On-Line Construction of Position Heaps

Gregory Kucherov

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Abstract

We propose a simple linear-time on-line algorithm for constructing a position heap for a string [EMOW11]. Our definition of position heap differs slightly from the one proposed in [EMOW11] in that it considers the suffixes ordered in the descending order of length. Our construction is based on classic suffix pointers and resembles the Ukkonen's algorithm for suffix trees [Ukk95]. Using suffix pointers, the position heap can be extended into the augmented position heap that allows for a linear-time string matching algorithm [EMOW11].

1 Introduction

The theory of string algorithms developed beautiful data structures for string matching and text indexing. Among them, *suffix tree* and *suffix array* are most widely used structures, providing efficient solutions for a wide range of applications [CR94, Gus97]. The DAWG (*Directed Acyclic Word Graph*) [BBH+85], also known as *suffix automaton* [Cro86], is another elegant structure that can be used both as a text index [BBH+85] or as a matching automaton [Cro88, CR94].

Recently, a new *position heap* data structure was proposed [EMOW11]. Similar to the suffix tree, DAWG or suffix array, position heap allows for a pre-processing of a text string in order to efficiently search for patterns in it. As for the above-mentioned data structures, a position heap for a string of length n can be constructed in time $O(n)$. Then all locations of a pattern of length m can be found in time $O(m + occ)$, where occ is the number of occurrences.

The construction algorithm of [EMOW11] processes the string from right to left, like the Weiner's algorithm does for suffix trees [Wei73]. Moreover, the construction requires a so-called dual heap, which is an additional trie on the same set of nodes. The position heap and its dual heap are constructed simultaneously.

To obtain a linear-time pattern matching algorithm of [EMOW11], the position heap should be post-processed in order to add some additional information, resulting in the *augmented position heap*. Most importantly, this information includes so-called *maximal-reach pointers* assigned to certain nodes. Computing these pointers makes use of the dual heap too.

In this paper, we propose a different construction of the position heap. First, we change the definition of the position heap by reversing the order of suffixes and thus allowing for a left-to-right traversal of the input string. The modified definition, however, preserves good properties of the position heap and does not affect the string matching algorithm proposed in [EMOW11]. For this modified definition, we propose an *on-line* algorithm for constructing the position heap. Our algorithm does not use the dual heap, replacing it by classic *suffix pointers* used for constructing suffix trees by the Ukkonen's algorithm [Ukk95] or for constructing the DAWG

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[BBH+85]. Our algorithm is simple and has some similarity with the Ukkonen's algorithm for suffix trees, as opposed to the Weiner's algorithm. We deliberately use some terminology of the Ukkonen's algorithm to underline this similarity.

We further show that the augmented position heap can be easily constructed using suffix pointers. Thus, we completely eliminate the use of the dual heap, replacing it by suffix pointers for constructing both the position heap and its augmented version. Even if this replacement does not provide an immediate improvement in space or running time, we believe that our construction is conceptually simpler and more natural.

Throughout the paper, we assume given a constant-size alphabet A. Positions of strings over A are numbered from 1, that is, a string w of *length* k is $w[1]\dots w[k]$. The length k of w is denoted by $|w|$. $w[i..j]$ denotes substring $w[i]\dots w[j]$.

A *trie* (term attributed to Fredkin [Fre60]) is a simple natural data structure for storing a set of strings. It is a tree with edges labeled by alphabet letters, such that for any internal node, the edges leading to the children nodes are labeled by distinct letters. In this paper, we assume the edges to be directed towards leaves, and call an edge labeled by a letter a an a-edge. A *label* of a node (*path label*) is the string formed by the letters labeling the edges of the path from the root to this node. Given a trie, a string w is said to be represented in the trie if it is a path label of some node. The corresponding node will then be denoted by $\overline{w}$.

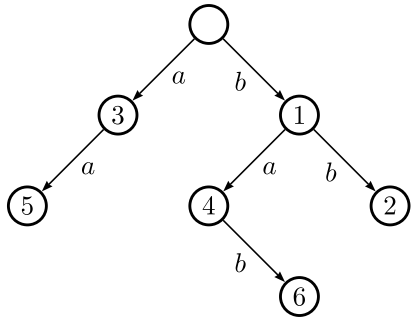
2 Definition of position heap

To define position heaps, we first need to introduce the *sequence hash tree* proposed by Coffman and Eve back in 1970 [CE70] as a data structure for implementing hash tables. Assume we are given an ordered set of strings $W = \{w_1, \dots, w_n\}$ and assume for now that no w_i is a prefix of w_j for any $j < i$. The sequence hash tree for W, denoted $SHT(W)$, is a trie defined by the following iterative construction. We start with the tree $SHT_0(W)$ consisting of a single root node *root*[1]. We then construct $SHT(W)$ by processing strings $w_1, \dots, w_k$ in this order and for each w_i, adding one node to the tree. By induction, assume that $SHT_i(W)$ is the sequence hash tree for $\{w_1, \dots, w_i\}$. To construct $SHT_{i+1}(W)$, we find the shortest prefix v of w_{i+1} which is *not* represented in $SHT_i(W)$. Note that by our assumption, such a prefix always exists. Let $v = v'a$, $a \in A$, i.e. v' is the longest prefix of w_{i+1} represented in $SHT_i(W)$. Then $SHT_{i+1}(W)$ is obtained from $SHT_i(W)$ by adding a new node as a child of v' connected to v' by an a-edge and pointing to w_{i+1}. After inserting all strings of W, we obtain $SHT(W)$, that is $SHT(W) = SHT_k(W)$. Thus, $SHT(W)$ is a trie of $n+1$ nodes such that a node pointing to w_i is labeled by some prefix of w_i. Note that the size of the sequence hash tree depends only on the number of strings in the set and does not depend on the length of these words. An example of sequence hash tree is given on Figure 1.

We now define the *position heap* of a string T. In [EMOW11], the position heap for T is defined as the sequence hash tree for the set of suffixes of T, where *the suffixes are ordered in the ascending order of length*, i.e. from right to left. This insures, in particular, the condition that no suffix is a prefix of a previously inserted suffix, and then no suffix is already represented in the position heap at the time of its insertion.

In this paper, we define the position heap of T to be the sequence hash tree for the set of suffixes of T, where *the suffixes are ordered in the descending order of length*, i.e. from left to right. From now on, we stick to this order. An immediate

[1]This definition agrees with the definition of [CE70] but is slightly different from that of [EMOW11] which defines the root to store w_1. The difference is insignificant, however.



1. *babbaab*
2. *bbab*
3. *ab*
4. *baaa*
5. *aabab*
6. *babaaba*

Figure 1: Sequence hash tree for the set of words shown on the right. Each node stores the rank of the corresponding word in the set.

observation is that the assumption of the suffix hash tree does not hold anymore, and it may occur that an inserted suffix is already represented in the position heap by an existing node. One easy way to cope with this is to systematically assume that T is ended by a special sentinel symbol \$, like it is generally assumed for the suffix tree.

On the other hand, as we will be interested in an on-line construction of the position heap, we will still need to construct the position heap for strings without the ending sentinel symbol. For that, we have to slightly change the definition of sequence hash tree of a set W, by allowing one node to point to several strings of W. The definition of the position heap extends then to any string, with the only difference that inserting a suffix may no longer lead to the creation of a node, but to inserting a new pointer to an existing node. This feature, however, will be used in a very restricted way, as the following observation shows.

Lemma 1 *Let W be a set of distinct strings. Then every node of $SHT(W)$ points to at most two strings of W.*

Proof: Straightforward from the definition of $SHT(W)$ and the fact that all strings are distinct. □

As a consequence of Lemma 1, a position heap contains two types of nodes, pointing respectively to one and two suffixes of T. The former will be called *regular nodes* and the latter *double nodes*. We naturally assume that a pointer to a suffix is simply the starting position of that suffix, therefore regular and double nodes store one and two string positions respectively. Hereafter we interchangeably refer to "suffixes" and "positions" when the underlying string is unambiguously defined.

Figure 2 provides an example of a position heap.

3 Properties of position heap

Denote by $PH(T)$ the position heap for a string $T[1..n]$ constructed as defined in the previous section. In the following theorem, we summarize some key properties of the position heap. Property *(i)* is a straightforward from the definition, and properties *(ii)*-*(iv)* have been established in [EMOW11] but remain valid for our definition of position heap when suffixes are inserted from left to right.

Theorem 1 ([EMOW11]) *Consider $PH(T[1..n])$. The following properties hold.*

(i) A substring $T[i..j]$ is represented in $PH(T)$ iff there exist in T occurrences of strings $T[i]$, $T[i..i+1]$, $T[i..i+2]$, ..., $T[i..j]$, appearing in this order.

(ii) The labels of all nodes of $PH(T)$ form a factorial set. That is, if a string is represented in $PH(T)$, all its substrings are represented too.

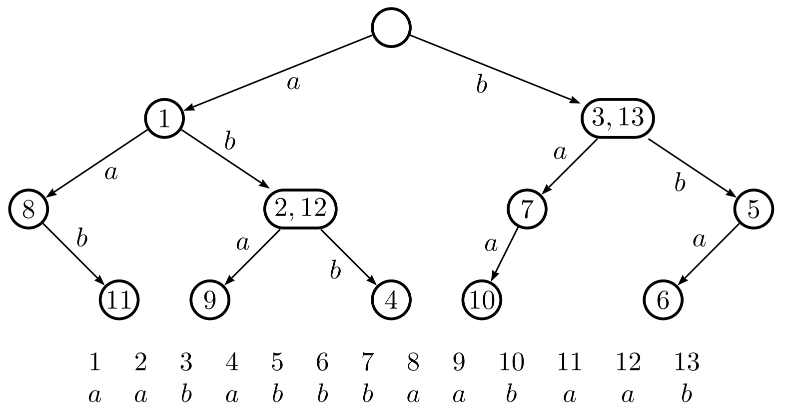


Figure 2: Position heap for string *aababbbaabaab*. Double nodes store pairs of positions.

(iii) The depth of $PH(T)$ is no more than $2h(T)$, where $h(T)$ is the length of the longest substring w of T which occurs $|w|$ times in T (possibly with overlap).

(iv) If a string w occurs in T at least $|w|$ times, then w is represented in $PH(T)$. Inversely, if w is not represented in $PH(T)$ and w' is the longest prefix of w which is represented, then w cannot occur in T more than $|w'|$ times.

Properties *(iii)* and *(iv)* show that the position heap of a string "adapts" to frequencies of its substrings. In particular, if a string is "frequent" (occurs as many times as it is long), then it is necessarily represented in the position heap. On the other hand, if it is not represented, it has less occurrences than its length. The latter property is crucial for obtaining a linear-time string matching algorithm of [EMOW11].

4 On-line construction algorithm

Let us have a closer look at the properties of double nodes of a position heap $PH(T)$. Each such node stores two positions i, j of T. Assume $i < j$, then positions i and j will be called the *primary* and the *secondary* positions respectively.

Lemma 2 *If j is the secondary position of some node of a position heap, then so is $j + 1$.*

Proof: Consider $PH(T)$ for some string $T[1..n]$. Assume i, j, $i < j$, are respectively primary and secondary positions of some node. This means that by the time the suffix $T[j..n]$ is inserted into $PH(T)$ during its construction, node $\overline{T[j..n]}$ already exists. By Theorem 1*(ii)*, node $\overline{T[j+1..n]}$ exists too. *A fortiori*, node $\overline{T[j+1..n]}$ exists when $T[j+1..n]$ is inserted into $PH(T)$. Therefore, $j + 1$ becomes the secondary position of that node after the insertion of suffix $T[j + 1..n]$. □

Lemma 2 implies that all positions of $T[1..n]$ are split into two intervals: primary positions $[1..s - 1]$, for some position s, and secondary positions $[s..n]$. Position s will be called *active secondary position*, or *active position* for short.

Assume we have constructed the position heap $PH(T[1..k])$ for some prefix $T[1..k]$ of the input string $T[1..n]$. Let us analyze the differences between $PH(T[1..k])$

and $PH(T[1..k+1])$ and the modifications that need to be made to transform the former into the latter.

Let s be the active position of $T[1..k]$. First observe that for suffixes $1,\dots,s-1$, no changes need to be made. Inserting each suffix $T[i..k]$ for $1 \le i \le s-1$ into $PH(T[1..k])$ led to the creation of a new node. This means that by the time this suffix was inserted into $PH(T[1..k])$, some prefix $T[i..\ell]$ of $T[i..k]$, $\ell \le k$, was not represented in the position heap, which led to the creation of a new node $\overline{T[i..\ell]}$ with the minimal such ℓ. This shows that inserting suffixes $1,\dots,s-1$ involve completely identical steps in the construction of both $PH(T[1..k])$ and $PH(T[1..k+1])$.

The situation is different for the secondary positions $s,\dots,k$. Each suffix $T[i..k]$ for $s \le i \le k$ was already represented in $PH(T[1..k])$ at the moment of its insertion, and then resulted in the addition of the secondary position i to the node $\overline{T[i..k]}$. When inserting the corresponding suffix $T[i..k+1]$ into the position heap $PH(T[1..k+1])$, two cases arise. In the *first case*, inserting the suffix $T[i..k+1]$ leads to the creation of the new node $\overline{T[i..k+1]}$ if this node does not exist yet. Position i then becomes the primary position of this new node. Observe that this only occurs when $PH(T[1..k])$ does not contain an $T[k+1]$-edge outgoing from the node $\overline{T[i..k]}$. It is easily seen that such an edge cannot appear by the time of insertion of $T[i..k+1]$ into $PH(T[1..k+1])$ if it is not already present in $PH(T[1..k])$. In the *second case*, node $\overline{T[i..k]}$ has an outgoing $T[k+1]$-edge in $PH(T[1..k])$, and in the construction of $PH(T[1..k+1])$, the secondary position i stored in this node should be "moved" to the child node $\overline{T[i..k+1]}$. It becomes then the secondary position of this node.

Observe now that if for a secondary position i, the corresponding node $\overline{T[i..k]}$ has an outgoing $T[k+1]$-edge, then so does the node $\overline{T[i+1..k]}$ storing the secondary position $i+1$. This can again be seen from the factorial property of the position heap (Theorem 1*(ii)*). This shows that the above two cases split the interval of secondary positions $[s..k]$ into two subintervals $[s..t-1]$ and $[t..k]$, such that node $\overline{T[i..k]}$ does not have an outgoing $T[k+1]$-edge for $i \in [s..t-1]$ and does have such an edge for $i \in [t..k]$.

The above discussion is summarized in the following lemma specifying the changes that have to be made to transform $PH(T[1..k])$ into $PH(T[1..k+1])$.

Lemma 3 *Given $T[1..n]$, consider $PH(T[1..k])$ for $k < n$. Let s be the active secondary position, stored in the node $\overline{T[s..k]}$. Let $t \ge s$ be the smallest position such that node $\overline{T[t..k]}$ has an outgoing $T[k+1]$-transition. To obtain $PH(T[1..k+1])$, $PH(T[1..k])$ should be modified in the following way:*

(i) for every node $\overline{T[i..k]}$, $s \le i \le t-1$, create a new child linked to $\overline{T[i..k]}$ by a $T[k+1]$-edge. Delete secondary position i from the node $\overline{T[i..k]}$ and assign it as a primary position to the new node $\overline{T[i..k+1]}$,

(ii) for every node $\overline{T[i..k]}$, $i \ge t$, move the secondary position i from node $\overline{T[i..k]}$ to node $\overline{T[i..k+1]}$.

We describe now the algorithm implementing the changes specified by Lemma 3. We augment $PH(T)$ with *suffix pointers* f defined in the usual way:

Definition 1 *For each node $\overline{T[i..j]}$ of $PH(T)$, a* suffix pointer *is defined by $f(\overline{T[i..j]}) = \overline{T[i+1..j]}$.*

Note that the definition is sound, as the node $\overline{T[i+1..j]}$ exists whenever the node $\overline{T[i..j]}$ exists, according to Theorem 1*(ii)*. For the root node, it will be convenient for us to define $f(root) = \perp$, where $\perp$ is a special node such that there is an a-edge between $\perp$ and *root* for every $a \in A$ (cf [Ukk95]). Figure 3 shows the position heap of Figure 2 supplemented by suffix pointers.

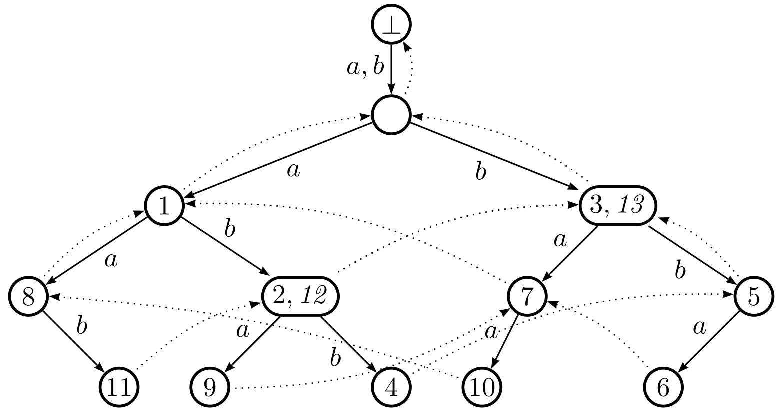


Figure 3: Position heap for string *aababbbaabaab* with suffix pointers (dotted arrows). Secondary positions are shown in italic.

We now begin to describe the on-line construction algorithm for $PH(T)$, given a text $T[1..n]$. Consider the node $\overline{T[s..k]}$ of $PH(T[1..k])$ storing the active secondary position s, that we call the *active node*. If the active secondary position does not exist (i.e. there is no secondary positions at all), then the active node is *root* and the active position is set to $k+1$. Observe that the nodes storing the other secondary positions $s+1, s+2, \ldots, n$ can be reached, in order, by following the chain of suffix pointers $f(\overline{T[s..n]}), f(f(\overline{T[s..n]})), \ldots$ until the root node is reached. Figure 3 provides an illustration.

This leads us to the main trick of our construction: *we will not store secondary positions at all, but only memorize the active secondary position and the active node*. The secondary positions can be easily recovered by traversing the chain of suffix pointers starting from the active node and incrementing the position counter after traversing each edge. Note also that if the input string T is ended by a unique sentinel symbol, the resulting position heap does not contain any secondary nodes and there is no need to recover them.

Having in mind that the secondary positions are not stored explicitly, the transformation of $PH(T[1..k])$ into $PH(T[1..k+1])$ specified by Lemma 3 is done by the following simple procedure. Starting from the active node, the algorithm traverses the chain of suffix pointers as long as the current node does not have an outgoing $T[k+1]$-edge. For each such node, a new node is created linked by a $T[k+1]$-edge to the current node. A suffix pointer to this new node is set from the previously created new node. Once the first node with an outgoing $T[k+1]$-edge is encountered, the algorithm moves to the node this edge leads to, sets the suffix pointer to this node, and assigns this node to be the active node for the following iteration. The correctness of the last assignment is stated in the following lemma.

Lemma 4 *Consider $PH(T[1..k])$ and let s be the active position, and $t \geq s$ be the smallest position such that node $\overline{T[t..k]}$ has an outgoing $T[k+1]$-edge. Then node $\overline{T[t..k+1]}$ is the active node of $PH(T[1..k+1])$.*

Proof: As it follows from Lemma 3, t is the largest secondary position of $T[1..k+1]$. □

Algorithm 1 provides a pseudo-code of the algorithm.

The correctness of Algorithm 1 follows from Lemmas 3, 4 and the discussion above. It is instructive, in addition, to observe the following:

Algorithm 1 On-line construction of the position heap $PH(T[1..n])$

```
create states root and ⊥
f(root) ←⊥
for all a ∈ A do
  set an a-edge from ⊥ to root
end for
currentnode ← root
currentsuffix ← 1
for i = 1 to n do
  lastcreatednode ← undefined
  while currentnode does not have an outgoing T[i]-edge do
    create a new node newnode pointing to currentsuffix
    set a T[i]-edge from currentnode to newnode
    if lastcreatednode ≠ undefined then
      f(lastcreatednode) ← newnode
    end if
    lastcreatednode ← newnode
    currentnode ← f(currentnode)
    currentsuffix ← currentsuffix + 1
  end while
  move currentnode to the target node of the outgoing T[i]-edge
  if lastcreatednode ≠ undefined then
    f(lastcreatednode) ← currentnode
  end if
end for
```

- it is easily seen that the suffix pointers of $T[1..k+1]$ are correctly updated. Indeed, the algorithm assigns to $\overline{T[i..k+1]}$ a suffix pointer to $\overline{T[i+1..k+1]}$ which is obviously correct. Note that for the active position s of $T[1..k]$, the created node $\overline{T[s..k+1]}$ does not get pointed to by any suffix pointer, which is correct, as $T[s-1..k+1]$ is not represented in $PH(T[1..k+1])$: the position $s-1$ is primary in $T[1..k]$ and therefore the node $\overline{T[s-1..k]}$, if it exists in $PH(T[1..k])$, does not get extended by a $T[k+1]$-edge (cf Lemma 3).

- since the depth of $\overline{T[s..k]}$ (s is the active position) in $PH(T[1..k])$ is $k+1-s$ and a traversal of a suffix link decrements the depth by 1 and increments the current position by 1, it follows that if the traversal of the suffix chain reaches the root node, the active position value becomes $k+1$, which is exactly what we need to start processing the next letter $T[k+1]$. This shows why Algorithm 1 correctly maintains *currentsuffix* and never needs to reset it at the beginning of the **for**-loop iteration.

It is easy to see that the running time of Algorithm 1 is linear in the length n of the input string. Since each iteration of the **while**-loop creates a node, this loop iterates exactly n times over the whole run of the algorithm. Trivially, the **for**-loop iterates n times too, and all the involved operations are constant time. Thus, the whole algorithm takes $O(n)$ time. The following theorem concludes the construction.

Theorem 2 *For an input string $T[1..n]$, Algorithm 1 correctly constructs $PH(T)$ on-line in time $O(n)$.*

5 Augmented position heap

Assume we have a text $T[1..n]$ for which we constructed the position heap $PH(T)$. We don't assume that T is ended by a unique letter, and therefore some nodes of $PH(T)$ are double nodes and store two positions of T, one primary and one secondary. Here we assume that the secondary positions *are* actually stored (or can be retrieved in constant time for each node). As explained in Section 4, even if the secondary positions are not stored during the construction of $PH(T)$, they can be easily recovered once the construction is completed.

[EMOW11] proposed a linear-time string matching algorithm using $PH(T[1..n])$, i.e. an algorithm that computes all occurrences of a pattern string in T in time $O(m + occ)$, where m is the pattern length and occ the number of occurrences. Describing this elegant algorithm is beyond the scope of this paper, we refer the reader to [EMOW11] for its description. Note that the algorithm itself applies without changes to our definition of position heap, as it does not depend in any way on the order that the suffixes of T are inserted.

However, the algorithm of [EMOW11] runs on $PH(T)$ enriched with some additional information. Let $\bar{i}$ denote the node of $PH(T)$ storing position i, $1 \leq i \leq n$. The extended data structure, called the *augmented position heap*, should allow the following queries to be answered in constant time:

- given a position i, retrieve the node $\bar{i}$,
- given two nodes $\bar{i}$ and $\bar{j}$, is $\bar{i}$ a (not necessarily immediate) ancestor of $\bar{j}$?
- given a position i of T, retrieve the node $\overline{T[i..i+\ell]}$, where $T[i..i+\ell]$ is the longest substring of T starting at position i and represented in $PH(T)$.

To answer the first query, [EMOW11] simply introduces an auxiliary array storing, for each position i, a pointer to the node $\bar{i}$. Maintaining this array during the construction of $PH(T)$ by Algorithm 1 is trivial: once a position is assigned to a newly created node (line 11 of Algorithm 1), a new entry of the array is created. If T is not ended by a unique symbol and then the final $PH(T)$ has secondary positions, those are easily recovered by traversing the chain of suffix pointers at the very end of the construction.

The second query can be also easily answered in constant time after a linear-time preprocessing of $PH(T)$. A solution proposed in [EMOW11] consists in traversing $PH(T)$ depth-first and storing, for each node, its discovery and finishing times [CLR99]. Then node $\bar{i}$ is an ancestor of node $\bar{j}$ if and only if the discovery and finishing time of $\bar{i}$ is respectively smaller and greater than the discovery and finishing time of $\bar{j}$.

A more space-efficient solution would be to use a balanced parenthesis representation of the tree topology of $PH(T)$, taking $2n$ bits, and link each node to the corresponding opening parenthesis. Then the corresponding closing parenthesis can be retrieved in constant time by the method of [MR01] using $o(n)$ auxiliary bits. This allows ancestor queries to be answered in constant time.

The third type of queries is answered by a precomputed function, called *maximal-reach pointer* [EMOW11]: for a position i of $T[1..n]$, define $mrp(i)$ to be the node $\overline{T[i..i+\ell]}$, where $T[i..i+\ell]$ is the longest prefix of $T[i..n]$ represented in $PH(T)$. Observe first that if i is a secondary position, then $mrp(i) = \bar{i}$. This is because a secondary position i is stored in node $\overline{T[i..n]}$, which trivially corresponds to the longest prefix starting at i. Therefore, as it is done in [EMOW11], mrp can be represented by pointers from node $\bar{i}$ to node $mrp(i)$ whenever these nodes are different. In our case, we have then to keep in mind that a maximal-reach pointer from a double node applies to the primary position of this node. Figure 4 provides an illustration.

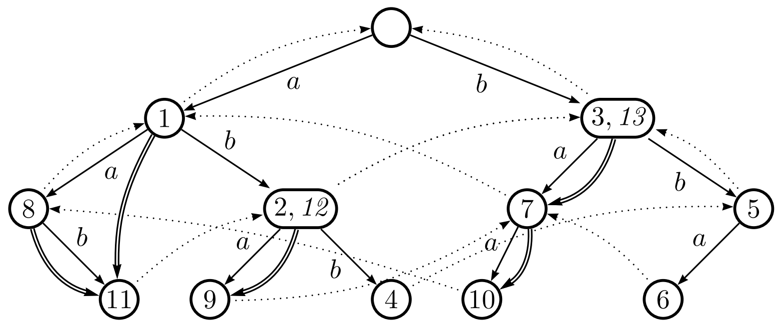


Figure 4: Position heap for string *aababbbaabaab* with suffix pointers and maximal-reach pointers *mrp* (double arrows). Only values for which $mrp(i) \neq \overline{i}$ are shown, namely $mrp(1) = \overline{11}$, $mrp(8) = \overline{11}$, $mrp(2) = \overline{9}$, $mrp(3) = \overline{7}$, $mrp(7) = \overline{10}$. Note that maximal reach pointers outgoing from double nodes are unambiguous as for all secondary positions i, we have $mrp(i) = \overline{i}$.

In [EMOW11], maximal-reach pointers are computed by an extra traversal of $PH(T)$, using an auxiliary *dual heap* structure on top of it (see Introduction). Here we show that maximal-reach pointers can be easily computed using suffix pointers instead of the dual heap. Thus, we completely get rid of the dual heap for constructing the augmented position heap, replacing it with suffix pointers.

We compute $mrp(i)$ iteratively for $i = 1, 2, \ldots, s-1$, where s is the active secondary position of $T[1..n]$. Assume we have computed $mrp(i)$ for some i and have to compute $mrp(i+1)$. Assume $mrp(i) = \overline{T[i..i+\ell]}$. It is easily seen that $T[i+1..i+\ell]$ is a prefix of the string represented by $mrp(i+1)$. To compute $mrp(k+1)$, we follow the suffix link $f(mrp(k))$ to reach $\overline{T[i+1..i+\ell]}$ and then keep extending the prefix $T[i+1..i+\ell]$ as long as it is represented in $PH(T)$. The resulting pseudo-code is given in Algorithm 2.

Algorithm 2 Linear-time computation of maximal-reach pointers $mrp(i)$

```
1: currentnode ← root
2: readhead ← 1
3: for i = 1 to n do
4:    while currentnode has an outgoing T[readhead]-edge and readhead ≤ n do
5:       move currentnode to the target node of the outgoing T[readhead]-edge
6:       readhead ← readhead + 1
7:    end while
8:    mrp(i) ← currentnode
9:    currentnode ← f(currentnode)
10: end for
```

It is very easy to see that Algorithm 2 works in time $O(n)$: the **while**-loop makes exactly n iterations overall, as each iteration increments the *readhead* counter.

The following property of Algorithm 2 is useful to observe: as soon as *readhead* gets the value $n+1$ (line 6), the node *currentnode* gets assigned to the active node of $PH(T[1..n])$ (line 9); at the subsequent iterations, the algorithm simply traverses the chain of suffix links and sets the maximal-reach pointer for each secondary position to be the node storing this position (lines 8-9).

6 Concluding remarks

We proposed a construction algorithm of a position heap of a string, under a modified definition of position heap compared to [EMOW11]. In contrast with the algorithm of [EMOW11] that processes the text right-to-left, our algorithm reads the string left-to-right and has the on-line property. Drawing a parallel to suffix trees, our algorithm can be compared to the Ukkonen's on-line algorithm [Ukk95], while the algorithm of [EMOW11] can be compared to the Weiner's algorithm [Wei73]. The similarity of our algorithm to the Ukkonen's algorithm goes beyond this parallel, as the two algorithms are also somewhat analogous in their design.

The $O(n)$ complexity bounds of both Algorithm 1 (Theorem 2) and Algorithm 2 are stated for a constant-size alphabet, otherwise a correcting factor $\log|A|$ should be introduced, similarly to the suffix tree construction.

The position heap is a smaller data structure than the suffix tree: it contains exactly $n + 1$ nodes whereas the suffix tree has n leaves and then up to $2n$ nodes. Still, the position heap allows for a linear-time string matching. The authors of [EMOW11] proposed algorithms for updating the position heap when the input string undergoes modifications (character insertions/deletions). These algorithms can be easily applied to our definition of position heap. Other interesting applications of position heap are still to be discovered.

It would be interesting to study further the properties of maximal-reach pointers. Note that their structure differs between our definition of position heap and the definition of [EMOW11]. It would be also interesting to exploit the "adaptiveness" of position heaps to substring frequencies, mentioned in Section 3.

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