INCREASING STABILITY FOR THE INVERSE PROBLEM FOR THE SCHRÖDINGER EQUATION

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Abstract. In this article, we study the increasing stability property for the determination of the potential in the Schrödinger equation from partial data. We shall assume that the inaccessible part of the boundary is flat and homogeneous boundary condition is prescribed on this part. In contrast to earlier works, we are able to deal with the case when potentials have some Sobolev regularity and also need not be compactly supported inside the domain.

1. Introduction

Let us consider the boundary value problem for the Schrödinger equation

$$(\Delta + k^2 + q(x))u(x) = 0, \text{ in } \Omega, \tag{1.1}$$

posed in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary. The boundary data

$$u(x) = f(x) \text{ on } \partial\Omega. \tag{1.2}$$

is assumed to be of the class $H^{\frac{1}{2}}(\partial\Omega)$, and q is real-valued and satisfies $q \in H^s(\Omega)$, for some $s > \frac{3}{2}$. Note, that, by Sobolev embedding, this yields, that the potentials are in fact Hölder continuous. Without loss of generality, we shall assume that the wave number $k \geq 1$.

For $N > 0$ and $s > \frac{3}{2}$, let us define the set of potentials

$$\mathcal{Q}_N := \{q : \|q\|_{H^s(\Omega)} \leq N\}.$$

In this article, we shall consider a bounded domain Ω with smooth boundary such that $\Omega \subset \{x : x_3 < 0\}$ and a part of the boundary Γ_0 (which we shall also refer as the inaccessible part of the boundary) is contained in the plane $\{x : x_3 = 0\}$. We shall assume that the support of f is contained in $\Gamma := \partial\Omega \setminus \Gamma_0$. Let

$$\mathcal{C}_q := \left\{ \left(u\Big|_{\Gamma}, \frac{\partial u}{\partial \nu}\Big|_{\Gamma} \right), \text{where } u \text{ is a solution to (1.1) and } u = 0 \text{ on } \Gamma_0 \right\}$$

denote the partial Cauchy data and $\frac{\partial u}{\partial \nu}|_\Gamma \in H^{-\frac{1}{2}}(\Gamma)$. We can define a distance in the set of partial Cauchy data as

$$\operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) := \max\Big\{ \max_{(f,g)\in\mathcal{C}_{q_1}} \min_{(\tilde f,\tilde g)\in\mathcal{C}_{q_2}} \frac{\|(f,g)-(\tilde f,\tilde g)\|_{H^{\frac12}\oplus H^{-\frac12}}}{\|(f,g)\|_{H^{\frac12}\oplus H^{-\frac12}}}, \max_{(f,g)\in\mathcal{C}_{q_2}} \min_{(\tilde f,\tilde g)\in\mathcal{C}_{q_1}} \frac{\|(f,g)-(\tilde f,\tilde g)\|_{H^{\frac12}\oplus H^{-\frac12}}}{\|(f,g)\|_{H^{\frac12}\oplus H^{-\frac12}}} \Big\},$$

where $\|(f,g)\|_{H^{\frac12}\oplus H^{-\frac12}} = (\|f\|^2_{H^{\frac12}(\Gamma)} + \|g\|^2_{H^{-\frac12}(\Gamma)})^{\frac12}$.

Our aim, here, is to address the question of stability of the recovery of the potential q from the knowledge of the partial Cauchy data $\mathcal{C}_q$ and to study the behaviour of the stability estimates as the wave number k grows. The unique identification of the potential q from $\mathcal{C}_q$ was established in the work [9].

Starting with the work [3] and following the impetus provided by the work [15], such problems started receiving intense consideration. The question of stability in the case of full data (and $k = 0$) was

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considered in [1] and a logarithmic stability estimate was established. It was also shown that this is the optimal result one can achieve. In the partial data case (with $k = 0$), a double logarithmic type stability estimate was established in [7] following the work [2] which dealt with the issue of unique identification. We would also like to refer to the work [6] in this context. In the case of domains under consideration (with $k = 0$), it was shown in [8] that a logarithmic type stability estimate can be established even from partial data.

In order to improve the logarithmic type stability estimates (which means that the problem is severely ill-posed and therefore inconvenient also from a numerical point of view) to Lipschitz-type stability estimates, the corresponding problem with $k \neq 0$ started receiving attention. It was found in many works (see [10–14]) and in the context of different models that a growing k tends to improve the stability, a property which was termed as increasing stability. In this article, we shall investigate this property in case of the domains stated above and endeavour to improve the logarithmic stability estimate established in [8].

We would like to remark that the property of increasing stability in similar domains was also studied in [13]. In that article, the author assumed the condition $\frac{\partial u}{\partial \nu} = 0$ on Γ_0 instead of $u = 0$ on Γ_0 that we have assumed here. Nevertheless our proof with minor modifications (see [9]) would also hold true in that case. Moreover, here we assume only Sobolev regularity of the potentials in contrast to the assumption of potentials in $C^1(\Omega)$ considered in [13]. We also do not assume that the difference of the potentials vanishes near the boundary $\partial\Omega$.

Our main result on the stability of recovery of the potential q from the Cauchy data $\mathcal{C}_q$ reads as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain as described above. Also assume that $R > 0$ be a large real number such that $\Omega \subset B(0, R)$. Let $\mathcal{C}_{q_1}, \mathcal{C}_{q_2}$ denote the partial Cauchy data corresponding to the potentials $q_1, q_2 \in \mathcal{Q}_N$ respectively. Then there exist constants $C, \tilde{\alpha}, \eta > 0$ such that*

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leq C\Big(e^{6Rk} \operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) + \frac{1}{(k + \frac{E}{5R})^{\tilde{\alpha}}}\Big)^{\frac{\eta}{2(1+s)}}, \tag{1.3}$$

where $E = |\log \operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})|$. The constant C depends on Ω, N and s only and the constants $\tilde{\alpha}, \eta$ depend on s only.

It can be observed from (1.3) that as the positive constant k grows, the term in the right-hand side with the logarithmic part in the denominator decays to zero and the first term (the Lipschitz part) dominates. Thus the logarithmic stability is improved to a Lipschitz-type stability estimate exhibiting the property of increasing stability.

The above result should also hold true, with minor modifications, for any dimension $n > 3$. To simplify the presentation in terms of the CGO solutions, we have restricted ourselves to the case $n = 3$.

In Section 2, we recollect some preliminary results that shall be necessary in the proof of the stability estimates. In Section 3, we introduce appropriate solutions to (1.1) and proceed to derive the desired stability estimates.

2. Some preliminary results

In this section, we recollect some preliminary results which we shall use later in the proofs. We begin by stating a result on the existence of complex geometric optics (CGO) solutions to (1.1).

Lemma 2.1. *(see [11, 15]) Let $s > \frac{3}{2}$. Assume that $\zeta = \operatorname{Re}\zeta + i \operatorname{Im}\zeta$ satisfies $|\operatorname{Re}\zeta|^2 = k^2 + |\operatorname{Im}\zeta|^2$ and $\operatorname{Re}\zeta \cdot \operatorname{Im}\zeta = 0$, that is, $\zeta \cdot \zeta = k^2$. Then there exist constants C_* and $C > 0$, independent of k, such that if $|\operatorname{Im}\zeta| > C_* \|q\|_{H^s(\Omega)}$, then there exists a solution u to (1.1) of the form*

$$u(x) = e^{i\zeta\cdot x}(1 + \psi(x))$$

where

$$\|\psi\|_{H^s(\Omega)} \leq \frac{C}{|\operatorname{Im}\zeta|}\|q\|_{H^s(\Omega)}.$$

In the next section, we shall choose ζ suitably so as to be able to use the above lemma to infer the existence of CGO solutions with the error terms satisfying the above estimates. We shall also need the following Green's identity which can be proved following [1, 11].

Proposition 2.2. *Let u_j and $\mathcal{C}_{q_j}$ be solution and Cauchy data for the equation* (1.1) *corresponding to the potential q_j $(j = 1, 2)$. Then*

$$\Big| \int_\Omega (q_2 - q_1)\ u_1 u_2\ dx \Big| \leq \Big\| \Big(u_1, \frac{\partial u_1}{\partial \nu} \Big) \Big\|_{H^{\frac{1}{2}} \oplus H^{-\frac{1}{2}}} \Big\| \Big(u_2, \frac{\partial u_2}{\partial \nu} \Big) \Big\|_{H^{\frac{1}{2}} \oplus H^{-\frac{1}{2}}} \ \mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}).$$

Using the equation (1.1), it can be proved (see [5, 11]) that

$$\begin{aligned} \Big\| \Big(u_l, \frac{\partial u_l}{\partial \nu} \Big) \Big\|_{H^{\frac{1}{2}} \oplus H^{-\frac{1}{2}}} &\leq Ck^2 \|u_l\|_{L^2(\Omega)} + C \|\nabla u_l\|_{L^2(\Omega)} \\ &\leq Ck^2 \|u_l\|_{H^1(\Omega)}. \end{aligned}$$

Using this together with the above proposition, we can derive

$$\Big| \int_\Omega (q_2 - q_1)\ u_1 u_2\ dx \Big| \leq Ck^4 \|u_1\|_{H^1(\Omega)} \|u_2\|_{H^1(\Omega)}\ \mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}). \tag{2.1}$$

In what follows, we shall also require the following quantitative version of the Riemann-Lebesgue lemma. For the proofs of the results, we refer to [4, 8].

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary and let $f \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. Let $\tilde{f}$ denote the extension of f to $\mathbb{R}^n$ by zero. Then there exist $\tilde{\delta} > 0$ and $C > 0$ such that*

$$\|\tilde{f}(\cdot - y) - \tilde{f}(\cdot)\|_{L^1(R)^n} \leq C|y|^\alpha,$$

for any $y \in \mathbb{R}^n$ with $|y| < \tilde{\delta}$.

Lemma 2.4. *Let $f \in L^1(\mathbb{R}^n)$ and suppose there exist constants $\tilde{\delta} > 0$, $C_0 > 0$ and $\alpha \in (0, 1)$ such that for $|y| < \tilde{\delta}$,*

$$\|f(\cdot - y) - f(y)\|_{L^1(\mathbb{R}^n)} \leq C_0 |y|^\alpha. \tag{2.2}$$

Then there exist constants $C > 0$ and $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, we have the inequality

$$|\mathcal{F}f(\xi)| \leq C(e^{-\frac{\epsilon^2 |\xi|^2}{4\pi}} + \epsilon^\alpha), \tag{2.3}$$

where the constant C depends on $C_0, \|f\|_{L^1}, n, \tilde{\delta}$ and α.

By assumption, the potentials $q \in H^s(\Omega)$ with $s > \frac{3}{2}$ and therefore there exists $\alpha > 0$ such that $q \in C^{0,\alpha}(\overline{\Omega})$. The conclusions of the Lemma 2.4, therefore, hold true for the potentials q.

3. CGO and the stability estimates

In this section, we shall construct appropriate solutions to (1.1) via CGO solutions as described in Lemma 2.1. In order to do so, we introduce a change of coordinates as follows (see also [4, 8, 9, 13]).
Given $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, the new coordinate representation is obtained by rotating the standard axes in a manner such that under the transformed coordinates, the representation of ξ, which we shall denote henceforth by $\tilde{\xi}$, is of the form

$$\tilde{\xi} = (\tilde{\xi}_1, 0, \tilde{\xi}_3), \text{ where } \tilde{\xi}_1 = (\xi_1^2 + \xi_2^2)^{\frac{1}{2}} \text{ and } \tilde{\xi}_3 = \xi_3.$$

Let $\tilde{x}$ denote the representation of x in this new coordinates. It is easy to see that for $x, y \in \mathbb{R}^3$, $\sum_{i=1}^3 x_i \cdot y_i = \sum_{i=1}^3 \tilde{x}_i \cdot \tilde{y}_i$.
In the transformed coordinates, let us choose

$$\begin{aligned} \tilde{\zeta}_1 &= \Big(-\frac{\tilde{\xi}_1}{2} + \tau \tilde{\xi}_3, -i \Big(|\xi|^2 (\frac{1}{4} + \tau^2) - k^2 \Big)^{\frac{1}{2}}, -\frac{\tilde{\xi}_3}{2} - \tau \tilde{\xi}_1 \Big), \\ \tilde{\zeta}_2 &= \Big(-\frac{\tilde{\xi}_1}{2} - \tau \tilde{\xi}_3, i \Big(|\xi|^2 (\frac{1}{4} + \tau^2) - k^2 \Big)^{\frac{1}{2}}, -\frac{\tilde{\xi}_3}{2} + \tau \tilde{\xi}_1 \Big), \end{aligned} \tag{3.1}$$

where τ is a positive real number. We note that in the original coordinates ζ_1, ζ_2 are of the form

$$\zeta_1 : \begin{cases} \zeta_{1,1} = (-\frac{\tilde{\xi_1}}{2} + \tau\tilde{\xi_3})\frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}} + i\frac{\xi_2}{\sqrt{\xi_1^2+\xi_2^2}}(|\xi|^2(\frac{1}{4}+\tau^2) - k^2)^{\frac{1}{2}} \\ \zeta_{1,2} = (-\frac{\tilde{\xi_1}}{2} + \tau\tilde{\xi_3})\frac{\xi_2}{\sqrt{\xi_1^2+\xi_2^2}} - i\frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}}(|\xi|^2(\frac{1}{4}+\tau^2) - k^2)^{\frac{1}{2}} \\ \zeta_{1,3} = -\frac{\tilde{\xi_3}}{2} - \tau\tilde{\xi_1} = -\frac{\xi_3}{2} - \tau(\xi_1^2+\xi_2^2)^{\frac{1}{2}} \end{cases} \tag{3.2}$$

$$\zeta_2 : \begin{cases} \zeta_{2,1} = (-\frac{\tilde{\xi_1}}{2} - \tau\tilde{\xi_3})\frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}} - i\frac{\xi_2}{\sqrt{\xi_1^2+\xi_2^2}}(|\xi|^2(\frac{1}{4}+\tau^2) - k^2)^{\frac{1}{2}} \\ \zeta_{2,2} = (-\frac{\tilde{\xi_1}}{2} - \tau\tilde{\xi_3})\frac{\xi_2}{\sqrt{\xi_1^2+\xi_2^2}} + i\frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}}(|\xi|^2(\frac{1}{4}+\tau^2) - k^2)^{\frac{1}{2}} \\ \zeta_{2,3} = -\frac{\tilde{\xi_3}}{2} + \tau\tilde{\xi_1} = -\frac{\xi_3}{2} + \tau(\xi_1^2+\xi_2^2)^{\frac{1}{2}} \end{cases}$$

where $\zeta_{i,j}$ denote the j-th coordinate of ζ_i. Let us also define the reflections of $\tilde{\zeta_i}$ on the plane $\xi_3 = 0$, i.e.

$$\begin{aligned} \tilde{\zeta}_1^* &= \Big(-\frac{\tilde{\xi}_1}{2} + \tau\tilde{\xi}_3, -i\Big(|\xi|^2(\frac{1}{4}+\tau^2) - k^2\Big)^{\frac{1}{2}}, \frac{\tilde{\xi}_3}{2} + \tau\tilde{\xi}_1\Big), \\ \tilde{\zeta}_2^* &= \Big(-\frac{\tilde{\xi}_1}{2} - \tau\tilde{\xi}_3, i\Big(|\xi|^2(\frac{1}{4}+\tau^2) - k^2\Big)^{\frac{1}{2}}, \frac{\tilde{\xi}_3}{2} - \tau\tilde{\xi}_1\Big). \end{aligned} \tag{3.3}$$

It is easy to see from (3.1)–(3.3) that for $j = 1, 2$,

$$|\text{Re } \zeta_j|^2 = |\xi|^2(\frac{1}{4}+\tau^2), \text{ and } |\text{Im } \zeta_j|^2 = |\xi|^2(\frac{1}{4}+\tau^2) - k^2,$$

and $\zeta_1 + \zeta_2, \zeta_1^* + \zeta_2^*, \zeta_1 + \zeta_2^*, \zeta_1^* + \zeta_2$ are real vectors. Also $\zeta_1 \cdot \zeta_1 = \zeta_2 \cdot \zeta_2 = \zeta_1^* \cdot \zeta_1^* = \zeta_2^* \cdot \zeta_2^* = k^2$.
We next extend the potentials q_1, q_2 onto the whole of $\mathbb{R}^3$ as even functions in x_3. Lemma 2.1 then guarantees the existence of CGO solutions to the equation (1.1) in $\mathbb{R}^3$ of the form $e^{i\zeta_j\cdot x}(1+w_j)$ and $e^{i\zeta_j^*\cdot x}(1+w_j^*)$ for $j = 1, 2$ with the remainder terms satisfying the estimate $\|w_j\|_{H^s(\Omega)} \leq \frac{C}{|\text{Im } \zeta_j|}\|q_j\|_{H^s(\Omega)}$.
Let us define

$$u_1(x) = e^{i\zeta_1\cdot x}(1+w_1) - e^{i\zeta_1^*\cdot x}(1+w_1^*), \ u_2(x) = e^{i\zeta_2\cdot x}(1+w_2) - e^{i\zeta_2^*\cdot x}(1+w_2^*). \tag{3.4}$$

It can be easily checked from the definitions (3.4) that the functions u_j $(j = 1, 2)$ satisfy the equations (1.1) in $\mathbb{R}^3_-$ with potentials q_1, q_2 respectively and $u_j(x) = 0$ on $x_3 = 0$.
With all this preparation in place, we now proceed to derive the stability estimates.

3.1. **Derivation of the stability estimates.** Let us denote $M = C_*N$. Then provided $|\text{Im } \zeta_j| > M$, the estimate

$$\|w_j\|_{H^s(\Omega)} \leq \frac{C}{|\text{Im } \zeta_j|}\|q_j\|_{H^s(\Omega)} \leq \frac{CN}{|\text{Im } \zeta_j|} \leq \frac{CN}{C_*N} = C,$$

holds true. Let $\Omega \subset B(0, R)$ for a fixed $R(>> 1)$ large enough. Then since $|e^{i\zeta_j\cdot x}| \leq e^{|Im\ \zeta_j||x|}$, we can write

$$\|u_j\|_{H^1(\Omega)} \leq 2e^{R[|\xi|^2(\frac{1}{4}+\tau^2)-k^2]^{\frac{1}{2}}}\|1+w_j\|_{H^s(\Omega)} \leq Ce^{R[|\xi|^2(\frac{1}{4}+\tau^2)-k^2]^{\frac{1}{2}}},$$

since $s > \frac{3}{2} > 1$. Using this in (2.1), we see that

$$\Big|\int_\Omega (q_2 - q_1)u_1u_2\ dx\Big| \leq Ck^4e^{2R\left[|\xi|^2(\frac{1}{4}+\tau^2)-k^2\right]^{\frac{1}{2}}}\ \text{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}), \tag{3.5}$$

provided $|\text{Im }\zeta_j| > M$, that is, $|\xi|^2\Big(\frac{1}{4}+\tau^2\Big) > M^2+k^2$.
Let us denote $q_0 = q_2 - q_1$. Using the definitions of u_1, u_2 from (3.4), we can write

$$\begin{aligned}
\int_\Omega q_0 u_1 u_2 \ dx &= \int_\Omega q_0(x)\Big[e^{i(\zeta_1+\zeta_2)\cdot x}(1+w_1)(1+w_2) + e^{i(\zeta_1^*+\zeta_2^*)\cdot x}(1+w_1^*)(1+w_2^*) \\
&\qquad - e^{i(\zeta_1+\zeta_2^*)\cdot x}(1+w_1)(1+w_2^*) - e^{i(\zeta_1^*+\zeta_2)\cdot x}(1+w_1^*)(1+w_2)\Big] \ dx \\
&= \int_\Omega q_0(x)[e^{-i\xi\cdot x} + e^{-i\xi^*\cdot x}] \ dx + \int_\Omega q_0(x) f(x, w_1, w_2, w_1^*, w_2^*) \ dx \\
&\qquad - \int_\Omega q_0(x)[e^{i(\zeta_1+\zeta_2^*)\cdot x} + e^{i(\zeta_1^*+\zeta_2)\cdot x}] \ dx \\
&= \mathcal{F}q_0(\xi) + \int_\Omega q_0(x) f(x, w_1, w_2, w_1^*, w_2^*) \ dx - \int_\Omega q_0(x)[e^{i(\zeta_1+\zeta_2^*)\cdot x} + e^{i(\zeta_1^*+\zeta_2)\cdot x}] \ dx,
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
f = e^{-i\xi\cdot x}(w_1+w_2+w_1w_2) + e^{-i\xi^*\cdot x}(w_1^*+w_2^*+w_1^*w_2^*) - e^{i(\zeta_1^*+\zeta_2)\cdot x}(w_1^*+w_2+w_1^*w_2) \\
- e^{i(\zeta_1+\zeta_2^*)\cdot x}(w_1+w_2^*+w_1w_2^*).
\end{aligned} \tag{3.7}$$

Next since $\zeta_1+\zeta_2^*, \tilde{\zeta}_1+\tilde{\zeta}_2^*, \zeta_1^*+\zeta_2$ and $\tilde{\zeta}_1^*+\tilde{\zeta}_2$ are real vectors, we note that

$$\begin{aligned}
(\zeta_1+\zeta_2^*)\cdot x &= (\tilde{\zeta}_1+\tilde{\zeta}_2^*)\cdot\tilde{x} = -\tilde{\xi}_1\tilde{x}_1 - 2\tau\tilde{\xi}_1\tilde{x}_3 = -[\xi'\cdot x' + 2\tau|\xi'|x_3], \\
(\zeta_1^*+\zeta_2)\cdot x &= (\tilde{\zeta}_1^*+\tilde{\zeta}_2)\cdot\tilde{x} = -\tilde{\xi}_1\tilde{x}_1 + 2\tau\tilde{\xi}_1\tilde{x}_3 = -[\xi'\cdot x' - 2\tau|\xi'|x_3],
\end{aligned}$$

where $\xi' = (\xi_1, \xi_2), x' = (x_1, x_2)$, and therefore we can write

$$\int_\Omega q_0(x)e^{i(\zeta_1+\zeta_2^*)\cdot x} \ dx = \mathcal{F}q_0(\xi', 2\tau|\xi'|) \text{ and } \int_\Omega q_0(x)e^{i(\zeta_1^*+\zeta_2)\cdot x} \ dx = \mathcal{F}q_0(\xi', -2\tau|\xi'|). \tag{3.8}$$

Using the version of Riemann-Lebesgue Lemma stated in Lemma 2.4, the terms in (3.8) can be estimated as

$$|\mathcal{F}q_0(\xi', 2\tau|\xi'|)| + |\mathcal{F}q_0(\xi', -2\tau|\xi'|)| \le C\Big[e^{-\frac{\epsilon^2(1+4\tau^2)|\xi'|^2}{4\pi}} + \epsilon^\alpha\Big], \text{ where } \alpha\in(0,1), \tag{3.9}$$

and for any $\epsilon < \epsilon_0$ with ϵ_0 defined as in Lemma 2.4.
Also for $|\xi|^2(\frac{1}{4}+\tau^2) > M^2+k^2$, we can use the estimates for the remainder terms w_j to derive

$$|\int_\Omega q_0 f(x, w_1, w_2, w_1^*, w_2^*)| \ dx \le C\|q_0\|_{L^2}\|f\|_{L^2} \le \frac{C}{[|\xi|^2(\frac{1}{4}+\tau^2)-k^2]^{\frac{1}{2}}}. \tag{3.10}$$

Combining the above estimates (3.5)–(3.10), it follows that provided $|\xi|^2(\frac{1}{4}+\tau^2) > M^2+k^2$ holds, we have
(3.11)

$$\begin{aligned}
&|\mathcal{F}q_0(\xi)| \le C\Big[k^4 e^{2R[|\xi|^2(\frac{1}{4}+\tau^2)-k^2]^{\frac{1}{2}}}\text{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2}) + e^{-\frac{\epsilon^2(1+4\tau^2)|\xi'|^2}{4\pi}} + \epsilon^\alpha + \frac{1}{[|\xi|^2(\frac{1}{4}+\tau^2)-k^2]^{\frac{1}{2}}}\Big] \\
\Rightarrow &|\mathcal{F}q_0(\xi)|^2 \le C\Big[k^8 e^{4R[|\xi|^2(\frac{1}{4}+\tau^2)-k^2]^{\frac{1}{2}}}\text{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2 + e^{-\frac{\epsilon^2(1+4\tau^2)|\xi'|^2}{2\pi}} + \epsilon^{2\alpha} + \frac{1}{|\xi|^2(\frac{1}{4}+\tau^2)-k^2}\Big].
\end{aligned}$$

Our strategy next is to estimate the H^{-1} norm of q_0 and then use the interpolation inequality to derive an estimate for the L^∞ norm of q_0. It will be worthwhile to note at this point that it is sufficient to derive the stability estimates when $dist(\mathcal{C}_{q_1},\mathcal{C}_{q_2}) < \delta$ for some $\delta\in(0,1)$ sufficiently small. The case when $dist(\mathcal{C}_{q_1},\mathcal{C}_{q_2}) \ge \delta$ easily follows from the continuous inclusions $L^\infty(\Omega)\hookrightarrow L^2(\Omega)\hookrightarrow H^{-1}(\Omega)$ using the bound N on the norm of the potentials. Therefore we shall henceforth focus on the case $dist(\mathcal{C}_{q_1},\mathcal{C}_{q_2}) < \delta$ where the choice of δ shall be made clear in the course of the proof.
Let us denote $E = |\log\ \text{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})|$ and for $\rho > 0$ to be chosen later, we set $Z_\rho := \{\xi\in\mathbb{R}^3 : |\xi'| <$

$\rho, |\xi_3| < \rho\}$. The integral over the higher frequency modes can be estimated using the bounds on the L^2-norms of the potentials q_1, q_2 and we can write

$$\|q_0\|^2_{H^{-1}} = \int_{Z_\rho} \frac{|\mathcal{F}q_0(\xi)|^2}{1+|\xi|^2}\, d\xi + \int_{Z_\rho^c} \frac{|\mathcal{F}q_0(\xi)|^2}{1+|\xi|^2}\, d\xi \leq \int_{Z_\rho} \frac{|\mathcal{F}q_0(\xi)|^2}{1+|\xi|^2}\, d\xi + \frac{C}{\rho^2}. \tag{3.12}$$

In order to estimate the integral over the lower frequency modes, that is, the first term in the right hand side of (3.12) we proceed as follows. Using (3.11), provided $|\xi|^2(\frac{1}{4}+\tau^2) > M^2 + k^2$, we can write

$$\int_{Z_\rho} \frac{|\mathcal{F}q_0(\xi)|^2}{1+|\xi|^2}\, d\xi \leq C\Big[\int_{Z_\rho} \frac{k^8 e^{4R[|\xi|^2(\frac{1}{4}+\tau^2)-k^2]^{\frac{1}{2}}}\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2 + \epsilon^{2\alpha} + \frac{1}{|\xi|^2(\frac{1}{4}+\tau^2)-k^2}}{1+|\xi|^2}\Big] d\xi + C\int_{-\rho}^{\rho}\int_{B'(0,\rho)} \frac{e^{-\frac{\epsilon^2(1+4\tau^2)|\xi'|^2}{2\pi}}}{1+|\xi|^2} d\xi. \tag{3.13}$$

Now we choose $\frac{1}{4}+\tau^2 = \frac{2k^2+(\frac{E}{5R})^2}{|\xi|^2}$. This would imply that $|\xi|^2(\frac{1}{4}+\tau^2)-k^2 = k^2+(\frac{E}{5R})^2$ and therefore we shall also have to choose E such that $(\frac{E}{5R})^2 > M^2$. This, in turn, is linked to the choice of the $\delta \in (0,1)$. In fact, choosing δ sufficiently small such that $\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2}) < \delta$, E can be made large enough to fulfil the condition. It will be worth noting that the choice of δ depends on the constants R and M only. Then

$$C\int_{Z_\rho} \frac{1}{|\xi|^2(\frac{1}{4}+\tau^2)-k^2} d\xi \leq \frac{C\rho^3}{k^2+(\frac{E}{5R})^2} \leq \frac{C\rho^3}{(k+\frac{E}{5R})^2}, \text{ and } C\int_{Z_\rho} \epsilon^{2\alpha} d\xi = C\rho^3\epsilon^{2\alpha}. \tag{3.14}$$

Also

$$e^{4R[|\xi|^2(\frac{1}{4}+\tau^2)-k^2]^{\frac{1}{2}}}\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2 = e^{4R[k^2+(\frac{E}{5R})^2]^{\frac{1}{2}}}\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2,$$

and therefore

$$C\int_{Z_\rho} k^8 e^{4R[|\xi|^2(\frac{1}{4}+\tau^2)-k^2]^{\frac{1}{2}}}\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2 d\xi \leq C\rho^3 k^8 e^{4R[k+\frac{E}{5R}]}\operatorname{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2. \tag{3.15}$$

To estimate the last term in the right-hand side of (3.13), we proceed as follows (see also [4]). We note that $1+4\tau^2 = \frac{8k^2+4(\frac{E}{5R})^2}{|\xi|^2} \geq \frac{2[k^2+(\frac{E}{5R})^2]}{|\xi|^2} \geq \frac{[k+\frac{E}{5R}]^2}{|\xi|^2}$ which implies

$$e^{-\frac{\epsilon^2(1+4\tau^2)|\xi'|^2}{2\pi}} \leq e^{-\frac{\epsilon^2[k+\frac{E}{5R}]^2|\xi'|^2}{2\pi|\xi|^2}}.$$

Also since $|\xi|^2 < 2\rho^2$, we have $e^{\frac{\epsilon^2[k+\frac{E}{5R}]^2|\xi'|^2}{2\pi\cdot 2\rho^2}} \leq e^{\frac{\epsilon^2[k+\frac{E}{5R}]^2|\xi'|^2}{2\pi|\xi|^2}}$ and therefore

$$e^{-\frac{\epsilon^2(1+4\tau^2)|\xi'|^2}{2\pi}} \leq e^{-\frac{\epsilon^2[k+\frac{E}{5R}]^2|\xi'|^2}{2\pi\cdot 2\rho^2}}.$$

Let us choose $\epsilon > 0$ such that $\epsilon^2 = \frac{1}{k+\frac{E}{5R}}$. If required, we can choose δ smaller again such that $\epsilon < \epsilon_0$ also holds. Then we can write

$$\begin{aligned}\int_{-\rho}^{\rho}\int_{B'(0,\rho)} \frac{e^{-\frac{\epsilon^2(1+4\tau^2)|\xi'|^2}{2\pi}}}{1+|\xi|^2} d\xi &\leq \int_{-\rho}^{\rho}\int_{B'(0,\rho)} \frac{e^{-\frac{\epsilon^2[k+\frac{E}{5R}]^2|\xi'|^2}{4\pi\rho^2}}}{1+|\xi|^2}\, d\xi' d\xi_3 \leq C\rho\int_0^\rho r e^{-\frac{k+\frac{E}{5R}}{4\pi\rho^2}r^2}\, d\xi' d\xi_3 \\ &\leq C\rho^2\Big[k+\frac{E}{5R}\Big]^{-\frac{1}{2}}\rho\Big[k+\frac{E}{5R}\Big]^{-\frac{1}{2}}\int_0^\infty u e^{-\frac{1}{4\pi}u^2}\, du \leq C\rho^3\Big[k+\frac{E}{5R}\Big].\end{aligned} \tag{3.16}$$

We shall now specify our choice of ρ. Since $\alpha \in (0,1)$, we have $C\rho^3\epsilon^{2\alpha} = C\rho^3\Big(\frac{1}{k+\frac{E}{5R}}\Big)^\alpha \geq C\rho^3\Big(\frac{1}{k+\frac{E}{5R}}\Big)$. Now we choose $\rho > 0$ such that $\rho^3 = \Big(k+\frac{E}{5R}\Big)^\beta$, where $\beta < \alpha(<1)$. Then $C\rho^3\Big(\frac{1}{k+\frac{E}{5R}}\Big)^\alpha = C\Big(\frac{1}{k+\frac{E}{5R}}\Big)^{\alpha-\beta}$ and also $\frac{C}{\rho^2} = C.\Big(\frac{1}{k+\frac{E}{5R}}\Big)^{\frac{2\beta}{3}}$ and $\frac{C\rho^3}{(k+\frac{E}{5R})^2} = C\Big(\frac{1}{k+\frac{E}{5R}}\Big)^{2-\beta}$.

Finally, let $\tilde\alpha = \min\{\alpha-\beta, 2-\beta, \frac{2\beta}{3}\}$. Then from (3.12)–(3.16) we derive

$$
\begin{aligned}
\|q_0\|^2_{H^{-1}} &\le C\rho^3 k^8 e^{4R\left[k+\frac{E}{5R}\right]}\ dist(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2 + C\Big(\frac{1}{k+\frac{E}{5R}}\Big)^{\tilde\alpha}\\
&= Ck^8\Big(k+\frac{E}{5R}\Big)^{\beta} e^{4R\left[k+\frac{E}{5R}\right]}\ dist(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2 + C\Big(\frac{1}{k+\frac{E}{5R}}\Big)^{\tilde\alpha}\\
&\le C\Big[k^8 e^{R\left[k+\frac{E}{5R}\right]} e^{4R\left[k+\frac{E}{5R}\right]}\ dist(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2 + C\Big(\frac{1}{k+\frac{E}{5R}}\Big)^{\tilde\alpha}\Big]\\
&\le C\Big[k^8 e^{5Rk} e^{E}\ dist(\mathcal{C}_{q_1},\mathcal{C}_{q_2})^2 + \frac{1}{(k+\frac{E}{5R})^{\tilde\alpha}}\Big]\\
&\le C\Big[e^{6Rk}\ dist(\mathcal{C}_{q_1},\mathcal{C}_{q_2}) + \frac{1}{(k+\frac{E}{5R})^{\tilde\alpha}}\Big].
\end{aligned}
\tag{3.17}
$$

As already discussed before, the estimate (3.17) also holds true when $dist(\mathcal{C}_{q_1},\mathcal{C}_{q_2}) \ge \delta$.

Using (3.17), we can now estimate the L^∞-norm of $q_0 = q_2 - q_1$ by using interpolation. To see this, we recall that given t_0, t, t_1 such that $t_0 < t_1$ and $t = (1-p)t_0 + pt_1$, where $p \in (0,1)$, the H^t-norm of a function q can be estimated (by the interpolation theorem) as

$$\|q\|_{H^t} \le \|q\|^{1-p}_{H^{t_0}} \cdot \|q\|^p_{H^{t_1}}.$$

In our case, we define $\eta > 0$ such that $s = \frac{3}{2} + 2\eta$ and choose $t_0 = -1, t_1 = s$ and $t_2 = \frac{3}{2} + \eta = s - \eta$. Thus we can write

$$t = (1-p)t_0 + pt_1, \text{ where } p = \frac{1+s-\eta}{1+s}.$$

Using the Sobolev embedding theorem and the interpolation theorem, we have

$$
\begin{aligned}
\|q_1-q_2\|_{L^\infty(\Omega)} &\le C\|q_1-q_2\|_{H^{\frac{3}{2}+\eta}(\Omega)} \le C\|q_1-q_2\|^{1-p}_{H^{-1}(\Omega)}\|q_1-q_2\|^p_{H^s(\Omega)} \le C\|q_1-q_2\|^{\frac{\eta}{1+s}}_{H^{-1}(\Omega)}\\
&\le C\Big(e^{6Rk}\ dist(\mathcal{C}_{q_1},\mathcal{C}_{q_2}) + \frac{1}{(k+\frac{E}{5R})^{\tilde\alpha}}\Big)^{\frac{\eta}{2(1+s)}},
\end{aligned}
\tag{3.18}
$$

which is the required stability estimate (1.3).

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