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RESEARCH REPORT
N° 7776
October 2011
Project-Team Mathfi

ISSN 0249-6399 ISRN INRIA/RR--7776--FR+ENG

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Research Report n° 7776 — October 2011 — 35 pages

Abstract: We study optimal stochastic control problems with jumps under model uncertainty. We rewrite such problems as (zero-sum) stochastic differential games of forward-backward stochastic differential equations. We prove general stochastic maximum principles for such games, both in the zero-sum case (finding conditions for saddle points) and for the non-zero sum games (finding conditions for Nash equilibria). We then apply these results to study optimal portfolio and consumption problems under model uncertainty. We combine the optimality conditions given by the stochastic maximum principles with Malliavin calculus to obtain a set of equations which determine the optimal strategies.

Key-words: Forward-backward SDEs, stochastic differential games, maximum principle, model uncertainty, optimal portfolio, jump diffusions

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CENTRE DE RECHERCHE

PARIS - ROCQUENCOURT

Jeux différentiels stochastiques et contrôle stochastique avec ambiguité de modèles

Résumé : On étudie des problèmes de contrôle stochastique de diffusions avec sauts avec ambiguité de modèles, que l'on réécrit comme des jeux différentiels stochastiques à somme nulle d'équations différentielles stochastiques "forward-backward". On démontre des principes du maximum stochastiques généraux pour de tels jeux, à la fois dans le cas à somme nulle (conditions d'existence de points selles) et dans le cas général (conditions pour un équilibre de Nash). Ces résultats sont appliqués à l'étude de problèmes de portefeuilles et consommation optimale avec ambiguité de modèle. En combinant les conditions d'optimalité obtenus par les principes du maximum stochastiques avec des techniques de calcul de Malliavin, on obtient la caractérisation des stratégies optimales.

Mots-clés : Equations différentielles stochastiques forward et rétrogrades, jeux différentiels stochastiques, principes du maximum stochastique, incertitude de modèle, portefeuille optimal, diffusions avec sauts

1 Introduction

One of the aftereffects of the financial crisis is the increased awareness of the need for more advanced modeling in mathematical finance, and a focus of attention is on the problem of *model uncertainty.* This paper is motivated by a topic of this type. We consider a stochastic system described by a general Itô-Lévy process controlled by an agent. The performance functional is expressed as the Q-expectation of an integrated profit rate plus a terminal payoff, where Q is a probability measure absolutely continuous with respect to the original probability measure P. We may regard Q as a *scenario measure* controlled by the market or the environment. If $Q = P$ the problem becomes a classical stochastic control problem of the type studied in [15]. If Q is uncertain, however, the agent might seek the strategy which maximizes the performance in the worst possible choice of Q. This leads to a *stochastic differential game* between the agent and the market. Our approach is the following: We write the performance functional as the value at time $t = 0$ of the solution of an associated *backward* stochastic differential equation (BSDE). Thus we arrive at a (zero sum) stochastic differential game of a system of *forward-backward* SDEs (FBSDEs) that we study by the maximum principle approach.

There are several papers of related content. Stochastic control of forward-backward SDEs (FBSDEs) has been studied in [16] and in [2] a maximum principle for stochastic differential g-expectation games of SDEs is developed. The papers [11], [18] and [19] also study optimal portfolio under model uncertainty by means of BSDEs, but the approaches there are strongly linked to the exponential utility case. A key feature of the current paper is that it applies to general utility functions and also general dynamics for the state process.

Our paper is organised as follows: in Section 2, we state general stochastic maximum principles for stochastic differential games, both in the zero-sum case (finding conditions for saddle points) and for the non-zero sum games (finding conditions for Nash equilibria). The proofs are given in Appendix A. In Section 3 we consider stochastic control problems under uncertainty. We formulate these problems as (zero sum) stochastic differential games of *forward-backward* SDEs (FBSDEs) and we study them by the maximum principle approach of Section 2. In Section 4 we apply these techniques to study an optimal portfolio and consumption problem under model uncertainty. Using the solution for linear *Malliavin–differential type equations* given in [16] we arrive at a set of equations which determine the optimal portfolio and consumption of the agent and the corresponding optimal portfolio scenario measure of the market.

2 Maximum principles for stochastic differential games of forward-backward stochastic differential equations

In this section, we formulate and prove a sufficient and a necessary maximum principle for general stochastic differential games (not necessarily zero-sum games) of forward-backward SDEs. Let $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space. Consider a controlled *forward* SDE of the form

$$dX(t) = dX^{(u)}(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB(t) \\ + \int_{\mathbb{R}} \gamma(t, X(t), u(t), \zeta)\tilde{N}(dt, d\zeta) \; ; \; X(0) = x \in \mathbb{R}. \tag{2.1}$$

where B is a Brownian motion, and $\tilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta)dt$ is an independent compensated Poisson random measure where ν is the Lévy measure of N such that $\int_{\mathbb{R}} \zeta^2 \nu(d\zeta) < \infty$. We assume that $I\!\!F = \{\mathcal{F}_t, t \geq 0\}$ is the natural filtration associated with B and N. Here $u = (u_1, u_2)$, where $u_i(t)$ is the control of player i ; $i = 1, 2$. We assume that we are given two subfiltrations

$$\mathcal{E}_t^{(i)} \subseteq \mathcal{F}_t \; ; \; t \in [0, T], \tag{2.2}$$

representing the information available to player i at time t ; $i = 1, 2$. We let $\mathcal{A}_i$ denote a given set of admissible control processes for player i, contained in the set of $\mathcal{E}_t^{(i)}$-predictable processes ; $i = 1, 2$, with values in $A_i \subset \mathbf{R}^d$, $d \geq 1$. Denote $\mathbb{U} = A_1 \times A_2$.

We consider the associated *backward* SDE's (i.e. BSDEs) in the unknowns $Y_i(t), Z_i(t), K_i(t, \zeta)$ of the form

$$\begin{aligned} dY_i(t) &= -g_i(t, X(t), Y_i(t), Z_i(t), K_i(t, \cdot), u(t))dt \\ &\quad + Z_i(t)dB(t) + \int_{\mathbb{R}} K_i(t, \zeta)\tilde{N}(dt, d\zeta) \; ; \; 0 \leq t \leq T \\ Y_i(T) &= h_i(X(T)) \; ; \; i = 1, 2. \end{aligned} \tag{2.3}$$

Here $g_i(t, y, z, k, u) : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \to \mathbb{R}$ and $h_i : \mathbb{R} \to \mathbb{R}$ are given functions such that the BSDEs (2.3) have unique solutions.

Let $f_i(t, x, u) : [0, T] \times \mathbb{R} \times \mathbb{U} \to \mathbb{R}$, $\varphi_i(x) : \mathbb{R} \to \mathbb{R}$ and $\psi_i(x) : \mathbb{R} \to \mathbb{R}$ be given profit rates, bequest functions and "risk evaluations" respectively, of player i ; $i = 1, 2$. Define

$$J_i(u) = E\left[\int_0^T f_i(t, X^{(u)}(t), u(t))dt + \varphi_i(X^{(u)}(T)) + \psi_i(Y_i(0))\right] \; ; \; i = 1, 2, \tag{2.4}$$

provided the integrals and expectations exist. We call $J_i(u)$ the *performance functional* of player i ; $i = 1, 2$.

A *Nash equilibrium* for the FBSDE game (2.1)-(2.4) is a pair $(\hat{u}_1, \hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ such that

$$J_1(u_1, \hat{u}_2) \leq J_1(\hat{u}_1, \hat{u}_2) \text{ for all } u_1 \in \mathcal{A}_1 \tag{2.5}$$

and

$$J_2(\hat{u}_1, u_2) \leq J_2(\hat{u}_1, \hat{u}_2) \text{ for all } u_2 \in \mathcal{A}_2. \tag{2.6}$$

Heuristically this means that player i has no incentive to deviate from the control $\hat{u}_i$, as long as player j $(j \neq i)$ does not deviate from $\hat{u}_j$; $i = 1, 2$. Therefore a Nash equilibrium is in some cases a likely outcome of a game. We now present a method to find it, based on the maximum principle for stochastic control. Our result may be regarded as an extension of the maximum principles for FBSDEs in [16] and for (forward) SDE games in [2].

Define the *Hamiltonians*

$$H_i(t, x, y, z, k, u_1, u_2, \lambda, p, q, r) \ : \ [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathcal{R}\times A_1\times A_2\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathcal{R} \to \mathbb{R}$$

of this game by

$$\begin{aligned} H_i(t, x, y, z, k, &u_1, u_2, \lambda, p, q, r) = f_i(t, x, u_1, u_2) + \lambda g_i(t, x, y, z, k, u_1, u_2) + pb(t, x, u_1, u_2) \\ &+ q\sigma(t, x, u_1, u_2) + \int_{\mathbb{R}} r(\zeta)\gamma(t, x, u_1, u_2, \zeta)\nu(d\zeta) \ ; \ i = 1, 2, \end{aligned} \tag{2.7}$$

where $\mathcal{R}$ is the set of functions from $\mathbb{R}_0$ into $\mathbb{R}$ such that the integral in (2.7) converges.

We assume that H_i is Fréchet differentiable $(\mathcal{C}^1)$ in the variables x, y, z, k, u i=1,2.

In the following, we are using the shorthand notation

$$\frac{\partial H_i}{\partial y}(t) = \frac{\partial H_i}{\partial y}(t, X(t), Y_i(t), Z_i(t), K_i(t, \cdot), u_1(t), u_2(t), \lambda_i(t), p_i(t), q_i(t), r_i(t, \cdot))$$

and similarly for the other partial derivatives of H_i.

To these Hamiltonians we associate a system of FBSDEs in the adjoint processes $\lambda_i(t)$, $p_i(t), q_i(t)$ and $r_i(t, \zeta)$ as follows:

(i) Forward SDE in $\lambda_i(t)$:

$$\begin{cases} d\lambda_i(t) &= \dfrac{\partial H_i}{\partial y}(t)dt + \dfrac{\partial H_i}{\partial z}(t)dB(t) + \displaystyle\int_{\mathbb{R}} \nabla_k H_i(t, \zeta)\tilde{N}(dt, d\zeta) \ ; \ 0 \leq t \leq T \\ \lambda_i(0) &= \psi_i'(Y_i(0)) \left(= \dfrac{d\psi_i}{dy}(Y_i(0)) \right). \end{cases} \tag{2.8}$$

(ii) Backward SDE in $p_i(t), q_i(t), r_i(t,\zeta)$:

$$\begin{cases} dp_i(t) &= -\dfrac{\partial H_i}{\partial x}(t)dt + q_i(t)dB(t) + \displaystyle\int_{\mathbb{R}} r_i(t,\zeta)\tilde{N}(dt,d\zeta) \ ; \ 0 \le t \le T \\ p_i(T) &= \varphi_i'(X(T)) + h_i'(X(T))\lambda_i(T). \end{cases} \tag{2.9}$$

See Appendix A for an explanation of the gradient operator $\nabla_k H_i(t,\zeta) = \nabla_k H_i(t,\zeta)(\cdot)$.

Theorem 2.1 (Sufficient maximum principle for FBSDE games) *Let* $(\hat{u}_1,\hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ *with corresponding solutions* $\hat{X}(t), \hat{Y}_i(t), \hat{Z}_i(t), \hat{K}_i(t), \hat{\lambda}_i(t), \hat{p}_i(t), \hat{q}_i(t), \hat{r}_i(t,\zeta)$ *of equations* (2.1), (2.3), (2.8) *and* (2.9) *for* $i = 1,2$. *Suppose that the following holds:*

- *(Concavity I) The functions* $x \to h_i(x), x \to \varphi_i(x), x \to \psi_i(x)$ *are concave,* i *= 1,2.*
- *(The conditional maximum principle)*

$$max_{v\in A_1}\{E[H_1(t,\hat{X}(t),\hat{Y}_1(t),\hat{Z}_1(t),\hat{K}_1(t,\cdot),v,\hat{u}_2(t),\hat{\lambda}_1(t),\hat{p}_1(t),\hat{q}_1(t),\hat{r}_1(t,\cdot)) \mid \mathcal{E}_t^{(1)}] \ ; \\ = E[H_1(t,\hat{X}(t),\hat{Y}_1(t),\hat{Z}_1(t),\hat{K}_1(t,\cdot),\hat{u}_1(t),\hat{u}_2(t),\hat{\lambda}_1(t),\hat{p}_1(t),\hat{q}_1(t),\hat{r}_1(t,\cdot)) \mid \mathcal{E}_t^{(1)}] \tag{2.10}$$

and similarly

$$max_{v\in A_2}\{E[H_2(t,\hat{X}(t),\hat{Y}_2(t),\hat{Z}_2(t),\hat{K}_2(t,\cdot),u_1(t),v,\hat{\lambda}_2(t),\hat{p}_2(t),\hat{q}_2(t),\hat{r}_2(t,\cdot)) \mid \mathcal{E}_t^{(2)}] \ ; \\ = E[H_2(t,\hat{X}(t),\hat{Y}_2(t),\hat{Z}_2(t),\hat{K}_2(t,\cdot),\hat{u}_1(t),\hat{u}_2(t),\hat{\lambda}_2(t),\hat{p}_2(t),\hat{q}_2(t),\hat{r}_2(t,\cdot)) \mid \mathcal{E}_t^{(2)}] \tag{2.11}$$

- *(Concavity II) (The Arrow conditions). The functions*

$$\hat{h}_1(x,y,z,k) := \max_{v_1\in A_1} E[H_1(t,x,y,z,k,v_1,\hat{u}_2(t),\hat{\lambda}_1(t),\hat{p}_1(t),\hat{q}_1(t),\hat{r}_1(t,\cdot)) \mid \mathcal{E}_t^{(1)}],$$

and

$$\hat{h}_2(x,y,z,k) := \max_{v_2\in A_2} E[H_2(t,x,y,z,k,\hat{u}_1(t),v_2,\hat{\lambda}_2(t),\hat{p}_2(t),\hat{q}_2(t),\hat{r}_2(t,\cdot)) \mid \mathcal{E}_t^{(2)}]$$

are concave for all t*, a.s.*

- *Moreover, assume the following growth conditions hold:*

$$
\begin{aligned}
E\Bigg[\int_0^T \Bigg\{ & \hat{p}_i^2(t)\left[(\sigma(t)-\hat{\sigma}(t))^2+\int_{\mathbb{R}}(r(t,\zeta)-\hat{r}(t,\zeta))^2\nu(d\zeta)\right] \\
& +(X(t)-\hat{X}(t))^2\left[\hat{q}_i^2(t)+\int_{\mathbb{R}}\hat{r}_i^2(t,\zeta)\nu(d\zeta)\right] \\
& +(Y_i(t)-\hat{Y}_i(t))^2\left[\left(\frac{\partial \hat{H}_i}{\partial z}\right)^2(t)+\int_{\mathbb{R}}\left\|\nabla_k \hat{H}_i(t,\zeta)\right\|^2\nu(d\zeta)\right] \\
& +\hat{\lambda}_1^2(t)\left[(Z_i(t)-\hat{Z}_i(t))^2+\int_{\mathbb{R}}(K_i(t,\zeta)-\hat{K}_i(t,\zeta))^2\nu(d\zeta)\right]\Bigg\}dt\Bigg] < \infty \text{ for } i=1,2.
\end{aligned}
\tag{2.12}
$$

Then $\hat{u}(t)=(\hat{u}_1(t),\hat{u}_2(t))$ *is a Nash equilibrium for* (2.1)-(2.4).

Remark 2.2 Above we have used the following shorthand notation:

If $i=1$, *then* $X(t)=X^{(u_1,\hat{u}_2)}(t)$ *and* $Y_1(t)=Y_1^{(u_1,\hat{u}_2)}(t)$ *are the processes corresponding to the control* $u(t)=(u_1(t),\hat{u}_2(t))$, *while* $\hat{X}(t)=X^{(\hat{u})}(t)$ *and* $\hat{Y}_1(t)=Y_1^{(\hat{u})}(t)$ *are those corresponding to the control* $\hat{u}(t)=(\hat{u}_1(t),\hat{u}_2(t))$. *An analogue notation is used for* $i=2$.

Moreover, we put

$$
\frac{\partial \hat{H}_i}{\partial x}(t)=\frac{\partial H_i}{\partial x}(t,\hat{X}(t),\hat{Y}_i(t),\hat{Z}_i(t),\hat{K}_i(t,\cdot),\hat{u}(t),\hat{\lambda}_i(t),\hat{p}_i(t),\hat{q}_i(t),\hat{r}_i(t,\cdot))
$$

and similarly with $\dfrac{\partial \hat{H}_i}{\partial z}(t)$ *and* $\nabla_k \hat{H}_i(t,\zeta), i=1,2.$

Proof. See Appendix A. □

It is also of interest to prove a version of the maximum principle which does not require the concavity conditions. One such version is the following *necessary maximum principle* (Theorem 2.3) which requires the following assumptions:

- For all $t_0 \in [0,T]$ and all bounded, $\mathcal{E}_t^{(i)}$-measurable random variables $\alpha_i(\omega)$, the control $\beta_i(t) := \chi_{(t_0,T)}(t)\alpha_i(\omega)$ belongs to $\mathcal{A}_i$; $i = 1, 2$ (2.13)
- For all $u_i, \beta_i \in \mathcal{A}_i$ with β_i bounded there exists $\delta_i > 0$ such that the control $\tilde{u}_i(t) := u_i(t) + s\beta_i(t)$; $t \in [0,T]$ belongs to $\mathcal{A}_i$ for all $s \in (-\delta_i, \delta_i)$; $i = 1, 2$. (2.14)
- The following derivative processes exist and belong to $L^2([0,T] \times \Omega)$: (2.15)

$$x_1(t) = \frac{d}{ds}X^{(u_1+s\beta_1,u_2)}(t) \mid_{s=0} ; \quad y_1(t) = \frac{d}{ds}Y_1^{(u_1+s\beta_1,u_2)}(t) \mid_{s=0}$$
$$z_1(t) = \frac{d}{ds}Z_1^{(u_1+s\beta_1,u_2)}(t) \mid_{s=0} ; \quad k_1(t,\zeta) = \frac{d}{ds}K_1^{(u_1+s\beta_1,u_2)}(t) \mid_{s=0}$$

and, similarly $x_2(t) = \frac{d}{ds}X^{(u_1,u_2+s\beta_2)}(t) \mid_{s=0}$ etc.

Note that since $X^{(u)}(0) = x$ for all u we have $x_i(0) = 0$ for $i = 1, 2$.

In the following we write

$$\frac{\partial b}{\partial x}(t) \text{ for } \frac{\partial b}{\partial x}(t, X(t), u(t)) \text{ etc.}$$

By (2.1) and (2.3) we have

$$\begin{aligned} dx_1(t) = {} & \left\{ \frac{\partial b}{\partial x}(t)x_1(t) + \frac{\partial b}{\partial u_1}(t)\beta_1(t) \right\} dt + \left\{ \frac{\partial \sigma}{\partial x}(t)x_1(t) + \frac{\partial \sigma}{\partial u_1}(t)\beta_1(t) \right\} dB(t) \\ & + \int_{\mathbb{R}} \left\{ \frac{\partial \gamma}{\partial x}(t,\zeta)x_1(t) + \frac{\partial \gamma}{\partial u_1}(t,\zeta)\beta_1(t) \right\} \tilde{N}(dt,d\zeta), \end{aligned} \tag{2.16}$$

$$\begin{aligned} dy_1(t) = {} & -\left\{ \frac{\partial g_1}{\partial x}(t)x_1(t) + \frac{\partial g_1}{\partial y}(t)y_1(t) + \frac{\partial g_1}{\partial z}(t)z_1(t) \right. \\ & \left. + \int_{\mathbb{R}} \nabla_k g_1(t,\zeta)k_1(t,\zeta)\nu(d\zeta) + \frac{\partial g_1}{\partial u_1}(t)\beta_1(t) \right\} dt \\ & + z_i(t)dB(t) + \int_{\mathbb{R}} k_1(t,\zeta)\tilde{N}(dt,d\zeta) \; ; \; 0 \le t \le T, \\ y_1(T) = {} & h_1'(X^{(u_1,u_2)}(T))x_1(T), \end{aligned} \tag{2.17}$$

and similarly for $dx_2(t), dy_2(t)$.

We are now ready to state a necessary maximum principle, which is an extension of Theorem 3.1 in [2] and Theorem 3.1 in [16]. In the sequel, $\frac{\partial H}{\partial v}$ means $\nabla_v H$.

Theorem 2.3 (Necessary maximum principle) *Suppose* $u \in \mathcal{A}$ *with corresponding solutions* $X(t), Y_i(t), Z_i(t), K_i(t,\zeta), \lambda_i(t), p_i(t), q_i(t), r_i(t,\zeta)$ *of equations* (2.1), (2.3), (2.8) *and* (2.9). *Suppose* (2.13), (2.14) *and* (2.15) *hold.*

Moreover, assume that

$$
\begin{aligned}
&E\Bigg[\int_0^T \Bigg\{p_i^2(t)\Bigg[\left(\frac{\partial\sigma}{\partial x}(t)x_i(t)+\frac{\partial\sigma}{\partial u_i}(t)\beta_i(t)\right)^2 \\
&+\int_{\mathbb{R}}\left(\frac{\partial\gamma}{\partial x}(t,\zeta)x_i(t)+\frac{\partial\gamma}{\partial u_i}(t,\zeta)\beta_i(t)\right)^2\nu(d\zeta)\Bigg] \\
&+x_i^2(t)(q_i^2(t)+\int_{\mathbb{R}} r_i^2(t,\zeta)\nu(d\zeta)) \\
&+\lambda_i^2(t)(z_i^2(t)+\int_{\mathbb{R}} k_i^2(t,\zeta)\nu(d\zeta)) \\
&+y_i^2(t)\left((\frac{\partial H_i}{\partial z})^2(t)+\int_{\mathbb{R}}\left\|\nabla_k H_i(t,\zeta)\right\|^2\nu(d\zeta)\right)\Bigg\}\Bigg]dt<\infty \text{ for } i=1,2.
\end{aligned}
\tag{2.18}
$$

Then the following are equivalent:

(i)

$$
\frac{d}{ds}J_1(u_1+s\beta_1,u_2)\mid_{s=0}=\frac{d}{ds}J_2(u_1,u_2+s\beta_2)\mid_{s=0}=0
$$

for all bounded $\beta_1 \in \mathcal{A}_1$, $\beta_2 \in \mathcal{A}_2$.

(ii)

$$
\begin{aligned}
&E\left[\frac{\partial}{\partial v_1}H_1(t,X(t),Y_1(t),Z_1(t),K_1(t,\cdot),v_1,u_2(t),\lambda_1(t),p_1(t)q_1(t),r_1(t,\cdot))\mid\mathcal{E}_t^{(1)}\right]_{v_1=u_1(t)} \\
&=E\left[\frac{\partial}{\partial v_2}H_2(t,X(t),Y_2(t),Z_2(t),K_2(t,\cdot),u_1(t),v_2,\lambda_2(t),p_2(t),q_2(t),r_2(t,\cdot))\mid\mathcal{E}_t^{(2)}\right]_{v_2=u_2(t)} \\
&=0.
\end{aligned}
$$

Proof. See Appendix A. □

The zero-sum game case. In the *zero-sum case* we have

$$
J_1(u_1,u_2)+J_2(u_1,u_2)=0. \tag{2.19}
$$

Then the Nash equilibrium $(\hat{u}_1, \hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ satisfying (2.5)-(2.6) becomes a *saddle point* for $J(u_1, u_2) := J_1(u_1, u_2)$. To see this, note that (2.5)-(2.6) imply that

$$J_1(u_1, \hat{u}_2) \leq J_1(\hat{u}_1, \hat{u}_2) = -J_2(\hat{u}_1, \hat{u}_2) \leq -J_2(\hat{u}_1, u_2)$$

and hence

$$J(u_1, \hat{u}_2) \leq J(\hat{u}_1, \hat{u}_2) \leq J(\hat{u}_1, u_2) \text{ for all } u_1, u_2.$$

From this we deduce that

$$\begin{aligned}\inf_{u_2 \in \mathcal{A}_2} \sup_{u_1 \in \mathcal{A}_1} J(u_1, u_2) &\leq \sup_{u_1 \in \mathcal{A}_1} J(u_1, \hat{u}_2) \leq J(\hat{u}_1, \hat{u}_2) \\ &\leq \inf_{u_2 \in \mathcal{A}_2} J(\hat{u}_1, u_2) \leq \sup_{u_1 \in \mathcal{A}_1} \inf_{u_2 \in \mathcal{A}_2} J(u_1, u_2). \end{aligned} \tag{2.20}$$

Since we always have $\inf \sup \geq \sup \inf$, we conclude that

$$\begin{aligned}\inf_{u_2 \in \mathcal{A}_2} \sup_{u_1 \in \mathcal{A}_1} J(u_1, u_2) &= \sup_{u_1 \in \mathcal{A}_1} J(u_1, \hat{u}_2) = J(\hat{u}_1, \hat{u}_2) \\ &= \inf_{u_2 \in \mathcal{A}_2} J(\hat{u}_1, u_2) = \sup_{u_1 \in \mathcal{A}_1} \inf_{u_2 \in \mathcal{A}_2} J(u_1, u_2). \end{aligned} \tag{2.21}$$

i.e. $(\hat{u}_1, \hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ is a *saddle point* for $J(u_1, u_2)$.

We now state the necessary maximum principle for the zero sum game problem:

Choose $g_i = g, h_i = h, f_1 = f = -f_2, \varphi_1 = \varphi = -\varphi_2$ and $\psi_1 = \psi = -\psi_2$; $i = 1, 2$. For $u = (u_1, u_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ define

$$J(u_1, u_2) = E\left[\int_0^T f(t, X^{(u)}(t), u(t))dt + \varphi(X^{(u)}(T)) + \psi(Y(0))\right], \tag{2.22}$$

where $X^{(u)}(t), Y(t) = Y_i(t), Z(t) = Z_i(t)$ and $K(t,\zeta) = K_i(t,\zeta)$ are defined by (2.1) and (2.3). In this case only one Hamiltonian is needed, namely (see (2.7)):

$$\begin{aligned} H(t, x, y, z, k, u_1, u_2, \lambda, p, q, r) = f(t, x, u_1, u_2) + \lambda g(t, x, y, z, k, u_1, u_2) + pb(t, x, u_1, u_2) \\ + q\sigma(t, x, u_1, u_2) + \int_{\mathbb{R}} r(\zeta)\gamma(t, x, u_1, u_2, \zeta)\nu(d\zeta). \end{aligned} \tag{2.23}$$

Let $\lambda = \lambda_i,\ p = p_i,\ q = q_i$ and $r = r_i$ $i = 1, 2$ be as in (2.8)-(2.9).

Theorem 2.4 (Necessary maximum principle for zero-sum forward-backward games) *Assume the conditions of Theorem 2.3 hold. Then the following are equivalent:*

(i)

$$\frac{d}{ds}J(u_1 + s\beta_1, u_2) \mid_{s=0} = \frac{d}{ds}J(u_1, u_2 + s\beta_2) \mid_{s=0} = 0 \tag{2.24}$$

for all bounded $\beta_1 \in \mathcal{A}_1, \beta_2 \in \mathcal{A}_2$.

(ii)

$$\begin{aligned} &E\left[\frac{\partial}{\partial v_1}H(t, X(t), Y(t), Z(t), K(t,\cdot), v_1, u_2(t), \lambda(t), p(t), q(t), r(t,\cdot)) \mid \mathcal{E}_t^{(1)}\right]_{v_1=u_1(t)} \\ &= E\left[\frac{\partial}{\partial v_2}H(t, X(t), Y(t), Z(t), K(t,\cdot), u_1(t), v_2, \lambda(t), p(t), q(t), r(t,\cdot)) \mid \mathcal{E}_t^{(2)}\right]_{v_2=u_2(t)} \\ &= 0. \end{aligned} \tag{2.25}$$

Proof. The proof is similar to the proof of Theorem 2.3 and is omitted. □

Corollary 2.5 *Let* $u = (u_1, u_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ *be a Nash equilibrium (saddle point) for the zero-sum game in Theorem 2.4. Then* (2.25) *holds.*

Proof. This follows from Theorem 2.4 by noting that if $u = (u_1, u_2)$ is a Nash equilibrium, then (2.24) holds by (2.21). □

Similarly we get

Theorem 2.6 *[Sufficient maximum principle for zero-sum forward-backward games]*

Let $H(t, x, y, z, k, u_1, u_2, \lambda, p, q, r)$ *be as in* (2.25). *Let* $(\hat{u}_1, \hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$*, with corresponding solutions* $\hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\zeta)$ *of equation* (2.1), (2.2), (2.8) *and* (2.9) *for* $g_i = g, h_i = h, f_i = f, \varphi_i = \varphi$ *and* $\psi_i = \psi$. *Suppose the following holds*

- *The functions*

$$x \to h(x),\ x \to \varphi(x) \ \text{ and } x \to \psi(x) \tag{2.26}$$

  are affine.

- *(The conditional maximum principle)*

$$\begin{aligned} &\max_{v_1 \in A_1} E[H(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), v_1, \hat{u}_2(t), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)) \mid \mathcal{E}_t^{(1)}] \\ &= E[H(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), \hat{u}_1(t), \hat{u}_2(t), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)) \mid \mathcal{E}_t^{(1)}] \end{aligned} \tag{2.27}$$

and

$$\min_{v_2 \in A_2} E[H(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), \hat{u}_1(t), v_2, \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)) \mid \mathcal{E}_t^{(2)}]$$
$$= E[H(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), \hat{u}_1(t), \hat{u}_2(t), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)) \mid \mathcal{E}_t^{(2)}]. \tag{2.28}$$

- *(The Arrow conditions) The function*

  $$\hat{h}(x,y,z,k) := \max_{v_1 \in A_1} E[H(t,x,y,z,k,v_1,\hat{u}_2(t),\hat{\lambda}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) \mid \mathcal{E}_t^{(1)}]$$

  is concave, and the function

  $$\overset{\vee}{h}(x,y,z,k) := \min_{v_2 \in A_2} E[H(t,x,y,z,k,\hat{u}_1(t),v_2,\hat{\lambda}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) \mid \mathcal{E}_t^{(2)}]$$

  is convex, for all $t \in [0,T]$, a.s.

- *The growth condition* (2.14) *holds with $p_i = p$ etc.*

Then $\hat{u}(t) = (\hat{u}_1(t), \hat{u}_2(t))$ is a saddle point for $J(u_1,u_2)$.

Proof. The proof is similar to the proof of Theorem 2.1 and is omitted. □

3 Stochastic control under model uncertainty

Let $X(t) = X_x^v(t)$ be a controlled Itô–Lévy process of the form

$$\begin{aligned} dX(t) &= b(t,X(t),v(t))dt + \sigma(t,X(t),v(t))dB(t) \\ &\qquad + \int_{\mathbb{R}} \gamma(t,X(t),v(t),\zeta)\tilde{N}(dt,d\zeta)\ ;\ 0 \le t \le T \\ X(0) &= x \in \mathbb{R} \end{aligned} \tag{3.1}$$

where $v(\cdot)$ is the control process.

We consider a *model uncertainty* setup, represented by a probability measure $Q = Q^\theta$ which is equivalent to P, with the Radon-Nikodym derivative on $\mathcal{F}_t$ given by

$$\frac{d(Q \mid \mathcal{F}_t)}{d(P \mid \mathcal{F}_t)} = G^\theta(t) \tag{3.2}$$

where, for $0 \leq t \leq T$, $G^\theta(t)$ is a martingale of the form

$$\begin{aligned} dG^\theta(t) &= G^\theta(t^-)[\theta_0(t)dB(t) + \int_{\mathbb{R}} \theta_1(t,\zeta)\tilde{N}(dt,d\zeta)] \\ G^\theta(0) &= 1. \end{aligned} \tag{3.3}$$

Here $\theta = (\theta_0, \theta_1)$ may be regarded as a *scenario control*. Let $\mathcal{A}_1$ denote a given family of admissible controls v and $\mathcal{A}_2$ denote a given set of admissible scenario controls θ such that $E[\int_0^T \{|\theta_0^2(t)| + \int_{\mathbb{R}} \theta_1^2(t,\zeta)\nu(d\zeta)\}dt] < \infty$ and $\theta_1(t,\zeta) \geq -1+\epsilon$ for some $\epsilon > 0$. Let $\mathcal{E}^{(1)}_{0\leq t\leq T}$ and $\mathcal{E}^{(2)}_{0\leq t\leq T}$ be given subfiltrations of $\mathcal{F}_{0\leq t\leq T}$, representing the information available to the controllers at time t. It is required that $v \in \mathcal{A}_1$ be $\mathcal{E}^1_t$-predictable, and $\theta \in \mathcal{A}_2$ be $\mathcal{E}^2_t$-predictable. We consider the stochastic differential game to find $(\hat{v}, \hat{\theta}) \in \mathcal{A}_1 \times \mathcal{A}_2$ such that

$$\sup_{v\in\mathcal{A}_1} \inf_{\theta\in\mathcal{A}_2} E_{Q^\theta}[W(v,\theta)] = E_{Q^{\hat{\theta}}}[W(\hat{v},\hat{\theta})] = \inf_{\theta\in\mathcal{A}_2} \sup_{v\in\mathcal{A}_1} E_{Q^\theta}[W(v,\theta)], \tag{3.4}$$

where

$$W(v,\theta) = U_2(X^v(T)) + \int_0^T U_1(s, X^v(s), v(s))ds + \int_0^T \rho(\theta(t))dt. \tag{3.5}$$

Here, $U_1 : [0,T]\times\mathbb{R}\times\mathcal{V} \to \mathbb{R}$ and $U_2 : \mathbb{R} \to \mathbb{R}$ are given functions, concave and increasing with a strictly decreasing derivative, and ρ is a convex function. The term $\Lambda(\theta) := E_{Q^\theta}[\int_0^T \rho(\theta(t))dt]$ can be seen as a penalty term, penalizing the difference between Q^θ and the original probability measure P.

Put

$$F(t,x,u) = U_1(t,x,v) + \rho(\theta);\ u = (v,\theta) = (c,\pi,\theta_0,\theta_1). \tag{3.6}$$

Then

$$E_{Q^\theta}[W(v,\theta)] = E[G^\theta(T)U_2(X^v(T)) + \int_0^T G^\theta(s)F(s,X^v(s),u(s))ds]. \tag{3.7}$$

We now define $Y(t) = Y^{v,\theta}(t)$ by

$$Y(t) = E[\frac{G^\theta(T)}{G^\theta(t)}U_2(X^v(T)) + \int_t^T \frac{G^\theta(s)}{G^\theta(t)}F(s,X^v(s),u(s))ds \mid \mathcal{F}_t];\ \ t \in [0,T]. \tag{3.8}$$

Then we recognize $Y(t)$ as the solution of the linear BSDE (see Lemma B.1)

$$\begin{aligned} dY(t) &= -[F(t,X^v(t),u(t)) + \theta_0(t)Z(t) + \int_{\mathbb{R}} \theta_1(t,\zeta)K(t,\zeta)\nu(d\zeta)]dt \\ &\qquad + Z(t)dB(t) + \int_{\mathbb{R}} K(t,\zeta)\tilde{N}(dt,d\zeta); 0 \leq t \leq T \\ Y(T) &= U_2(X^v(T)). \end{aligned} \tag{3.9}$$

Note that

$$Y(0) = Y^{v,\theta}(0) = E_{Q^\theta}[W(v,\theta)]. \tag{3.10}$$

Therefore the problem (3.4) can be written

$$\sup_{v\in\mathcal{A}_1} \inf_{\theta\in\mathcal{A}_2} Y^{v,\theta}(0) = Y^{\hat{v},\hat{\theta}}(0) = \inf_{\theta\in\mathcal{A}_2} \sup_{v\in\mathcal{A}_1} Y^{v,\theta}(0), \tag{3.11}$$

where $Y^{v,\theta}(t)$ is given by the forward-backward system (3.1) & (3.9). This is a zero-sum *stochastic differential game (SDG) of forward-backward SDEs* of the form (2.22) with $f = \varphi = 0$ and $\psi = Id$.

Proceeding as in Section 2, define the Hamiltonian

$$H : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_0 \times \mathcal{R} \times A_1 \times A_2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \to \mathbb{R}$$

by

$$\begin{aligned} H(t,x,y,z,k,v,\theta,\lambda,p,q,r) = {} & [F(t,x,u) + \theta_0 z + \int_{\mathbb{R}} \theta_1(\zeta)k(\zeta)\nu(d\zeta)]\lambda \\ & + b(t,x,v)p + \sigma(t,x,v)q + \int_{\mathbb{R}} \gamma(t,x,v,\zeta)r(\zeta)\nu(d\zeta). \end{aligned} \tag{3.12}$$

where $\mathcal{R}$ is the set of functions $r : \mathbb{R}_0 \to \mathbb{R}$ such that (3.12) converge. Define a pair of FBSDEs in the adjoint processes $\lambda(t), p(t), q(t), r(t,\zeta)$ as follows:

Forward SDE for $\lambda(t)$:

$$\begin{aligned} d\lambda(t) &= \frac{\partial H}{\partial y}(t)dt + \frac{\partial H}{\partial z}(t)dB(t) + \int_{\mathbb{R}} \nabla_k H(t,\zeta)\tilde{N}(dt,d\zeta) \\ &= \lambda(t)\theta_0(t)dB(t) + \lambda(t)\int_{\mathbb{R}} \theta_1(t,\zeta)(\cdot)\tilde{N}(dt,d\zeta);\ t \in [0,T] \\ \lambda(0) &= 1 \end{aligned} \tag{3.13}$$

Backward SDE for $p(t), q(t), r(t,\zeta)$:

$$\begin{aligned} dp(t) &= -\frac{\partial H}{\partial x}(t)dt + q(t)dB(t) + \int_{\mathbb{R}} r(t,\zeta)\tilde{N}(dt,d\zeta) \\ &= -\{\frac{\partial F}{\partial x}(t) + p(t)\frac{\partial b}{\partial x}(t) + q(t)\frac{\partial \sigma}{\partial x}(t) + \int_{\mathbb{R}} r(t,\zeta)\frac{\partial \gamma}{\partial x}(t,\zeta)\nu(d\zeta)\}dt \\ &\quad + q(t)dB(t) + \int_{\mathbb{R}} r(t,\zeta)\tilde{N}(dt,d\zeta); \quad t \in [0,T] \\ p(T) &= \lambda(T)U_2'(X(T)). \end{aligned} \tag{3.14}$$

Here we have used the abbreviated notation

$$\frac{\partial H}{\partial y}(t) = \frac{\partial H}{\partial y}(t, X(t), Y(t), Z(t), K(t,\cdot), v(t), \theta(t), \lambda(t), p(t), q(t), r(t,\cdot))$$

and similarly for the other partial derivatives. We now present a necessary maximum principle for the forward-backward stochastic differential game (3.1), (3.9), (3.11) by adapting Theorem 2.4 to this case.

Theorem 3.1 *Suppose that the conditions of Theorem 2.3 hold. Let* $(\hat{v}, \hat{\theta}) \in \mathcal{A}_1 \times \mathcal{A}_2$, *with corresponding solutions* $\hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)$ *of equations* (3.1), (3.9), (3.14) *and* (3.13). *Suppose* (3.11) *holds, together with* (2.12). *Then the following holds:*

$$\begin{aligned} E[\hat{\lambda}(t)\frac{\partial U_1}{\partial v}(t,\hat{X}(t),\hat{v}(t)) + \hat{p}(t)\frac{\partial b}{\partial v}(t,\hat{X}(t),\hat{v}(t)) \\ +\hat{q}(t)\frac{\partial \sigma}{\partial v}(t,\hat{X}(t),\hat{v}(t)) + \int_{\mathbb{R}} \hat{r}(t,\zeta)\frac{\partial \gamma}{\partial v}(t,\hat{X}(t),\hat{v}(t),\zeta)\nu(d\zeta) \mid \mathcal{E}_t^{(1)}] = 0 \\ E[\hat{\lambda}(t)(\frac{\partial \rho}{\partial \theta_0}(\hat{\theta}(t)) + \hat{Z}(t)) \mid \mathcal{E}_t^{(2)}] = 0 \\ E[\hat{\lambda}(t)(\nabla_{\theta_1} F(t,\hat{X}(t),\hat{u}(t)) + \int_{\mathbb{R}} (\cdot)\hat{K}(t,\zeta)\nu(d\zeta)) \mid \mathcal{E}_t^{(2)}] = 0. \end{aligned}$$

Note that both $\nabla_{\theta_1} F$ and $\int_{\mathbb{R}} (\cdot)\hat{K}(t,\zeta)\nu(d\zeta)$ are linear functionals, the latter being defined by the action

$$\varphi \to \int_{\mathbb{R}} \varphi(\zeta)\hat{K}(t,\zeta)\nu(d\zeta)$$

for all bounded continuous functions $\varphi : \mathbb{R}_0 \mapsto \mathbb{R}$.

4 Portfolio and consumption problem under model uncertainty

We now apply this to the following portfolio and consumption problem under model uncertainty. Consider a financial market consisting of a bond with unit price $S_0(t) = 1\ ;\ \ 0 \le t \le T$, and a stock, with unit price $S(t)$ given by

$$dS(t) = S(t^-)[b_0(t)dt + \sigma_0(t)dB(t) + \int_{\mathbb{R}} \gamma_0(t,\zeta)\tilde{N}(dt,d\zeta)], \qquad (4.1)$$

where $b_0(t) = b_0(t,\omega)$, $\sigma_0(t) = \sigma_0(t,\omega)$ and $\gamma_0(t,\zeta) = \gamma_0(t,\zeta,\omega)$ are given $\{\mathcal{F}_t\}$-predictable processes such that $\gamma_0 \geq -1+\epsilon$ for some $\epsilon > 0$ and

$$E[\int_0^T \{|b_0(t)| + \sigma_0^2(t) + \int_{\mathbb{R}} \gamma_0^2(t,\zeta)\nu(d\zeta)\}dt] < \infty.$$

Note that this system is non–Markovian since the coefficients are random processes.

We introduce the state price density $\Gamma(t)$ defined by

$$\Gamma(t) := \exp(\int_0^t -\frac{b_0(s)}{\sigma_0(s)}dB(s) - \frac{1}{2}\int_0^t (\frac{b_0(s)}{\sigma_0(s)})^2 ds). \tag{4.2}$$

Let $X(t) = X^v(t)$ be the wealth process corresponding to a portfolio $\pi(t)$ and a consumption rate $c(t)$, i.e.

$$\begin{cases} dX(t) & = \pi(t)[b_0(t)dt + \sigma_0(t)dB(t) + \int_{\mathbb{R}} \gamma_0(t,\zeta)\tilde{N}(dt,d\zeta)] - c(t)dt, \quad t \in [0,T] \\ X(0) & = x \in \mathbb{R}, \end{cases} \tag{4.3}$$

and put $v = (\pi, c)$. We consider the stochastic differential game (3.4)-(3.5). For $i = 1,2$, I_i will denote the inverse of U_i', in the sense that

$$I_i(y) = \begin{cases} (U_i')^{-1}(y); & 0 \leq y \leq y_i \\ 0 & y > y_i \end{cases} \tag{4.4}$$

where $y_i = \lim_{x\to 0^+} U_i'(x)$. We assume that $\rho'(\theta)$ has an inverse.

We have seen in Section 3, that the problem (3.4)-(3.5) can be written as

$$\sup_{v\in\mathcal{A}_1} \inf_{\theta\in\mathcal{A}_2} Y^{v,\theta}(0) = Y^{\hat{v},\hat{\theta}}(0) = \inf_{\theta\in\mathcal{A}_2} \sup_{v\in\mathcal{A}_1} Y^{v,\theta}(0), \tag{4.5}$$

where $Y(t) = Y^{v,\theta}(t)$ is given by equation (3.9) and (4.3).

We now apply the necessary maximum principle given by Theorem 3.1. The Hamiltonian for the problem (4.5) is, by (3.12),

$$\begin{aligned} H(t,x,y,z,k,v,\theta,\lambda,p,q,r) = {} & [U_1(t,c) + \rho(\theta) + \theta_0 z + \int_{\mathbb{R}} \theta_1(\zeta)k(\zeta)\nu(d\zeta)]\lambda \\ & +(\pi b_0(t) - c)p + \pi\sigma_0(t)q + \pi\int_{\mathbb{R}} \gamma_0(t,\zeta)r(\zeta)\nu(d\zeta). \end{aligned}$$

The forward SDE for $\lambda(t) = \lambda_\theta(t)$ and the BSDE for $p(t), q(t), r(t,\zeta)$ are (see (3.13)- (3.14))

$$\begin{aligned} d\lambda(t) &= \lambda(t)[\theta_0(t)dB(t) + \int_{\mathbb{R}} \theta_1(t,\zeta)\tilde{N}(dt,d\zeta)];\ t \in [0,T] \\ \lambda(0) &= 1 \end{aligned} \tag{4.6}$$

$$\begin{aligned} dp(t) &= q(t)dB(t) + \int_{\mathbb{R}} r(t,\zeta)\tilde{N}(dt,dz);\ t \in [0,T] \\ p(T) &= \lambda(T)U_2'(X(T)). \end{aligned} \tag{4.7}$$

Maximizing H with respect to (c,π) gives the following first order conditions:

$$E[\lambda(t) \mid \mathcal{E}_t^{(1)}]\frac{\partial U_1}{\partial c}(t,c(t)) = E[p(t) \mid \mathcal{E}_t^{(1)}] \tag{4.8}$$

$$E[b_0(t)p(t) + \sigma_0(t)q(t) + \int_{\mathbb{R}} \gamma_0(t,\zeta)r(t,\zeta)\nu(d\zeta) \mid \mathcal{E}_t^{(1)}] = 0 \tag{4.9}$$

Minimizing H with respect to $\theta = (\theta_0, \theta_1)$ gives the following first order conditions:

$$\frac{\partial \rho}{\partial \theta_0}(\theta(t)) + E[Z(t) \mid \mathcal{E}_t^{(2)}] = 0 \tag{4.10}$$

$$\nabla_{\theta_1}\rho(\theta(t))(\cdot) + E[\int_{\mathbb{R}} (\cdot)K(t,\zeta)\nu(d\zeta) \mid \mathcal{E}_t^{(2)}] = 0 \tag{4.11}$$

We now restrict ourselves to the case when there are no jumps, i.e. $\tilde{N} = \nu = K = \theta_1 = 0$ and $\mathcal{E}_t^{(1)} = \mathcal{E}_t^{(2)} = \mathcal{F}_t$. For simplicity of notation, we write θ instead of θ_0. Then equations (4.6)-(4.11) simplify to:

$$\lambda(t) = \exp(\int_0^t \theta(s)dB(s) - \int_0^t \frac{1}{2}\theta^2(s)ds) \tag{4.12}$$

$$p(t) = E[\lambda(T)U_2'(X(T)) \mid \mathcal{F}_t]\ ; \tag{4.13}$$

$$\lambda(t)\frac{\partial U_1}{\partial c}(t,c(t)) = p(t) \tag{4.14}$$

$$b_0(t)p(t) + \sigma_0(t)q(t) = 0 \tag{4.15}$$

$$\rho'(\theta(t)) + Z(t) = 0 \tag{4.16}$$

and by the generalized Clark-Ocone formula [1],

$$q(t) = E[D_t(\lambda(T)U_2'(X(T))) \mid \mathcal{F}_t], \tag{4.17}$$

where D_t denotes the Malliavin derivative at t with respect to $B(\cdot)$. (See e.g. [7]).

The FBSDEs (4.3)-(3.9) simplify to:

$$dX(t) = \pi(t)[b_0(t)dt + \sigma_0(t)dB(t)] - c(t)dt, \quad 0 \le t \le T$$
$$X(0) = x > 0 \tag{4.18}$$
$$dY(t) = -[U_1(t, c(t)) + \rho(\theta(t)) + \theta(t)Z(t)]dt + Z(t)dB(t); \; 0 \le t \le T$$
$$Y(T) = U_2(X(T)). \tag{4.19}$$

Put

$$R = p(T) = \lambda(T)U_2'(X(T)). \tag{4.20}$$

Then (4.15) can be written

$$b_0(t)E[R \mid \mathcal{F}_t] + \sigma_0(t)E[D_t R \mid \mathcal{F}_t] = 0. \tag{4.21}$$

Following [16] we call this a Malliavin-differential type equation in the unknown random variable R. By Theorem A.1 in [16], the general solution of this equation is $R = R_\beta(T)$; where

$$R_\beta(t) = \beta\Gamma(t); \; 0 \le t \le T, \tag{4.22}$$

for some constant β, where $\Gamma(t)$ is defined in (4.2). Note that $R_\beta(t)$ is a martingale. Hence since $p(T) = R_\beta(T)$, we get by (4.13) that

$$p(t) = R_\beta(t); \; 0 \le t \le T. \tag{4.23}$$

Modulo the unknown constant β we can now find the *optimal terminal wealth* $X_\beta(T)$ by (4.20) as follows:

$$X_\beta(T) = I_2(\frac{\beta\Gamma(T)}{\lambda(T)}), \tag{4.24}$$

Similarly the *optimal consumption rate* is, by (4.14),

$$c(t) = c_\beta(t) = I_1(t, \frac{\beta\Gamma(t)}{\lambda(t)}); \quad 0 \le t \le T \tag{4.25}$$

The *optimal scenario parameter* is, by (4.16)

$$\theta(t) = \theta^\beta(t) = (\rho')^{-1}(-Z_\beta(t)); \quad 0 \le t \le T \tag{4.26}$$

where $(Y_\beta(t), Z_\beta(t))$ is the solution of the corresponding BSDE (4.19), i.e.

$$dY_\beta(t) = -[U_1(t, c_\beta(t)) + \rho(\theta(t)) + \theta(t)Z_\beta(t)]dt + Z_\beta(t)dB(t); \; 0 \le t \le T$$
$$Y_\beta(T) = U_2(I_2(\frac{\beta\Gamma(T)}{\lambda(T)})). \tag{4.27}$$

Let us consider the case when

$$U_1 = c = 0 \text{ (no consumption) and } \rho(\theta) = \frac{1}{2}\theta^2. \tag{4.28}$$

Substituting (4.26) into (4.27), we get

$$\begin{cases} dY_\beta(t) &= \frac{1}{2}\theta^2(t)dt - \theta(t)dB(t)\ ;\ 0 \le t \le T \\ Y_\beta(T) &= U_2(I_2(\frac{\beta\Gamma(T)}{\lambda(T)})). \end{cases} \tag{4.29}$$

Integrating (4.29), and using (4.12) at $t = T$, we get

$$-\frac{1}{2}\int_0^T \theta^2(s)ds + \int_0^T \theta(s)dB(s) = Y_\beta(0) - U_2(I_2(\frac{\beta\,\Gamma(T)}{\lambda(T)})). \tag{4.30}$$

Taking exponentials in (4.30) we obtain

$$\lambda(T) = \exp\left(\int_0^T \theta(s)dB(s) - \frac{1}{2}\int_0^T \theta^2(s)ds\right) = \frac{\exp Y_\beta(0)}{\exp(U_2(I_2(\frac{\beta\,\Gamma(T)}{\lambda(T)})))}. \tag{4.31}$$

Therefore $\lambda(t)$ is given as the solution of the BSDE (or more precisely SDE with terminal condition)

$$\begin{cases} d\lambda(t) &= \lambda(t)\theta(t)dB(t)\ ;\ 0 \le t \le T \\ \lambda_\theta(T) &= L \end{cases} \tag{4.32}$$

where $L = L(\beta, Y_\beta(0))$ is the solution of the equation:

$$L\exp(U_2(I_2(\frac{\beta\,\Gamma(T)}{L}))) = \exp Y_\beta(0). \tag{4.33}$$

By the generalized Clark-Ocone formula [1] this gives

$$\lambda(t)\theta(t) = E[D_tL \mid \mathcal{F}_t]\ ;\ 0 \le t \le T. \tag{4.34}$$

By (4.6) and (4.34), we have:

$$\begin{cases} d\lambda(t) &= E[D_tL \mid \mathcal{F}_t]dB(t)\ ;\ 0 \le t \le T \\ \lambda(0) &= 1 \end{cases} \tag{4.35}$$

and

$$\theta(t) = \frac{E[D_tL \mid \mathcal{F}_t]}{\lambda(t)}\ ;\ 0 \le t \le T. \tag{4.36}$$

Note that $E[L] = 1$ by the martingale property of $\lambda(t)$.

It remains to determine β and $Y_\beta(0)$. To this end, we consider the equation (4.18) for $X(t)$ as a BSDE as follows:

Put

$$\tilde{Z}_\beta(t) = \pi(t)\sigma_0(t).$$

Then

$$\pi(t) = \frac{\tilde{Z}_\beta(t)}{\sigma_0(t)} \tag{4.37}$$

and (4.18) becomes, using (4.24),

$$dX(t) = \frac{b_0(t)}{\sigma_0(t)}\tilde{Z}_\beta(t)dt + \tilde{Z}_\beta(t)dB(t); \tag{4.38}$$

$$X(T) = I_2(\frac{\beta\,\Gamma(T)}{L}). \tag{4.39}$$

The solution of this linear BSDE is

$$\begin{aligned} X(t) &= E[I_2(\frac{\beta\,\Gamma(T)}{L})\exp(\int_t^T -\frac{1}{2}(\frac{b_0(s)}{\sigma_0(s)})^2 ds - \int_t^T \frac{b_0(s)}{\sigma_0(s)}dB(s)) \mid \mathcal{F}_t] \\ &= E[I_2(\frac{\beta\,\Gamma(T)}{L})\frac{\Gamma(T)}{\Gamma(t)} \mid \mathcal{F}_t]. \end{aligned} \tag{4.40}$$

In particular, putting $t = 0$, we get

$$x = E[I_2(\frac{\beta\,\Gamma(T)}{L})\Gamma(T)]. \tag{4.41}$$

Finally, by taking expectation in (4.30), we deduce that

$$Y_\beta(0) = E\left[U_2(I_2(\frac{\beta\,\Gamma(T)}{L})) - \frac{1}{2}\int_0^T \theta^2(s))ds\right] \tag{4.42}$$

which, together with (4.41) gives the value of β and the solution $Y_\beta(0) = Y^{\hat{\pi},\hat{\theta}}(0)$ of (3.11).

We summarize what we have proved

Theorem 4.1 *Consider the problem to find $(\hat{\pi}, \hat{\theta})$ such that*

$$\sup_{\pi\in\mathcal{A}_1}\inf_{\theta\in\mathcal{A}_2} E_{Q^\theta}[W(v,\theta)] = E_{Q^{\hat{\theta}}}[W(\hat{\pi},\hat{\theta})] = \inf_{\theta\in\mathcal{A}_2}\sup_{v\in\mathcal{A}_1} E_{Q^\theta}[W(\pi,\theta)], \tag{4.43}$$

with

$$W(\pi,\theta) = U_2(X^\pi(T)) + \int_0^T \theta(t)^2 dt \tag{4.44}$$

where

$$dX(t) = \pi(t)[b_0(t)dt + \sigma_0(t)dB(t)], \quad 0 \le t \le T$$
$$X(0) = x > 0. \tag{4.45}$$

This problem is equivalent to

$$\sup_{\pi\in\mathcal{A}_1}\inf_{\theta\in\mathcal{A}_2} Y^{\pi,\theta}(0) = Y^{\hat{\pi},\hat{\theta}}(0) = \inf_{\theta\in\mathcal{A}_2}\sup_{\pi\in\mathcal{A}_1} Y^{v,\theta}(0), \tag{4.46}$$

where $Y = Y^{\pi,\theta}$ is given by

$$dY(t) = -[\frac{1}{2}\theta(t)^2 + \theta(t)Z(t)]dt + Z(t)dB(t); \; 0 \le t \le T$$
$$Y(T) = U_2(X(T)). \tag{4.47}$$

Then, the optimal scenario parameter is

$$\hat{\theta} = E[D_t L \mid \mathcal{F}_t](1 + \int_0^t E[D_s L \mid \mathcal{F}_s]dB(s))^{-1}.$$

The optimal portfolio $\hat{\pi}$ is given by

$$\hat{\pi} = \frac{D_t \hat{X}(t)}{\sigma_0(t)}$$

where $\hat{X}(t)$ is the optimal state process given by (4.40)*, with β and $Y_\beta(0)$ given by* (4.41)-(4.42) *with $\theta = \hat{\theta}$, and hence $L = L(\beta, Y_\beta(0))$ given by* (4.33).

Proof. The argument above shows that, by the necessary maximum principle (Theorem 3.1), if there is an optimal pair $(\hat{\pi}, \hat{\theta})$, then it is given as in the theorem.

Conversely, if we define $(\hat{\pi}, \hat{\theta})$ as in the theorem, we can show that $(\hat{\pi}, \hat{\theta})$ must be optimal, as follows:

Fix an arbitrary $\pi \in \mathcal{A}_1$ in the BSDE (4.47). Then, proceeding as in [19], by the comparison theorem for BSDEs, we obtain the minimal value $Y^{\pi,\hat{\theta}}(0)$ and its minimizer $\hat{\theta}$ simply by minimizing the driver of (4.47), i.e. by minimizing for each t and ω the function:

$$\theta \mapsto \frac{1}{2}\theta^2 + \theta Z(t).$$

This gives

$$\hat{\theta}(t) = -Z(t), \tag{4.48}$$

which is identical to (4.16). Substituting this into (4.47), we have reduced the original game problem to the following FBSDE control problem:

Find $\hat{\pi} \in \mathcal{A}_1$ such that

$$\sup_{\pi \in \mathcal{A}_1} Y^{\pi}(0) = Y^{\hat{\pi}}(0), \tag{4.49}$$

where

$$\begin{aligned} dY^{\pi}(t) &= \frac{1}{2} Z(t)^2 dt + Z(t) dB(t);\ 0 \leq t \leq T \\ Y^{\pi}(T) &= U_2(X^{\pi}(T)) \end{aligned} \tag{4.50}$$

and $X^{\pi}(t)$ given in (4.45). This problem is of the type discussed in [16]. If we apply the sufficient maximum principle (Theorem 2.3) of that paper, we get that the optimal $\hat{\pi}$ is given as the maximizer π of the associated Hamiltonian:

$$H_0(t, x, y, z, \pi, \lambda, p, q) := -\frac{1}{2} \lambda z^2 + \pi(p\, b_0(t) + q \sigma_0(t)). \tag{4.51}$$

This gives the equation

$$p(t)\, b_0(t) + q(t) \sigma_0(t) = 0, \tag{4.52}$$

which is (4.15). Moreover, again by Theorem 2.3 in [16], the equation for the associated process $\lambda(t)$ is

$$d\lambda(t) = -Z(t)\lambda(t) dB(t) = \lambda(t)\theta(t) dB(t), \tag{4.53}$$

$$\lambda(0) = 1 \tag{4.54}$$

which is (4.12). We conclude that, since the pair $(\hat{\pi}, \hat{\theta})$ of Theorem 4.1 does indeed satisfy the sufficient conditions (4.48), (4.52), and (4.53), it also satisfies all the conditions of the sufficient maximum principle of Theorem 2.3 in [16] and hence the pair is optimal. □

The logarithmic utility case. In this case, substituting $U_2(x) = \ln x$ and $I_2(x) = \frac{1}{x}$ in the general formulas above, we get:

$$\beta = \frac{1}{x} \tag{4.55}$$

$$L = \frac{\Gamma(T)^{1/2}}{E[\Gamma(T)^{1/2}]} \tag{4.56}$$

$$\hat{\theta} = E[D_t L \mid \mathcal{F}_t](1 + \int_0^t E[D_s L \mid \mathcal{F}_s] dB(s))^{-1} \tag{4.57}$$

$$Y_\beta(0) = \ln x + E\left[\int_0^T (\frac{1}{2}(\frac{b_0(s)}{\sigma_0(s)})^2 - \theta^2(s))ds\right] \tag{4.58}$$

$$\hat{X}(t) = x\frac{E[\Gamma(T)^{1/2} \mid \mathcal{F}_t]}{E[\Gamma(T)^{1/2}]\Gamma(t)}. \tag{4.59}$$

The case with no model uncertainty. In this case, $\theta = 0$ and $\lambda = 1$ and the problem reduces to maximizing

$$Y(0) = E[\int_O^T U_1(t, c(t))dt + U_2(X(T))]$$

which is a classical optimal portfolio/consumption problem. Then the optimal terminal wealth $X(T)$ is given by :

$$X_\beta(T) = I_2(\beta\Gamma(T))$$

and by (4.25), and the optimal consumption rate $c(t)$ is given by

$$c_\beta(t) = I_1(t, \beta\Gamma(t)).$$

To find the unknown β, we consider the equation (4.18) for $X(t)$ as a BSDE as follows: Put

$$\tilde{Z}_\beta(t) = \pi(t)\sigma_0(t).$$

Then

$$\pi(t) = \frac{\tilde{Z}_\beta(t)}{\sigma_0(t)} \tag{4.60}$$

and (4.18) becomes, using (4.24),

$$dX(t) = (\frac{b_0(t)}{\sigma_0(t)}\tilde{Z}_\beta(t) - I_1(t, \beta\Gamma(t)))dt + \tilde{Z}_\beta(t)dB(t); \tag{4.61}$$

$$X(T) = I_2(\beta\Gamma(T)) \tag{4.62}$$

The solution of this linear BSDE is

$$X(t) = E[I_2(\beta.\Gamma(T))\frac{\Gamma(T)}{\Gamma(t)} + \int_t^T \frac{\Gamma(s)}{\Gamma(t)} I_1(s, \beta.\Gamma(s))ds \mid \mathcal{F}_t].$$

Putting $t = 0$, we get

$$x = E[I_2(\beta\Gamma(T))\Gamma(T) + \int_0^T \Gamma(s)I_1(s, \beta\Gamma(s))ds]$$

and this equation determines β. We thus recover by a completely different method the results obtained by the classical martingale method, (see e.g. [5], Chapter 3).

A Proofs of the maximum principles for FB-SDE games

We first recall some basic concepts and results from Banach space theory. Let V be an open subset of a Banach space $\mathcal{X}$ with norm $\|\cdot\|$ and let $F : V \to \mathbb{R}$.

(i) We say that F has a directional derivative (or Gâtaux derivative) at $x \in X$ in the direction $y \in \mathcal{X}$ if

$$D_y F(x) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(F(x + \varepsilon y) - F(x))$$

exists.

(ii) We say that F is a Fréchet differentiable at $x \in V$ if there exists a linear map

$$L := \mathcal{X} \to \mathbb{R}$$

such that

$$\lim_{\substack{h \to 0 \\ h \in \mathcal{X}}} \frac{1}{\|h\|}|F(x + h) - F(x) - L(h)| = 0.$$

In this case we call L the *gradient* (or Fréchet derivative) of F at x and we write

$$L = \nabla_x F.$$

(iii) If F is Fréchet differentiable, then F has a directional derivative in all directions $y \in \mathcal{X}$ and

$$D_y F(x) = \nabla_x F(y).$$

Proof of Theorem 2.1 (Sufficient maximum principle). We first prove that

$$J_1(u_1, \hat{u}_2) \leq J_1(\hat{u}_1, \hat{u}_2) \text{ for all } u_1 \in \mathcal{A}_1.$$

To this end, fix $u_1 \in \mathcal{A}_1$ and consider

$$\Delta := J_1(u_1, \hat{u}_2) - J_1(\hat{u}_1, \hat{u}_2) = I_1 + I_2 + I_3, \tag{A.1}$$

where

$$I_1 = E\left[\int_0^T \{f_1(t, X(t), u(t)) - f_1(t, \hat{X}(t), \hat{u}(t))\} dt\right] \tag{A.2}$$

$$I_2 = E[\varphi_1(X(T)) - \varphi_1(\hat{X}(T))] \tag{A.3}$$

$$I_3 = E[\psi_1(Y_1(0)) - \psi_1(\hat{Y}_1(0))]. \tag{A.4}$$

By (2.7) we have

$$\begin{aligned} I_1 = E\Bigg[\int_0^T \{&H_1(t) - \hat{H}_1(t) - \hat{\lambda}_1(t)(g_1(t) - \hat{g}_1(t)) - \hat{p}_1(t)(b(t) - \hat{b}(t)) \\ &-\hat{q}_1(t)(\sigma(t) - \hat{\sigma}(t)) - \int_{\mathbb{R}} \hat{r}_1(t,\zeta)(\gamma(t,\zeta) - \hat{\gamma}(t,\zeta))\nu(d\zeta)\Bigg\} dt\Bigg] \end{aligned} \tag{A.5}$$

By concavity of φ_1, (2.9) and the Itô formula,

$$
\begin{aligned}
I_2 &\leq E[\varphi_1'(\hat{X}(T))(X(T)-\hat{X}(T))] \\
&= E[\hat{p}_1(T)(X(T)-\hat{X}(T))] \\
&\quad - E[\hat{\lambda}_1(T)h_1'(\hat{X}(T))(X(T)-\hat{X}(T))] \\
&= E\left[\int_0^T \hat{p}_1(t^-)(dX(t)-d\hat{X}(t)) + \int_0^T (X(t^-)-\hat{X}(t^-))d\hat{p}_1(t)\right. \\
&\quad + \int_0^T \hat{q}_1(t)(\sigma(t)-\hat{\sigma}(t))dt \\
&\quad \left. + \int_0^T \int_{\mathbb{R}} \hat{r}_1(t,\zeta)(\gamma(t,\zeta)-\hat{\gamma}(t,\zeta))\nu(d\zeta)dt\right] \\
&\quad - E[\hat{\lambda}_1(T)h_1'(\hat{X}(T))(X(T)-\hat{X}(T))] \\
&= E\left[\int_0^T \hat{p}_1(t)(b(t)-\hat{b}(t))dt + \int_0^T (X(t)-\hat{X}(t))\left(-\frac{\partial \hat{H}_1}{\partial x}(t)\right)dt\right. \\
&\quad + \int_0^T \hat{q}_1(t)(\sigma(t)-\hat{\sigma}(t))dt \\
&\quad \left. + \int_0^T \int_{\mathbb{R}} \hat{r}_1(t,\zeta)(\gamma(t,\zeta)-\hat{\gamma}(t,\zeta))\nu(d\zeta)dt\right] \\
&\quad - E[\hat{\lambda}_1(T)h_1'(\hat{X}(T))(X(T)-\hat{X}(T))].
\end{aligned} \tag{A.6}
$$

By concavity of ψ_1, (2.8), and concavity of φ_1 :

$$
\begin{aligned}
I_3 &= E[\psi_1(Y_1(0)) - \psi_1(\hat{Y}_1(0))] \\
&\leq E[\psi_1'(\hat{Y}_1(0))(Y_1(0) - \hat{Y}_1(0))] \\
&= E[\hat{\lambda}_1(0)(Y_1(0) - \hat{Y}_1(0))] \\
&= E[(Y_1(T) - \hat{Y}_1(T))\hat{\lambda}_1(T)] \\
&- \Bigg\{ E\Bigg[\int_0^T (Y_1(t^-) - \hat{Y}_1(t^-))d\hat{\lambda}_1(t) + \int_0^T \hat{\lambda}_1(t^-)(dY_1(t) - d\hat{Y}_1(t)) \\
&\quad + \int_0^T \frac{\partial \hat{H}_1}{\partial z}(t)(Z_1(t) - \hat{Z}_1(t))dt \\
&\quad + \int_0^T \int_{\mathbb{R}} \nabla_k \hat{H}_1(t,\zeta)(K_1(t,\zeta) - \hat{K}_1(t,\zeta))\nu(d\zeta)dt \Bigg] \Bigg\} \\
&= E[(h_1(X(T)) - h_1(\hat{X}(T)))\hat{\lambda}_1(T)] \\
&- \Bigg\{ E\Bigg[\int_0^T \frac{\partial \hat{H}_1}{\partial y}(t)(Y_1(t) - \hat{Y}_1(t))dt \\
&\quad + \int_0^T \hat{\lambda}_1(t)(-g_1(t) + \hat{g}_1(t))dt \\
&\quad + \int_0^T \frac{\partial \hat{H}_1}{\partial z}(t)(Z_1(t) - \hat{Z}_1(t))dt \\
&\quad + \int_0^T \int_{\mathbb{R}} \nabla_k \hat{H}_1(t,\zeta)(K_1(t,\zeta) - \hat{K}_1(t,\zeta))\nu(d\zeta)dt \Bigg] \Bigg\} \\
&\leq E[\hat{\lambda}_1(T)h_1'(\hat{X}(T))(X(T) - \hat{X}(T))] \\
&- \Bigg\{ E\Bigg[\int_0^T \frac{\partial \hat{H}_1}{\partial y}(t)(Y_1(t) - \hat{Y}_1(t))dt \\
&\quad + \int_0^T \hat{\lambda}_1(t)(-g_1(t) + \hat{g}_1(t))dt \\
&\quad + \int_0^T \frac{\partial \hat{H}_1}{\partial z}(t)(Z_1(t) - \hat{Z}_1(t))dt \\
&\quad + \int_0^T \int_{\mathbb{R}} \nabla_k \hat{H}_1(t,\zeta)(K_1(t,\zeta) - \hat{K}_1(t,\zeta))\nu(d\zeta)dt \Bigg] \Bigg\}. \qquad \text{(A.7)}
\end{aligned}
$$

Adding (A.5), (A.6) and (A.7) we get

$$
\begin{aligned}
\Delta &= I_1 + I_2 + I_3 \\
&\leq E\left[\int_0^T \left\{ H_1(t) - \hat{H}_1(t) - \frac{\partial \hat{H}_1}{\partial x}(t)(X(t) - \hat{X}(t)) \right.\right. \\
&- \frac{\partial H_1}{\partial y}(t)(Y_1(t) - \hat{Y}_1(t)) - \frac{\partial \hat{H}_1}{\partial z}(t)(Z_1(t) - \hat{Z}_1(t)) \\
&\left.\left. - \int_{\mathbb{R}} \nabla_k \hat{H}_1(t,\zeta)(K_1(t,\zeta) - \hat{K}_1(t,\zeta))\nu(d\zeta) \right\} dt \right]
\end{aligned} \tag{A.8}
$$

Since $\hat{h}_1(x,y,z,k)$ is concave, it follows by a standard separating hyperplane argument (see e.g. [21], Chap.5, Sect. 23) that there exists a supergradient $a = (a_0, a_1, a_2, a_3(\cdot)) \in \mathbb{R}^3 \times \mathcal{R}$ for $\hat{h}_1(x,y,z,k)$ at $x = \hat{X}(t), y = \hat{Y}_1(t), z = \hat{Z}_1(t^-)$ and $k = \hat{K}_1(t^-,\cdot)$ such that if we define

$$
\begin{aligned}
\varphi_1(x,y,z,k) := \hat{h}_1(x,y,z,k) &- \hat{h}_1(\hat{X}(t^-), \hat{Y}_1(t^-), \hat{Z}_1(t^-), \hat{K}_1(t,\cdot)) \\
&- [a_0(x - \hat{X}(t)) + a_1(y - \hat{Y}_1(t)) + a_2(z - \hat{Z}_1(t)) \\
&+ \int_{\mathbb{R}} a_3(\zeta)(k(\zeta) - \hat{K}(t,\zeta))\nu(d\zeta) \Big]
\end{aligned}
$$

then

$$\varphi_1(x,y,z,k) \leq 0 \text{ for all } x,y,z,k.$$

On the other hand we clearly have

$$\varphi_1(\hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}_1(t,\cdot)) = 0.$$

It follows that

$$
\begin{aligned}
\frac{\partial \hat{H}_1}{\partial x}(t) &= \frac{\partial \hat{h}_1}{\partial x}(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot)) = a_0 \\
\frac{\partial \hat{H}_1}{\partial y}(t) &= \frac{\partial \hat{h}_1}{\partial y}(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot)) = a_1 \\
\frac{\partial \hat{H}_1}{\partial z}(t) &= \frac{\partial \hat{h}_1}{\partial z}(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot)) = a_2 \\
\nabla_k \hat{H}_1(t,\zeta) &= \nabla_k \hat{h}_1(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot)) = a_3.
\end{aligned}
$$

Combining this with (A.8) we get

$$
\begin{aligned}
\Delta &\leq \hat{h}_1(X(t), Y_1(t), Z_1(t), K_1(t,\cdot)) \\
&- \hat{h}_1(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot)) \\
&- \frac{\partial \hat{h}_1}{\partial x}(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot))(X(t) - \hat{X}(t)) \\
&- \frac{\partial \hat{h}_1}{\partial y}(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot))(Y_1(t) - \hat{Y}_1(t)) \\
&- \frac{\partial \hat{h}_1}{\partial z}(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot))(Z_1(t) - \hat{Z}_1(t)) \\
&- \int_{\mathbb{R}} \nabla_k \hat{h}_1(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot))(K_1(t,\zeta) - \hat{K}_1(t,\zeta))\nu(d\zeta) \\
&\leq 0 \text{ since } \hat{h}_1 \text{ is concave.}
\end{aligned}
$$

Hence

$$J_1(u_1, \hat{u}_2) \leq J_1(\hat{u}_1, \hat{u}_2) \quad \text{for all } u_1 \in \mathcal{A}_1.$$

The inequality

$$J_2(\hat{u}_1, u_2) \leq J_2(\hat{u}_1, \hat{u}_2) \text{ for all } u_2 \in \mathcal{A}_2$$

is proved similarly. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.3 (Necessary maximum principle) Consider

$$
\begin{aligned}
D_1 &:= \frac{d}{ds} J_1(u_1 + s\beta_1, u_2) \mid_{s=0} \\
&= E\left[\int_0^T \left\{\frac{\partial f_1}{\partial x}(t)x_1(t) + \frac{\partial f_1}{\partial u_1}(t)\beta_1(t)\right\} dt + \varphi_1'(X^{(u_1,u_2)}(T))x_1(T) + \psi_1'(Y_1(0))y_1(0)\right].
\end{aligned}
\tag{A.9}
$$

By (2.9), (2.12) and the Itô formula,

$$
\begin{aligned}
&E[\varphi_1'(X^{(u_1,u_2)}(T))x_1(T)] \\
&\quad = E[p_1(T)x_1(T)] - E[h_1'(X^{(u_1,u_2)}(T))\lambda_1(T)] \\
&\quad = E\left[\int_0^T \left\{p_1(t^-)dx_1(t) + x_1(t^-)dp_1(t) + q_1(t)\left[\frac{\partial\sigma}{\partial x}(t)x_1(t) + \frac{\partial\sigma}{\partial u_1}(t)\beta_1(t)\right]dt\right.\right. \\
&\qquad \left.\left. + \int_{\mathbb{R}} r_1(t,\zeta)\left[\frac{\partial\gamma}{\partial x}(t,\zeta)x_1(t) + \frac{\partial\gamma}{\partial u_1}(t,\zeta)\beta_1(t,\zeta)\right]\nu(d\zeta)dt\right\}\right] \\
&\quad - E[h_1'(X^{(u_1,u_2)}(T))\lambda_1(T)] \\
&\quad = E\left[\int_0^T \left\{p_1(t)\left[\frac{\partial b}{\partial x}(t)x_1(t) + \frac{\partial b}{\partial u_1}(t)\beta_1(t)\right]\right.\right. \\
&\qquad + x_1(t)\left(-\frac{\partial H_1}{\partial x}(t)\right) + q_1(t)\left[\frac{\partial\sigma}{\partial x}(t)x_1(t) + \frac{\partial\sigma}{\partial u_1}(t)\beta_1(t)\right] \\
&\qquad \left.\left. + \int_{\mathbb{R}} r_1(t,\zeta)\left[\frac{\partial\gamma}{\partial x}(t,\zeta)x_1(t) + \frac{\partial\gamma}{\partial u_1}(t,\zeta)\beta_1(t,\zeta)\right]\nu(d\zeta)\right\}dt\right] \\
&\quad - E[h_1'(X^{(u_1,u_2)}(T))\lambda_1((T)].
\end{aligned}
\tag{A.10}
$$

By (2.8), (2.12) and the Itô formula

$$
\begin{aligned}
&E[\psi_1'(Y_1(0))y_1(0)] = E[\lambda_1(0)y_1(0)] \\
&\quad = E[\lambda_1(T)y_1(T)] - E\left[\int_0^T \{\lambda_1(t^-)dy_1(t) + y_1(t^-)d\lambda_1(t)\right. \\
&\qquad \left. + \frac{\partial H_1}{\partial z}(t)z_1(t)dt + \int_{\mathbb{R}} \nabla_k H_1(t,\zeta)k_1(t,\zeta)\nu(d\zeta)dt\right] \\
&\quad = E[\lambda_1(T)h_1'(X^{(u_1,u_2)}(T))] \\
&\qquad - E\left[\int_0^T \left\{\lambda_1(t)\left[-\frac{\partial g_1}{\partial x}(t)x_1(t) - \frac{\partial g_1}{\partial y}(t)y_1(t) - \frac{\partial g_1}{\partial z}(t)z_1(t)\right.\right.\right. \\
&\qquad \left. - \int_{\mathbb{R}} \nabla_k g_1(t,\zeta)k_1(t,\zeta)\nu(d\zeta) - \frac{\partial g_1}{\partial u_1}(t)\beta_1(t)\right] \\
&\qquad \left.\left. + \frac{\partial H_1}{\partial y}(t)y_1(t) + \frac{\partial H_1}{\partial z}(t)z_1(t) + \int_{\mathbb{R}} \nabla_k H_1(t,\zeta)k_1(t,\zeta)\nu(d\zeta)\right\}dt\right].
\end{aligned}
\tag{A.11}
$$

Adding (A.10) and (A.11) we get, by (A.9),

$$
\begin{aligned}
D_1 = E\Bigg[\int_0^T \Bigg\{ & \Bigg[\frac{\partial f_1}{\partial x}(t) + p_1(t)\frac{\partial b}{\partial x}(t) + q_1(t)\frac{\partial \sigma}{\partial x}(t) \\
& + \int_{\mathbb{R}} r_1(t,\zeta)\frac{\partial \gamma}{\partial x}(t,\zeta)\nu(d\zeta) - \frac{\partial H_1}{\partial x}(t) + \lambda_1(t)\frac{\partial g_1}{\partial x}(t)\Bigg] x_1(t) \\
& + \left[-\frac{\partial H_1}{\partial y}(t) + \lambda_1(t)\frac{\partial g_1}{\partial y}(t)\right] y_1(t) \\
& + \left[-\frac{\partial H_1}{\partial z}(t) + \lambda_1(t)\frac{\partial g_1}{\partial z}(t)\right] z_1(t) \\
& + \int_{\mathbb{R}} \left[-\nabla_k H_1(t,\zeta) + \lambda_1(t)\nabla_k g_1(t,\zeta)\right] k_1(t,\zeta)\nu(d\zeta) \\
& + \Bigg[\frac{\partial f_1}{\partial u_1}(t) + p_1(t)\frac{\partial b}{\partial u_1}(t) + q_1(t)\frac{\partial \sigma}{\partial u_1}(t) \\
& + \int_{\mathbb{R}} r_1(t,\zeta)\frac{\partial \gamma}{\partial u_1}(t,\zeta)\nu(d\zeta) + \frac{\partial g_1}{\partial u_1}(t)\Bigg] \beta_1(t)\Bigg\} dt\Bigg] \\
= E\Bigg[\int_0^T & \frac{\partial H_1}{\partial u_1}(t)\beta_1(t)dt\Bigg] \\
= E\Bigg[\int_0^T & E\left[\frac{\partial H_1}{\partial u_1}(t)\beta_1(t) \mid \mathcal{E}_t^{(1)}\right] dt\Bigg]. \qquad \text{(A.12)}
\end{aligned}
$$

If $D_1 = 0$ for all bounded $\beta_1 \in \mathcal{A}_1$, then this holds in particular for β_1 of the form in **(a1)**, i.e.

$$\beta_1(t) = \chi_{(t_0,T]}(t)\alpha_1(\omega),$$

where $\alpha_1(\omega)$ is bounded and $\mathcal{E}_{t_0}^{(1)}$-measurable. Hence

$$E\left[\int_{t_0}^T E\left[\frac{\partial H_1}{\partial u_1}(t) \mid \mathcal{E}_t^{(1)}\right] \alpha_1 dt\right] = 0.$$

Differentiating with respect to t_0 we get

$$E\left[\frac{\partial H_1}{\partial u_1}(t_0)\alpha_1\right] = 0 \text{ for a.a. } t_0.$$

Since this holds for all bounded $\mathcal{E}_{t_0}^{(1)}$-measurable random variables α_1 we conclude that

$$E\left[\frac{\partial H_1}{\partial u_1}(t) \mid \mathcal{E}_t^{(1)}\right] = 0 \text{ for a.a. } t \in [0,T].$$

A similar argument gives that

$$E\left[\frac{\partial H_2}{\partial u_2}(t) \mid \mathcal{E}_t^{(2)}\right] = 0$$

provided that

$$D_2 := \frac{d}{ds} J_2(u_1, u_2 + s\beta_2) \mid_{s=0} = 0 \text{ for all bounded } \beta_2 \in \mathcal{A}_2.$$

This shows that (i) $\Rightarrow$ (ii). The argument above can be reversed, to give that (ii) $\Rightarrow$ (i). We omit the details. □

B Linear BSDEs with jumps

Lemma B.1 *[Linear BSDEs with jumps]. Let F be a $\mathcal{F}_T$-measurable and square-integrable random variable. Let β and ξ_0 be bounded predictable processes and ξ_1 a predictable process such that $\xi_1(t,\zeta) \geq C_1$ with $C_1 > -1$ and $|\xi_1(t,\zeta)| \leq C_2(1 \wedge |\zeta|)$ for a constant $C_2 \geq 0$. Let φ be a predictable process such that $E[\int_0^T \varphi^2(t)dt] < \infty$. Then the linear BSDE*

$$\begin{aligned} dY(t) &= -[\varphi(t) + \beta(t)Y(t) + \xi_0(t)Z(t) + \int_{\mathbb{R}} \xi_1(t,\zeta)K(t,\zeta)\nu(d\zeta)]dt \\ &\quad + Z(t)dB(t) + \int_{\mathbb{R}} K(t,\zeta)\tilde{N}(dt,d\zeta); 0 \leq t \leq T \\ Y(T) &= F \end{aligned} \tag{B.1}$$

has the unique solution

$$Y(t) = E[F\ \Upsilon(t,T) + \int_t^T \Upsilon(t,s)\varphi(s)ds \mid \mathcal{F}_t]; \quad 0 \leq t \leq T, \tag{B.2}$$

where $\Upsilon(t,s); 0 \leq t \leq s \leq T$; is defined by

$$\begin{aligned} d\Upsilon(t,s) &= \Upsilon(t,s^-)[\beta(s)ds + \xi_0(s)dB(s) + \int_{\mathbb{R}} \xi_1(s,\zeta)\tilde{N}(ds,d\zeta)];\ t \leq s \leq T \\ \Upsilon(t,t) &= 1 \end{aligned} \tag{B.3}$$

i.e.

$$\begin{aligned} \Upsilon(t,s) &= \exp(\int_t^s \{\beta(u) - \frac{1}{2}\xi_0^2(u)\}du + \int_t^s \xi_0(u)dB(u) \\ &+ \int_t^s \int_{\mathbb{R}} \{\ln(1+\xi_1(u)) - \xi_1(u)\}\nu(d\zeta)du + \int_t^s \int_{\mathbb{R}} \ln(1+\xi_1(u))\tilde{N}(du,d\zeta)). \end{aligned} \tag{B.4}$$

Hence

$$\Upsilon(t,s) = \frac{\Upsilon(0,s)}{\Upsilon(0,t)}, \quad \Upsilon(t,T) = \frac{\Upsilon(0,T)}{\Upsilon(0,t)}.$$

Proof. For completeness we give the proof, but it is also given in [22]. Existence and uniqueness follow by general theorems for BSDEs with Lipschitz coefficients. See e.g. [22]. Hence it only remains to prove that if we define $Y(t)$ to be the solution of (B.1), then (B.2) holds. To this end, define

$$\Upsilon(s) = \Upsilon(0,s).$$

Then by the Itô formula (see e.g. [15], Ch.1)

$$\begin{aligned}
d(\Upsilon(t)Y(t)) =& \Upsilon(t^-)dY(t) + Y(t^-)d\Upsilon(t) + d[\Upsilon Y](t) \\
=& \Upsilon(t^-)[-\{\varphi(t) + \beta(t)Y(t) + \xi_0(t)Z(t) + \int_{\mathbb{R}} \xi_1(t,\zeta)K(t,\zeta)\nu(d\zeta)\}dt + Z(t)dB(t) \\
&+ \int_{\mathbb{R}} K(t,\zeta)\tilde{N}(dt,d\zeta)] + Y(t^-)\Upsilon(t^-)\{\beta(t)dt + \xi_0(t)dB(t) + \int_{\mathbb{R}} \xi_1(t,\zeta)\tilde{N}(dt,d\zeta)\} \\
&+ \Upsilon(t)\xi_0(t)Z(t)dt + \int_{\mathbb{R}} \Upsilon(t^-)\xi_1(t,\zeta)K(t,\zeta)\tilde{N}(dt,d\zeta) \\
=& -\Upsilon(t)\varphi(t)dt + (Z(t) + \xi_0(t)Y(t))\Upsilon(t)dB(t) \\
&+ \int_{\mathbb{R}} \xi_1(t,\zeta))\Upsilon(t^-)(Y(t^-) + K(t,\zeta))\tilde{N}(dt,d\zeta).
\end{aligned}$$

Hence, $\Upsilon(t)Y(t) + \int_0^t \Upsilon(s)\varphi(s)ds$ is a martingale and therefore

$$\Upsilon(t)Y(t) + \int_0^t \Upsilon(s)\varphi(s)ds = E[F\Upsilon(T) + \int_0^T \Upsilon(s)\varphi(s)ds \mid \mathcal{F}_t]$$

or

$$Y(t) = E[F\frac{\Upsilon(T)}{\Upsilon(t)} + \int_t^T \frac{\Upsilon(s)}{\Upsilon(t)}\varphi(s)ds \mid \mathcal{F}_t],$$

as claimed.

□

Acknowledgments. We thank Olivier Menoukeu Pamen and Marie-Claire Quenez for helpful comments.

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