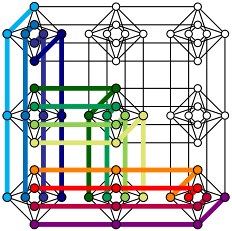
get $\bar{f}(s) = \sum_{v \in V_G} \left(\sum_{q \in M(v)} \overline{H}_q s_q + \sum_{\{q,p\} \in E(M(v))} \bar{J}_{qp} s_q s_p \right) + \sum_{\substack{\{v,w\} \in E_G \\ \{q,p\} = M(\{v,w\})}} J_{vw} s_q s_p$

with $s \in \{-1, 1\}^{V_H}$

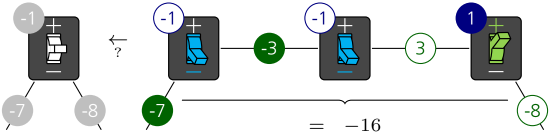
→ New coefficients $\bar{J}$, $\overline{H}$

- Problem graph $G = (V_G, E_G)$
- Hardware graph $H = (V_H, E_H)$
- Embedding $M : V_G \to 2^{V_H}$





Weight Distribution



- Obvious requirements:

$$H_v = \sum_{q \in M(v)} \overline{H}_q$$

$$\bar{J} < \mathbb{O}$$

- Equivalence of solutions for synchronized variables:

$$f(t) = \bar{f}(s) + \mathit{offset} \quad \Leftrightarrow \quad t_v = s_q \ \forall q \in M(v) \ \forall v \in V_G$$

→ How to choose coefficients to guarantee optimality?

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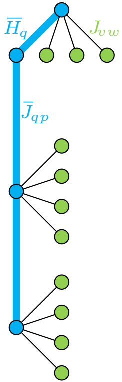
Weight Distribution



$$\bar{f}_v(s,t) = \sum_{q \in M(v)} \overline{H}_q s_q + \sum_{\{q,p\} \in E(M(v))} \overline{J}_{qp} s_q s_p + \sum_{\substack{q \in M(v) \\ w \in d_q(v)}} J_{vw} s_q t_w$$

- For synchronization

$$\underset{s \in \{-1,1\}^{M(v)}}{\operatorname{argmin}} \bar{f}_v(s,t) \subseteq \{-\mathbb{1}, \mathbb{1}\} \quad \forall t \in \{-1,1\}^{d(v)}$$

- Then $\forall s \in \{-1,1\}^{M(v)} \setminus \{-\mathbb{1}, \mathbb{1}\}$ $\forall t \in \{-1,1\}^{d(v)}$:

$$\bar{f}_v(s,t) > \min \left\{ \bar{f}_v(-\mathbb{1}, t), \bar{f}_v(\mathbb{1}, t) \right\}$$

→ Exponential number of brute force constraints on $\overline{J}$, $\overline{H}$

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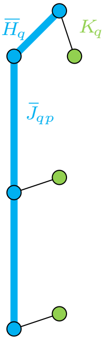
Simplifications



$$\bar{\bar{f}}_v(s, t) = \sum_{q \in M(v)} \overline{H}_q s_q - \bar{J} \sum_{\{q,p\} \in E(M(v))} s_q s_p + \sum_{q \in M(v)} K_q s_q t_q$$

- Restrict on $J > \mathbb{O}$ (symmetry in $t \in \{-1, 1\}^{d(v)}$)
- Combine the incoming strengths $K_q = \sum_{w \in d_q(v)} J_{vw}$
- Set $\bar{J}_{qp} = -\bar{J}$ for all $\{q, p\} \in E(M(v))$
- Start with weight $H_v = 0$, hence $\overline{H}_q = 0$

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Reduction

$$\bar{J} > \max_{\substack{\emptyset \neq S \subset M(v) \\ S \text{ is fundamental cut}}} \frac{\min\left\{\sum_{q \in S} K_q, \sum_{q \in M(v) \setminus S} K_q\right\}}{|\delta(S)|}$$

- Lower bounds on $\bar{J}$ to be minimized
- Related to graph property called expansion for zero weight

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  - → easy to compute for trees, like in embedding setting

Reduction

$$\bar{J} > \max_{\substack{\emptyset \neq S \subset M(v) \\ S \text{ is fundamental cut}}} \frac{\min\left\{\sum_{q \in S}(K_q - \overline{H}_q), \sum_{q \in M(v)\setminus S}(K_q + \overline{H}_q)\right\}}{|\delta(S)|}$$

- Lower bounds on $\bar{J}$ to be minimized
- Related to graph property called expansion for zero weight
  - easy to compute for trees, like in embedding setting
- Reintroducing weight
  - advantage of specific unequal distribution

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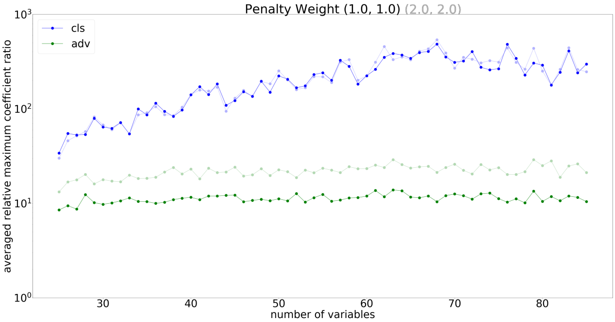
- Lower bounds on $\bar{J}$ to be minimized
- Related to graph property called expansion for zero weight
  - ➔ easy to compute for trees, like in embedding setting
- Reintroducing weight
  - advantage of specific unequal distribution
  - ➔ still easy to compute
- Minimizing all coefficients with $\bar{J} \geq \|\overline{H}\|_\infty$ and/or $\bar{J} \in \mathbb{N}$, $\overline{H} \in \mathbb{N}^{M(v)}$
  - work in progress

Preliminary Results



- Yields better scaling in coefficient ratio of embedded Ising
- Influence of penalty weight

→ Larger success probabilities expected

Questions?

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