An Exposition of Discrete Morse Theory and Applications

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ABSTRACT

An Exposition of Discrete Morse Theory and Applications

Lingfeng Lu

The classical Morse theory is a powerful tool to study topological properties of a smooth manifold by examining critical points of some differentiable functions on that manifold. Robin Forman developed a discrete variant of Morse theory by adapting it on abstract simplicial complexes that resulted in a new theory with wide applications in other fields of mathematics, computer science, data science, and others. In this thesis, we present Forman's construction of discrete Morse theory, as well as its main theorems such as the Collapse theorem, discrete Morse inequalities, the theorem for cancelling critical simplices, and some additional topics. We also discuss some applications of discrete Morse theory with a major focus on the concept of persistence in topological data analysis. While many results exist in the literature, we wrote our own proofs, added more details and explanations, and adapted it to our own settings with a strong topological flavor whenever possible.

Acknowledgments

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I want to thank my parents, Chunlei and Xinyu, for everything they have ever given me, especially their support for my decision of pursuing graduate study in pure mathematics. I can never repay them enough, so the best I can do is to make them proud.

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I also want to thank my closest friends, Lingxin Zhao, Ruize Xu, Jialin Zhang, William Zhang, Zean Tang, for their continuing support and companionship.

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Chapter 1

Introduction

The classical Morse theory, named after Marston Morse ([8]), was developed to recover and analyze the topology of a manifold by studying critical points of differentiable functions on that manifold. Many fundamental results on smooth manifolds were later proven and summarized by John Milnor ([28]). Our main focus of this thesis, the discrete Morse theory, is a combinatorial equivalent of the classical Morse theory developed by Robin Forman ([15]) by adapting some core concepts of the classical Morse theory and applying them on complexes. It has wide applications throughout different areas such as geometry, computer science ([18]) and data analysis ([30]). When my supervisor Dr. Stancu first introduced me to this topic, it immediately caught my attention. I have very strong interest in geometry and topology, and combinatorics was my favorite field in discrete mathematics. I thought this could be a great opportunity for me to explore a relatively new theory while pushing forward my study to the next level.

This thesis serves as an exposition of the discrete Morse theory. We follow Forman's construction ([15]) and Scoville's recently-published book ([34]) on the theory and present a variety of results. In Chapter 2, we review mathematical concepts in different fields which are needed for later chapters. In Chapter 3, we discuss the main object on which the theory is built on, abstract simplicial complexes, following by a special type of homotopy and then basics of simplical homology. Chapter 4 and 5 are the main bodies of this thesis. In Chapter 4, we begin by discussing some fundamental results in classical Morse theory, of which we also discuss their discrete counterparts. Then we present Forman's construction of the discrete Morse theory with topics such as discrete Morse functions and (discrete) gradient vector fields. The chapter ends with a section dedicated to some main theorems in the discrete Morse theory as well as their proofs. Chapter 5 discusses some additional constructions and results that revolve around the discrete Morse theory. In Chapter 6, we take a detour and present a relatively new concept called persistence, and draw connections between it and the discrete Morse theory. We finish the thesis with Chapter 7, mentioning a few interesting topics that are closely related to what we have discussed earlier.

Chapter 2

Preliminaries

In this chapter, we review some mathematical concepts that will be appearing throughout the thesis. We will not give definition to some terms that only come up occasionally, but readers should be very familiar with most of them.

2.1 Algebra

2.1.1 Vector Spaces

A vector space over a field F is a set V together with two operations, addition and multiplication by scalars in F, that satisfies certain axioms. We assume familiarity with basic definitions and notations.

Definition 2.1. Let V, W be vector spaces over some field F. A map $f : V \to W$ is said to be a **linear transformation** or **vector space homomorphism** if for any $u, v \in V$, and any $c \in F$, we have

$$f(u+v) = f(u) + f(v)$$

$$f(cu) = cf(u).$$

If f is bijection, then it is called a **(linear) isomorphism**.

Definition 2.2. If f is a linear transformation, the **kernel** of f is defined as

$$Ker(f) = \{v \in V : f(v) = \mathbf{0}\},$$

and the **image** of f is defined as

$$Im(f) = \{w \in W : w = f(v), v \in V\}.$$

The dimension of $Ker(f)$ is called the **nullity** of f, which is denoted $\mathrm{null}(f)$, and the dimension of $Im(f)$ is called the **rank** of f, which is denoted $\mathrm{rank}(f)$.

The following theorem is a fundamental and very useful result concerning the rank and the nullity of a linear transformation.

Theorem 2.1.1 (Rank-nulity theorem)**.** *Let V, W be vector spaces over a field F, and let $f : V \to W$ be a linear transformation. Then*

$$rank(f) + null(f) = dim(V).$$

2.1.2 Groups

A group is a non-empty set of elements together with an associative binary operation under which the set is closed, and such that there exists an identity element and every element has an inverse. A semi-group will have the same properties except the existence of an inverse for each element. Again, we assume familiarity with basic definitions and notations, for which readers may refer to [19].

Definition 2.3. A group G is said to be **abelian** if $ab = ba$ for all $a, b \in G$.

Remark. A vector space is an abelian group with respect to the first operation, the addition, a semi-group with respect to the second operation, the multiplication, with certain natural distributive properties between the two operations as for example satisfied by $\mathbb{R}$ with the usual operations.

Definition 2.4. A subset H of a group G is said to be a **subgroup** of G if it is a group itself under the same operation of G.

The next well-known theorem suggests that subgroups can be formed by intersecting existing subgroups. We will omit the proof as it's very straightforward.

Theorem 2.1.2. *Let G be a group. Then the intersection of any two subgroups of G is again a subgroup of G.*

Definition 2.5. Let $G, \bar{G}$ be groups. A map $f : G \to \bar{G}$ is a **(group) homomorphism** if for all $a, b \in G$,

$$f(ab) = f(a)f(b),$$

that is f preserves the group operation denoted here multiplicatively. A homomorphism from a group to itself is called an **automorphism**. A bijective homomorphism is called a **(group) isomorphism**.

The following is a fundamental property of homomorphisms:

Lemma 2.1.3. *A homomorphism $f : G \to \bar{G}$ maps the identity of G to the identity of $\bar{G}$.*

Remark. The definitions for kernel and rank of a group homomorphism are almost identical to those of a linear transformation, so we will not repeat them here. Throughout the paper, we will only use the term homomorphism in the context of groups, but we will use homomorphism and linear transformation interchangeably in the context of vector spaces.

We often come across problems in which one needs to show that a certain homomorphism is an isomorphism, that is to show that it is injective and surjective. A common method to show that a homomorphism is injective is through studying its kernel. This method, of course, can also be used for the same purpose for linear transformations.

Theorem 2.1.4. *A homomorphism $f : G \to \bar{G}$ is injective if and only if its kernel is trivial, that is $Ker(f) = \{e\}$, where e is the identity element of G.*

Proof. The forward direction follows from the fact that a homomorphism maps the identity to the identity. For the backward direction, let $a, b \in \bar{G}$ and suppose that $f(a) = f(b)$. Then, by Lemma 2.1.3,

$$\bar{e} = f(a)f(a)^{-1} = f(a)f(b)^{-1} = f(a)f(b^{-1}) = f(ab^{-1}).$$

Thus $ab^{-1} = e$, i.e. $a = b$.

□

Definition 2.6. Let G be a group. A subgroup H of G is called a **normal subgroup** if $aH = Ha$ for all $a \in G$, where $aH = \{ah : h \in H\}$ and $Ha = \{ha : h \in H\}$.

Corollary 2.1.5. *Every subgroup of an abelian group is normal.*

Proof. This immediately follows from the definition of an abelian group.

□

Definition 2.7. Let G be a group and H be a normal subgroup of G. The set $G/H = \{aH : a \in G\}$ is a group under the operation $(aH)(bH) = (ab)H$ and is called the **quotient group of G by H**.

One way to interpret G/H is that it is the group formed by (classes of) elements of G with H becoming the identity: for any $a \in G$, $(aH)H = (aH)(eH) = (ae)H = aH$ and $H(aH) = (eH)(aH) = (ea)H = aH$. Two elements of G determine, or are in, the same class if $aH = bH$ as sets.

2.2 Topology

We will primarily use [21] and [31] as references to establish a couple of important topological relations. Some familiarity with topological spaces is assumed.

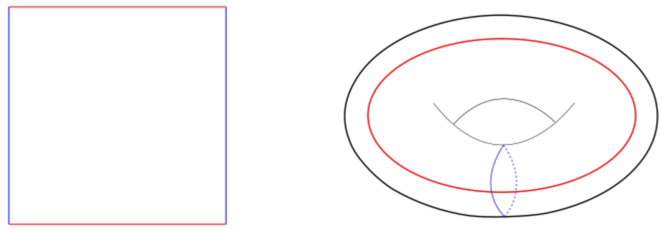

Figure 2.1: Identifying opposite sides of a square to obtain a torus

2.2.1 Homeomorphism

Definition 2.8. Let X and Y be topological spaces. A **homeomorphism** between X and Y is a continuous function $f : X \to Y$ whose inverse is also continuous. If such function exists, we say that X is **homeomorphic** to Y.

In other words, homeomorphism is the topological variant of isomorphism, preserving all the topological properties between spaces.

Definition 2.9. Let X be a topological space and let $\sim$ be an equivalence relation on X. The **quotient set** $X^* = X/\sim$ is the collection of equivalence class $[x]$ of $x \in X$. The **quotient space** under $\sim$ is the quotient set X^* equipped with the **quotient topology**, which is a collection of subsets of X^* whose pre-images are open under the surjective map $x \to [x]$.

Example 2.2.1. By identifying opposite sides of a square, we obtain a quotient space that is homeomorphic to the torus. A visualization is given in Figure 2.1.

2.2.2 Homotopy

Another topological equivalence relation is *homotopy equivalence*. It is broader (thus weaker) than homeomorphism, but still carries many important topological invariants.

Definition 2.10. Let f and f' be continuous maps from X to Y. We say that f is **homotopic** to f', denoted $f \simeq f'$, if there is a continuous map $F : X \times [0,1] \to Y$ such that

$$F[x,0] = f(x) \text{ and } F[x,1] = f'(x)$$

for each $x \in X$. The map F is called the **homotopy** between f and f'.

Definition 2.11. A **homotopy equivalence** from X to Y is a continuous map $f : X \to Y$ such that there is a continuous map $g : Y \to X$ and

$$g \circ f \simeq \mathbb{1}_X,$$
$$f \circ g \simeq \mathbb{1}_Y.$$

If such f exists, we say that X and Y have the same **homotopy type**, also denoted as $X \simeq Y$.

One of the methods for detecting homotopy equivalence is through a special case called *deformation retraction*. One can think of deformation retraction as a shrinking or expanding process during a unit time interval. Formally, we have the definition of deformation retraction as follows.

Definition 2.12. A **deformation retraction** of a space X onto a subspace A is a parametrized family of maps $f_t : X \to X$, $t \in [0,1]$, such that $f_0 = \mathbb{1}_X$, $f_1(X) = A$ and $f_t|_A = \mathbb{1}_A$ for all t. We also say A is a **deformation retract** of X.

Theorem 2.2.2. *If A is a deformation retract of X, then A has the same homotopy type as X.*

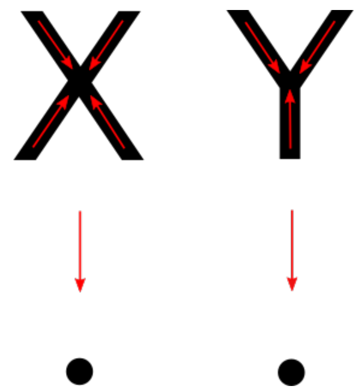

Figure 2.2: Both X and Y deformations retract to a point

Example 2.2.3. Consider placing letters X and Y on a plane, so that they can be viewed as subspaces of $\mathbb{R}^2$. Then both X and Y can deformation retract to their center points as shown in Figure 2.2, so they both have the homotopy type of a point. Since homotopy equivalence is an equivalence relation, we can conclude that X and Y have the same homotopy type.

2.3 Order and Relation

In a later chapter, we will present a geometric view of relations between simplices of an abstract simplicial complex (defined in the next chapter) that is based on a *partial order* relation. For reference, we give here some relevant definitions. Some basic definitions such as *relation* and *equivalence relation* are assumed.

Definition 2.13. Let R be a relation on a set A. We say R is **reflexive** if $a \sim a$ for all $a \in A$; R is **transitive** if $a \sim b$ and $b \sim c$ implies $a \sim c$ for all $a, b, c \in A$; R is **antisymmetric** if, for all $a, b \in A$, $a \sim b$ and $b \sim a$ implies $a = b$.

Definition 2.14. A **partially ordered set** (or **poset**) is a set P associated with a relation that is reflexive, antisymmetric and transitive.

Definition 2.15 (Covering relation)**.** Let x, y be elements of a partially ordered set X. We say that y **covers** x if $x < y$ and there is no element $z \in X$ such that $x < z < y$.

2.4 Differential Geometry

Before presenting some aspects of discrete Morse theory, we will discuss briefly the classical smooth Morse theory. The latter is built to study the topology of a smooth *manifold* through certain differentiable functions on the manifold, more in Section 4.1. Here, we will give some relevant definitions even if they are not meant to be completely rigorous as some familiarity with differentiable geometry is already assumed.

Definition 2.16. An n-**dimensional manifold** is a topological space M such that every point in M has a neighborhood that is homeomorphic to the n-dimensional Euclidean space. The manifold is said to be a differentiable manifold if M is additionally endowed with a global differentiable structure.

Differentiable can mean different things to different authors. For simplicity, we assume smoothness so each map from a neighborhood as above to the n-dimensional Euclidean space is differentiable infinitely many times.

Example 2.4.1. A 2-dimensional manifold is often referred as a *surface*. A torus and a Klein bottle are both examples of 2-dimensional manifolds, though quite different in that one is orientable and the other is not.

Like other structures, manifolds have its own structural-preserving mapping. However, the definition of such a map requires some basic knowledge of *differentiable* functions on manifolds. We refer readers who are unfamiliar with these topics to [26], while stating the definition here.

Definition 2.17. Let M, N be manifolds. A differentiable function $f : M \to N$ is a **diffeomorphism** if it is a bijection and its inverse function is also differentiable. If such function exists, we say that M is **diffeomorphic** to N.

We can say, again, that M is an n-dimensional differentiable manifold if every point in M has a neighborhood that is diffeomorphic to an open set of the n-dimensional Euclidean space.

2.5 Cell Complexes

Although commonly referred as CW-complexes, we do not discuss either closure-compact (C) or "weak" topology (W) in this paper. We use the term *cell complexes* to emphasise the role of *cell* in the construction.

Definition 2.18. A **cell complex** is a topological space built recursively from **cells** of various dimension, which are homeomorphic to closed balls of the same dimension, by gluing them together along their boundaries with some specific restrictions:

- A 0-complex X^0 is just a collection of 0-cells (vertices);
- An n-complex X^n is obtained from an $(n-1)$-complex X^{n-1} by attaching n-cells D_i^n to it following some **attaching maps** $f_i^n : \partial D_i^n \to X^{n-1}$, which are continuous.

This means that X^n is the quotient space of the disjoint union $X^{n-1} \sqcup_i D_i^n$ under the equivalence relation $x \sim f_i^n(x)$ for $x \in \partial D_i^n$. This can be written as

$$\left\{X^{n-1} \bigsqcup_i D_i^n\right\} / _{\{x \sim f_i^n(x)\}}.$$

Example 2.5.1. The n-sphere S^n is a cell complex consists of a single 0-cell and a single n-cell. For example, S^2 can be obtained through the following steps:

(i) Let $X^0 = x_0$, a single 0-cell;

(ii) Let $X^1 = X^0$, that is with no 1-cell added;

(iii) Let $X^2 = \left\{X^1 \bigsqcup_i D^2\right\} / _{\{x \sim f(x)\}}$, where $f(x) = x_0$ for all $x \in \partial D^2$.

2.5.1 Euler Characteristic

Here we mention one topological invariant that is important to our work, the *Euler characteristic*. It has some nice properties and can be used as a simple tool to identify a space.

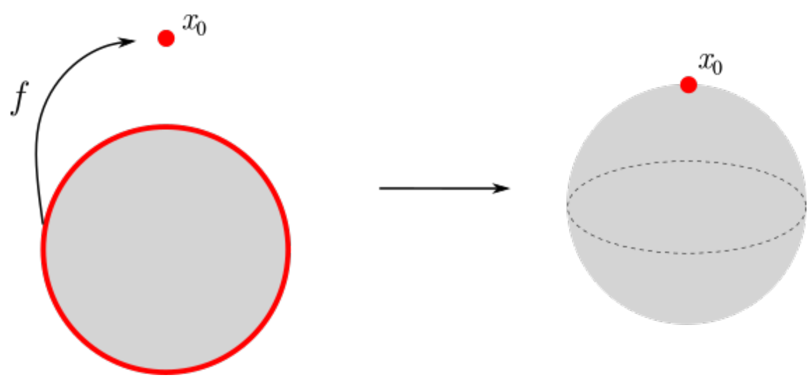

Figure 2.3: A 2-sphere as a cell complex

Definition 2.19. Let C be a cell complex. The **Euler characteristic** of C, denoted $\chi(C)$, is defined by

$$\chi(C) = \sum_{i=0}^{n} (-1)^i k_i,$$

where k_i is the number of i-dimensional cells in C.

In the next chapter we will discuss homology, which is a homotpy invariant (Theorem 3.3.6). In a general setting, the Euler characteristic can be written in terms of ranks of homology groups. We refer interested readers to [21] for these proofs. As such, the next proposition follows nicely.

Proposition 2.5.2. *Let X and Y be cell complexes. If $X \simeq Y$, then $\chi(X) = \chi(Y)$.*

2.5.2 Graphs

We often refer to a 1-complex as a *graph*, which consists of vertices (0-cells) and edges (1-cells) that each connects a pair of vertices. Vertices of a graph can represent objects of some kind, and an edge connecting two vertices represents that these objects are in some relation. We will be discussing one particular *directed graph*, so here we give some relevant definitions and present a well-known result of it that will come in handy later.

Definition 2.20. A **directed graph** or **digraph** is a graph whose edges are associated with some directions. For vertices v_i and v_j of a directed graph, we write $v_i \to v_j$ if there is an edge with a direction from v_i to v_j, that is an edge *leaving* v_i and *entering* v_j.

Definition 2.21. A **path** on a digraph is a sequence of vertices that follow directions of those edges connecting them. A **cycle** is a path that begins and ends on the same vertex. A directed graph is said to be **acyclic** if it contains no cycle.

Proposition 2.5.3. *Every acyclic digraph has a vertex with no edge entering it.*

Proof. Suppose there is no such vertex. Pick a vertex v, travel in the opposite direction of the entering edge, and visit the next vertex. Since every vertex has an entering edge, we can keep travelling in this way and always move to a vertex we have yet to visit. After travelling through all vertices, since the last vertex has an entering edge, the next step will necessarily take us to a vertex we have already visited, thus creating a cycle.

□

Chapter 3

Simplicial Complexes

In this chapter, we will discuss some fundamentals of the object of interest in the discrete Morse theory.

3.1 Basics of Simplicial Complexes

Our building blocks here are objects called *simplices*, singular *simplex* which is essentially a generalized triangle. A common definition for an *n-dimensional simplex* or *n-simplex* $\Delta^{(n)}$ is the following: it is a subset of $\mathbb{R}^{n+1}$ such that

$$\Delta^{(n)} = \left\{ \sum_{i=1}^{n+1} t_i e_i : 0 \leq t_i \leq 1 \text{ and } \sum t_i = 1 \right\},$$

i.e. it is the convex hull of (linearly independent) $n+1$ points that become its vertices. Here e_i are the unit vectors in the positive direction of the axes of coordinates. A *simplicial complex* is a cell complex such that each closed n-cell is a copy of an n-simplex and the non-empty intersection of two simplices is also a simplex.

However, under these definitions, we can only work on some rather restricted complexes. For example, we cannot intersect edges without counting the intersection as a vertex. Thus, we want to loose the restriction a bit and emphasize on the combinatorial aspect of the complex. Formally, we have:

Definition 3.1. Let $[v_n] = \{v_0, v_1, \ldots, v_n\}$ be a collection of $n+1$ vertices. An **abstract simplicial complex** K **on** $[v_n]$ is a collection of subsets of $[v_n]$, excluding $\emptyset$, such that

(a) if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$;

(b) $\{v_i\} \in K$ for every $v_i \in [v_n]$.

The set $[v_n]$ is called the **vertex set** of K. A subset of the vertex set with cardinality of $i+1$ is called an ***i*-dimensional simplex** or ***i*-simplex**. A ***c*-vector** of K is the vector $\vec{c}_K = \left(c_1, c_2, \ldots, c_{dim(K)}\right)$, where c_i is the number of i-simplices of K for $0 \leq i \leq dim\,(K)$.

Remark. From this point, whenever the term *simplicial complex* is used, it refers to the definition above.

It is an easy task to find the Euler Characteristic of a simplicial complex under this setting. For example, if K has a c-vector of $(2, 1, 1)$, then $\chi\,(K) = 2 - 1 + 1 = 2$.

Definition 3.2. Let K be a simplicial complex. A **subcomplex** L of K is a subset of K that is also a simplicial complex. If $\sigma \in K$ is a simplex, we write $\bar{\sigma}$ for the subcomplex **generated** by σ, that is $\bar{\sigma} = \{\tau \in K : \tau \in \sigma\}$. If $\sigma, \tau \in K$ and $\tau \subseteq \sigma$, then we say τ is a **face** of σ, and σ is a **coface** of τ. If a simplex is not properly contained in any other simplex, then it is called a **facet**.

To simplify notation and minimize the number of brackets we use, we shall describe a simplex by concatenating its vertices (0-simplices). For example, a standard 3-simplex, i.e. a tetrahedron whose vertices are v_0, v_1, v_2, v_3, is denoted as $v_0v_1v_2v_3$.

It it apparent from these definitions that the "shape" of a simplicial complex is not relevant; it is the relation among simplices that is interesting to us. For example, both of simplicial complexes in Figure 3.1 are visualizations of $\{v_0, v_1, v_2, v_3, v_4, v_0v_1, v_0v_2, v_1v_2, v_1v_3, v_2v_3, v_0v_1v_2\}$.

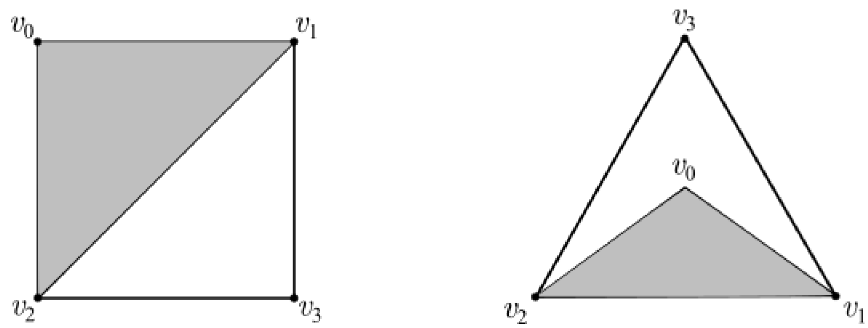

Figure 3.1: Different realizations of the same simplicial complex

Since any simplex is contained in some facet, one can also use a list of facets to describe a simplicial complex. The information of any simplex can be retrieved from the facet in which the simplex is contained.

Example 3.1.1. Both simplicial complexes (which are actually the same simplicial complex we mentioned above) in Figure 3.1 can be written as the collection of their facets: $\{v_1v_3, v_2v_3, v_0v_1v_2\}$.

Definition 3.3. Let K and L be disconnected simplicial complexes. The **join** of K and L, denoted $K * L$, is the simplicial complex defined by

$$K * L \coloneqq \{\sigma \cup \tau : \sigma \in K, \tau \in L\}.$$

The **simplicial cone over** K, denoted CK, is the join of K and a single vertex.

3.2 Simple Homotopy

Previously, we presented the definition of homotopy under a general topological setting. For simplicial complexes specifically, we use a particular kind of homotopy called *simple homotopy*, originated from [35], to determine if two simplicial complexes are the "same". Interested readers can refer to [11] for a comprehensive coverage on the subject, as well as some closely related theories. Intuitively, the action of simple homotoping can be viewed as a form of deformation retraction, in the sense that it involves some "squashing" and "stretching".

Definition 3.4. Let K be a simplicial complex. Suppose there are simplices of K, $\sigma^{(p-1)}$ and τ^p, such that σ is a face of τ with no other cofaces. Then $K - \{\sigma, \tau\}$ is a simplicial complex called an **elementary collapse** of K. The action of **collapsing** is denoted by $K \searrow K - \{\sigma, \tau\}$. On the other hand, suppose $\sigma^{(p-1)}$ and $\tau^{(p)}$ are simplices not in K, where σ is a face of τ and all other faces of τ are in K. Then $K \cup \{\sigma, \tau\}$ is a simplicial complex called an **elementary expansion** of K. The action of **expanding** is denoted by $K \nearrow K \cup \{\sigma, \tau\}$. For either of these situations, the pair of simplices $\{\sigma, \tau\}$ is called a **free pair**.

Definition 3.5. Let K and L be simplicial complexes. We say K and L have the same **simple homotopy type**, denoted $K \sim L$, if there is a sequence of elementary collapses and expansions through which we can obtain L from K.

It is clear that simple homotopy is an equivalence relation: K has the same simple homotopy type as itself; if $K \sim L$, then we can replace every elementary collapse with elementary expansion, and vise versa, to obtain K from L; if $K \sim L$ and $L \sim J$, then by "concatenating" these sequences of actions, we will be able to obtain J from K.

Proposition 3.2.1. *Given two simplicial complexes K and L, if $K \sim L$, then $K \simeq L$.*

This should not be surprising, as an elementary collapse or expansion can also be seen as a deformation retraction. This is more clear with the alternative definition of elementary collapse and expansion given in [11]. This proposition, together with Proposition 2.5.2, also directly implies the next one:

Proposition 3.2.2. *If $K \sim L$, then $\chi(K) = \chi(L)$.*

Definition 3.6. A simplicial complex K is **collapsible** if there is a sequence of elementary collapses such that

$$K = K_0 \searrow K_1 \searrow \dots K_{n-1} \searrow K_n = \{v\}$$

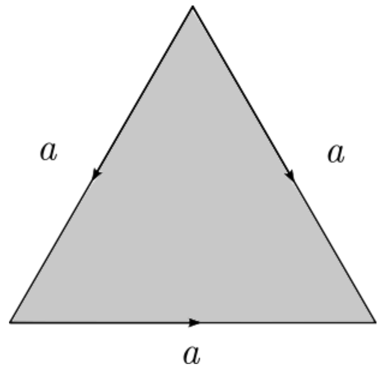

Figure 3.2: The Dunce Hat

for some vertex $v \in K$.

Proposition 3.2.3 ([34])**.** *The simplicial cone over any simplicial complex is collapsible.*

Note that, a simplicial complex being collapsible implies that it has the simple homotopy type of a point (usually called *contractible*), but having the simple homotopy type of a point does not necessarily mean it is necessarily collapsible. This is due to the fact that being collapsible only allows elementary collapses, while being the same simple homotopy type allows elementary expansion as well.

Example 3.2.4. The *Dunce Hat* D is a classic example of a space that is contractible but not collapsible. One way to represent D is by identifying sides of a triangle in the way shown in Figure 3.2, usually written as aaa^{-1}.

The contractibility of the D can be shown with the *Steifert-Van Kampen Theorem*. Details of the theorem can be found in [31] and we will leave the proof to the reader. On the other hand, Scoville ([34]) gives a simplicial view of D, and one can see that D is not collapsible by observing that it does not have any free pair. Despite this, Zeeman proved that $D \times I$, where I is the unit interval, is collapsible. Furthermore, he showed that the product of any contractible 2-dimensional complex with I is collapsible. We refer readers to [36] for this beautiful work.

3.3 Simplicial Homology

Definition 3.7. A **chain complex** $\mathcal{C}$ is a sequence of vector spaces C_i along with homomorphisms $\partial_i : C_i \to C_{i-1}$ between them:

$$\cdots \xrightarrow{\partial_{i+2}} C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} = 0,$$

with the property that $\partial_{n-1} \circ \partial_n = 0$.

Although commonly constructed as groups, in this thesis we study simplicial homology in the context of vector spaces over $\mathbb{Z}_2$. That is, let $\Bbbk^n = \langle v_0, v_1, \ldots, v_{n-1} \rangle$ denote the vector space generated by n elements with its additive operation being addition modulo 2. Let K be a simplicial complex, and K_i be the collection of i-simplices of K. Recall that c_i is the number of i-simplices of K, so elements of K_i generate a vector space $\Bbbk^{c_i}$. In addition, each homomorphism is taken to be a *boundary operator*, which maps a simplex to the union of its codimensional-1 faces. Formally:

Definition 3.8. Let $\sigma = v_1 v_2 \ldots v_n$ be a simplex of a simplicial complex. The **boundary operator** $\partial_i : \Bbbk^{c_i} \to \Bbbk^{c_{i-1}}$ is defined as the following:

$$\partial_i(\sigma) = \begin{cases} 0 & \text{if } i = 0 \\ \sum_{0 \le j \le i} \sigma_0 \sigma_1 \cdots \hat{\sigma}_j \cdots \sigma_i & \text{otherwise,} \end{cases}$$

where $\hat{\sigma}_j$ means excluding σ_j.

Example 3.3.1. The Möbius strip $\mathcal{M}$ is a surface homeomorphic to the complex obtained from a square by identifying one pair of opposite sides with a twist (Figure 3.3).

The two red sides of the square that are not "glued" together become a single bounding circle. To verify this, we directly compute the boundary of $\mathcal{M}$ with a simplicial representation shown in Figure 3.4.

Since $\mathcal{M}$ is a surface, we will compute its 2-dimensional boundary:

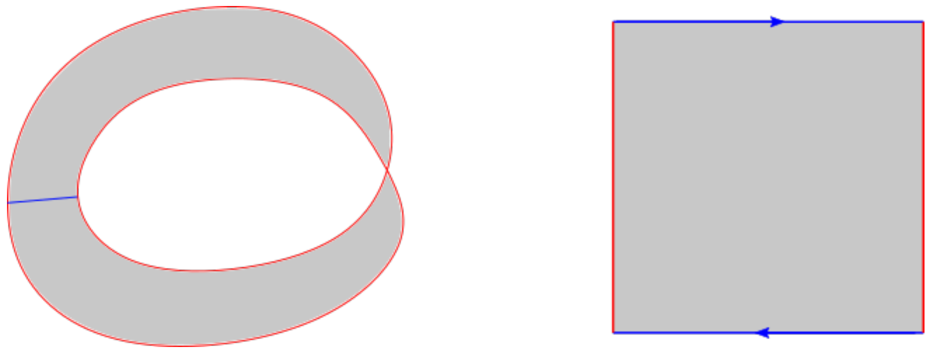

Figure 3.3: The Möbius strip

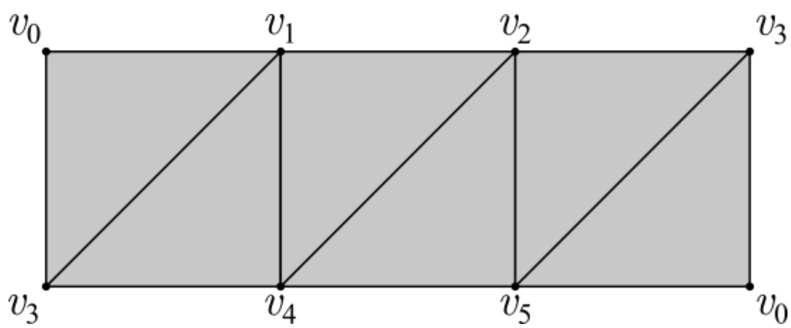

Figure 3.4: The Möbius strip as a simplicial complex

$$
\begin{aligned}
\partial_2(\mathcal{M}) &= \partial(v_0v_1v_3 + v_1v_3v_4 + v_1v_2v_4 + v_2v_4v_5 + v_2v_3v_5 + v_3v_5v_0) \\
&= v_1v_3 + v_0v_3 + v_0v_1 + v_3v_4 + v_1v_4 + v_1v_3 + v_2v_4 + v_1v_4 + v_1v_2 \\
&\quad + v_4v_5 + v_2v_5 + v_2v_4 + v_3v_5 + v_2v_5 + v_2v_3 + v_5v_0 + v_3v_0 + v_3v_5 \\
&= v_0v_1 + v_1v_2 + v_2v_3 + v_3v_4 + v_4v_5 + v_5v_0.
\end{aligned}
$$

Terms other than the ones in the last equality appeared twice in the previous step, so they sum up to 0. Note that the result gives a sequence of 1-simplices that forms a complex that is homeomorphic to a circle, hence verifying the statement at the beginning.

We should also check that the boundary operator suits the definition of a chain complex.

Proposition 3.3.2. *The boundary operator ∂_i satisfies the property $\partial_{i-1} \circ \partial_i = 0$.*

The detailed proof of this proposition will be omitted, one can check and see that each term will appear twice in the sum, so we get 0 in the end because of modulo 2 arithmetic.

Hence the resulting chain complex is of the form

$$\cdots \xrightarrow{\partial_{i+1}} \Bbbk^{c_i} \xrightarrow{\partial_i} \Bbbk^{c_{i-1}} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_2} \Bbbk^{c_1} \xrightarrow{\partial_1} \Bbbk^{c_0} \xrightarrow{\partial_0} 0.$$

Now we are ready to define *simplicial homology*.

Definition 3.9. Let K be a simplicial complex. Let $\mathcal{C}$ be the chain complex in which each chain $\Bbbk^{c_i}$ is a vector space generated by elements of K_i. For each i, we define the following:

- $Z_i = Z_i(K) = Ker(\partial_i) \subset \Bbbk^{c_i}$, the vector space of i-**cycles**;
- $B_i = B_i(K) = Im(\partial_{i+1}) \subset \Bbbk^{c_i}$, the vector space of i-**boundaries**.

The i-th **simplicial homology vector space** of K is definied as $H_i = H_i(K) = Z_i/B_i$. Its elements are in the form of equivalence classes $[z] = \{z + w : w \in Im(\partial_{i+1})\}$, where $z \in Ker(\partial_i)$.

Example 3.3.3. Consider the simplicial complex in Figure 3.5 which can be written as the collection of facets

$$K = \{v_0v_1v_2, v_0v_1v_3, v_0v_2v_3, v_1v_2v_3, v_1v_3v_5, v_3v_4v_5, v_4v_5v_6, v_2v_4v_6, v_0v_2v_6\}.$$

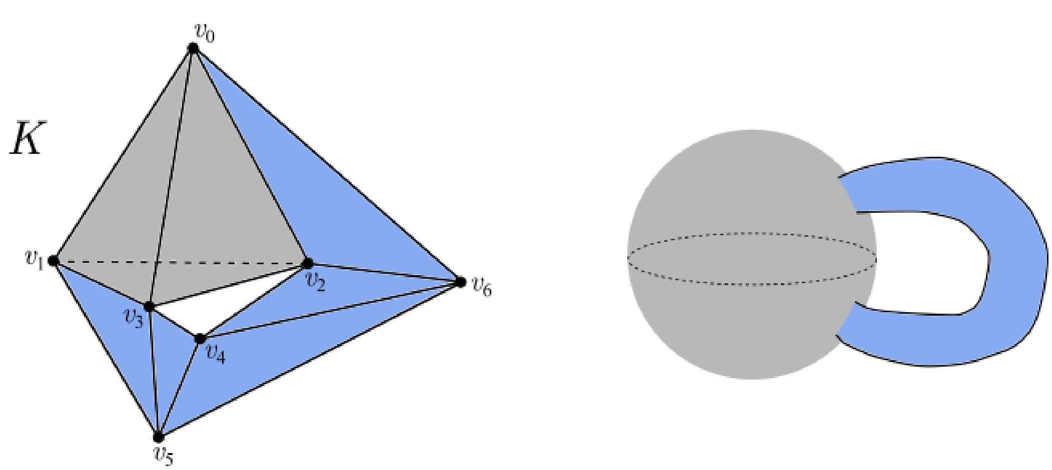

Figure 3.5: A simplicial complex homeomorphic to a 2-sphere with a band attached

Observe that the c-vector of K is $(7, 15, 9)$. To calculate its homology vector spaces, we first obtain from K the chain complex

$$0 \xrightarrow{\partial_3} \Bbbk^9 \xrightarrow{\partial_2} \Bbbk^{15} \xrightarrow{\partial_1} \Bbbk^7 \xrightarrow{\partial_0} 0.$$

Starting with $i = 2$, since $Im\,(\partial_3) = 0$, we have

$$H_2(K) = Z_2/B_2 = Ker\,(\partial_2)\,/Im\,(\partial_3) = Ker\,(\partial_2)\,,$$

and $Ker\,(\partial_2) = \langle v_0v_1v_2 + v_0v_1v_3 + v_0v_2v_3 + v_1v_2v_3\rangle$. Hence, $H_2(K) \cong \Bbbk^1$.

For $i = 1$, observe that $Ker\,(\partial_1)$ is generated by $\{e_p + e_q + e_r\}$, where e_p, e_q and e_r are edges that form a boundary of a 2-simplex. On the other hand, $Im\,(\partial_2)$ has the same generators as $Ker\,(\partial_1)$, excluding $v_2v_3 + v_2v_4 + v_3v_4$. Hence,

$$H_1(K) = \langle v_2v_3 + v_2v_4 + v_3v_4\rangle \cong \Bbbk^1.$$

Finally, $Ker\,(\partial_0)$ is the entire $\Bbbk^7$ in the chain complex, and $Im\,(\partial_1)$ is generated by $\{v_p + v_q\}$, where v_p and v_q are vertices of the same 1-simplex. Since any 2 vertices of K are connected by some sequence of edges, the quotient implies that all vertices are equivalent. Therefore,

$$H_0(K) \cong \Bbbk^1.$$

Alternatively, we can also define the *i-th simplicial homology* of K to be the vector space

$$H_i\,(K) := \Bbbk^{\mathrm{null}(\partial_i) - \mathrm{rank}(\partial_{i+1})}.$$

Definition 3.10. The integer $\mathrm{null}\,(\partial_i) - \mathrm{rank}\,(\partial_{i+1})$ is called the ***i*-th Betti number** of K, denoted $b_i\,(K)$.

It it worth noting that Betti numbers could be different depending on the choice of field over which the vector field is. In this thesis, however, the field will always be $\mathbb{Z}_2$.

Example 3.3.4. The simplicial complex K in Figure 3.5 has Betti numbers $b_0(K) = 1$, $b_1(K) = 1$ and $b_2(K) = 1$. We can arrive at the same result by observing the geometric structure of K (or that of the object to its right): it is connected, i.e. having only 1 component; it has a 1-dimensional circular hole created by v_2v_3, v_2v_4 and v_3v_4; it also has a 2-dimensional cavity created by $v_0v_1v_2$, $v_0v_1v_3$, $v_0v_2v_3$ and $v_1v_2v_3$.

Like Euler characteristics, Betti numbers are preserved by the action of elementary expanding and collapsing.

Proposition 3.3.5 ([34])**.** *Let K and K' be simplicial complexes with the same simple homotopy type. Then $b_d(K) = b_d(K')$ for all $d = 0, 1, 2, \ldots$.*

Proposition 3.3.5 is a result of the following fundamental theorem in homology theory, which we have previously mentioned in Subsection 2.5.1. It shall be referenced later so we state it here while referring readers to [21] for a proof.

Theorem 3.3.6 ([21])**.** *If $f : X \to Y$ and $g : Y \to X$ are homotopy equivalences, then*

$$H_i(X) \cong H_i(Y)$$

for all i.

Chapter 4

Discrete Morse Theory

This chapter serves to present certain definitions and fundamental results in discrete Morse theory. We will start with Section 4.1, presenting some basics and results of the classical Morse theory to motivate the discussion in the discrete case. In Section 4.2, we present the construction of the discrete Morse theory while providing our own proofs for some results. In Section 4.3, we selected some main theorems of the discrete Morse theory, providing our own proofs and examples in combination with [15] and [34].

4.1 Classical Morse Theory

Morse theory is used to find topological invariants of manifolds by examining critical points of some chosen functions. We begin by giving some relative definitions and an example. Throughout the section, we use M to denote a smooth manifold unless specified otherwise.

Definition 4.1. Let $f : M \to \mathbb{R}$ be a smooth function. A point $x \in X$ is a **critical point** if

$$\frac{\partial f}{\partial x_1}(x) = \cdots = \frac{\partial f}{\partial x_n}(x) = 0.$$

A critical point x is **non-degenerate** if the Hessian of f at x is non-degenerate, i.e. has no zero eigenvalues.

Definition 4.2. Let $f : M \to \mathbb{R}$. For any $a \in \mathbb{R}$, the **sublevel set** M_a is defined as

$$M_a = \{x \in M : f(x) \leq a\}.$$

Generally speaking, the behavior of the function around its *degenerate* critical points (i.e. where the Hessian of f is singular) are difficult to observe. This gives the motivation for choosing the following family of nice functions:

Definition 4.3. A map $f : M \to \mathbb{R}$ is a **Morse function** if all of its critical points are non-degenerate.

The idea of Morse function goes back to Marston Morse, ([29]), who studied the topology of a manifold with the information of its critical points. A well-known result from this usage of Morse function was *Reeb's Sphere Theorem.*

Theorem 4.1.1 (Reeb's Sphere Theorem)**.** *Let M be a compact manifold. Suppose there exists a Morse function on M with exactly* 2 *critical points. Then M is homeomorphic to a sphere.*

One might-be-surprising result is that Morse functions are *generic* on differentiable manifolds. This is explained in depth by the next two propositions. Readers can find more details and proofs in [3].

Proposition 4.1.2 ([3])**.** *Let $f : M \to \mathbb{R}$ be a smooth function and let $k \in \mathbb{N}$. Then on any compact subset of M, f and its derivatives of order less than or equal to k can be approximated by a Morse function uniformly.*

Proposition 4.1.3 ([3])**.** *Let M be a compact manifold. Then, the set of Morse functions on M is a dense open subset of $C^\infty(M)$.*

One of the main results of classical Morse theory was the following theorem:

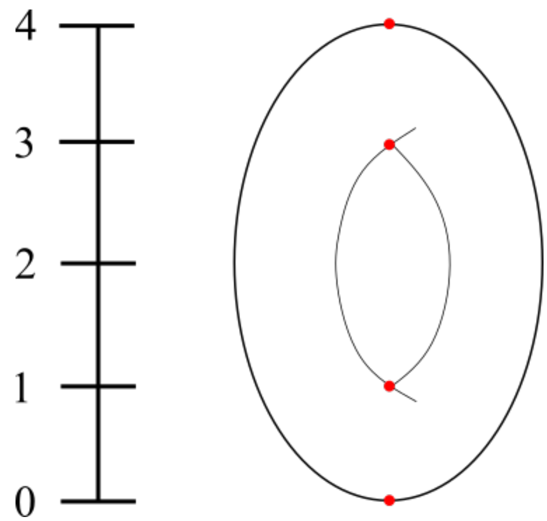

Figure 4.1: A torus with height function

Theorem 4.1.4 ([28]). *Let $f : M \to \mathbb{R}$ be a Morse function. Let $a < b$ and suppose that $f^{-1}[a, b]$ is compact and contains no critical points of f. Then M_a is diffeomorphic to M_b. Furthermore, M_a is a deformation retract of M_b, so that the inclusion map $\iota : M_a \to M_b$ is a homotopy equivalence.*

Example 4.1.5. Consider a torus T in the 3-dimensional Euclidean space with a height function $h : T \to \mathbb{R}$ in Figure 4.1. The four critical points are marked in red, having height $0, 1, 3$ and 4. Sublevel sets of T induced by h have the same homotopy type between two adjacent critical points. Figure 4.2 shows the sublevel set T_b deformation retracts to T_a, where $1 < a < b < 3$, by "smooshing down those tubes".

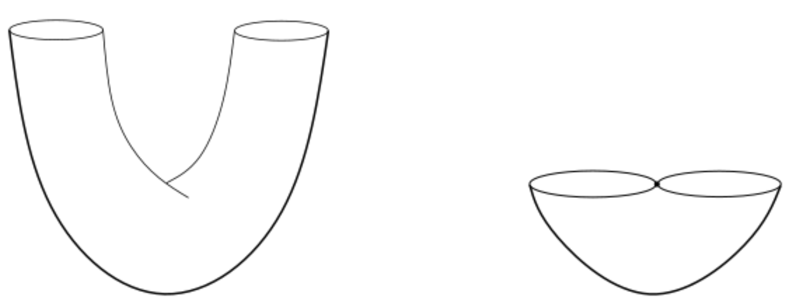

Figure 4.2: Deformation retraction of sublevel sets of a torus between critical points

For more details on the classical Morse theory, we direct the reader to the previously mentioned references.

4.2 Discrete Morse Theory

Now, let us turn our attention to the main topic of this thesis: discrete Morse theory. The name suggests how the theory is related to classical Morse theory. However, it is not a simple discretization of the smooth case. The objects of our interest are now simplicial complexes, and instead of critical points, we study so-called *critical simplices* which will be defined in a moment. With discrete Morse theory, we can learn topological information about a complicated simplicial complex by studying an equivalent, that is of the same simple homotopy type, but "simpler" simplicial complex. The idea is to collapse from one to the other. Given a complicated simplicial complex, the process could be tedious if we draw out which pair of simplices is being collapsed at each stage. We shall present a cleaner description later, called *gradient vector field*. But first, we need to define a family of functions on simplicial complexes that corresponds to the classical family of Morse function on a smooth manifold. Then, we shall build the theory on that, relate it to some topological concepts, and present the main theorems of discrete Morse theory. While we follow on existing results, we present as much as possible our own proofs.

4.2.1 Discrete Morse Function

Definition 4.4. Let K be a simplicial complex. A map $f : K \to \mathbb{R}$ is a **discrete Morse function** if for every p-simplex $\sigma \in K$, we have

$$\left|\left\{\tau^{(p-1)} < \sigma : f(\tau) \geq f(\sigma)\right\}\right| \leq 1$$

and

$$\left|\left\{\tau^{(p+1)} > \sigma : f(\tau) \leq f(\sigma)\right\}\right| \leq 1.$$

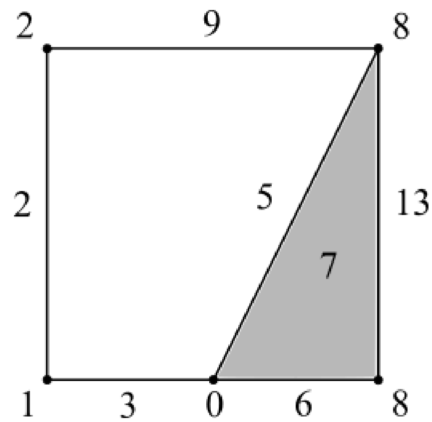

Figure 4.3: A discrete Morse function on a simplicial complex

The general rule of thumb is that, higher dimensional simplices get assigned higher values and lower dimensional simplices get assigned lower values, with at most one "exception" being allowed for each simplex.

Example 4.2.1. The function f that assigns values to simplices of the simplicial complex shown in Figure 4.3 is a discrete Morse function. One should note that the 2-simplex has two 0-dimensional faces (vertices) whose values under f are greater than that of itself. This causes no problem as the definition of discrete Morse function only has restrictions on codimension-1 faces and cofaces.

In classical Morse theory, we examine critical points of a Morse function. Here we do the same, but instead of critical points, we study *critical simplices*. A simplex is critical if it does not admit any "exception" in the sense we mentioned above in the definition of discrete Morse function. Formally, we have:

Definition 4.5. A p-simplex $\sigma \in K$ is said to be **critical** with respect to a discrete Morse function f if

$$|\{\tau^{(p-1)} < \sigma : f(\tau) \geq f(\sigma)\}| = 0$$

and

$$|\{\tau^{(p+1)} > \sigma : f(\tau) \leq f(\sigma)\}| = 0.$$

If σ is critical, then $f(\sigma) \in \mathbb{R}$ is its corresponding **critical value**. Any simplex that is not critical is said to be **regular**, and its value under f is its **regular value**.

Lemma 4.2.2. *For any simplicial complex K, there is a discrete Morse function such that every simplex of K is critical.*

Proof. For any simplex $\sigma^{(p)} \in K$, define $f(\sigma) = p$, so there is no "exception" at any simplex. Then it follows that f is a discrete Morse function that makes every simplex of K critical.

□

The following lemma is a simple yet very important observation. It plays a key role in many parts of the discrete Morse theory, and we shall use it quite often later.

Lemma 4.2.3 (Exclusion lemma, [15])**.** *Let $f : K \to \mathbb{R}$ be a discrete Morse function and $\sigma \in K$ be a regular simplex. Then, exactly one of the following two conditions holds:*

(i) There exists $\tau^{(p+1)} > \sigma$ such that $f(\tau) \leq f(\sigma)$;

(ii) There exists $\nu^{(p-1)} < \sigma$ such that $f(\nu) \geq f(\sigma)$.

Proof. By the definition of the regular simplex, we know that at least one of these conditions is true. Now suppose both are true, that is, there exist $\nu^{(p-1)} = v_0v_1 \ldots v_{p-1}$ and $\tau^{(p+1)} = v_0v_1 \ldots v_pv_{p+1}$ such that $f(\tau) \leq f(\sigma) \leq f(\nu)$. Let $\tilde{\sigma} = v_0v_1 \ldots v_{p-1}v_{p+1}$, so that $\nu < \tilde{\sigma} < \tau$. Since f is a discrete Morse function, we can only have at most 1 exception at each simplex, then we must have $f(\nu) < f(\tilde{\sigma})$ and $f(\tilde{\sigma}) < f(\tau)$. Hence

$$f(\tau) \leq f(\sigma) \leq f(\nu) < f(\tilde{\sigma}) < f(\tau),$$

and we have a contradiction.

□

Here is an example of how we utilize the exclusion lemma:

Corollary 4.2.4. *If f is a discrete Morse function, then f has at least one critical simplex.*

Proof. Suppose f has no critical simplex. Then, there must exist at least one 1-simplex, say v_iv_j, since otherwise v_i and v_j will be critical. Since v_i is regular, we have $f(v_i) \geq f(v_iv_j)$. Then, by the exclusion lemma, $f(v_iv_j) > f(v_j)$. Now, if v_j is not a face of any other 1-simplex, v_j is critical; if v_j is a face of some other 1-simplex, then we repeat the process and, eventually, we will get a critical 0-simplex.

□

A specific kind of discrete Morse function called *excellent* is "nicer" in the sense that every critical simplex gets assigned a distinct value. It is natural to expect that this kind of discrete Morse function will bring ease on constructions and calculations. Fortunately, we have a generous way to relate every ordinary discrete Morse function to an excellent one, and we will show how in the next subsection. But first, we state its definition here:

Definition 4.6. A discrete Morse function is said to be **excellent** if it is $1-1$ on its critical simplices.

4.2.2 Forman Equivalence

Recall that regular homotopy, which was defined for functions, represents some kind of "sameness" between them, and, similarly, for spaces. We have defined earlier the simple homotopy for simplicial complexes, so we now need a notion of "sameness" for discrete Morse functions.

Definition 4.7. Let f and g be discrete Morse functions on a simplicial complex K. We say f and g are **Forman equivalent** if for every pair of simplices $\sigma^{(p)} < \tau^{(p+1)}$ of K, $f(\sigma) < f(\tau)$ if and only if $g(\sigma) < g(\tau)$.

Given the "if and only if" condition in the definition above, it is straightforward to verify that Forman equivalence is an equivalence relation.

Now, as promised, we present a way to relate every ordinary discrete Morse function to an excellent one, and we do so through Forman equivalence.

Lemma 4.2.5 ([34])**.** *Let $f : K \to \mathbb{R}$ be a discrete Morse function. Then there exists an excellent discrete Morse function $g : K \to \mathbb{R}$ that is Forman equivalent to f.*

Proof. Let $\sigma_1, \sigma_2 \in K$ be critical simplices of f such that $f(\sigma_1) = f(\sigma_2)$. Note if no such simplices exist, then f is excellent by itself, so we may assume that they exist. Now, define $g : K \to \mathbb{R}$ by setting $g(\sigma) = f(\sigma)$ for all $\sigma \neq \sigma_2$, and $g(\sigma_2) = f(\sigma_2) + \epsilon$ where $\epsilon > 0$ and $f(\sigma_2) + \epsilon$ is less than the smallest value of f that is greater than $f(\sigma_2)$. This way, $g(\sigma_1) \neq g(\sigma_2)$ and g is Forman equivalent to f since none of the inequalities induced by f is changed by g (also it implies that σ_1 and σ_2 remain as critical simplices under g). If no other two critical simplices have the same value under f, then we are done. Otherwise, repeat the construction for any two of such simplices. Since there are only finitely many of them, eventually we will be able to construct an excellent discrete Morse function that is Forman equivalent to f.

□

With this lemma, we are now able to assume any given discrete Morse function to be an excellent one.

4.2.3 Gradient Vector Field

Definition 4.8. Let f be a discrete Morse function on a simplicial complex K. The **gradient vector field induced by** f, denoted V_f, is defined by

$$V_f := \{(\sigma^{(p)}, \tau^{(p+1)}) : \sigma < \tau, f(\sigma) \geq f(\tau)\}.$$

If $(\sigma, \tau) \in V_f$, then (σ, τ) is called a **vector** or an **arrow**. For a vector (σ, τ), σ is the **tail**, while τ is the **head**.

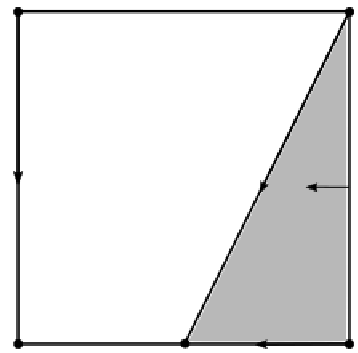

Figure 4.4: The gradient vector field induced by a discrete Morse function

Example 4.2.6. Figure 4.4 shows the gradient vector field induced by the discrete Morse function from Example 4.2.1.

One quickly notices that a vector consists of a pair of regular simplices. Hence, Lemma 4.2.3 introduces certain rules on vectors of a gradient vector field.

Lemma 4.2.7. *Let $f : K \to \mathbb{R}$, so it induces a gradient vector field (a set of vectors) on K. Then for any simplex $\sigma \in K$, exactly one of the following holds:*

(i) σ is the tail of exactly one vector;

(ii) σ is the head of exactly one vector;

(iii) σ is neither the head nor the tail of any vector.

Proof. If σ is critical, then by Definition 4.8, it does not belong to any vector, so we have (iii). If σ is regular, by Lemma 4.2.3, we have exclusively either (i) or (ii).

□

Ayala, Fernández and Vilches ([5]) proved the following theorem, showing that Forman equivalence preserves the induced gradient vector field.

Theorem 4.2.8 ([5], Theorem 3.1)**.** *Let $f, g : K \to \mathbb{R}$ be discrete Morse functions. Then f and g are Forman equivalent if and only if they induce the same gradient vector field.*

Now, suppose we have a simplicial complex on which we can add a set of vectors that satisfies exactly one of the three conditions above. This, however, does not guarantee the existence of a discrete Morse function which induces the given vector field. Some other conditions must be satisfied as well. Regardless, such vector field is very important in developing the theory, so we give its formal definition here:

Definition 4.9. Let K be a simplicial complex. A **discrete vector field** V on K is a set of pairs of simplices $(\sigma^{(p)}, \tau^{(p+1)})$ in K such that $\sigma < \tau$ and neither simplex is in any other pair.

It is straightforward to see that every gradient vector field is a discrete vector field – the result follows from the definition of a discrete Morse function. And, as we have mentioned, the converse fails to be always true. It turns out that such failure happens when the discrete vector field contains a "closed path".

Definition 4.10. Let V be a discrete vector field on a simplicial complex K. A **gradient path**, or V**-path** when context is clear, is a sequence of simplices

$$\left(\tau_{-1}^{(p+1)},\right) \sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}, \tau_1^{(p+1)}, \sigma_2^{(p)}, \ldots, \tau_{k-1}^{(p+1)}, \sigma_k^{(p)}$$

of K such that:

- the sequence begins either at a simplex that is not in any pair of V $(\tau_{-1}^{(p+1)})$, or a simplex $\sigma_0^{(p)}$ such that $(\sigma_0^{(p)}, \tau_0^{(p+1)}) \in V$;
- $(\sigma_i^{(p)}, \tau_i^{(p+1)}) \in V$ for $0 \leq i \leq k-1$;
- for $0 \leq i, j \leq k-1$, $\tau_{i-1}^{(p+1)} > \sigma_i^{(p)}$ and $\sigma_i^{(p)} \neq \sigma_j^{(p)}$.

Note that a gradient path can both begin and end on a simplex that is not in any pair of V. If $k \neq 0$, we say the gradient path is **non-trivial**; a gradient path that is not properly contained in any other gradient path is called **maximal**. Finally, a gradient path is **closed** if it begins and ends on the same simplex.

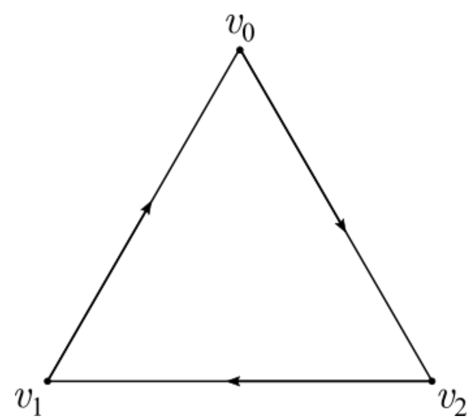

Figure 4.5: A discrete vector field that is not a gradient vector field

Theorem 4.2.9 ([34]). *A discrete vector field is a gradient vector field induced by some discrete Morse function if and only if it contains no non-trivial closed gradient path.*

We shall provide a proof in the next section. For now, we use the following example to help visualize this theorem.

Example 4.2.10. Consider the simplicial complex with a discrete vector field V in Figure 4.5. Clearly, V has a closed path. Then, by Theorem 4.2.9, it cannot be a gradient vector field induced by some discrete Morse function f. Otherwise, we will have the following inequality on the closed path:

$$f(v_0) \geq f(v_0v_1) > f(v_2) \geq f(v_2v_1) > f(v_1) \geq f(v_1v_0) > f(v_0),$$

which is a contradiction.

4.2.4 The Hasse Diagram

The *Hasse diagram* is a graphical representation of a finite partially ordered set. Instead of tracing through all elements in a set to determine how they are related, the structure of a Hasse diagram provides a clear summary of those relations.

Definition 4.11. Let P be a finite partially ordered set. The **Hasse diagram** $\mathcal{H}$ of P is constructed in the following way:

- Each element of P corresponds to a vertex of $\mathcal{H}$;
- An edge exists between 2 vertices if and only if they form a covering relation (see Definition 2.15).

Now, there is a natural partial order on a simplicial complex K: we relate two simplices if one is a codimension-1 face of the other. In fact, this is a covering relation as well. To start building the Hasse diagram, we arrange all simplices of the same dimension on a same row, and a row is placed higher if it corresponds to a higher dimension. To incorporate a discrete vector field V on K in its Hasse diagram, we draw an upward arrow along the edge from σ to τ if (σ, τ) is a pair in V. For a purpose that will be clear in a moment, also draw a downward arrow on all other edges. The resulting graph is called a **directed Hasse diagram** on K induced by V, denoted $\mathcal{H}_V$. A sequence of upward and downward arrows that begins and ends on the same vertex is called a **directed cycle**.

Example 4.2.11. Let us consider a simple example of a 2-simplex

$$K = \{v_1, v_2, v_3, v_1v_2, v_1v_3, v_2v_3, v_1v_2v_3\},$$

and introduce the discrete vector field

$$V = \{(v_2, v_1v_2), (v_3, v_1v_3), (v_2v_3, v_1v_2v_3)\}$$

on K. Then the resulting directed Hasse diagram (Figure 4.6) will have upward arrows from v_2 to v_1v_2, v_3 to v_1v_3 and v_2v_3 to $v_1v_2v_3$.

We will now present a few results about Hasse diagram in order to proceed in proving Theorem 4.2.9.

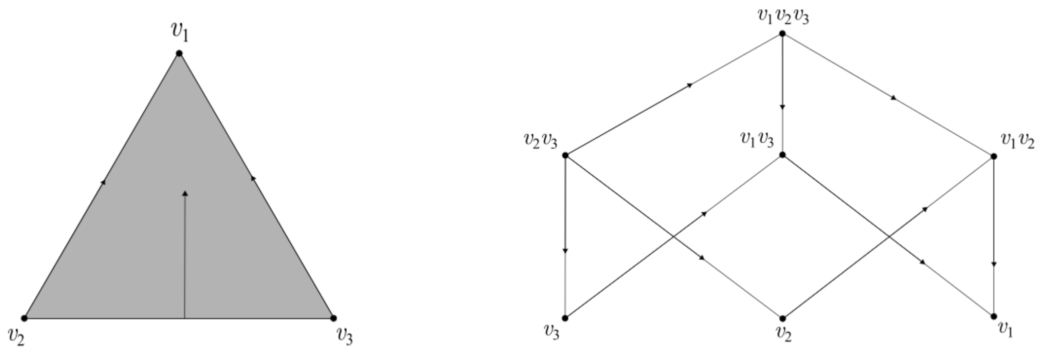

Figure 4.6: Directed Hasse diagram on a 2-simplex

Lemma 4.2.12 ([34]). *Let K be a simplicial complex and let V be a discrete vector field. If the Hasse diagram $\mathcal{H}_V$ contains a directed cycle, then the directed cycle is contained in exactly two levels.*

Proof. It is clear by definition that we cannot have any directed cycle in a single level, since there cannot be any covering relation between two simplices of the same dimension.

Suppose there is a directed cycle that ranges in more than two levels. Then at some point, we must have two consecutive upward arrows, i.e. there must be simplices $\sigma^{(p-1)}, \tau^{(p)}$ and $\nu^{(p+1)}$ such that $(\sigma, \tau) \in V$ and $(\tau, \nu) \in V$. But this is not possible, by the definition of V, since τ is in two different pairs. Therefore, we can only have directed cycles contained in exactly two levels.

□

Theorem 4.2.13 ([34]). *Let K be a simplicial complex, V be a discrete vector field on K, and $\mathcal{H}_V$ be the directed Hasse diagram on K induced by V. Then, there is no non-trivial closed V-path if and only if there is no directed cycle in $\mathcal{H}_V$.*

Proof. ($\Longrightarrow$) Suppose that there is a directed cycle in $\mathcal{H}_V$, then by Lemma 4.2.12, it must be contained in exactly two levels. So, the directed cycle will look like

$$\sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}, \tau_0^{(p+1)}, \sigma_2^{(p)}, \ldots, \tau_k^{(p+1)}, \sigma_{k+1}^{(p)} = \sigma_0^{(p)},$$

i.e. an alternating sequence of upward and downward arrows, since there cannot be any edge within the same level of $\mathcal{H}_V$. Within this cycle, every upward arrow, that is every pair of simplices $(\sigma_i^{(p)}, \tau^{(p+1)})$, represents an element of V; every downward arrow satisfies the condition $\tau^{(p+1)} > \sigma_i^{(p)}$. Therefore, this directed cycle is also a closed V-path on K.

($\Longleftarrow$) Like the forward direction, suppose that there is a closed V-path on K

$$\sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}, \tau_0^{(p+1)}, \sigma_2^{(p)}, \ldots, \tau_k^{(p+1)}, \sigma_{k+1}^{(p)} = \sigma_0^{(p)}.$$

Starting at $\sigma_0^{(p)} \in \mathcal{H}_V$. There is an upward arrow from $\sigma_0^{(p)}$ to $\tau^{(p+1)}$ since $(\sigma_0^{(p)}, \tau_0^{(p+1)}) \in V$. An edge with a downward arrow exists from $\tau_0^{(p+1)}$ to $\sigma_1^{(p)}$, since $\tau_0^{(p+1)} > \sigma_1^{(p)}$ and $(\sigma_1^{(p)}, \tau_0^{(p+1)}) \notin V$, as $\tau_0^{(p+1)}$ cannot be in two different pairs. Continuing in this way, we see that the closed V-path gives a directed cycle in $\mathcal{H}_V$.

□

The general version of next lemma can be found in [7]. We will present the lemma and its proof in a modified way that fits a directed Hasse diagram.

Lemma 4.2.14. *Let $[v_n]$ be the vertex set of a Hasse diagram $\mathcal{H}$. Then, there exists $f : [v_n] \to \mathbb{R}$ such that $f(v_i) > f(v_j)$ whenever there is an arrow going from v_i to v_j if and only if $\mathcal{H}$ does not have any directed cycle.*

Proof. For the forward direction, suppose there exists a directed cycle

$$v_{i_0}, v_{i_1}, \ldots, v_{i_k} = v_{i_0}.$$

Then we have $f(v_{i_0}) > f(v_{i_1}) > \cdots > f(v_{i_k}) = f(v_{i_0})$, a contradiction.

We show the backward direction constructively. Suppose that $\mathcal{H}$ does not have any directed cycle. Let $W = \{w_1, \ldots, w_i, \ldots w_n\}$ be a sequence of n elements. To start, choose a vertex v_i that does not have any arrow entering it. The existence of such vertex is guaranteed by Proposition 2.5.3. Let $w_1 = v_i$, and remove v_i together with arrows that are leaving it.

The remaining graph also does not have any directed cycle, then we repeat until there is no vertex left. Suppose there is an arrow from w_i to w_j with $i > j$. But this means that w_j was chosen and removed before w_i, so there cannot be any arrow entering w_j when it was chosen, and we have a contradiction. So $i < j$. Define $f(v_i) = n - k$ for $w_k = v_i$, and we obtain a function with the desired properties.

□

Remark. Note that this f must be strictly decreasing along any directed path.

Now, we are ready to present the **proof of Theorem 4.2.9**.

Proof of Theorem 4.2.9. ($\Longrightarrow$) If a discrete vector field is a gradient vector field, then it cannot contain any non-trivial closed gradient path by the argument in Example 4.2.10.

($\Longrightarrow$) Suppose the discrete vector field V contains no non-trivial closed gradient path. Then by Theorem 4.2.13, there is no directed cycle in $\mathcal{H}_V$. By Lemma 4.2.14, there exists some function $f : K \to \mathbb{R}$ such that $f(\sigma) > f(\tau)$ whenever there is an arrow going from σ to τ (note that σ and τ are vertices of $\mathcal{H}_V$). The function f is in fact a discrete Morse function. This follows from the fact that for any $(\sigma, \tau) \in V$, neither simplex can be in any other pair. Let V_f is the gradient vector field induced by f. If $(\sigma, \tau) \in V$, then $\sigma < \tau$ and $f(\sigma) > f(\tau)$, so $V \subseteq V_f$; If $(\sigma, \tau) \in V_f$, then $\sigma < \tau$, and neither is in any other pair by Lemma 4.2.7, so $V_f \subseteq V$. Hence, $V = V_f$.

□

4.2.5 Level Subcomplex

Here, we state a few more definitions that give more ways to describe and interact with simplicial complexes in a notion similar to classic Morse theory.

Definition 4.12. Let K be a simplicial complex. For any $\alpha, \beta \in K$, the **interval** $[\alpha, \beta]$ is the subset of K given by

$$[\alpha, \beta] \coloneqq \{\gamma \in K : \alpha \subseteq \gamma \subseteq \beta\}.$$

This allows us to bring together a set of simplices that are in face relations with others. Note that under this definition, $[\alpha, \beta] \neq \emptyset$ if and only if $\alpha \subseteq \beta$, and in particular, $[\alpha, \alpha] = \{\alpha\}$.

The next definition corresponds to sublevel sets in classical Morse theory:

Definition 4.13. Let $f : K \to \mathbb{R}$ be a discrete Morse function. For any $c \in \mathbb{R}$, the **level subcomplex** $K(c)$ is defined as

$$K(c) = \bigcup_{\substack{\tau \in K \\ f(\tau) \leq c}} \bigcup_{\sigma \leq \tau} \sigma,$$

that is all simplices $\tau \in K$ such that $f(\tau) \leq c$, as well as all faces of τ.

Same as for sublevel sets, it is clear that $K(a) \subseteq K(b)$ if and only if $a < b$. The equality of these sets happens when there is no τ such that $f(\tau) \in (a, b]$.

Example 4.2.15. Consider the simplicial complex K in Figure 4.7 with values of a discrete Morse function labeled on its simplices. The right hand side shows the level subcomplex $K(4)$. Note that $K(4)$ contains the 0-simplex to which 9 is assigned. This is because that 0-simplex is a face of the 1-simplex to which 4 is assigned.

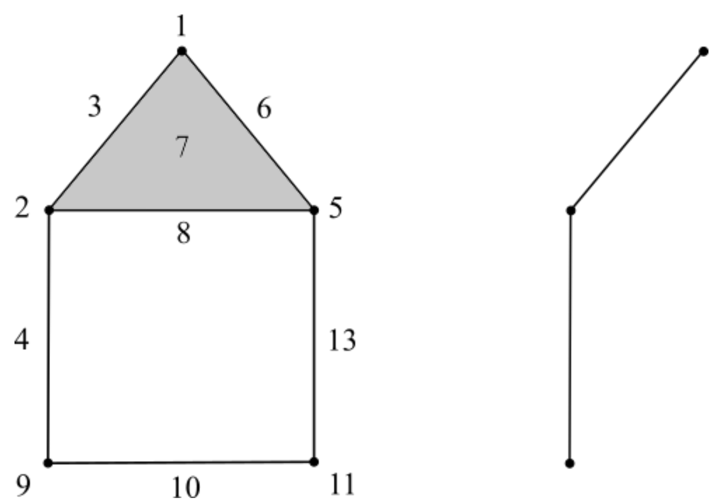

Figure 4.7: A level subcomplex of a simplicial complex under a discrete Morse function

4.3 Main Theorems of Discrete Morse Theory

In this section, we present some main results in discrete Morse theory. In Subsection 4.3.1, we present the *collapse theorem*, which is the discrete equivalent of Theorem 4.1.4, in Subsection 4.3.1 along with a couple of theorems that describe the homotopy equivalence between a simplicial complex and certain cell complexes. In Subsection 4.3.2, we present two *discrete Morse inequalities* that describe the relationship between the Betti numbers and the number of critical values of a simplicial complex. After that, we present how to "optimize" a discrete Morse function by reducing the number of critical simplices in Subsection 4.3.3. Finally, in Subsection 4.3.4, we discuss the homotopy between discrete Morse functions and its effect on their gradient vector spaces. The section is built on [15] with some of our own proofs.

4.3.1 The Collapse Theorem

As we mentioned, the collapse theorem is the discrete equivalent of Theorem 4.1.4. Intuitively, it says that the topological structure of the subcomplex stays the same unless new critical simplices are introduced.

Theorem 4.3.1 (Collapse Theorem, [34])**.** *Let $f : K \to \mathbb{R}$ be a discrete Morse function. If f has no critical values on the interval $[a, b] \in \mathbb{R}$, then $K(b) \searrow K(a)$.*

Here, we make an important observation: we may *perturb* f by a bit so that it is $1-1$ without changing $K(a)$ or $K(b)$. This is done in a similar fashion as was done in the proof of Lemma 4.2.5, by changing the value of f on some simplices by a small enough ϵ without changing which simplices are critical and which are regular. The formal statement can be found in [34] (Lemma 4.25). So, we may assume f is $1-1$. Now, we will present the proof of the theorem based on works in [15] and [34] with details added by ourselves.

Proof. Let σ be a regular simplex in K. First of all, if $f(\sigma) \notin [a, b]$ for all $\sigma \in K$, then $K(a) = K(b)$ since there isn't any simplex between $K(a)$ and $K(b)$, and we are done. So, suppose there is at least one σ such that $f(\sigma) \in [a, b]$. Then, we can partition the interval $[a, b]$ into smaller ones, and with some relabelling and abuse of notations, assume that there is exactly one σ with $f(\sigma) \in [a, b]$. Note that, by definition, $\sigma \in K(b)$. By the exclusion lemma (Lemma 4.2.3), exactly one of the following holds:

(i) there exists $\tau^{(p+1)} > \sigma$ such that $f(\tau) \leq f(\sigma)$;

(ii) there exists $\nu^{(p-1)} < \sigma$ such that $f(\nu) \geq f(\sigma)$.

If (i) is true, then $f(\tau) < a$, since σ is the only simplex with $f(\sigma) \in [a, b]$. So $\tau \in K(a)$. But $\sigma < \tau$, so $\sigma \in K(a)$, which can only happen when $K(a) = K(b)$, and there is nothing to prove.

Suppose that (ii) is true. Again, by the exclusion lemma, (i) cannot be true. Then, for all $\tau^{(p+1)} > \sigma$, we have $f(\tau) > f(\sigma)$. So $f(\tau) > b$. Following this, we claim that given any coface $\tilde{\tau}$ (of any dimension) of σ, we have $f(\tilde{\tau}) > b$. Let us look at the case when $\tilde{\tau}$ is of dimension $p + 2$, the rest will follow by induction. Suppose there is some $\tilde{\tau}^{(p+2)} > \sigma$ such that $f(\tilde{\tau}) \leq b$. Then $f(\tilde{\tau}) < a$. But $f(\tau^{(p+1)}) > b > a > f(\tilde{\tau})$ for all $\tau > \sigma$, and $\tilde{\tau}$ has at least two $(p + 1)$-faces which have σ as a face, so $\tilde{\tau}$ will have two "exceptions", and that is not allowed. It follows from the claim that $\sigma \notin K(a)$.

By our assumption, and the fact that σ is again the only simplex with $f(\sigma) \in [a, b]$, there exists $\nu^{(p-1)} < \sigma$ with $f(\nu) > b > f(\sigma)$. Here, note that $\nu \in K(b)$, since $\nu < \sigma$ and $f(\sigma) \leq b$. If $\tilde{\nu}^{(p-1)} \neq \nu$ is any other face of σ, then by the definition of discrete Morse function with at most one "exception" allowed at each simplex, we must have $f(\tilde{\nu}) < f(\sigma)$, and thus $f(\tilde{\nu}) < a$. So $\tilde{\nu}$ and all its faces are contained in $K(a)$ by definition. If $\tilde{\sigma}^{(p)} \neq \sigma$ is any other coface of ν, then, again by the definition of a discrete Morse function, we must have $f(\tilde{\sigma}) > f(\nu) > b$. By a similar argument as in the claim earlier in the proof, given any $\tilde{\sigma}$ (of any dimension) such that $\tilde{\sigma} > \nu$, we have $f(\tilde{\sigma}) > b$. So, $\nu \notin K(a)$ and ν does not have any coface in $K(a)$.

All and all, we see that $K(b) = K(a) \sqcup \nu \sqcup \sigma$ where $\{\nu, \sigma\}$ is a free pair. Therefore,

$$K(b) \searrow K(b) - \{\nu, \sigma\} = K(a).$$

□

We state another important theorem in discrete Morse theory here. It will be used together with the collapse theorem to prove the homotopy theorem we mentioned earlier.

Theorem 4.3.2 ([15]). *Let $f : K \to \mathbb{R}$ be a discrete Morse function. Suppose $\sigma^{(p)}$ is the only critical simplex with $f(\sigma) \in [a, b]$, then there is an attaching map $F : \partial B^p \to K(a)$ such that $K(b)$ is homotopy equivalent to*

$$K(a) \sqcup B^p /_{\{x \sim F(x) : x \in \partial B^p\}},$$

where B^p is the Euclidean unit ball of dimension p.

A discussion of the theorem can be found in [16]. The proof is very similar to the one above, using the fact that the interior of a p-simplex is homeomorphic to a Euclidean p-ball, and the homeomorphism implies homotopy equivalence. A complete proof can be found in [15].

Theorem 4.3.3 ([34]). *Let $f : K \to \mathbb{R}$ be a discrete Morse function and let m_p be the number of critical p-simplices of K. Then K is homotopy equivalent to a cell complex with exactly m_p p-cells.*

Proof. Again, we may assume f is $1-1$. Let $[a, b]$ be an interval such that $f(\sigma_i^{(p)}) \in [a, b]$ for all critical p-simplices σ_i, $1 \leq i \leq m_p$. Then, partition $[a, b]$ into m_p subintervals so that each subinterval $[a_i, b_i]$ contains only σ_i. From here, Theorem 4.3.1 and Theorem 4.3.2 tell us that each $[a_i, b_i]$ gives a homotopy equivalence between $K(b_i)$ and $K(a_i)$ with a p-cell attached. The theorem follows from taking the union of all these level subcomplexes.

□

4.3.2 Discrete Morse Inequalities

There are two inequalities that we will be presenting: the *weak discrete Morse inequality* and the *strong discrete Morse inequality*. We will first present our own proof for the second statement in the weak inequality; the first statement will be discussed after we present the strong inequality.

Theorem 4.3.4 (Weak discrete Morse inequalities, [34])**.** *Let f be a discrete Morse function with m_i different critical values in dimension i, $i = 1, 2, \dots, n = dim(K)$. Then*

(i) $b_i \leq m_i$ *for all* $i = 0, 1, \dots, n$*;*

(ii) $\sum_{i=0}^{n} (-1)^i m_i = \chi(K)$.

Proof. (ii) By the power of Lemma 4.2.5, we shall assume that f is excellent, so K has m_i critical i-simplices. Let d_i be the number of regular i-simplices that forms a gradient vector with an $(i+1)$-simplex, and recall that c_i is the total number of i-simplices in K. For any dimension i, we have the following equality:

$$c_i = m_i + d_{i,-} + d_{i,+},$$

where $d_{i,-}$ is the number of regular i-simplices that each forms a gradient vector with a regular $(i-1)$-simplex, and $d_{i,+}$ is the number of regular i-simplices that each forms a gradient vector with a regular $(i+1)$-simplex. For $i = 0$, since there is no lower dimension, $d_{0,-} = 0$; similarly for $i = n$, since there is no higher dimension, $d_{n,+} = 0$. Now, for $1 \leq i \leq n-1$, we also have $d_{i,+} = d_{i+1,-}$ by the definition of the gradient vector, which is formed by a pair of regular simplices. So it follows that

$$\begin{aligned}\sum_{i=0}^{n}(-1)^{i}c_{i} &= \sum_{i=0}^{n}(-1)^{i}m_{i}+\sum_{i=0}^{n}(-1)^{i}d_{i,-}+\sum_{i=0}^{n}(-1)^{i}d_{i,+}\\ &= \sum_{i=0}^{n}(-1)^{i}m_{i}+d_{0,-}+(d_{1,-}-d_{0,+})+(d_{2,-}-d_{1,+})+\cdots\\ &\quad +(d_{n,-}-d_{n-1,+})+d_{n,+}\\ &= \sum_{i=0}^{n}(-1)^{i}m_{i}.\end{aligned}$$

Therefore, we have $\sum_{i=0}^{n}(-1)^{i}m_{i}=\sum_{i=0}^{n}(-1)^{i}c_{i}=\chi(K)$.

□

Before we move on to the strong inequality, there are a few corollaries that need to be addressed. Together with Theorem 4.3.3, they will provide a straightforward proof for the strong inequality.

Corollary 4.3.5 ([33], Corollary 4.24)**.** *If topological spaces X and Y have the same homotopy type, then $H_n(X)\cong H_n(Y)$, where the isomorphism is induced by any homotopy equivalence.*

Remark. An immediate result of this corollary is that the simplicial complex and the cell complex in Theorem 4.3.3 will have the same Betti numbers.

Corollary 4.3.6. *Let K be a simplicial complex. Then, for each $p=0,1,2,\ldots,n,n+1,\ldots$, we have*

$$b_p-b_{p-1}+b_{p-2}-\cdots+(-1)^{p}b_0\leq c_p-c_{p-1}+c_{p-2}-\cdots+(-1)^{p}c_0,$$

where b_i is the i-th Betti number of K (in our case, with respect to $\mathbb{Z}_2$) and c_i is the number of i-simplices of K.

The proof for this corollary is pure algebra: we have

$$\begin{aligned}\sum_{i=0}^{p}(-1)^{p-i}\,b_i &= \mathrm{null}\,(\partial_p) - \mathrm{rank}\,(\partial_{p+1}) - \mathrm{null}\,(\partial_{p-1}) + \mathrm{rank}\,(\partial_p)\\ &\quad + \mathrm{null}\,(\partial_{p-2}) - \mathrm{rank}\,(\partial_{p-1}) - \cdots + (-1)^p\,(\mathrm{null}\,(\partial_0) - \mathrm{rank}\,(\partial_1))\\ &= \sum_{i=0}^{p}(-1)^{p-i}\,(\mathrm{null}\,(\partial_i) + \mathrm{rank}\,(\partial_i)) - \mathrm{rank}\,(\partial_{p+1})\\ &= \sum_{i=0}^{p}(-1)^{p-i}\,c_i - \mathrm{rank}\,(\partial_{p+1})\\ &\le \sum_{i=0}^{p}(-1)^{p-i}\,c_i,\end{aligned}$$

where the second equality follow from that $\mathrm{null}\,(\partial_0) = 0$, while the last inequality follows from that $\mathrm{rank}\,(\partial_{p+1}) \geq 0$.

Remark. With Theorem 4.3.3, Corollary 4.3.5 and Corollary 4.3.6, given a simplicial complex, we may assume that $c_i = m_i$ for all i, and this assumption does not affect the Betti numbers.

Now, we can state the strong inequality:

Theorem 4.3.7 (Strong Discrere Morse inequality, [34])**.** *Let $f : K \to \mathbb{R}$ be a discrete Morse function. For each $p = 0, 1, \ldots, n, n+1, \ldots,$ we have*

$$b_p - b_{p-1} + b_{p-2} - \cdots + (-1)^p\, b_0 \leq m_p - m_{p-1} + m_{p-2} - \cdots + (-1)^p\, m_0.$$

Proof. The inequality follows immediately from remarks of Corollary 4.3.5 and Corollary 4.3.6.

□

Now, we have a straightforward **proof for the inequalities in Theorem 4.3.4**.

Proof. Suppose we want to prove the inequality for certain i. By the strong discrete Morse inequality, we have

$$\begin{aligned}
b_i - b_{i-1} + \cdots + (-1)^i b_0 &\leq m_i - m_{i-1} + \cdots + (-1)^i m_0 \\
b_i - \left(b_{i-1} - \cdots + (-1)^i b_0\right) &\leq m_i - \left(m_{i-1} - \cdots + (-1)^i m_0\right) \\
b_i &\leq m_i - \left(m_{i-1} - \cdots + (-1)^i m_0\right) + \left(b_{i-1} - \cdots + (-1)^i b_0\right) \\
b_i &\leq m_i,
\end{aligned}$$

since

$$b_{i-1} - b_{i-2} + \cdots + (-1)^i b_0 \leq m_{i-1} - m_{i-2} + \cdots + (-1)^i m_0,$$

and the sum of terms after m_i in the third inequality is non-positive.

□

4.3.3 Cancelling Critical Simplices

Whenever we have an inequality, we are interested in how to achieve its optimality. Theorem 4.3.4 (i) is no exception. It tells us that, theoretically, we could potentially reduce the number of critical simplices in a simplicial complex with the lower bound of Betti number in respective dimensions. To study this matter, we begin with the following definitions:

Definition 4.14. [34] Let K be an n-dimensional simplicial complex and $f : K \to \mathbb{R}$ be a discrete Morse function. The **discrete Morse vector of** f is defined as

$$\vec{f} \coloneqq \left(m_0^f, m_1^f, \ldots, m_n^f\right),$$

where m_i^f is the number of critical i-simplices of f. A discrete Morse vector $\vec{f}$ is said to be **optimal** if $\sum_{i=0}^n m_i^f$ is minimal, i.e. for any other discrete Morse function g on K, we have $\sum_{i=0}^n m_i^f \leq \sum_{i=0}^n m_i^g$.

Definition 4.15. A discrete Morse vector is said to be **perfect** if $\vec{f} = (b_0, b_1, \ldots, b_n)$, where

b_i is the i-th Betti number.

In other words, a discrete Morse function f produces a perfect discrete Morse vector if it gives equality in Theorem 4.3.4 (i). The concept of a *perfect Morse function* appeared first in the classical Morse theory, where the inequality concerns the number of critical points. In general, not every manifold can admit a perfect Morse function ([2]). Here in the discrete case, it is also true that not every simplicial complex admits a perfect discrete Morse function. For example, Ayala, Fernández, and Vilches ([6]) showed that a connected simplicial complex that is homologically trivial and non-collapsible does not admit a perfect discrete Morse function. The following proposition gives some complexes that do admit perfect discrete Morse functions.

Proposition 4.3.8. *There exists a perfect discrete Morse function on Δ^n for all $n = 0, 1, 2, \ldots$.*

Proof. The proof simply follows from the fact that the simplicial cone of an n-simplex is an $(n+1)$-simplex and Proposition 3.2.3. It is worth noting that the statement is also true for the n-sphere since they are homeomorphic to the n-simplex.

□

Proposition 4.3.9. *A perfect discrete Morse vector is unique and optimal.*

Proof. The uniqueness follows from the fact that Betti numbers are unique for a given simplicial complex. The optimality follows from the weak discrete Morse inequality: for any given discrete Morse function f, we have $b_i \leq m_i^f$ for all $0 \leq i \leq n$, so $\sum_{i=0}^{n} b_i \leq \sum_{i=0}^{n} m_i^f$.

□

These definitions tell us that a discrete Morse function is "better" if it induces less critical simplices. It makes a lot of sense if we think about what we have discussed in Section 3.2 and Subsection 4.3.1: the Collapse theorem tells us that if there is no critical value in certain interval, we can simplify the simplicial complex without changing its simple

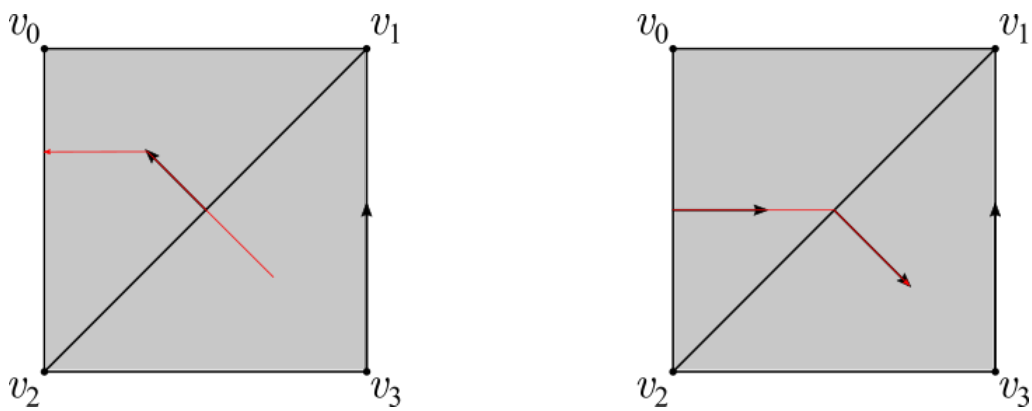

Figure 4.8: Cancelling critical simplices through reversing gradient path

homotopy type; so, given a simplicial complex, if we can find a perfect discrete Morse function on it, then we will be able to simplify its structure to the maximum degree without changing its topological properties. Then, given a discrete Morse function, how can we make it better in this sense? One of the methods available is called, as the title of this subsection suggests, *cancelling critical simplices*. The idea originates in [15] (Chapter 11), and we will present it in combination with [16] and [34]. Let us first demonstrate this method with the following example:

Example 4.3.10. Consider the gradient vector field V_f on the left-side of Figure 4.8. Observe there is a non-closed gradient path

$$v_1v_2v_3, v_1v_2, v_0v_1v_2, v_0v_2,$$

where $v_1v_2v_3$ and v_0v_2 are critical simplices. Now, we "reverse" this path to get a new gradient path, while keeping everything else the same:

$$v_0v_2, v_0v_1v_2, v_1v_2, v_1v_2v_3.$$

Note here, both $v_1v_2v_3$ and v_0v_2 are parts of the new gradient vector field, thus no longer critical.

The method in the example is summarized in the next proposition, reversing paths to construct a new gradient vector field with less critical simplicies.

Proposition 4.3.11 (Cancelling critical simplices, [16], [34])**.** *Let V be the gradient vector field (induced by a discrete Morse function f) on K. Suppose that $\sigma^{(p)}$ and $\tau^{(p+1)}$ are critical simplices, and there is exactly one V-path γ from a boundary (codimensional-1) simplex of τ to σ. Then, there exists another gradient vector field $\bar{V}$ (induced by a different discrete Morse function g) on K such that $\bar{V} - \gamma = V - \gamma$, that has the same critical simplices except that σ and τ are no longer critical.*

Proof. The gradient path γ has the following form:

$$\gamma_0^{(p)}, \tau_0^{(p+1)}, \gamma_1^{(p)}, \dots, \gamma_{n-1}^{(p)}, \tau_{n-1}^{(p+1)}, \gamma_n = \sigma,$$

where $\gamma_0^{(p)} < \tau$ is a codimensional-1 face of τ (in the boundary of τ). Define $\bar{V}$ in the following way:

(a) $\bar{V} - \gamma = V - \gamma$;

(b) $(\gamma_0, \tau) \in \bar{V}$;

(c) $(\gamma_{i+1}, \tau_i) \in \bar{V}$ for $i = 0, 1, \dots, n-1$.

Note that σ and τ are no longer critical with $\bar{V}$. By (a), we see that $\bar{V}$ and V agree on everything except for the gradient path γ. So the critical simplices of $\bar{V} - \gamma$ are exactly those of $V - \gamma$. Moreover, $\bar{V} - \gamma$ cannot contain any closed gradient path since V cannot contain any, as V is induced by a discrete Morse function (Theorem 4.2.9). Suppose there is a closed gradient path of $\bar{V}$ on γ, that is, without loss of generality, there exists $\gamma_j^{(p)} = \gamma_i^{(p)}$ for some $0 < i < j < n$. However, this implies that

$$\gamma' \coloneqq \gamma_0, \tau_0, \dots, \gamma_i = \gamma_j, \tau_j, \dots, \gamma_{n-1}, \tau_{n-1}, \gamma_n = \sigma$$

is a gradient path of V from a boundary simplex of τ to σ, and $\gamma' \neq \gamma$, which contradicts the fact that γ is unique. Hence, $\bar{V}$ is a gradient vector field.

□

Remark. As we mentioned, the method is essentially reversing the unique gradient path from a boundary simplex of a critical simplex to another critical simplex. In fact, this reversed gradient path is unique as well, which can be shown in a similar fashion as in the proof above.

4.3.4 Homotopy between Discrete Morse Functions

To finish up this chapter, we present a result which shows that we can "move" from one discrete Morse function to any other discrete Morse function through homotopy.

Definition 4.16. A discrete Morse function is said to be **flat** if we have $f(\sigma) = f(\tau)$ whenever (σ, τ) is a regular pair induced by f.

The next proposition and its proof show a technique called *flattening* through which we can transform a given discrete Morse function into a flat one that is Forman equivalent to the original one.

Proposition 4.3.12 ([34]). *Let $f : K \to \mathbb{R}$ be a discrete Morse function and V_f be the gradient vector field induced by f. Then there exists a flat discrete Morse function on K that is Forman equivalent to f.*

For the proof, we use the following lemma:

Lemma 4.3.13. *Let $f : K \to \mathbb{R}$ be a discrete Morse function. If there exists simplices $\sigma^{(p)} < \tau^{(p+2)}$ such that $f(\sigma) \geq f(\tau)$, then both σ and τ are not critical.*

Proof. By the nature of a simplicial complex, if $\sigma^{(p)} < \tau^{(p+2)}$, then there exists $\eta^{(p+1)}$ such that $\sigma < \eta < \tau$. Suppose σ is critical. Then τ cannot be critical, otherwise we will have

$f(\sigma) < f(\eta) < f(\tau)$, which is a contradiction. Similarly, if τ is critical, then σ cannot be critical. So, σ and τ cannot both be critical.

On the other hand, suppose that σ is not critical. If τ is not critical either, then we are done. So suppose that τ is critical. Then there exists $\eta^{(p+1)}$ such that $f(\tau) > f(\eta) > f(\sigma)$, which is a contradiction. The other direction is similar. Therefore, both σ and τ are not critical.

$\square$

Now, we turn our attention to;

Proof of Proposition 4.3.12. The **flattening** of f is a function g defined on K as the following:

$$g(\sigma) := \begin{cases} f(\tau) & \text{if } (\sigma, \tau) \in V_f \text{ for some } \tau, \\ f(\sigma) & \text{otherwise.} \end{cases}$$

First of all, we see that every regular pair f is also a regular pair of g by the definition of g, so if we can show that every critical simplex of f is also critical for the function g, then we are done. Also, for any $\sigma \in K$, we have $g(\sigma) \leq f(\sigma)$: when σ is regular, $f(\sigma) \geq f(\tau) = g(\sigma)$; when σ is critical, $g(\sigma) = f(\sigma)$. Let $\tau^{(p)}$ be a critical simplex of f. Then for any $\sigma^{(p-1)} < \tau < \eta^{(p+1)}$, we have $f(\sigma) < f(\tau) < f(\eta)$. Since τ is critical, we have $g(\tau) = f(\tau) > f(\sigma) \geq g(\sigma)$. We also need to show that $g(\eta) > g(\tau)$. If η is not a tail of a vector (see Definition 4.8) of V_f, then $g(\eta) = f(\eta) > f(\tau) = g(\tau)$, and we are finished. So suppose that it is, i.e. there exists some $\gamma^{(p+2)} > \eta$ such that $f(\eta) \geq f(\gamma)$. Then, by definition, $g(\eta) = f(\gamma)$. If indeed $g(\eta) > g(\tau)$, then $f(\gamma) = g(\eta) > g(\tau) = f(\tau)$. We use proof by contradiction and suppose that $f(\tau) \geq f(\gamma)$. Then, Lemma 4.3.13 tells us that τ is not critical, which is a contradiction. Therefore, $g(\eta) > g(\tau)$. Hence, $V_f = V_g$, and by Theorem 4.2.8, f and g are Forman equivalent.

$\square$

The following lemma is the main result of this subsection. It describes a specific version of what is known as the **straight-line homotopy** between functions. With the power of Proposition 4.3.12, we may assume flat discrete Morse functions.

Lemma 4.3.14 ([34])**.** *Let $f, g : K \to \mathbb{R}$ be flat discrete Morse functions. Define*

$$h_t(\sigma) \coloneqq (1-t)f(\sigma) + tg(\sigma)$$

for all $\sigma \in K$ and $t \in [0,1]$. Then h_t is a discrete Morse function on K for all $t \in [0,1]$. Furthermore, for any $t \in (0,1)$, we have $V_{h_t} = V_f \cap V_g$.

Proof. For any $\sigma^{(p)} \in K$, we have

$$|\tau^{(p-1)} < \sigma : f(\tau) \geq f(\sigma)| \leq 1$$

and

$$|\eta^{(p-1)} < \sigma : g(\eta) \geq g(\sigma)| \leq 1.$$

Since both f and g are flat, equalities are attained when there is exception (see Definition 4.4 for "exception").

If such exception happens at the same $(p-1)$-simplex for f and g, say τ_0, then

$$\begin{aligned} h_t(\tau_0) &= (1-t)f(\tau_0) + tg(\tau_0) \\ &= (1-t)f(\sigma) + tg(\sigma) \\ &= h_t(\sigma), \end{aligned}$$

and no other exceptions in dimension $p-1$ are possible.

If such exception happens at τ_1 for f and at τ_2 for g, $\tau_1 \neq \tau_2$, then exceptions are only possible at τ_1 and τ_2 for h_t. But

$$
\begin{aligned}
h_t(\tau_1) &= (1-t)f(\tau_1) + tg(\tau_1) \\
&< (1-t)f(\sigma) + tg(\sigma) \\
&= h_t(\sigma)
\end{aligned}
$$

since $g(\tau_1) < g(\sigma)$, and similarly,

$$
\begin{aligned}
h_t(\tau_2) &= (1-t)f(\tau_2) + tg(\tau_2) \\
&< (1-t)f(\sigma) + tg(\sigma) \\
&= h_t(\sigma)
\end{aligned}
$$

since $f(\tau_2) < f(\sigma)$. Hence, h_t does not have any exceptions in this case. In addition, it does not have any exceptions if neither f nor g has any.

In a similar fashion, we can prove the same for $(p+1)$-simplices. Consequently, h_t is a discrete Morse function, and particularly, a flat one.

Now, we show that $V_{h_t} = V_f \cap V_g$. Let $(\sigma^{(p)}, \tau^{(p+1)}) \in V_{h_t}$. Then $h_t(\sigma) = h_t(\tau)$, since h_t is flat. Since f and g are flat as well, we have $f(\sigma) \leq f(\tau)$ and $g(\sigma) \leq g(\tau)$. Suppose that either $f(\sigma) < f(\tau)$ or $g(\sigma) < g(\tau)$ is true. Then, we must have $h_t(\sigma) = (1-t)f(\sigma) + tg(\sigma) < (1-t)f(\tau) + tg(\tau) = h_t(\tau)$, which is a contradiction. So, $f(\sigma) = f(\tau)$ and $g(\sigma) = g(\tau)$, hence $(\sigma, \tau) \in V_f$ and $(\sigma, \tau) \in V_g$. On the other hand, let $(\sigma^{(p)}, \tau^{(p+1)}) \in V_f \cap V_g$. Since both f and g are flat, we have $f(\sigma) = f(\tau)$ and $g(\sigma) = g(\tau)$. So it is straight forward that $h_t(\sigma) = (1-t)f(\sigma) + tg(\sigma) = (1-t)f(\tau) + tg(\tau) = h_t(\tau)$. Hence $(\sigma, \tau) \in V_{h_t}$.

$\square$

Remark. By Theorem 4.2.8, all h_t's are Forman equivalent on K.

Here is a simple example illustrating Lemma 4.3.14:

Example 4.3.15. Let K be the simplicial complex in Example 3.1.1. Let $f, g : K \to \mathbb{R}$

be discrete Morse functions and let $h_t = (1-t)f + tg$, $0 < t < 1$. We know from Lemma 4.3.14 that h_t is a discrete Morse function on K for all t. Observe that f and g only differ at two simplices: the bottom 1-simplex and the bottom-right 0-simplex. The resulting h_t will then have a value between 2 and 3 on the 1-simplex and a value between 1 and 2 on the 0-simplex, making both simplices critical. As shown in Figure 4.9, h_t indeed induces the same gradient vector field as $V_f \cap V_g$.

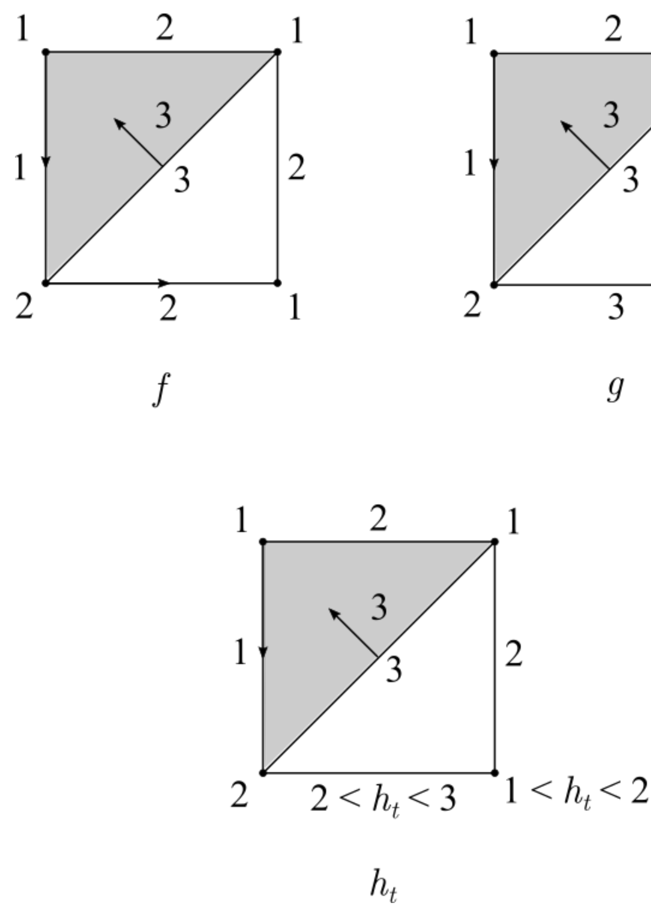

Figure 4.9: h_t induces a gradient vector field of $V_f \cap V_g$.

Chapter 5

More on Discrete Morse Theory

In this chapter, we dive deeper into topics in discrete Morse theory. We will first look at the set of all discrete Morse functions on a given simplicial complex in two different ways in Section 5.1, then we will revisit the induced gradient vector field by viewing it as a function in Section 5.2. Finally, in Section 5.3, we will present a way to simplify calculations of the simplicial homology by reducing the size of chains.

5.1 Morse Complex

Morse Complex was first studied by Chari and Joswig in [10] under the name of *complex of discrete Morse functions* (associated with the given simplicial complex). It served as a mean to study the set of all possible discrete Morse functions on a given simplicial complex. This section is closely connected to Subsection 2.5.2, 4.2.3 and 4.2.4. We begin with a term in graph theory.

Definition 5.1. A **matching** on a directed graph is a set of arrows such that no two of them enter or leave the same vertex.

Remark. By Lemma 4.2.3, we know that the collection of upward arrows of a directed Hasse diagram of a gradient vector field forms a matching, since otherwise, the simplex represented by the shared vectex will have two exceptions, thus the function fails to be a discrete Morse function.

Definition 5.2. A **discrete Morse matching** is an acyclic matching of a directed Hasse diagram.

Remark. This definition makes sense because of Theorem 4.2.9.

Example 5.1.1. The highlighted arrows in Figure 5.1 form a discrete Morse matching on the directed Hasse diagram in Example 4.2.11.

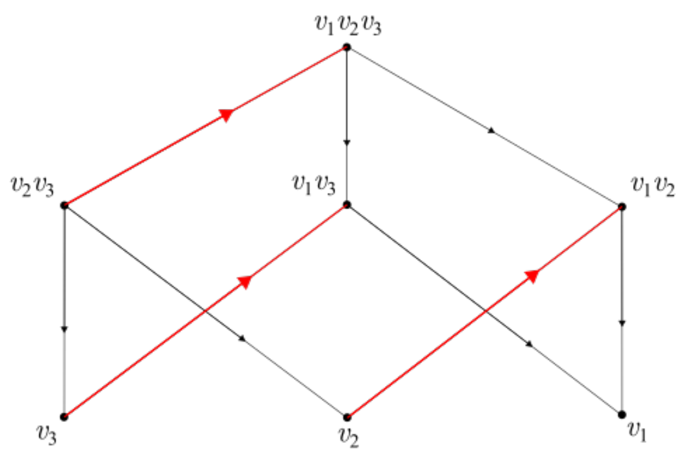

Figure 5.1: Discrete Morse matching on a directed Hasse diagram

We can clearly see that there can exist multiple discrete Morse matchings given some directed Hasse diagram. In fact, the collection of all such discrete Morse matchings form a simplicial complex.

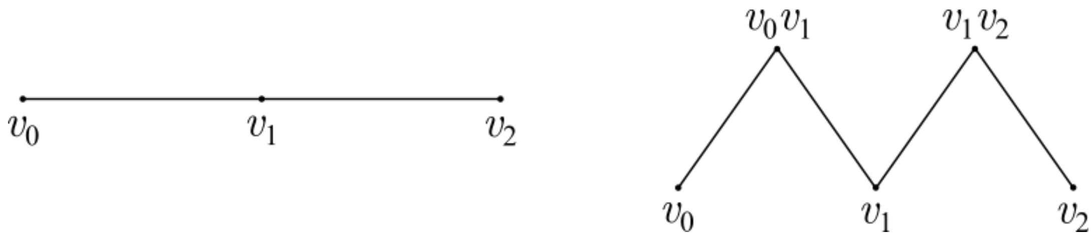

Figure 5.2: A simplicial complex and its (undirected) Hasse diagram

Definition 5.3. Let K be a simplicial complex. Consider $\{\mathcal{H}_V\}$, the collection of directed Hasse diagrams induced by discrete Morse functions on K. The **Morse complex** of K, denoted $\mathcal{M}(K)$, is the simplicial complex constructed in the following way: each vertex (0-simplex) of $\mathcal{M}(K)$ is given by an upward arrow of some $\mathcal{H}_V$; an n-simplex $\sigma = v_0v_1 \dots v_{n-1}$ exists if the collection of upward arrows corresponding to vertices of σ form a non-empty discrete Morse matching.

We can readily check that this definition does indeed give a simplicial complex. Clearly, every vertex is contained in $\mathcal{M}(K)$; if σ is a p-simplex, and $\tau \subseteq \sigma$, then σ corresponds to a non-empty discrete Morse matching, and any non-empty subset of a non-empty discrete Morse matching is again a non-empty discrete Morse matching.

Example 5.1.2. Consider the simplicial complex K on the left-side of Figure 5.2. On the right-side is its (undirected) Hasse diagram.

There are 4 possible upward arrows, each on an edge of the Hasse diagram. So, $\mathcal{M}(K)$ has 4 vertices, $\{v_0, v_0v_1\}, \{v_1, v_0v_1\}, \{v_1, v_1v_2\}$, and $\{v_2, v_1v_2\}$. To determine any higher dimensional simplices, we check which upward arrows form some non-empty discrete Morse matching. First, observe that there can be at most be two upward arrows at the same directed Hasse diagram, any more upward arrows will contradict Lemma 4.2.7, so there is no simplex in dimension 2 or higher. By the same lemma, there can be at most one upward arrow entering or leaving the same vertex, hence we obtain three 1-simplices of $\mathcal{M}(K)$, shown in Figure 5.3.

There is an alternative definition for a Morse complex, which builds the complex from

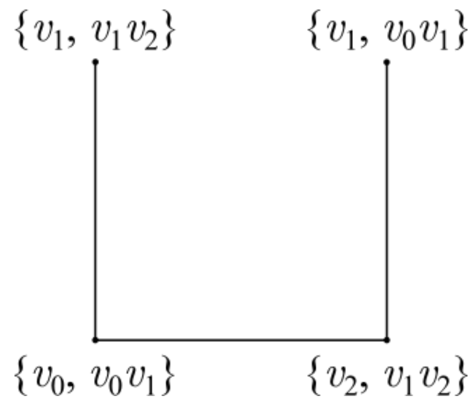

Figure 5.3: The Morse complex of the simplicial complex in Example 5.1.2

possible discrete Morse functions directly. We will state it and show that it is equivalent to the one above.

Definition 5.4. A discrete Morse function is said to be **primitive** if it has only one regular pair, which is called a **primitive pair**.

Definition 5.5. Let f and g be discrete Morse functions. We write $f \leq g$ if every regular pair of f is also a regular pair of g. Let h_i and h_j be discrete Morse functions. We say h_i and h_j are **compatible** is there exists a discrete Morse function h such that $h_i \leq h$ and $h_j \leq h$.

Definition 5.6 (Alternative definition for Morse complex)**.** The **Morse complex** $\mathcal{M}(K)$ is the simplicial complex whose n-simplices are given by $n+1$ compatible primitive discrete Morse functions on K.

Example 5.1.3. Consider the same simplicial complex from Example 5.1.2. We want to find all possible compatible primitive discrete Morse functions. Each primitive discrete Morse function by itself is trivially compatible, so we have four 0-simplices. By Lemma 4.2.3, the following primitive pairs are compatible: $\{v_0, v_0v_1\}$ and $\{v_1, v_1v_2\}$, $\{v_0, v_0v_1\}$ and $\{v_2, v_1v_2\}$, $\{v_1, v_0v_1\}$ and v_2, v_1v_2, each giving a 1-simplex. Hence, we obtain the same Morse complex as in Example 5.1.2.

This process of constructing the Morse complex can be seen as laying primitive discrete Morse functions on top of one another, compatibly, to build new discrete Morse functions.

Proposition 5.1.4. *Definition 5.3 and Definition 5.6 are equivalent.*

Proof. In Definition 5.3, a 0-simplex is created if and only if there is an upward arrow in some directed Hasse diagram, if and only if there is a primitive discrete Morse function that induces the directed Hasse diagram, which is exactly what Definition 5.6 describes. Again, starting with Definition 5.3, an n-simplex corresponds to a collection of upward arrows that form a non-empty acyclic matching of a directed Hasse diagram, which means no closed path and no two upward arrows entering or leaving the same vertex. Then, Lemma 4.2.3 and Theorem 4.2.9 imply that upward arrows give compatible primitive pairs, as stated by Definition 5.6.

□

Proposition 5.1.5. *For $n \geq 1$, there does not exist any simplicial complex K such that $\mathcal{M}(K) = \Delta^{(n)}$.*

Proof. Suppose that $\mathcal{M}(K) = \Delta^n$. Since Δ^n has $(n+1)$ 0-simplices and one n-simplex among others, K has a total of $n+1$ primitive discrete Morse functions and they are all compatible. But this is not possible. On any simplex of K with dimension ≥ 1, there are at least two ways of forming a primitive pair, and they cannot be compatible, hence a contradiction.

□

At this point, readers should expect very high complexity of Morse complexes when the dimension of the given simplicial complex is high. Chari and Joswig studied combinatorial and topological properties of Morse complexes of various simplicial complexes such as graphs, a circle, and simplices. We refer readers to [10] for their work.

Even with such complexity, a Morse complex and its underlying simplicial complex share similar algebraic structures. Capitelli and Minian showed that a simplicial complex is uniquely determined by its set of discrete Morse functions, i.e. its Morse complex, up to simplicial isomorphism. We state their main theorem here, and refer the readers to [9] for the proof.

Theorem 5.1.6 (Capitelli and Minian)**.** *Let K and L be simplicial complexes. If $\mathcal{M}(K)$ is isomorphic to $\mathcal{M}(L)$, then K is isomorphic to L.*

In [27], Lin and Scoville classified automorphism groups of the Morse complex. They proved the following theorem:

Theorem 5.1.7 (Lin and Scoville)**.** *Let K be a finite, connected (abstract) simplicial complex. Then*

$$\mathrm{Aut}(\mathcal{M}(K)) \cong \begin{cases} \mathrm{Aut}(K) & \text{if } K \neq \partial\Delta^{(n)}, C_{(n)} \\ \mathrm{Aut}(C_{(2n)}) & \text{if } K = C_{(n)} \\ \mathrm{Aut}(K) \times \mathbb{Z}_2 & \text{if } K = \partial\Delta^{(n)} \end{cases},$$

where $\partial\Delta^{(n)}$ is the boundary of an n-simplex and $C_{(n)}$ is the circle of length n.

Here is a special family of a Morse complex:

Definition 5.7. Let K be a simplicial complex. The **pure Morse complex of discrete Morse functions** of K, denoted $\mathcal{M}_{pure}(K)$, is the subcomplex of $\mathcal{M}(K)$ generated by facets of maximal dimension.

Generally, a simplicial complex is said to be *pure* if all of its facets have the same dimension. One thus might think that Definition 5.7 is not necessary since a Morse complex is a simplicial complex by definition. However, we shall see in the next example that the Morse complex of a pure simplicial complex is not necessarily pure as a simplicial complex.

Example 5.1.8. Let K be the simplicial complex in Figure 5.4. It is a pure simplicial complex since all of its facets are 1-simplices. However, its Morse complex is not pure. The gradient vector field on the bottom left corresponds to a 2-dimensional facet of $\mathcal{M}(K)$. The one on the bottom right corresponds to a 1-simplex of $\mathcal{M}(K)$, but it is not contained in any 2-simplex since adding an arrow on the middle edge, regardless of its direction, will no longer give a discrete Morse function.

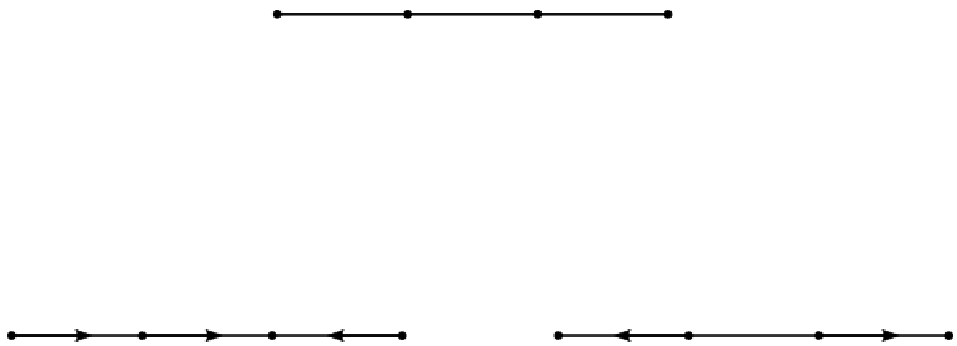

Figure 5.4: A pure simplicial complex whose Morse complex is not pure

Ayala, Fernández, and Vilches showed that the pure Morse complex of a tree is collapsible (Definition 3.6). They also proved some nice properties of the pure Morse complex of an arbitrary connected graph, we refer interested readers to [4] for details.

5.2 Gradient Vector Revised

In this section, we redefine the induced gradient vector field in the form of a function on a simplicial complex. We will use the new definition and the definition from Section 4.2.3 interchangeably throughout this section. Recall notations from Section 3.1 and Section 3.3.

Definition 5.8 (Gradient vector field redefined)**.** Let $f : K \to \mathbb{R}$ be a discrete Morse function. Define the function $V_p : \Bbbk^{c_p} \to \Bbbk^{c_{p+1}}$ by

$$V_p(\sigma) := \begin{cases} \tau & \text{if there exists } \tau > \sigma \text{ such that } f(\tau) \le f(\sigma); \\ 0 & \text{otherwise.} \end{cases}$$

It then follows from the definition of $\Bbbk^{c_p}$ that $V_p\left(\sum \sigma\right) = \sum V_p(\sigma)$. The collection of functions $V = \{V_p\}$, $p = 0, 1, \ldots, \dim(K)$ is called the **gradient vector field induced by** f.

Proposition 5.2.1 ([34])**.** *Let V be a gradient vector (in the sense of Definition 5.8) on a simplicial complex K and let $\sigma^{(p)} \in K$. Then, we have*

(i) $V_{i+1} \circ V_i = 0$ for all integers $i \ge 0$;

(ii) $|\{\tau^{(p-1)} : V(\tau) = \sigma\}| \leq 1$;

(iii) σ *is critical if and ony if* $\sigma \notin Im(V)$ *and* $V(\sigma) = 0$.

Proof. If $V_p(\sigma) = 0$, then (i) holds trivially. If $V_p(\sigma) \neq 0$, then σ is the tail of an arrow of V_f and there exists $\nu^{(p+1)} \in K$, which is the head of the same arrow, such that $V(\sigma) = \nu$. But Lemma 4.2.3 tells us that there cannot be any $\gamma^{(p+2)} > \nu$ such that $f(\gamma) \leq f(\nu)$, hence $V_{p+1} \circ V_p = 0$. Lemma 4.2.3 also directly implies (ii). For (iii), if σ is critical, then by definition, $V(\sigma) = 0$, and $\sigma \notin Im(V)$ since for all $\tau^{(p-1)} < \sigma$, $f(\tau) < f(\sigma)$. Conversely, $\sigma \notin Im(V)$ implies that σ is not a tail of an arrow and $V(\sigma) = 0$ implies that σ is either a tail of an arrow or a critical simplex. Hence, together we have that σ is critical.

□

This definition of gradient vector field can be viewed as a *flow*, sending a simplex from the tail of an arrow to the head. To formally study it this way, Forman ([15]) defined what is known as the *discrete gradient flow*:

Definition 5.9. Let V be a gradient vector field on a simplicial complex K. The **(discrete) gradient flow** or simply **flow** induced by V is defined as $\Phi_p : \Bbbk^{c_p} \to \Bbbk^{c_p}$, $\Phi_p(\sigma) = \sigma + \partial_{p+1}(V_p(\sigma)) + V_{p-1}(\partial_p(\sigma))$.

The term is well defined, since by definition, both $\partial_{p+1}(V_p(\sigma))$ and $V_{p-1}(\partial_p)(\sigma))$ give a collection of p-simplices. Hence, we write $\Phi(\sigma) = \sigma + \partial(V(\sigma)) + V(\partial(\sigma))$ when p is clear from the context. It is also clear from the definition that Φ is a linear transformation from a vector space to itself.

Example 5.2.2. Consider the simplicial complex K with the gradient vector field V in Figure 5.5.

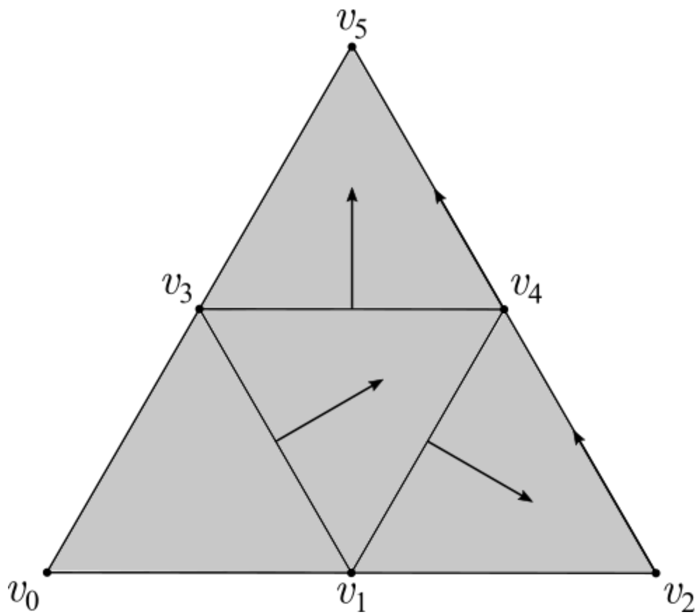

Figure 5.5: Gradient vector field in Example 5.2.2

Let us calculate gradient flows of some simplices:

$$\begin{aligned}
\Phi(v_1v_3) &= v_1v_3 + \partial(v_1v_3v_4) + V(v_3 + v_1) \\
&= v_1v_3 + v_3v_4 + v_1v_4 + v_1v_3 + 0 \\
&= v_3v_4 + v_1v_4 \\
\Phi(v_1v_4) &= v_1v_4 + \partial(v_1v_2v_4) + V(v_1 + v_4) \\
&= v_1v_4 + v_2v_4 + v_1v_4 + v_1v_2 + 0 + v_4v_5 \\
&= v_2v_4 + v_1v_2 + v_4v_5 \\
\Phi(v_4v_5) &= v_4v_5 + \partial(0) + V(v5 + v_4) \\
&= v_4v_5 + 0 + 0 + v_4v_5 \\
&= 0.
\end{aligned}$$

Here is a useful technical result about the gradient flow and the boundary operator:

Lemma 5.2.3 ([15]). *Let K be a simplicial complex, ∂ the boundary operator, and Φ a gradient flow on K. Then $\Phi\partial = \partial\Phi$.*

Proof. Let $\sigma \in K$. Then

$$\Phi(\partial(\sigma)) = \partial(\sigma) + \partial(V(\partial(\sigma))) + V(\partial(\partial(\sigma)))$$

and

$$\begin{aligned}\partial(\Phi(\sigma)) &= \partial(\sigma + \partial(V(\sigma)) + V(\partial(\sigma))) \\ &= \partial(\sigma) + \partial(\partial(V(\sigma))) + \partial(V(\partial(\sigma))).\end{aligned}$$

Recall that $\partial \circ \partial = 0$ (Proposition 3.3.2, hence $V(\partial(\partial(\sigma))) = 0$ and $\partial(\partial(V(\sigma))) = 0$. Therefore, $\Phi(\partial(\sigma)) = \partial(\Phi(\sigma))$.

□

Flows are well-studied objects in the field of dynamical systems. Whenever we have a dynamical system, naturally, we are interested in its stable (and unstable) states.

Definition 5.10. Let $f : X \to X$ be a function. We say f **stabilizes** at x if there exists m such that $f^{m+1}(x) = f^m(x)$.

Of course, this implies that if f stabilizes at x, then $f^{m'}(x) = f^m(x)$ for any $m' > m$. Since $\Phi_p : \Bbbk^{c_p} \to \Bbbk^{c_p}$, we can apply this idea to the gradient vector field, and we see in the next proposition that a gradient flow behaves really well on a simplicial complex.

Proposition 5.2.4. *Let Φ be the flow induced by a gradient vector field V on a simplicial complex K. Then Φ stabilizes at every vertex of K.*

Proof. Without loss of generality, suppose that there is a single vertex v at which Φ does not stabilize, that is $\Phi^m(v) \neq \Phi^{m+1}(v)$ for any m. Since Φ stabilizes at every other vertex v_i', take $M = \max\{m_i\}_{i=1}^n$, where m_i is the smallest number for which $\Phi^{m_i}(v_i') = \Phi^{m_i+1}(v_i')$. Since in particular, $\Phi_0 : \Bbbk^{c_0} \to \Bbbk^{c_0}$, we may assume that for an arbitrary m, $\Phi^m(v) = \sum_{i \in I} v_i'$, where

I is an index subset of $\{1, 2, \ldots, n\}$. Then

$$\begin{aligned}
\Phi^{M+m}(v) = \Phi^M(\Phi^m(v)) &= \Phi^M\left(\sum_{i\in I} v_i'\right) \\
&= \sum_{i\in I} \Phi^M(v_i') \\
&= \sum_{i\in I} \Phi^{M+1}(v_i') \\
&= \Phi^{M+1}\left(\sum_{i\in I} v_i'\right) \\
&= \Phi^{M+1}(\Phi^m(v)) \\
&= \Phi^{M+m+1}(v),
\end{aligned}$$

which contradicts that Φ does not stabilize at v. Therefore, Φ stabilizes at every vertex.

□

In fact, the general result is that the gradient flow stabilizes at every linear combination of simplices of a simplicial complex.

Theorem 5.2.5 ([15]). *Let $\sigma \in \Bbbk^{c_p}$ for $p = 0, 1, \ldots, \dim(K)$. Then there exists N such that $\Phi^i(\sigma) = \Phi^j(\sigma)$ for all $i, j \geq N$.*

To prove this theorem, we need a few technical results.

Proposition 5.2.6 ([34]). *Let $\sigma_1, \ldots, \sigma_r$ be the p-dimensional simplices of a simplicial complex K and write $\Phi(\sigma_i) = \sum_j a_{ij}\sigma_j$ where a_{ij} is either 0 or 1 (so it it written as a linear combination over the field $\mathbb{Z}_2$). Then $a_{ii} = 1$ if and only if σ_i is critical. Furthermore, if $a_{ij} = 1$, then $f(\sigma_j) < f(\sigma_i)$.*

Proof. By Proposition 5.2.1 (iii), for any σ_i, $1 \leq i \leq r$, we know exactly one of the following is satisfied: σ_i is critical, $\sigma_i \in Im(V)$, or $V(\sigma_i) \neq 0$.

Suppose σ_i is critical, then $V(\sigma_i) = 0$ and $f(\tau) < f(\sigma_i)$ for any $\tau^{(p-1)} < \sigma_i$. In addition, $V(\tau) = 0$ or $V(\tau) = \tilde{\sigma}_\tau \neq \sigma_i$ with $f(\tilde{\sigma}_\tau) \leq f(\tau) < f(\sigma_i)$. Then

$$\begin{aligned}
\Phi(\sigma_i) &= \sigma_i + 0 + V(\partial(\sigma_i)) \\
&= \sigma_i + V\sum_\tau \tau \\
&= \sigma_i + \sum_\tau V(\tau) \\
&= \sigma_i + \sum_\tau \tilde{\sigma_\tau},
\end{aligned}$$

where $\tilde{\sigma}_\tau$ appears in the sum if and only if $f(\tilde{\sigma}_\tau) < f(\sigma_i)$. Conversely, suppose σ_i is not critical, so it is either the head or the tail of an arrow. If σ_i is a head, then $\partial(V(\sigma_i)) = \partial(0) = 0$ and $V(\partial(\sigma_i)) = \sigma_i$ contains σ_i, so

$$\begin{aligned}
\Phi(\sigma_i) &= \sigma_i + \partial(V(\sigma_i)) + V(\partial(\sigma_i)) \\
&= \sigma_i + \sigma_i + \cdots + 0.
\end{aligned}$$

By Lemma 4.2.3, there cannot be another σ_i in the sum, hence $a_{ii} = 0$. If σ_i is a tail, then $\partial(V(\sigma_i))$ contains σ_i, and by Lemma 4.2.3, there cannot be another σ_i in either $\partial(V(\sigma_i))$ or $V(\partial(\sigma_i))$, so we have $a_{ii} = 0$. We have now proven the case when σ_i is critical.

Suppose that $\sigma_i \in Im(V)$. Then there exists $\nu^{(p-1)} < \sigma_i$ such that $V(\nu) = \sigma_i$. We know that $V \circ V = 0$ from Proposition 5.2.1 (i), then

$$\begin{aligned}
\Phi(\sigma_i) &= \sigma_i + V(\partial(\sigma_i)) + \partial(V(\sigma_i)) \\
&= \sigma_i + V(\partial(\sigma_i)) + \partial(V(V(\nu))) \\
&= \sigma_i + \sum_\tau V(\tau),
\end{aligned}$$

where $\tau^{(p-1)} < \sigma_i$. By Proposition 5.2.1 (ii) (or rather, Lemma 4.2.3), ν is the only codimension-1 face of σ_i such that $V(\nu) = \sigma_i$. Hence,

$$\Phi(\sigma_i) = \sigma_i + V(\nu) + \sum_{\tau \neq \nu} V(\tau) = \sigma_i + \sigma_i + \sum_{\tau \neq \nu} V(\tau) = \sum_{\tau \neq \nu} V(\tau).$$

For any τ, we have either $V(\tau) = 0$ or $V(\tau) = \tilde{\sigma}_i$, where $\tilde{\sigma}_i^{(p)} > \tau$ and $f(\tilde{\sigma}_i) \leq f(\tau) < f(\sigma_i)$. Thus,

$$\Phi(\sigma_i) = \sum_{f(\tilde{\sigma}_i) < f(\sigma_i)} \tilde{\sigma}_i.$$

Finally, suppose that $V(\sigma_i) = \tau^{(p+1)} \neq 0$. For each $\nu^{(p-1)} < \sigma_i$, by Definition 5.8 and Lemma 4.2.3, either $V(\nu) = 0$ or $V(\nu) = \tilde{\sigma}_j$, where $\tilde{\sigma}_j > \nu, \tilde{\sigma}_j \neq \sigma_i$ and $f(\tilde{\sigma}_j) \leq f(\nu) < f(\sigma_i)$. Then

$$V(\partial(\sigma_i)) = \sum V(\nu) = \sum \tilde{\sigma}_j.$$

On the other hand, Lemma 4.2.3 implies that for any $\tilde{\sigma}_k < \tau$, $\tilde{\sigma}_k \neq \sigma_i$, we must have $f(\tilde{\sigma}_k) < f(\tau) \leq f(\sigma_i)$. Then

$$\partial(V(\sigma_i)) = \partial(\tau) = \sigma_i + \sum \tilde{\sigma}_k.$$

Together, we have

$$\Phi(\sigma_i) = \sigma_i + V(\partial(\sigma_i)) + \partial(V(\sigma_i)) = \sigma_i + \sum \tilde{\sigma}_j + \sigma_i + \sum \tilde{\sigma}_k = \sum \tilde{\sigma}_j + \sum \tilde{\sigma}_k.$$

Note that there might be overlap between $\{\tilde{\sigma}_j\}$ and $\{\tilde{\sigma}_k\}$, and these terms will cancel out in the calculation. Regardless, every term that appears in the final sum has its value under f less than $f(\sigma_i)$.

□

Lemma 5.2.7. *Let $\sigma \in K$ and write $c = V(\partial(\sigma))$. If σ is critical, then $\Phi^m(\sigma) = \sigma + c + \Phi(c) + \Phi^2(c) + \cdots + \Phi^{m-1}(c)$.*

Proof. The proof is a straightforward induction. For the base case, since σ is critical,

$$\Phi(\sigma) = \sigma + V(\partial(\sigma)) + \partial(V(\sigma)) = \sigma + c + 0 = \sigma + c.$$

For the induction step, suppose that $\Phi^m(\sigma) = \sigma + c + \Phi(c) + \Phi^2(c) + \cdots + \Phi^{m-1}(c)$. Then $\Phi^{m+1}(\sigma) = \sigma + c + \Phi(c) + \Phi^2(c) + \cdots + \Phi^m(c)$, and we are done.

□

Now we are ready to present the **proof of Theorem 5.2.5**:

Proof of Theorem 5.2.5. We show that Φ stabilizes at every simplex, and the theorem follows from linearity of Φ. Let $r_\sigma = \{\tilde{\sigma}^{(p)} \in K : f(\tilde{\sigma}) < f(\sigma)\}$. We use induction on $|r_\sigma|$.

For $|r_\sigma| = 0$, $f(\tilde{\sigma}) \geq f(\sigma)$ for all $\tilde{\sigma}$. We claim that either $\Phi(\sigma) = \sigma$ or $\Phi(\sigma) = 0$. To prove this claim, we first observe that σ cannot be the tail of an arrow. If σ was the tail of an arrow, then there exists $\tau^{(p+1)} > \sigma$ such that $f(\tau) \leq f(\sigma)$. But this means that $f(\tilde{\sigma}) \geq f(\sigma) \geq f(\tau)$ for all $\tilde{\sigma}$, in particular, for any codimension-1 faces of τ. Hence, it contradicts Lemma 4.2.3. That means σ is either a critical simplex or the head of an arrow. If σ is crtical, then $\partial(V(\sigma)) = \partial(0) = 0$; in addition, any codimension-1 face ν of σ cannot be the tail of an arrow since $f(\tilde{\sigma}) \geq f(\sigma) > f(\nu)$, so $V(\partial(\sigma)) = 0$. Hence, $\Phi(\sigma) = \sigma$. On the other hand, if σ is the head of an arrow, then there exists $\nu^{(p-1)} < \sigma$ such that $f(\nu) \geq f(\sigma)$. By Definition 5.8, $\partial(V(\sigma)) = \partial(0) = 0$; also, Lemma 4.2.3 implies that $f(\tilde{\sigma}) \geq f(\sigma) > f(\tilde{\nu})$ for any $\tilde{\nu}^{(p-1)} < \sigma$ other than ν, so ν cannot be the tail of an arrow, hence $V(\partial(\sigma)) = V(\nu) = \sigma$. Therefore, $\Phi(\sigma) = \sigma + \sigma = 0$. This claim is now proved and it gives the base case.

For the induction step, suppose that Φ stabilizes at any σ' for $|r_{\sigma'}| = 0, 1, \ldots, n-1$. Let $\sigma \in K$ such that $|r_\sigma| = n$. Consider cases when σ is regular and when σ is critical. Here we present the proof when σ is regular. The proof for the case of a critical σ is very technical, we refer readers to [34]. There, Lemma 5.2.7 is used to state the sufficiency of proving the existence of some N such that $\Phi^N(c) = 0$, where $c = V(\partial(\sigma))$.

If σ is regular, then we can write $\Phi(\sigma) = \sum_{f(\tilde{\sigma}_k)<f(\sigma)} \tilde{\sigma}_k$, $1 \leq k \leq n$, by Proposition

5.2.6. For any $\tilde{\sigma} \in r_\sigma$, we have $|r_{\tilde{\sigma}}| < |r_\sigma|$ since $\tilde{\sigma} \notin r_{\tilde{\sigma}}$. By the induction hypothesis, for each $\tilde{\sigma}$, there exists $N_{\tilde{\sigma}}$ such that $\Phi^i(\tilde{\sigma}) = \Phi^j(\tilde{\sigma})$ for all $i, j \geq N_{\tilde{\sigma}}$. Then, by linearity, $\Phi^N(\sigma) = \Phi^M(\sigma)$ for any $N, M > \max_{\tilde{\sigma}} N_{\tilde{\sigma}}$.

□

5.3 Morse Homology

In Subsection 4.3.2 we related simplicial homology theory and discrete Morse theory in the form of discrete Morse inequalities. In this section, we present a deeper connection between the two, using discrete Morse theory to simplify the computation of simplicial homology.

Definition 5.11. Let $\mathcal{C} = \{(C_i, d_i)\}$ and $\mathcal{C}' = \{(C_i', d_i')\}$ be chain complexes. A **chain map** between $\mathcal{C}$ and $\mathcal{C}'$ is a collection of maps $f = \{f_i : \mathcal{C} \to \mathcal{C}'\}$ such that $d_i' \circ f_i = f_{i-1} \circ d_i$ for all i, i.e. the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"] & C_i \arrow[r, "d_i"] \arrow[d, "f_i"] & C_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\
\cdots \arrow[r] & C_{i+1}' \arrow[r, "d_{i+1}'"] & C_i' \arrow[r, "d_i'"] & C_{i-1}' \arrow[r] & \cdots
\end{tikzcd}$$

is satisfied.

Define $f_* : H_\bullet(C_\bullet) \to H_\bullet(C_\bullet')$, $f_*([z]) = [f(z)]$. Then it is a homomorphism (linear transformation) between homology vector spaces induced by the chain map f: let $[z], [z'] \in H_\bullet(C_\bullet)$, then $f_*([z] + [z']) = [f(z + z')] = [f(z)] + [f(z')] = f_*([z]) + f_*([z'])$. The following is a technical lemma about the induced homomorphism.

Lemma 5.3.1. *Let $f : \mathcal{C} \to \mathcal{C}'$ and $g : \mathcal{C}' \to \mathcal{C}''$ be chain maps. Then for each i,*

(i) $(g_i \circ f_i) = (g_i)_* \circ (f_i)_*$*;*

(ii) if $\mathbb{1}_{C_i}$ *is the identity map on* C_i*, then* $(\mathbb{1}_{C_i})_* = \mathbb{1}_{H_\bullet(C_i)}$*.*

Proof. For (i), we have

$$(g_i \circ f_i)_*([z]) = [g_i \circ f_i(z)] = [g_i(f_i(z))] = (g_i)_*([f_i(z)]) = (g_i)_*((f_i)_*([z])) = (g_i)_* \circ (f_i)_*([z]).$$

For (ii), we have

$$(\mathbb{1}_{C_i})_*([z]) = [\mathbb{1}_{C_i}(z)] = [z] = \mathbb{1}_{H_\bullet(C_i)}([z]).$$

□

One of the chain complexes we study here is $(\Bbbk^{c_i}, \partial_i)$, from which we developed the simplicial homology. The other chain complex, which is derived from discrete Morse theory, is a *flow complex.*

Definition 5.12. Let $f : K \to \mathbb{R}$ be a discrete Morse function, where K is an n-dimensional simplicial complex. Define $\Bbbk_p^\Phi(K) = \{c \in \Bbbk^{c_p} : \Phi(c) = c\}$. Because of the linearity of Φ, we can restrict the boundary operator $\partial_p : \Bbbk^{c_p} \to \Bbbk^{c_{p-1}}$ to $\partial_p : \Bbbk_p^\Phi \to \Bbbk_{p-1}^\Phi$. Then we obtain a chain complex $(\Bbbk_i^\Phi, \partial_i)$:

$$\cdots \longrightarrow \Bbbk_{i+1}^\Phi \xrightarrow{\partial_{i+1}} \Bbbk_i^\Phi \xrightarrow{\partial_i} \Bbbk_{i-1}^\Phi \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_1} \Bbbk_0^\Phi \xrightarrow{\partial_0} 0,$$

which is called the **flow complex** of K (associated to f).

Now, we present the first main theorem of the section, which says that the homology vector spaces obtained from the flow complex are the same as the simplicial homology vector spaces.

Theorem 5.3.2 ([17])**.** *For all $i \geq 0$, $H_i(K) \cong H_i(\Bbbk_\bullet^\Phi)$.*

Proof. By Theorem 5.2.5, we know that, for any p, every element in $\Bbbk^{c_p}$ eventually stabilizes at some element in $\Bbbk_p^\Phi$. Then, we obtain a map $\Phi^\infty : \Bbbk^{c_\bullet} \to \Bbbk_\bullet^\Phi$. We show that the induced map $\Phi_*^\infty : H_i(K) \to H_i(\Bbbk_\bullet^\Phi)$ is an isomorphism, particularly whose inverse is the chain map induced by inclusion.

By definition, $\Bbbk_\bullet^\Phi \subseteq \Bbbk^{c_\bullet}$, so we have a inclusion map $\iota : \Bbbk_\bullet^\Phi \to \Bbbk^{c_\bullet}$. Then $\Phi^\infty \circ \iota = \mathbb{1}_{\Bbbk_\bullet^\Phi}$, by Lemma 5.3.1, we have that $\Phi_*^\infty \circ \iota_* = \mathbb{1}_{H_i(\Bbbk_\bullet^\Phi)}$. Now, let $[z] \in H_i(K)$, then

$$\iota_* \circ \Phi_*^\infty([z]) = \iota_*([\Phi^\infty(z)] = [\iota(\Phi^\infty(z))] = [\Phi^\infty(z)].$$

We show that $[\Phi^\infty(z)] = [z]$, so that $\iota_* \circ \Phi_*^\infty = \mathbb{1}_{H_i(K)}$. By Theorem 5.2.5, we know that there exists N such that $\Phi^N(z) = \Phi^\infty(z)$. We will show that $\Phi(z) \in [z]$, so by applying Φ repeatedly, we get $\Phi^N(z) \in [z]$. By definition, $\Phi(z) = z + \partial(V(z)) + V(\partial(z))$. Note that $\partial(V(z)) \in Im(\partial)$ naturally, while $V(\partial(z)) = 0$ since $z \in Ker(\partial)$. Then $\Phi(z) \in [z]$, $[\Phi^\infty(z)] = [\Phi^n(z)] = [z]$. Hence, $\iota_* \circ \Phi_*^\infty = \mathbb{1}_{H_i(K)}$.

□

Theorem 4.3.3 tells us that a simplicial complex K is homotopy equivalent to a cell complex whose p-cells are in a bijection with critical p-simplices of K. This suggests that the set of critical simplices may contain enough information for us to recover the topological structure of K.

Definition 5.13. Let K be a simplicial complex with gradient vector field V. The **p-critical complex of K with respect to V**, denoted $\mathcal{M}_p$, is the vector space generated by the critical p-simplices of V.

Note that $\mathcal{M}_p$ is a subspace of $\Bbbk^p$, so Φ^∞ can be naturally restricted to $\mathcal{M}_p$. The following is the second main theorem of this section, which tells us that vector spaces generated by the critical simplices are the same as those of the flow complex:

Theorem 5.3.3 ([17]). *For each p, the map $\Phi^\infty : \mathcal{M}_p \to \Bbbk_p^\Phi$ is an isomorphism.*

Proof. We show that Φ^∞ is both surjective and injective. Let $c \in \Bbbk_p^\Phi$, and write $c = \sum_{\sigma \in K_p} a_\sigma \sigma$, where $a_\sigma \in \mathbb{Z}_2$. Now, let $\tilde{c} = \sum_{\sigma \text{ critical}} a_\sigma \sigma$, so $\tilde{c} \in \mathcal{M}_p$ is the sum of critical simplices in the expansion of c. Consider the components of $\Phi(\tilde{c})$. Since any σ that appears in the expansion of $\tilde{c}$ is critical, $\partial(V(\sigma)) = 0$. At the same time, observe that any non-zero components in the expansion of $V(\partial(\sigma))$ is not critical. We apply the same argument to Φ^∞ and conclude that

$$\Phi^\infty(\tilde{c}) = \sum_{\sigma \text{ critical}} a_\sigma \Phi^\infty(\sigma) = \sum_{\sigma \text{ critical}} a_\sigma(\sigma + V_{a_\sigma}) = \sum_{\sigma \text{ critical}} a_\sigma \sigma = \tilde{c}.$$

Since $\Phi(\Phi^\infty(\tilde{c}) - c) = \Phi(\Phi^\infty(\tilde{c})) - \Phi(c) = \Phi^\infty(\tilde{c}) - c$, we have $\Phi^\infty(\tilde{c}) - c \in \Bbbk_p^\Phi$. Then Proposition 5.2.6 implies that $\Phi^\infty(\tilde{c}) - c = 0$, hence Φ^∞ is surjective.

We use Theorem 2.1.4 to show that Φ^∞ is injective. Let $c \in \mathcal{M}_p$ such that $\Phi^\infty(c) = 0$. Then

$$\Phi^\infty(c) = \sum_{\sigma \text{ critical}} a_\sigma \Phi^\infty(\sigma) = 0.$$

A similar argument as in the surjective case using Proposition 5.2.6 implies that $c = 0$.

□

Thus, the flow complex should be equivalent to the chain complex formed by $\mathcal{M}_p$ and some proper boundary operator, giving isomorphic homology vector spaces. The desired boundary operator is described by the next theorem. Ultimately, we have a simplified chain complex, formed by critical simplices, from which we obtain the same homology vector spaces as the simplicial homology vector spaces.

Theorem 5.3.4 ([34]). *Let V be a gradient vector field on a simplicial complex K. For each $\sigma \in \mathcal{M}_p$, define*

$$\partial_p(\sigma) = \sum_{\beta^{(p-1)} \text{ critical}} \delta_{\sigma,\beta}\beta,$$

where

$$\delta_{\sigma,\beta} := \begin{cases} 0 & \text{if the number of } V\text{-paths from a maximal face of } \sigma \text{ to } \beta \text{ is even;} \\ 1 & \text{if it is odd} \end{cases}$$

Then $\mathcal{M} = \{(\mathcal{M}_i, \partial_i)\}$ is a chain complex, which we call a ***chained critical complex****, and $H_i(\mathcal{M}) \cong H_i(K)$.*

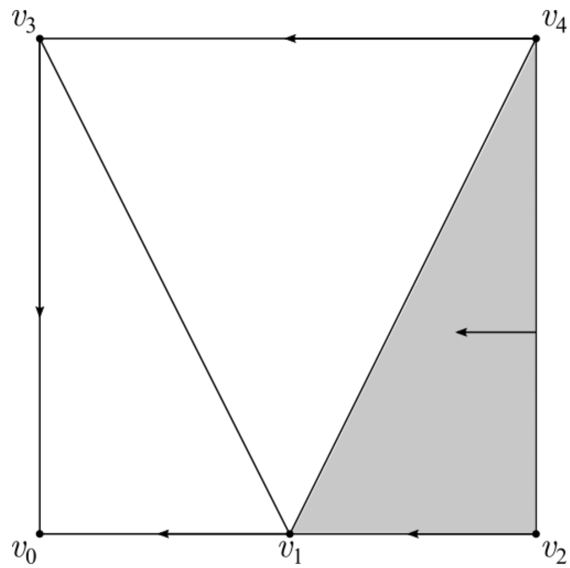

Figure 5.6: The simplicial complex with a gradient vector field in Example 5.3.5

For a proof of this theorem and an analogous case in the classical Morse theory, we refer readers to [25].

Example 5.3.5. Consider the simplicial complex K in Figure 5.6. The c-vector of K is $(5, 7, 1)$, so if we want to calculate the simplicial homology of K, we would work on the chain complex

$$\mathbb{k}^5 \longrightarrow \mathbb{k}^7 \longrightarrow \mathbb{k}^1 \longrightarrow 0.$$

However, with the additional gradient vector field V, we can simplify this calculation with Theorem 5.3.4.

The critical simplices of V are v_0, v_1v_3 and v_1v_4. Then we have

$$\mathcal{M}_1^2 \longrightarrow \mathcal{M}_0^1 \longrightarrow 0,$$

and we can already see that it is a much "smaller" chain complex.

We count that there are 2 gradient paths from v_1v_3 to v_0 and 2 gradient paths from v_1v_4 to v_0, so $\partial_1(v_1v_3) = 0$ and $\partial_1(v_1v_4) = 0$. Hence, $Ker(\partial_1) = \langle v_1v_3, v_1v_4 \rangle$. In addition, $Im(\partial_2) = 0$ trivially. As a result, $H_1(\mathcal{M}) = Ker(\partial_1) / Im(\partial_2) \cong \mathbb{k}^2$, while $H_0(\mathcal{M})$ is clearly isomorphic to $\Bbbk^1$. Therefore, by Theorem 5.3.4, we have $b_1(K) = 2$ and $b_0(K) = 1$, which correspond to the two 1-dimensional holes and the connectedness of K.

Chapter 6

Persistent Homology

6.1 Motivation

As we saw earlier, the main objects of interest of discrete Morse theory are simplicial complexes. In practice, simplicial complexes are commonly studied in a *filtration* built over some given data space to capture the topological feature of the date space, which are often hidden from naked eyes.

Definition 6.1. A **filtration** of a set is a nested sequence of its subsets, and each element in the sequence is called a **level**.

Let K be a simplicial complex, then the nested sequence of level subcomplexes

$$\emptyset = K^0 \subseteq K^1 \subseteq \cdots \subseteq K^{n-1} \subseteq K^n = K$$

forms a filtration of K.

It is often not sufficient to only compute the homology and Betti numbers at certain level of a filtration ([20]), as we cannot distinguish between topological features of the original space and the *noise* that appears and disappears quickly ([37]). To see this, we use an example of the *homological sequence*.

Definition 6.2. Let $f : K \to \mathbb{R}$ be a discrete Morse function with m critical values $c_1^* < c_2^* < \cdots < c_m^*$. Define $B_k^f(i) = b_k(K(c_i^*))$ for all $0 \leq k \leq n$ and $1 \leq i \leq m$. Then $\{B_k^f(i)\}_{i=1}^m$ is the k**-th homological sequence of** f.

For more on homological sequences, we refer the interested readers to [1], in which the authors studied its properties on finite simplicial complexes, particularly collapsible 2-dimensional simplicial complexes, and its connection to lattice walk.

Definition 6.3. Let $f, g : K \to \mathbb{R}$ be discrete Morse functions with m critical values. Then f and g are said to be **homologically equivalent** if $B_k^f(i) = B_k^g(i)$ for all $0 \leq k \leq n$ and $1 \leq i \leq m$.

Example 6.1.1. Consider the two discrete Morse functions f and g on the same simplicial complex shown in Figure 6.1.

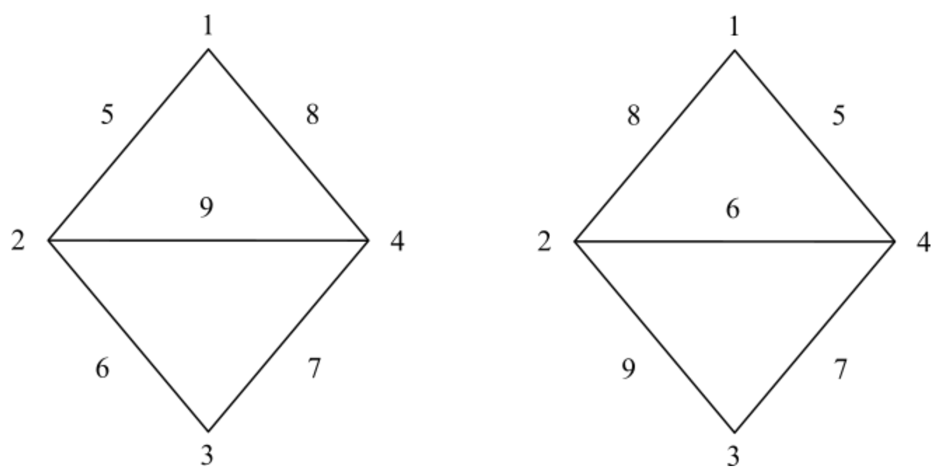

Figure 6.1: Two discrete Morse functions that give the same homological sequence

Observe that all simplices are critical under either function. If we construct filtrations by taking level subcomplexes at critical values, both f and g give the same homological sequence on K:

i :	1	2	3	4	5	6	7	8	9
B_0 :	1	2	3	4	3	2	1	1	1
B_1 :	0	0	0	0	0	0	0	1	2

thus they are homologically equivalent.

Despite this, with a closer examination of the two filtrations, we shall see that cycles were formed very differently under each function. A 1-cycle was created by f at K_8, but was destroyed at K_9 by the two newly-formed 1-cycles, resulting in $B_1^f(8) = 1$ and $B_1^f(9) = 2$. On the other hand, under g, a 1-cycle was created at K_8 and another 1-cycle was created at K_9 while nothing was destroyed, resulting in $B_1^g(8) = 1$ and $B_1^g(9) = 2$, same as that of f.

With this example, we clearly see that computing homology at an individual level does not produce enough information about either the topological structure of the underlying simplicial complex or the possible discrete Morse functions that can be applied.

6.2 Persistence

6.2.1 Persistent Homology

The concept of *persistence* was first introduced in [14] to provide a solution for the issue we discussed in the previous section. It has since then become a powerful tool in the field of *topological data analysis*. Recall some notions of simplicial homology from Section 3.3.

Definition 6.4. Let $\mathcal{F}$ be a filtration of a simplicial complex K. We write $Z_k^i = Z_k(K^i)$ and $B_k^i = B_k(K^i)$, where K^i is the i-th level of $\mathcal{F}$. If $p = j - i$, then the **p-persistent k-dimensional homology group** of K^i is defined as

$$H_k^{i,j} = Z_k^i / \left(B_k^j \cap Z_k^i\right).$$

This group is well-defined, as both B_k^j and Z_k^i are subgroups of C_k^j, the free abelian group with basis of the set of k-simplices of K^j, so their intersection is again a normal subgroup. With this definition, we are looking for non-bounding cycles (holes) that stay non-bounding for certain length of time, or in our case, certain number of levels in a filtration, i.e. those that *persist*.

Alternatively, within the filtration, if $\iota : K^{i-1} \to K^i, \iota(\alpha) = \alpha$ is the inclusion map, then it induces a homomorphism $H_k(K^{i-1}) \to H_k(K^i)$ ([13]). Write $H_k^i = H_k(K^i)$ and compose these homomorphisms, we obtain maps $\iota_k^{i,j} : H_k^i \to H_k^j$, where $j - i = p$.

Definition 6.5 (Alternative definition for persistent homology)**.** Under the settings above, the **p-persistent k-dimensional persistent homology vector spaces** are images of homomorphisms induced by the composed inclusion, that is $H_k^{i,j} = Im\left(\iota_k^{i,j}\right)$, $j - i = p$.

To simplify terms, when the context is clear, we will say *k-th persistent homology group* or *vector space* without mentioning *p-persistent*. The same applies to the following term:

Definition 6.6. The **k-dimensional persistent Betti numbers** $\beta_k^{i,j}$ are the ranks of $H_k^{i,j}$.

As we saw in Example 6.1.1, non-bounding cycles (we will just say "cycles" for short) are *created* and *destroyed* when moving through levels of a filtration. Elements of the persistent homology group $H_k^{i,j}$ consist of k-th homology classes of K^i that are still alive at K^j, i.e. k-cycles created at or before K^i that are still non-bounding when entering K^j. Persistent Betti numbers count such distinct homology classes. One way to visualize persistent Betti numbers is through plotting **multiplicities**, which is defined as

$$\mu_k^{i,j} = \left(\beta_k^{i,j-1} - \beta_k^{i,j}\right) - \left(\beta_k^{i-1,j-1} - \beta_k^{i-1,j}\right).$$

The difference in the first parenthesis counts cycles that were created at or before K^i and were destroyed when entering K^j; the difference in the second parenthesis counts cycles that were created at or before K^{i-1} and were destroyed when entering K^j. Together, we have number of cycles that were created precisely at K^i and were destroyed when entering K^j. If we adapt the common filtration construction by adding one simplex at a time, at most one cycle can be created at each level ([22]).

Definition 6.7. The **k-dimensional persistence diagram** of the filtration $\mathcal{F}$ is the set of points $(i, j) \in \mathbb{R}^2$ such that $\mu_k^{i,j} = 1$.

Since $\mu_k^{i,j}$ is defined with $i < j$, all points of the persistence diagram lie above the diagonal. Given some point (i, j), consider points that lie in its upper left quadrant with the horizontal baseline removed, that is (m, l) such that $k \leq i$ and $l > j$. Then, each (m, l) represents a cycle that was created before or at K^i and was destroyed when entering K^j or later, so the total number of such points is in fact the number of non-bounding cycles present throughout K^i to K^j. Hence, the k-th persistent Betti number can be written as

$$\beta_k^{i,j} = \sum_{m \leq i, l > j} \mu_k^{m,l}.$$

In addition, the lifetime, or the **persistence**, of each k-cycle can be read from the k-th persistent diagram by observing the vertical distance between its multiplicity point and the diagonal. These properties show that the persistent diagram encodes all the information about the persistent homology.

We use a simple example to help illustrate the concept of persistent diagram. While the computation of persistent Betti numbers of complicated simplicial complexes is not our focus here, we refer readers to [38] for algorithms concerning this topic.

Example 6.2.1. Let K be the a standard 2-simplex constructed by adding one simplex at a time. The sequence in which the simplices are added is labeled on each simplex of K as shown in Figure 6.2, and this gives a filtration of K. Although less important in this example, readers should notice that every simplex will be critical if we see the labelling as a discrete Morse function.

Following this construction, we see that the 0-cycle created at K^2 was destroyed with the addition of the 1-simplex at K^4, so we expect to have $\mu_0^{2,4} = 1$. Indeed,

$$\mu_0^{2,4} = (\beta_0^{2,3} - \beta_0^{2,4}) - (\beta_0^{1,3} - \beta_0^{1,4}) = (2 - 0) - (1 - 0) = 1.$$

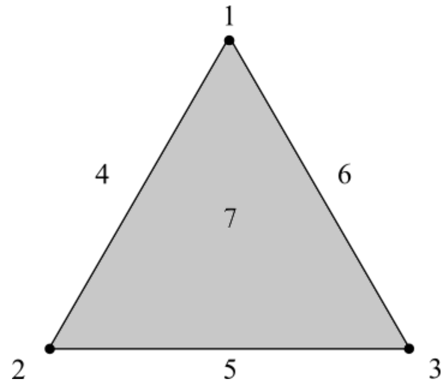

Figure 6.2: A 2-simplex formed by adding 1 simplex at a time

Similarly, we see that the 0-cycle created at K^3 was destroyed at K^5, and

$$\mu_0^{3,5} = (\beta_0^{3,4} - \beta_0^{3,5}) - (\beta_0^{2,4} - \beta_0^{2,5}) = (2-1) - (0-0) = 1.$$

Moving on to 1-dimension, a 1-cycle was created at K^6 and was destroyed (filled up) at K^7. Unsurprisingly,

$$\mu_1^{6,7} = (\beta_1^{6,6} - \beta_1^{6,7}) - (\beta_1^{5,6} - \beta_1^{5,7}) = (1-0) - (0-0) = 1.$$

Finally, the component created at K^1 was not destroyed until the additional 0-space at the end (K^8). As we can calculate,

$$\mu_0^{1,8} = (\beta_0^{1,7} - \beta_0^{1,8}) - (\beta_0^{0,7} - \beta_0^{0,8}) = (1-0) - (0-0) = 1.$$

The 0-dimensional persistence diagram is presented in Figure 6.3.

We can calculate all persistent Betti numbers in each dimension directly from the diagram. For example, in 0-dimension, there are 2 multiplicity points (m, l) such that $m \leq 3$ and $l > 4$, so $\beta_0^{3,4} = 2$; Similarly, in 1-dimension, there is no point in the upper left quadrant of $(3, 5)$, hence $\beta_1^{3,5} = 0$.

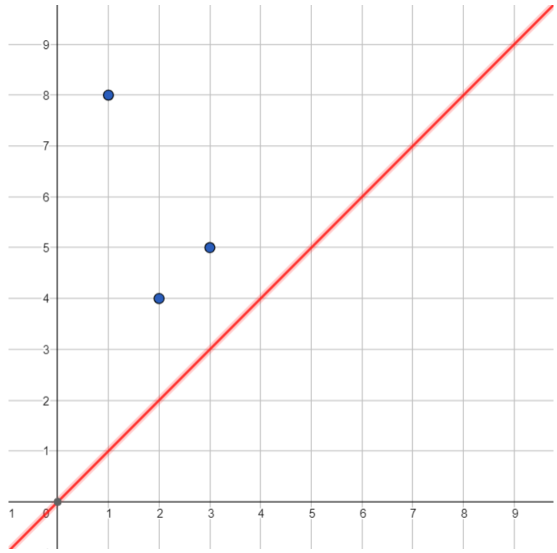

Figure 6.3: 0-dimensional persistent diagram of K in Example 6.2.1

6.3 Connection with Discrete Morse Theory

We now present the connection between discrete Morse theory and persistent homology. Having seen Section 5.3, readers should expect some simplification of computation of the persistent homology if we could construct some specific critical complex. However, for persistent homology, the underlying geometric object is a filtration of simplicial complexes instead of a single one. So to apply the theory, some adjustments need to be made. Mischaikow and Nanda studied this issue in [30] through modifications to the discrete Morse matching (Section 5.1). Let M be a discrete Morse matching. We can write $M = (\mathcal{A}, w : \mathcal{Q} \to \mathcal{K})$, where $\mathcal{A}$, $\mathcal{Q}$ and $\mathcal{K}$ partition the set of simplices of K: $\mathcal{A}$ denotes the set of critical simplices, $\mathcal{Q}$ denotes the set of tails and $\mathcal{K}$ denotes the set of heads. The map w is a bijection that describes how tails and heads are paired: for $\sigma \in \mathcal{Q}$ and $\tau \in \mathcal{K}$, if $w(\sigma) = \tau$, then (σ, τ) is a regular pair. Consider a filtration $\mathcal{F}$ of a simplicial complex K.

$$K^0 \subseteq K^1 \subseteq \cdots \subseteq K^{n-1} \subseteq K^n = K.$$

Definition 6.8. A **filtered discrete Morse matching** on $\mathcal{F}$ is a collection of the discrete

Morse matching $M^i = (\mathcal{A}^i, w^i : \mathcal{Q}^i \to \mathcal{K}^i)$ on each level K^i with the additional structures

- $\mathcal{A}^i \subseteq \mathcal{A}^{i+1}, \mathcal{Q}^i \subseteq \mathcal{Q}^{i+1}, \mathcal{K}^i \subseteq \mathcal{K}^{i+1}$
- $w^i = w^{i+1}|_{\mathcal{Q}^i}$

for each $i \in \{0, \ldots, n\}$. The final level of the filtered discrete Morse matching $(\mathcal{A}^n, w^n : \mathcal{Q}^n \to \mathcal{K}^n)$ is denoted by $(\mathcal{A}, w : \mathcal{Q} \to \mathcal{K})$.

If we construct $\mathcal{F}$ by increasing critical values, then $\{\mathcal{A}^i\}$ forms a filtration of the corresponding chained critical complex. Ultimately, Mischaikow and Nanda proved the following theorem, which can be used to greatly reduce the computation for persistent homology of a given simplicial complex.

Theorem 6.3.1 ([30]). *For all i, j, k, l, we have*

$$H_k^{i,j}(K^l) \cong H_k^{i,j}(\mathcal{A}^l).$$

We highlight some key concepts used in the proof. In Section 5.3, we discussed that a chain map induces homomorphisms between homology vector spaces. A similar result can be shown for persistent homology, for which additional structures on a chain map are needed.

Definition 6.9. Let $\phi, \psi : \mathcal{C} \to \mathcal{C}'$ be chain maps. A sequence of homomorphisms $\Theta = \{\Theta_k : C_k \to C'_k\}$ is a **chain homotopy between ϕ and ψ** if

$$\Theta_{k-1} \circ \partial_k + \partial'_{k+1} \circ \Theta_k = \phi_k - \psi_k$$

on C_k.

This terminology makes sense in the way that for $[z] \in Z_k/B_k$,

$$\begin{aligned}(\phi_k)_*([z]) - (\psi_k)_*([z]) = ((\phi_k)_* - (\psi_k)_*)([z]) &= (\Theta_{k-1} \circ \partial_k + \partial'_{k+1} \circ \Theta_k)([z]) \\ &= [\Theta_{k-1} \circ \partial_k(z) + \partial'_{k+1} \circ \Theta_k(z)] \\ &= [\Theta_{k-1}(\partial_k(z)) + \partial'_{k+1}(\Theta_k(z))] \\ &= [0 + \partial'_{k+1}(\Theta_k(z))] \\ &= [0].\end{aligned}$$

That is ϕ and ψ induce the same map on homology.

Now, consider simplicial complexes K and K', and let $\mathcal{F}$ and $\mathcal{F}'$ be filtrations of K and K' respectively. We have $C(K^l)$ and $C(K'^l)$ as chain complexes induced by inclusions.

Definition 6.10. A **filtered chain map** $\Phi : \mathcal{F} \to \mathcal{F}'$ is a sequence of chain maps $\{\phi^l_\bullet : C_\bullet(K^l) \to C_\bullet(K'^l)\}$ such that the following diagram commutes:

$$\begin{tikzcd} C_\bullet(K^l) \arrow[r, hook] \arrow[d, "\phi^l_\bullet"] & C_\bullet(K^{l+1}) \arrow[d, "\phi^{l+1}_\bullet"] \\ C_\bullet(K'^l) \arrow[r, hook] & C_\bullet(K'^{l+1}) \end{tikzcd}$$

Given any chain map Φ, there exists a collection of homomorphisms $\{\phi^{i,j,l}_\bullet : H^{i,j}_\bullet(K^l) \to H^{i,j}_\bullet(K'^l)\}$ defined by

$$\phi^{i,j,l}_\bullet(z) = \phi^{l+j-i}_\bullet \circ \iota^{i,j,l}_\bullet(z), z \in Z_\bullet(K^l),$$

where $\iota^{i,j,l} : C_\bullet(K^l) \to C_\bullet(K^{l+j-i})$ is the homomorphism induced by composition of nested inclusion maps.

Definition 6.11. Let $\Phi, \Psi : \mathcal{F} \to \mathcal{F}'$ be filtered chain maps. A **filtered chain homotopy between Φ and Ψ** is a collection of chain homotopies between each $\phi^k_\bullet$ and $\psi^k_\bullet$. If such collection exists, we say that Φ is **filtered chain homotopic to Ψ**.

Similar to the classic homotopy theory, two maps are said to be **filtered chain homo-**

topy equivalences of each other if their compositions are filtered chain homotopic to the identity map. Theorem 3.3.6 can also be extended to persistent homology:

Proposition 6.3.2 ([30]). *If there exist filtered chain homotopy equivalences* $\Phi : \mathcal{F} \to \mathcal{F}'$ *and* $\Psi : \mathcal{F}' \to \mathcal{F}$*, then*

$$H_k^{i,j}(K^l) \cong H_k^{i,j}(K'^l)$$

for all i, j, k, l*.*

The proof of this proposition follows from the fact that their induced homomorphisms on persistent homology groups are inverses of each other. Hence the key step in proving Theorem 6.3.1 is to construct maps $\Phi : \mathcal{F} \to \{\mathcal{A}^i\}$ and $\Psi : \{\mathcal{A}^i\} \to \mathcal{F}$ that are filtered chain homotopy equivalences. We refer readers to [30] for details.

Chapter 7

Related Topics

In this chapter, we close the thesis by mentioning in a less rigorous manner some works in topics closely related to discrete Morse theory and persistence.

7.1 Evasiveness

We have all taken surveys and questionnaires that require us to answer only "Yes" or "No", and each question depends on the answer to the previous question. In mathematics, this type of sequence is formally regarded as a *decision tree algorithm*. In the context of simplicial complex, let S be a standard n-simplex (Δ^n) and let σ be a face of S. These Boolean questions often take the form of "Is vertex v_i contained in σ?" in order to determine if σ is contained in some subcomplex M of S. For any decision tree algorithm A, i.e. the sequence of questions asked, the number of questions that must be asked to determine if σ is in M is denoted by $Q(\sigma, A, M)$. The *complexity* of M, $c(M)$, is then defined by

$$c(M) = \inf_{A} \sup_{\sigma} Q(\sigma, A, M).$$

The subcomplex M is said to be *evasive* if $c(M) = n+1$, and *nonevasive* otherwise. On the other hand, for a given decision tree algorithm A, any σ with $Q(\sigma, A, M) = n+1$ is called an *evader* of A.

Some interesting results have been proven under this setting. In [24], Kahn, Saks, and Sturtevant showed that if M is nonevasive, then M is both collapsible and contractible. Forman quantified evasiveness and linked it with discrete Morse theory. We state one of his main theorems here.

Theorem 7.1.1. *For any decision tree algorithm A, let V denote the vector field consisting of pairs of non-empty faces of S which cannot be distinguished by A until the last question. Then, V is a gradient vector field on S.*

The key observation in understanding this theorem is that evaders come in pairs: by the time one gets to the $(n+1)$-th question, there will be two possible simplices σ_1, σ_2 with $Q(\sigma_1, A, M) = Q(\sigma_2, A, M) = n+1$ no matter what the answer to the $(n+1)$-th question is. Without loss of generality, let us assume that "No" gives simplex σ_1 and "Yes" gives simplex σ_2. Then $\sigma_1 < \sigma_2$ and (σ_1, σ_2) can be viewed as a vector of a gradient vector field.

In the field of computer science, evasiveness is often used to study connectivity and complexity within networks. Discrete Morse theory can provide an alternative way of studying these topics.

7.2 Dynamical Systems

Dynamical systems is a field in which one studies different states over time. Typical topics studied include fixed points, periodic motions, and flows. As we have mentioned, some topics in Section 5.2 such as gradient flow can be seen as components of a dynamical system. However, such a dynamical system is often too restricted given the nature of a discrete Morse function. In particular, there can be at most one "flow" going through any given simplex. Jost and Yaptieu ([23]) developed a generalized version of the gradient flow, which allows multiple inward or outward arrows (but not both at the same time) at a simplex, thus supporting more complicated dynamical systems. They also constructed the proper boundary operator, so that one may recover the homology of the underlying complex.

7.3 Čech and Rips Complexes

Given some data set in a Euclidean n-dimensional space $\mathbb{E}^n$, which is usually referred as a *point cloud data*, there are many methods to generate a simplicial complex. Two of the most natural ones, *Čech complex* and *Rips complex*, are generated through distances among data points. Let $\mathcal{X} \in \mathbb{E}^n$ be a collection of data points.

Definition 7.1. The **Čech complex**, $\mathcal{C}_\epsilon$, is the simplicial complex whose k-simplices are determined by the $(k+1)$-element subcollection of $\mathcal{X}$ with $\epsilon/2$-neighborhoods of all elements having a non-empty intersection.

Definition 7.2. The **Vietoris-Rips complex**, or **Rips complex** for short, denoted $\mathcal{R}_\epsilon$, is the simplicial complex whose K-simplices are determined by the $(k+1)$-element subcollection of $\mathcal{X}$ with elements being pairwise within distance ϵ.

In the field of *topological data analysis*, both Čech complex and Rips complex have been studied to reveal topological information of data sets that cannot be seen. Particularly, the *Nerve Theorem* ([32], Theorem 2) states that $\mathcal{C}_\epsilon$ has the same homotopy type as the union of closed balls of radius $\epsilon/2$ about points in $\{x_\alpha\}$. However, direct computations for Čech complexes are difficult, especially when the data set is large, while Rips complexes are computationally less intensive. This is because, by definition, Rips complexes are completely determined by the combinatorics of their 1-skeleton. With this advantage, Silva and Ghrist ([12]) showed that pairs of appropriate Rips complexes can be used to approximate a given Čech complex.

Theorem 7.3.1 ([12], Theorem 2.5)**.** *Let $\mathcal{X}$ be a data set in $\mathbb{E}^n$. Then there is a chain of inclusions*

$$\mathcal{R}_\epsilon \hookrightarrow \mathcal{C}_{\epsilon'} \hookrightarrow \mathcal{R}_{\epsilon'}$$

whenever

$$\frac{\epsilon'}{\epsilon} \geq \sqrt{\frac{2n}{n+1}}.$$

The concept of persistence can be naturally applied to the inclusion induced map $\iota_* : H_\bullet(\mathcal{R}_\epsilon) \to H_\bullet(\mathcal{R}_{\epsilon'})$ with the increasing radius. Hence, Theorem 7.3.1 provides another reason why persistence is advantageous for capturing more topological information.

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