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Abstract. This paper aims at obtaining a 3D fundamental solution for unsaturated soils under dynamic loadings in Laplace transform domain using the method of Hörmander. These solutions can be used, afterwards, in a convolution quadrature method (CQM)-based boundary element formulations in order to model the wave propagation phenomena in such media in time domain.

Introduction. In compacted fills or in arid climate areas where soils are submitted to wetting-drying cycles such as groundwater recharge, surface runoff and evapo-transpiration, fine-grained soils are not saturated with water, and contain some air. Due to capillary effects and soil-clay adsorption, the pore water is no more positive, and is submitted to suction. Wave propagation in unsaturated soils and the dynamic response of such media are of great interest in geophysics, soil and rock mechanics, and many earthquake engineering problems. However, in geomechanics, the behavior of such media including more than two phases is not consistent with the principles and concepts of classic saturated soil mechanics.

From the mechanical point of view, an unsaturated porous medium can be represented as a three-phase (gas, liquid, and solid), or three-component (water, dry air, and solid) system in which two phases can be classified as fluids (i.e. liquid and gas). The liquid phase is considered to be pure water containing dissolved air and the gas phase is assumed to be a binary mixture of water vapor and 'dry' air.

In this paper first of all, a set of fully coupled governing differential equations of hydro-mechanical behaviour of unsaturated porous media including the equilibrium, air and water transfer equations subjected to dynamic loadings is presented based on the suction-based mathematical model presented by [3,4]. In this model, the effect of deformations on the suction distribution in the soil skeleton and the inverse effect are included in the formulation via a suction-dependent formulation of state surfaces of void ratio and degree of saturation. The linear constitutive law is assumed. The mechanical and hydraulic properties of porous media are assumed to be suction dependent. In this formulation, the solid skeleton displacements u_i, water pressure p_w and air pressure p_a are presumed to be independent variables.

Secondly, the associated fundamental solution in Laplace transform domain is presented using the method of Hörmander (1963) [5] for 3D $u_i - p_w - p_a$ formulation of unsaturated porous media. In this case that the fundamental solution is known only in the frequency domain and it seems too difficult to obtain the time-dependent fundamental solution in an explicit analytical form by an inverse transformation of the frequency domain results; the convolution integral in the BIE can be numerically approximated by a new approach called "Operational Quadrature Methods" developed by [1,2]. In this formulation, the convolution integral is numerically approximated by a quadrature formula whose weights are determined by the Laplace transform of the fundamental solution and a linear multistep method [6, 7].

Governing Equations. Governing differential equations consist of mass conservation equations of liquid and gaseous phases, the equilibrium equation of the skeleton associated with water and air flow equations and constitutive relation. The assumption of infinitesimal transformation and incompressibility of solid matrix is considered.

Solid Skeleton. The equilibrium equation and the constitutive law for the soil's solid skeleton including the effect of suction are written [3]:

$$(\sigma_{ij} - \delta_{ij} p_a)_{,j} + p_{a,i} + f_i = \rho \ddot{u}_i \qquad (1)$$

$$(\sigma_{ij} - \delta_{ij} p_a) = (\lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}) - F^s_{ij}(p_a - p_w) \qquad (2)$$

where μ, λ are Lame coefficients, $p_{\alpha=w,a}$ is the water or air pressure, δ_{ij} is the Kronecker delta and F^s_{ij} is the suction modulus matrix:

$$F^s = D.D^{suc-1} \qquad (3)$$

in which D^{suc} is a vector obtained from the state surface of void ratio (e) which is a function of the independent variables of $\sigma - p_a$ and $p_a - p_w$.

$$D^{suc-1} = \partial e / \left((1+e)\partial(p_a - p_w)\right) \qquad (4)$$

The elasticity matrix (D) can be presented by using the bulk modulus and the tangent modulus

$$D = D(K_0, E_t) = D(\sigma - p_a, p_a - p_w) \qquad (5)$$

where E_t is tangent elastic modulus which can be evaluated as

$$E_t = E_l + E_s \qquad (6)$$

in which E_l is the elastic modulus in absence of suction and

$$E_s = m_s(p_a - p_w) \qquad (7)$$

m_s being a constant, E_s represents the effect of suction on the elastic modulus. K_0 is the bulk modulus of an open system and evaluated from the surface state of void ratio

$$K_0^{-1} = (1+e)\partial e / \partial(\sigma - p_a) \qquad (8)$$

Mass Conservation of Water. The mass of the water in a representative elementary volume can be written as

$$\dot{w}^w_{i,i} = -S_w \dot{\varepsilon}_{ii} + C_{ww} \dot{p}_w + C_{wa} \dot{p}_a \qquad (9)$$

where $C_{ww} = (ng_1 - C_w n S_w)$; $C_{wa} = C_{aw} = -ng_1$.

In this equation, $w_i^{\alpha=w,a}$ is the displacement of water or air relative to solid, $S_{\alpha=w,a}$ is the degree of saturation relative to water or air, C_α is the compressibility of water or air $C_{\alpha=w,a} = \mathrm{d}\rho_\alpha / (\rho_\alpha \mathrm{d} p_\alpha)$ and $g_1 = dS_w / d(p_a - p_w)$.

Mass Conservation of Air. With the same approach presented before, the mass conservation of the air can be written as

$$\dot{w}^a_{i,i} = -S_a \dot{\varepsilon}_{ii} + C_{wa} \dot{p}_w + C_{aa} \dot{p}_a \qquad (10)$$

where $C_{aa} = (ng_1 - C_a n S_a)$; $C_{wa} = C_{aw} = -ng_1$.

Flow Equation for the Water. Based on generalized Darcy's law for describing the balance of the forces acting on the liquid phase of the representative elementary volume, the water velocity in the unsaturated soil takes the following form:

$$-p_{w,i} = \rho_w \ddot{\mathbf{u}} + \dot{\mathbf{w}}^{\mathbf{w}} / k_w - \rho_w \mathbf{g} \qquad (11)$$

in which k_w denotes the water permeability in an unsaturated soil.

Flow Equation for the Air. With the same approach presented for the water based on generalized Darcy's law, the air velocity in the unsaturated soil takes the following form:

$$-p_{a,i} = \rho_a \ddot{\mathbf{u}} + \dot{\mathbf{w}}^{\mathbf{a}} / k_a - \rho_a \mathbf{g} \qquad (12)$$

in which k_a denotes the air permeability in an unsaturated soil.

Summary of the Governing Differential Equations in Laplace Domain. By introducing (2) into (1), (11) into (9) and (12) into (10) and by applying the Laplace transform assuming $u_{i(t=0)} = w^w_{i(t=0)} = w^a_{i(t=0)} = 0$ and $p_{w(t=0)} = p_{a(t=0)} = 0$, we obtain the final set of governing equations in Laplace transform domain:

$$(\lambda + \mu)\tilde{u}_{\beta,\alpha\beta} + \tilde{\mu} u_{\alpha,\beta\beta} + F^s \tilde{p}_{w,\alpha} + (1 - F^s)\tilde{p}_{a,\alpha} - \rho.s^2.\tilde{u}_\alpha + \tilde{f}_\alpha = 0 \qquad (13)$$

$$-s.\theta_1.\tilde{u}_{\alpha,\alpha} + k_w \tilde{p}_{w,\alpha\alpha} + C_{ww}.s.\tilde{p}_w + C_{wa}.s.\tilde{p}_a = 0 \qquad (14)$$

$$-s.\theta_2.\tilde{u}_{\alpha,\alpha} + C_{wa}.s.\tilde{p}_w + k_a \tilde{p}_{a,\alpha\alpha} + C_{aa}.s.\tilde{p}_a = 0 \qquad (15)$$

where $\theta_1 = (S_w - \rho_w.k_w.s)$ and $\theta_2 = (S_a - \rho_a.k_a.s)$.

We would like to rewrite compactly the transformed coupled differential equation system Eqs. (13), (14) and (15) into the following matrix form:

$$\tilde{\mathbf{B}}^* \left[\tilde{u}_\alpha, \tilde{p}_w, \tilde{p}_a\right]^T + \left[\tilde{f}_\alpha, 0, 0\right]^T = 0 \tag{16}$$

with the not self-adjoint operator $\tilde{\mathbf{B}}^*$:

$$\tilde{\mathbf{B}}^* = \begin{bmatrix} (\mu\Delta - \rho s^2)\delta_{ij} + (\lambda+\mu)\partial_i\partial_j & F^s\partial_i & (1-F^s)\partial_i \\ -s\theta_1\partial_j & k_w\Delta + C_{ww}s & C_{wa}s \\ -s\theta_2\partial_j & C_{wa}s & k_a\Delta + C_{aa}s \end{bmatrix} \tag{17}$$

In equation (16), the partial derivative $(\ )_{,i}$ is denoted by ∂_i and $\Delta = \partial_{ii}$ is the Laplacian operator. Note the operators $\tilde{\mathbf{B}}^*$ in (17) are not self adjoint. Therefore, for the deduction of fundamental solutions, the adjoint operator to $\tilde{\mathbf{B}}^*$ has to be used:

$$\tilde{\mathbf{B}} = \begin{bmatrix} (\mu\Delta - \rho s^2)\delta_{ij} + (\lambda+\mu)\partial_i\partial_j & s\theta_1\partial_i & s\theta_2\partial_j \\ -F^s\partial_j & k_w\Delta + C_{ww}s & C_{wa}s \\ -(1-F^s)\partial_j & C_{wa}s & k_a\Delta + C_{aa}s \end{bmatrix} \tag{18}$$

Fundamental Solutions. Here, the fundamental solution associated with the operator (18) is derived in the Laplace transform domain. Mathematically spoken a fundamental solution is a solution of the equation $\mathbf{BG} + \mathbf{I}\delta(x-y)\delta(t-\tau) = \mathbf{0}$ where the matrix of fundamental solutions is denoted by $\mathbf{G}$, the identity matrix by $\mathbf{I}$ and the matrix differential operator by $\mathbf{B}$. These solutions can be used in a time-dependent convolution quadrature-based BE formulation which needs only Laplace transform fundamental solutions. In this study, because the operator type of the governing equations is an elliptical operator, the explicit 3D Laplace transform domain fundamental solution can be derived by using the method Hörmander [5]. The idea of this method is to reduce the highly complicated operator given in (18) to simple well known operators. In this method, in the Laplace transform domain, the first stage is to find the matrix of cofactors $\tilde{\mathbf{B}}^{co}$ to calculate the inverse matrix of $\tilde{\mathbf{B}}$ ($\tilde{\mathbf{B}}^{-1} = \tilde{\mathbf{B}}^{co} / \det(\tilde{\mathbf{B}})$). For the second stage, we assume that φ is a scalar solution to the equation

$$\det(\tilde{\mathbf{B}})\mathbf{I}\varphi + \mathbf{I}\delta(x-\xi) = \mathbf{0} \leftrightarrow \tilde{\mathbf{B}}\tilde{\mathbf{B}}^{co}\varphi + \delta(x-\xi) = \mathbf{0} \tag{19}$$

Consequently, we get

$$\tilde{\mathbf{G}} = \tilde{\mathbf{B}}^{co}\varphi \tag{20}$$

Following Hörmander's idea, first, the determinants of the operator $\tilde{\mathbf{B}}^*$ are calculated:

$$\det(\tilde{\mathbf{B}}^*) = \mu^2(\lambda+2\mu)k_w k_a(\Delta-\lambda_1^2)^2(\Delta-\lambda_2^2)(\Delta-\lambda_3^2)(\Delta-\lambda_4^2)$$

(21)

in which the coefficients $\lambda_1^2, \lambda_2^2, \lambda_3^2$ and λ_4^2 are the roots, where one of its roots is the $\lambda_1^2 = \rho s^2/\mu$ which is related to the shear wave velocity propagating through the medium. The remained three roots $\lambda_2^2, \lambda_3^2, \lambda_4^2$ must be determined as these which satisfy

$$\begin{aligned}
\lambda_2^2+\lambda_3^2+\lambda_4^2 &= \frac{\rho s^2 + F^s\rho_w s^2 + \rho_a(1-F^s)s^2}{(\lambda+2\mu)} - \frac{C_{aa}s}{k_a} - \frac{C_{ww}s}{k_w} - \frac{S_w F_s s}{(\lambda+2\mu)k_w} - \frac{S_a(1-F^s)s}{(\lambda+2\mu)k_a} \\
\lambda_2^2\lambda_3^2+\lambda_2^2\lambda_4^2+\lambda_3^2\lambda_4^2 &= \frac{-\rho C_{aa}s^3}{(\lambda+2\mu)k_a} - \frac{\rho C_{ww}s^3}{(\lambda+2\mu)k_w} - \frac{\rho_w(F^sC_{aa}-(1-F^s)C_{wa})s^3}{(\lambda+2\mu)k_a} + \frac{(C_{ww}C_{aa}-C_{wa}^2)s^2}{k_w k_a} \\
&\quad - \frac{\rho_a(-F^sC_{wa}+(1-F^s)C_{ww})s^3}{(\lambda+2\mu)k_w} + \frac{S_w(F^sC_{aa}-C_{wa}(1-F^s))s^2}{(\lambda+2\mu)k_w k_a} + \frac{S_a(-F^sC_{wa}+C_{ww}(1-F^s))s^2}{(\lambda+2\mu)k_w k_a} \\
\lambda_2^2\lambda_3^2\lambda_4^2 &= \frac{\rho(C_{ww}C_{aa}-C_{wa}^2)s^4}{(\lambda+2\mu)k_w k_a}
\end{aligned} \tag{22}$$

These three roots correspond to the three compressional waves which are affected by the degree of saturation and the spatial distribution of fluids within the medium.

Secondly, by introducing the determinant, the scalar equation corresponding to (19) is given by

$$(\Delta-\lambda_1^2)(\Delta-\lambda_2^2)(\Delta-\lambda_3^2)(\Delta-\lambda_4^2)\Phi + \delta(x-\xi) = 0 \tag{23}$$

in which Φ is an interim operator, i.e.

$$\Phi = \mu^2(\lambda+2\mu)k_w k_a(\Delta-\lambda_1^2)\varphi \tag{24}$$

Equation (23) can be expressed as either of four equations (25), (26), (27) and (28):

$$(\Delta-\lambda_1^2)\varphi_1+\delta(x-\xi)=0;\quad \varphi_1=(\Delta-\lambda_2^2)(\Delta-\lambda_3^2)(\Delta-\lambda_4^2)\Phi \tag{25}$$

$$(\Delta-\lambda_2^2)\varphi_2+\delta(x-\xi)=0;\quad \varphi_2=(\Delta-\lambda_1^2)(\Delta-\lambda_3^2)(\Delta-\lambda_4^2)\Phi \tag{26}$$

$$(\Delta-\lambda_3^2)\varphi_3+\delta(x-\xi)=0;\quad \varphi_3=(\Delta-\lambda_1^2)(\Delta-\lambda_2^2)(\Delta-\lambda_4^2)\Phi \tag{27}$$

$$(\Delta-\lambda_4^2)\varphi_4+\delta(x-\xi)=0;\quad \varphi_4=(\Delta-\lambda_1^2)(\Delta-\lambda_2^2)(\Delta-\lambda_3^2)\Phi \tag{28}$$

The above differential equations are of the familiar Helmholtz type. The fundamental solution of Helmholtz differential equations for an only r-dependent fully symmetric two-dimensional domain is

$$\varphi_i=\exp(-\lambda_i r)/4\pi r,\quad i=1,2,3,4 \tag{29}$$

By definition of φ_1, φ_2, φ_3 and φ_4, it is deduced:

$$\Phi=\frac{1}{(\lambda_3^2-\lambda_4^2)(\lambda_2^2-\lambda_1^2)}\left[\frac{\varphi_3-\varphi_2}{\lambda_3^2-\lambda_2^2}-\frac{\varphi_3-\varphi_1}{\lambda_3^2-\lambda_1^2}+\frac{\varphi_4-\varphi_1}{\lambda_4^2-\lambda_1^2}-\frac{\varphi_4-\varphi_2}{\lambda_4^2-\lambda_2^2}\right] \tag{30}$$

Replacing equation (29) into (30), one obtains

$$\varphi=\frac{1}{4\pi r}\left\{\begin{array}{l}\dfrac{\exp(-\lambda_1 r)}{(\lambda_1^2-\lambda_3^2)(\lambda_1^2-\lambda_4^2)(\lambda_1^2-\lambda_2^2)}+\dfrac{\exp(-\lambda_2 r)}{(\lambda_2^2-\lambda_4^2)(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_1^2)}+\\ \dfrac{\exp(-\lambda_3 r)}{(\lambda_3^2-\lambda_2^2)(\lambda_3^2-\lambda_1^2)(\lambda_3^2-\lambda_4^2)}+\dfrac{\exp(-\lambda_4 r)}{(\lambda_4^2-\lambda_1^2)(\lambda_4^2-\lambda_2^2)(\lambda_4^2-\lambda_3^2)}\end{array}\right\} \tag{31}$$

in which the argument $r=|x-\xi|$ denotes the distance between a load point and an observation point.

Finally, we can determine the components of fundamental solution tensor by applying the matrix of cofactors $\tilde{\mathbf{B}}^{*\,\infty}$ to the scalar function φ which are:

Displacement caused by a Dirac force in the solid:

$$\begin{aligned}\tilde{G}_{ij}=\frac{1}{4\pi\mu}\Bigg(&\frac{-(\lambda+\mu)\Lambda^2}{\rho s^2}\frac{(\lambda_1^2-K_{ss1}^2)(\lambda_1^2-K_{ss2}^2)}{(\lambda_1^2-\lambda_3^2)(\lambda_1^2-\lambda_4^2)(\lambda_1^2-\lambda_2^2)}(R_1+R_2\lambda_1+R_3\lambda_1^2)\exp(-\lambda_1 r)+\\ &\frac{-(\lambda+\mu)\Lambda^2}{\rho s^2}\frac{(\lambda_2^2-K_{ss1}^2)(\lambda_2^2-K_{ss2}^2)}{(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)(\lambda_2^2-\lambda_1^2)}(R_1+R_2\lambda_2+R_3\lambda_2^2)\exp(-\lambda_2 r)+\\ &\frac{-(\lambda+\mu)\Lambda^2}{\rho s^2}\frac{(\lambda_3^2-K_{ss1}^2)(\lambda_3^2-K_{ss2}^2)}{(\lambda_3^2-\lambda_2^2)(\lambda_3^2-\lambda_1^2)(\lambda_3^2-\lambda_4^2)}(R_1+R_2\lambda_3+R_3\lambda_3^2)\exp(-\lambda_3 r)+\\ &\frac{-(\lambda+\mu)\Lambda^2}{\rho s^2}\frac{(\lambda_4^2-K_{ss1}^2)(\lambda_4^2-K_{ss2}^2)}{(\lambda_4^2-\lambda_1^2)(\lambda_4^2-\lambda_2^2)(\lambda_4^2-\lambda_3^2)}(R_1+R_2\lambda_4+R_3\lambda_4^2)\exp(-\lambda_4 r)\Bigg)+\frac{\delta_{ij}}{4\pi\mu r}\exp(-\lambda_1 r)\end{aligned} \tag{32a}$$

with $R_1=3x_\alpha x_\beta/r^5-\delta_{\alpha\beta}/r^3$, $R_2=3x_\alpha x_\beta/r^4-\delta_{\alpha\beta}/r^2$, $R_3=x_\alpha x_\beta/r^3$, $\Lambda^2=\rho s^2/(\lambda+2\mu)$ and

$$K_{ss1}^2+K_{ss2}^2=\frac{-S_wF_s s}{(\lambda+\mu)k_w}+\frac{-S_a(1-F^s)s}{(\lambda+\mu)k_a}-\frac{(k_wC_{aa}+C_{ww}k_a)s}{k_wk_a}+\frac{\rho_wk_wF^s s^2}{(\lambda+\mu)k_w}+\frac{\rho_ak_a(1-F^s)s^2}{(\lambda+\mu)k_a}$$

$$K_{ss1}^2K_{ss2}^2=s^2\left(\begin{array}{l}\dfrac{-C_{wa}^2+C_{ww}C_{aa}}{k_wk_a}+\dfrac{S_wF_sC_{aa}}{(\lambda+\mu)k_wk_a}+\dfrac{-S_wC_{wa}(1-F^s)}{(\lambda+\mu)k_wk_a}+\dfrac{S_a\left(-F^sC_{wa}+C_{ww}(1-F^s)\right)}{(\lambda+\mu)k_wk_a}-\dfrac{\rho_wk_wF^sC_{aa}s}{(\lambda+\mu)k_wk_a}+\dfrac{\rho_ak_aF^sC_{wa}s}{(\lambda+\mu)k_wk_a}+\\ \dfrac{\rho_wk_wC_{wa}(1-F^s)s}{(\lambda+\mu)k_wk_a}+\dfrac{-\rho_ak_aC_{ww}(1-F^s)s}{(\lambda+\mu)k_wk_a}\end{array}\right)$$

Water pressure caused by a Dirac force in the solid: (32b)

$$\tilde{G}_{4j}=\frac{-F^sr_{,i}}{4\pi(\lambda+2\mu)k_wr^2}\left(\begin{array}{l}\dfrac{(1+r\lambda_2)\exp(-\lambda_2 r)}{(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)}\left(\lambda_2^2+\dfrac{C_{aa}F^ss-C_{wa}(1-F^s)s}{F^sk_a}\right)+\dfrac{(1+r\lambda_3)\exp(-\lambda_3 r)}{(\lambda_3^2-\lambda_2^2)(\lambda_3^2-\lambda_4^2)}\left(\lambda_3^2+\dfrac{C_{aa}F^ss-C_{wa}(1-F^s)s}{F^sk_a}\right)+\\ \dfrac{(1+r\lambda_4)\exp(-\lambda_4 r)}{(\lambda_4^2-\lambda_2^2)(\lambda_4^2-\lambda_3^2)}\left(\lambda_4^2+\dfrac{C_{aa}F^ss-C_{wa}(1-F^s)s}{F^sk_a}\right)\end{array}\right)$$

Air pressure caused by a Dirac force in the solid: (32c)

$$\tilde{G}_{5j}=\frac{-(1-F^s)r_{,i}}{4\pi(\lambda+2\mu)k_a r^2}\left(\begin{array}{l}\frac{(1+r\lambda_2)\exp(-\lambda_2 r)}{(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)}\left(\lambda_2^2+\frac{C_{ww}(1-F^s)s-C_{wa}F^s s}{(1-F^s)k_w}\right)+\frac{(1+r\lambda_3)\exp(-\lambda_3 r)}{(\lambda_3^2-\lambda_2^2)(\lambda_3^2-\lambda_4^2)}\left(\lambda_3^2+\frac{C_{ww}(1-F^s)s-C_{wa}F^s s}{(1-F^s)k_w}\right)\\+\frac{(1+r\lambda_4)\exp(-\lambda_4 r)}{(\lambda_4^2-\lambda_2^2)(\lambda_4^2-\lambda_3^2)}\left[\lambda_4^2+\frac{C_{ww}(1-F^s)s-C_{wa}F^s s}{(1-F^s)k_w}\right]\end{array}\right)$$

Displacement caused by a Dirac source in the water fluid:

$$\tilde{G}_{i4}=\frac{(S_w-k_w\rho_w s).s.r_{,i}}{4\pi(\lambda+2\mu)k_w r^2}\left(\begin{array}{l}\frac{(1+r\lambda_2)\exp(-\lambda_2 r)}{(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)}\left(\lambda_2^2-K_{\alpha w}\right)+\frac{(1+r\lambda_3)\exp(-\lambda_3 r)}{(\lambda_3^2-\lambda_2^2)(\lambda_3^2-\lambda_4^2)}\left(\lambda_3^2-K_{\alpha w}\right)+\\\frac{(1+r\lambda_4)\exp(-\lambda_4 r)}{(\lambda_4^2-\lambda_2^2)(\lambda_4^2-\lambda_3^2)}\left(\lambda_4^2-K_{\alpha w}\right)\end{array}\right)$$

(32d)

where $K_{\alpha w}=\left(C_{wa}(S_a-\rho_a k_a s)s-C_{aa}(S_w-\rho_w k_w s)s\right)/k_a(S_w-\rho_w k_w s)$.

Displacement caused by a Dirac source in the air fluid:

$$\tilde{G}_{i5}=\frac{(S_a-k_a\rho_a s).s.r_{,i}}{4\pi(\lambda+2\mu)k_a r^2}\left(\begin{array}{l}\frac{(1+r\lambda_2)\exp(-\lambda_2 r)}{(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)}\left(\lambda_2^2-K_{\alpha a}\right)+\frac{(1+r\lambda_3)\exp(-\lambda_3 r)}{(\lambda_3^2-\lambda_2^2)(\lambda_3^2-\lambda_4^2)}\left(\lambda_3^2-K_{\alpha a}\right)\\+\frac{(1+r\lambda_4)\exp(-\lambda_4 r)}{(\lambda_4^2-\lambda_2^2)(\lambda_4^2-\lambda_3^2)}\left(\lambda_4^2-K_{\alpha a}\right)\end{array}\right) \quad (32e)$$

where $K_{\alpha a}=\left(C_{wa}(S_w-\rho_w k_w s)s-C_{ww}(S_a-\rho_a k_a s)s\right)/k_w(S_a-\rho_a k_a s)$.

Water pressure caused by a Dirac source in the water fluid: (32f)

$$\tilde{G}_{44}=\frac{1}{4\pi k_w r}\left(\frac{\exp(-\lambda_2 r)}{(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)}\left(\lambda_2^2-K_w^2\right)\left(\lambda_2^2-\Lambda_w^2\right)+\frac{\exp(-\lambda_3 r)}{(\lambda_3^2-\lambda_2^2)(\lambda_3^2-\lambda_4^2)}\left(\lambda_3^2-K_w^2\right)\left(\lambda_3^2-\Lambda_w^2\right)+\frac{\exp(-\lambda_4 r)}{(\lambda_4^2-\lambda_2^2)(\lambda_4^2-\lambda_3^2)}\left(\lambda_4^2-K_w^2\right)\left(\lambda_4^2-\Lambda_w^2\right)\right)$$

with $K_w^2\Lambda_w^2=-\rho C_{aa}s^3/\left(\lambda+2\mu\right)k_a$ and $K_w^2+\Lambda_w^2=\frac{-S_a\left(1-F_s\right)s}{(\lambda+2\mu)k_a}-\frac{C_{aa}s}{k_a}+\frac{\rho_a k_a\left(1-F_s\right)s^2}{(\lambda+2\mu)k_a}+\frac{\rho s^2}{(\lambda+2\mu)}$.

Air pressure caused by a Dirac source in the air fluid: (32g)

$$\tilde{G}_{55}=\frac{1}{4\pi k_a r}\left(\frac{\exp(-\lambda_2 r)}{(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)}\left(\lambda_2^2-K_a^2\right)\left(\lambda_2^2-\Lambda_a^2\right)+\frac{\exp(-\lambda_3 r)}{(\lambda_3^2-\lambda_2^2)(\lambda_3^2-\lambda_4^2)}\left(\lambda_3^2-K_a^2\right)\left(\lambda_3^2-\Lambda_a^2\right)+\frac{\exp(-\lambda_4 r)}{(\lambda_4^2-\lambda_2^2)(\lambda_4^2-\lambda_3^2)}\left(\lambda_4^2-K_a^2\right)\left(\lambda_4^2-\Lambda_a^2\right)\right)$$

with $K_a^2\Lambda_a^2=-\rho C_{ww}s^3/\left(\lambda+2\mu\right)k_w$ and $K_a^2+\Lambda_a^2=\frac{-S_w F_s s}{(\lambda+2\mu)k_w}-\frac{C_{ww}s}{k_w}+\frac{\rho_w k_w F_s s^2}{(\lambda+2\mu)k_w}+\frac{\rho s^2}{(\lambda+2\mu)}$.

Air pressure caused by a Dirac source in the water fluid: (32h)

$$\tilde{G}_{54}=\frac{s}{4\pi(\lambda+2\mu)k_w k_a r}\left(\begin{array}{l}\frac{\exp(-\lambda_2 r)}{(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)}\left(-\left((\lambda+2\mu)C_{wa}+(S_w-\rho_w k_w s)(1-F^s)\right)\lambda_2^2+\rho C_{wa}s^2\right)+\\\frac{\exp(-\lambda_3 r)}{(\lambda_3^2-\lambda_2^2)(\lambda_3^2-\lambda_4^2)}\left(-\left((\lambda+2\mu)C_{wa}+(S_w-\rho_w k_w s)(1-F^s)\right)\lambda_3^2+\rho C_{wa}s^2\right)\\+\frac{\exp(-\lambda_4 r)}{(\lambda_4^2-\lambda_2^2)(\lambda_4^2-\lambda_3^2)}\left(-\left((\lambda+2\mu)C_{wa}+(S_w-\rho_w k_w s)(1-F^s)\right)\lambda_4^2+\rho C_{wa}s^2\right)\end{array}\right)$$

Water pressure caused by a Dirac source in the air fluid: (32i)

$$\tilde{G}_{45}=\frac{s}{4\pi(\lambda+2\mu)k_wk_ar}\left(\begin{array}{l}\frac{\exp(-\lambda_2 r)}{(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)}\left(-\left((\lambda+2\mu)C_{wa}+(S_a-\rho_ak_as)F^s\right)\lambda_2^2+\rho C_{wa}s^2\right)+\\ \frac{\exp(-\lambda_3 r)}{(\lambda_3^2-\lambda_2^2)(\lambda_3^2-\lambda_4^2)}\left(-\left((\lambda+2\mu)C_{wa}+(S_a-\rho_ak_as)F^s\right)\lambda_3^2+\rho C_{wa}s^2\right)+\end{array}\right.$$
$$\left.\frac{\exp(-\lambda_4 r)}{(\lambda_4^2-\lambda_2^2)(\lambda_4^2-\lambda_3^2)}\left(-\left((\lambda+2\mu)C_{wa}+(S_a-\rho_ak_as)F^s\right)\lambda_4^2+\rho C_{wa}s^2\right)\right)$$

Analytical verification of the fundamental solutions. Limiting Case: Elastodynamic: Having derived the fundamental solution, at this stage, it is of interest to verify the validity of these solutions in the limiting case of elastodynamic. Letting k_w and k_a approach infinity and ρ_w, ρ_a and F_s equal zero, the unsaturated fundamental solutions presented in this study take the form of the elastodynamic fundamental solutions [8]:

$$\tilde{G}_{ij}^*=\tilde{G}_{ij}=\frac{1}{4\pi\rho C_2^2}\left(a\delta_{ij}-b\frac{x_ix_j}{r^2}\right) \tag{33}$$

$$\tilde{G}_{\alpha 4}=\tilde{G}_{\alpha 5}=\tilde{G}_{4j}=\tilde{G}_{5j}=0 \tag{34}$$

where $a=\left(1/r+C_2/s.r^2+C_2^2/s^2.r^3\right)\exp\left(-s.r/C_2\right)-C_2^2\left(C_1/s.r^2+C_1^2/s^2.r^3\right)\exp\left(-s.r/C_1\right)/C_1^2$, $b=\left(1/r+3C_2/s.r^2+3C_2^2/s^2.r^3\right)\exp\left(-s.r/C_2\right)-C_2^2\left(1/r+3C_1/s.r^2+3C_1^2/s^2.r^3\right)\exp\left(-s.r/C_1\right)/C_1^2, C_1^2=\left(\lambda+2\mu\right)/\rho$ and $C_2^2=\mu/\rho$.

Conclusion. In this paper, firstly coupled governing differential equations of a porous medium saturated by two compressible fluids (water and air) subjected to dynamic loadings are presented based on the poromechanics theory in the frame of the suction-based mathematical model presented by Gatmiri [3] and Gatmiri et al. [4]. After that, the associated fundamental solution in Laplace transformed domain is presented by the use of the method of Hörmander for 3D $u_i-p_w-p_a$ formulation of unsaturated porous media. The derived Laplace transform domain fundamental solutions can be directly implemented in time domain BEM in which the convolution integral is numerically approximated by a new approach so-called "Operational Quadrature Methods" developed by Lubich [1, 2] to model the dynamic behaviour of unsaturated porous media. This enables one to develop more effective numerical hybrid BE/FE methods to solve 3D non-linear wave propagation problems in the near future.

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