GKW representation theorem and linear BSDEs under restricted information. An application to risk-minimization.

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Abstract

In this paper we provide Galtchouk-Kunita-Watanabe representation results in the case where there are restrictions on the available information. This allows to prove existence and uniqueness for linear backward stochastic differential equations driven by a general càdlàg martingale under partial information. Furthermore, we discuss an application to risk-minimization where we extend the results of Föllmer and Sondermann (1986) to the partial information framework and we show how our result fits in the approach of Schweizer (1994).

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1 Introduction

This paper provides two main contributions. First, we prove Galtchouk-Kunita-Watanabe representation results in the case where there are restrictions on the available information and we show an application to risk-minimization. Second, as an important consequence, we prove existence and uniqueness for linear backward stochastic differential equations (in short BSDEs) driven by a general càdlàg martingale under partial information.
For BSDEs driven by a general càdlàg martingale beyond the Brownian setting, there exist very few results in literature (see [5] and more recently [1] and [2], as far as we are aware). Here we study for the first time such a general case in the situation where there

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are restrictions on the available information, that represents an interesting issue arising in many financial problems. Mathematically, this means to consider an additional filtration $\mathbb{H}$ smaller than the full information flow $\mathbb{F}$. A typical example arises when $\mathcal{H}_t = \mathcal{F}_{(t-\tau)^+}$ where $\tau \in (0,T)$ is a fixed delay and $(t-\tau)^+ := \max\{0, t-\tau\}$ and T denotes a time horizon.
We start our investigation by considering BSDEs of the form

$$Y_t = \xi - \int_t^T Z_s \mathrm{d}M_s - (O_T - O_t), \quad 0 \le t \le T, \tag{1.1}$$

driven by a square-integrable (càdlàg) martingale $M = (M_t)_{0 \le t \le T}$, where $T > 0$ is a fixed time horizon, $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$[1] denotes the terminal condition and $O = (O_t)_{0 \le t \le T}$ is a square-integrable $\mathbb{F}$-martingale with $O_0 = 0$, satisfying a suitable orthogonality condition that we will make more precise in the next section.
We look for a solution (Y, Z) to equation (1.1) under partial information, where $Y = (Y_t)_{0 \le t \le T}$ is a càdlàg $\mathbb{F}$-adapted process and $Z = (Z_t)_{0 \le t \le T}$ is an $\mathbb{H}$-predictable process such that $\mathbb{E}\left[\int_0^T |Z_t|^2 \mathrm{d}\langle M \rangle_t\right] < \infty$.
To this aim, we prove a Galtchouk-Kunita-Watanabe decomposition in the case where there are restrictions on the available information. More precisely, we obtain that every random variable $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ can be uniquely written as

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} \mathrm{d}M_t + O_T, \quad \mathbb{P} - \text{a.s.}, \tag{1.2}$$

where $H^{\mathcal{H}} = (H^{\mathcal{H}})_{0 \le t \le T}$ is an $\mathbb{H}$-predictable process such that $\mathbb{E}\left[\int_0^T |H_t^{\mathcal{H}}|^2 \mathrm{d}\langle M \rangle_t\right] < \infty$. To the authors' knowledge such a decomposition has not been proved yet in the existing literature. We will see that decomposition (1.2) allows to construct a solution to the BSDE (1.1) and ensures its uniqueness in this setting.
Moreover, we are able to provide an explicit characterization of the integrand process $H^{\mathcal{H}}$ given in decomposition (1.2) in terms of the one appearing in the classical Galtchouk-Kunita-Watanabe decomposition, by using $\mathbb{H}$-predictable (dual) projections.
Finally, we discuss a financial application. More precisely, we study the problem of hedging a contingent claim in the case where investors acting in the market have partial information. Since the market is incomplete we choose the risk-minimization approach, a quadratic hedging method which keeps the replicability constraint and relaxes the self financing condition, see [6] and [14] for further details. As in [6] and [13], we consider the case where the price process is a martingale under the real world probability measure. In [6], under the case of full information, the authors provide the risk-minimizing hedging strategy in terms of the classical Galtchouk-Kunita-Watanabe decomposition. Here, by using the Galtchouk-Kunita-Watanabe decomposition under partial information, we extend this result to the case where there are restrictions on the available information. Finally, thanks to the explicit representation of the integrand process $H^{\mathcal{H}}$ appearing in

[1] The space $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ denotes the set of all $\mathcal{F}_T$-measurable real-valued random variables H such that $\mathbb{E}\left[|H|^2\right] = \int_\Omega |H|^2 \mathrm{d}\mathbb{P} < \infty$.

decomposition (1.2), we find the same expression for the optimal strategy in terms of the Radon-Nikodym derivative of two $\mathbb{H}$-predictable dual projections, that is proved in [13]. The paper is organized as follows. In Section 2 we give the definition of solution to BSDEs under partial information. Section 3 is devoted to prove existence and uniqueness results for the solutions, which are obtained by applying the Galtchouk-Kunita-Watanabe decomposition adapted to the restricted information setting. The explicit representation of the integrand process $H^{\mathcal{H}}$ appearing in (1.2) can be found in Section 4. Finally, an application to risk-minimization is given in Section 5.

2 Setting

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} := (\mathcal{F}_t)_{0\le t\le T}$, where $\mathcal{F}_t$ represents the full information at time t. We assume that $\mathcal{F}_T = \mathcal{F}$. Then we consider a subfiltration $\mathbb{H} := (\mathcal{H}_t)_{0\le t\le T}$ of $\mathbb{F}$, i.e. $\mathcal{H}_t \subseteq \mathcal{F}_t$, for each $t \in [0, T]$, corresponding to the available information level. We remark that both filtrations are assumed to satisfy the usual hypotheses of completeness and right-continuity, see e.g. [12].
For simplicity we only consider the one-dimensional case. Extensions to several dimensions are straightforward and left to the reader. The data of the problem are:

- an $\mathbb{R}$-valued square-integrable (càdlàg) $\mathbb{F}$-martingale $M = (M_t)_{0\le t\le T}$ with $\mathbb{F}$-predictable quadratic variation process denoted by $\langle M\rangle = (\langle M, M\rangle)_{0\le t\le T}$;
- a terminal condition $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.

Definition 2.1. *A solution of the BSDE*

$$Y_t = \xi - \int_t^T Z_s \mathrm{d}M_s - (O_T - O_t), \quad 0 \le t \le T, \tag{2.1}$$

with data $(\xi, \mathbb{H})$ under partial information, where $O = (O_t)_{0\le t\le T}$ is a square-integrable $\mathbb{F}$-martingale with $O_0 = 0$, satisfying the orthogonality condition

$$\mathbb{E}\left[O_T \int_0^T \varphi_t \mathrm{d}M_t\right] = 0, \tag{2.2}$$

for all $\mathbb{H}$-predictable processes $\varphi = (\varphi_t)_{0\le t\le T}$ such that $\mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$, is a couple (Y, Z) of processes with values in $\mathbb{R}\times\mathbb{R}$, satisfying (2.1) such that

- *$Y = (Y_t)_{0\le t\le T}$ is a càdlàg $\mathbb{F}$-adapted process;*
- *$Z = (Z_t)_{0\le t\le T}$ is an $\mathbb{H}$-predictable process such that $\mathbb{E}\left[\int_0^T |Z_t|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$.*

Remark 2.2. *The orthogonality condition given in* (2.2) *is weaker than the classical strong orthogonality condition, see e.g [11] or [12]. Indeed, set $N_t = \int_0^t \varphi_s \mathrm{d}M_s$, for each $t \in [0, T]$, where φ is an $\mathbb{H}$-predictable process such that $\mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$. If*

$$\langle O, M\rangle_t = 0 \quad \mathbb{P} - \text{a.s.}, \quad \forall t \in [0, T],$$

then

$$\langle O, N\rangle_t = \int_0^t \varphi_s \mathrm{d}\langle O, M\rangle_s = 0 \quad \mathbb{P}-\text{a.s.}, \quad \forall t \in [0,T].$$

Consequently, $O \cdot N$ is an $\mathbb{F}$-martingale null at zero, that implies

$$\mathbb{E}\left[O_t N_t\right] = 0, \quad \forall t \in [0,T],$$

and in particular condition (2.2).

Remark 2.3. *Since for any $\mathbb{H}$-predictable process φ, the process $\mathbf{1}_{(0,t]}(s)\varphi_s$, with $t \le T$, is $\mathbb{H}$-predictable, condition* (2.2) *implies that for every $t \in [0,T]$*

$$\mathbb{E}\left[O_T \int_0^t \varphi_s \mathrm{d}M_s\right] = 0,$$

and by conditioning with respect to $\mathcal{F}_t$ (note that O is an $\mathbb{F}$-martingale), we have

$$\mathbb{E}\left[O_t \int_0^t \varphi_s \mathrm{d}M_s\right] = \mathbb{E}\left[\int_0^t \varphi_s \mathrm{d}\langle M, O\rangle_s\right] = 0 \quad \forall t \in [0,T].$$

From this last equality, we can argue that in the case of full information, i.e., $\mathcal{H}_t = \mathcal{F}_t$, for each $t \in [0,T]$, condition (2.2) *is equivalent to the strong orthogonality condition between O and M (see e.g. Lemma 2 and Theorem 36, Chapter IV, page 180 of [12] for a rigorous proof).*

3 Existence and uniqueness for linear BSDEs under partial information

Our aim is to investigate existence and uniqueness of a solution to the BSDE (2.1) with data $(\xi, \mathbb{H})$ driven by the general martingale M in the sense of Definition 2.1. This requires to prove a Galtchouk-Kunita-Watanabe representation result under restricted information.
We introduce the linear subspace $\mathcal{L}_T^{\mathcal{H}}$ of $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ given by all random variables η of the form

$$\left\{ U_0 + \int_0^T \varphi_t \mathrm{d}M_t \,\middle|\, U_0 \in \mathcal{H}_0,\ \varphi \text{ is } \mathbb{H}-\text{predictable with } \mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M\rangle_t\right] < \infty \right\}. \tag{3.1}$$

Lemma 3.1. *The set $\mathcal{L}_T^{\mathcal{H}}$ is a closed subspace of $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.*

Proof. Let $U_0^n \in \mathcal{H}_0$ and $(\varphi^n)_{n\in\mathbb{N}}$, with $\varphi^n = (\varphi_t^n)_{0\le t\le T}$, be a sequence of $\mathbb{H}$-predictable processes satisfying $\mathbb{E}\left[\int_0^T |\varphi_t^n|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$ such that the sequence

$$\eta^n = U_0^n + \int_0^T \varphi_t^n \mathrm{d}M_t, \quad n \in \mathbb{N},$$

converges to some random variable $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, as n goes to infinity. By taking the conditional expectation with respect to $\mathcal{H}_0$, we have

$$U_0^n = \mathbb{E}\left[\eta^n \middle| \mathcal{H}_0\right] \longrightarrow \mathbb{E}\left[\eta \middle| \mathcal{H}_0\right], \quad \text{as } n \to \infty.$$

We set $U_0 = \mathbb{E}\left[\eta | \mathcal{H}_0\right]$. Since $(\eta^n - U_0^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, it follows that

$$\mathbb{E}\left[\int_0^T (\varphi_t^n - \varphi_t^m)^2 \mathrm{d}\langle M\rangle_t\right] \longrightarrow 0, \quad \text{as } n, m \to \infty.$$

Consequently, $(\varphi^n)_{n\in\mathbb{N}}$ converges in $L^2(\Omega, \mathrm{d}\langle M\rangle \otimes \mathrm{d}\mathbb{P})$[2] to some process $\varphi = (\varphi_t)_{0\le t\le T} \in L^2(\Omega, \mathrm{d}\langle M\rangle \otimes \mathrm{d}\mathbb{P})$. Finally, since there is a subsequence converging $\mathrm{d}\langle M\rangle \otimes \mathrm{d}\mathbb{P}$-a.e., the limit φ is necessarily an $\mathbb{H}$-predictable process. □

Proposition 3.2. *Let $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. There exists a unique decomposition of the form*

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} \mathrm{d}M_t + O_T, \quad \mathbb{P}-\text{a.s.}, \tag{3.2}$$

where $U_0 \in \mathcal{H}_0$, $H^{\mathcal{H}}$ is an $\mathbb{H}$-predictable process such that $\mathbb{E}\left[\int_0^T |H_t^{\mathcal{H}}|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$ and O is a square-integrable $\mathbb{F}$-martingale with $O_0 = 0$ such that $\mathbb{E}\left[O_T \cdot \eta\right] = 0$, for every $\eta \in \mathcal{L}_T^{\mathcal{H}}$. Moreover $U_0 = \mathbb{E}\left[\xi | \mathcal{H}_0\right]$ and $\mathbb{E}\left[O_T | \mathcal{H}_0\right] = 0$.

Proof. The existence and uniqueness property of decomposition (3.2) is clearly ensured by the orthogonal projection of the random variable ξ onto the space $\mathcal{L}_T^{\mathcal{H}}$, that is closed in virtue of Lemma 3.1. Since $(U_0 + \int_0^{\cdot} H_t^{\mathcal{H}} \mathrm{d}M_t)$ is an $\mathbb{F}$-martingale, by taking the conditional expectation of ξ with respect to $\mathcal{H}_0$ in (3.2), we have

$$\begin{aligned}
\mathbb{E}\left[\xi | \mathcal{H}_0\right] &= \mathbb{E}\left[U_0 + \int_0^T H_t^{\mathcal{H}} \mathrm{d}M_t \middle| \mathcal{H}_0\right] + \mathbb{E}\left[O_T | \mathcal{H}_0\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[U_0 + \int_0^T H_t^{\mathcal{H}} \mathrm{d}M_t \middle| \mathcal{F}_0\right] \middle| \mathcal{H}_0\right] + \mathbb{E}\left[\mathbb{E}\left[O_T | \mathcal{F}_0\right] | \mathcal{H}_0\right] \\
&= \mathbb{E}\left[U_0 | \mathcal{H}_0\right],
\end{aligned}$$

where in the last equality we have used the fact that $\mathbb{E}\left[O_T | \mathcal{F}_0\right] = O_0 = 0$. Consequently $\mathbb{E}\left[O_T | \mathcal{H}_0\right] = 0$ and $U_0 = \mathbb{E}\left[U_0 | \mathcal{H}_0\right] = \mathbb{E}\left[\xi | \mathcal{H}_0\right]$. This concludes the proof. □

Theorem 3.3. *Given data $(\xi, \mathbb{H})$, there exists a unique couple (Y, Z) which solves the BSDE (2.1) according to Definition 2.1.*

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[2] The space $L^2(\Omega, \mathrm{d}\langle M\rangle \otimes \mathrm{d}\mathbb{P})$ denotes the set of all $\mathbb{F}$-adapted processes $\varphi = (\varphi_t)_{0\le t\le T}$ such that

$$\|\varphi\|_{L^2(\Omega, \mathrm{d}\langle M\rangle\otimes\mathrm{d}\mathbb{P})} := \left(\mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M\rangle_t\right]\right)^{\frac{1}{2}} < \infty.$$

Proof. **Existence.** Let $\mathcal{L}_T^{\mathcal{H}}$ be the linear subspace of $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ introduced in (3.1). Given $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, we know by Proposition 3.2 that there exists a unique decomposition of the form

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} \mathrm{d}M_t + A_T, \quad \mathbb{P} - \text{a.s.},$$

where in particular A is a square-integrable $\mathbb{F}$-martingale with $A_0 = 0$ orthogonal to all the elements in $\mathcal{L}_T^{\mathcal{H}}$. We use this result to construct a solution to the BSDE (2.1). We consider the orthogonal projection of $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ onto this space:

$$P_{\mathcal{L}_T^{\mathcal{H}}}(\xi) := U_0 + \int_0^T H_t^{\mathcal{H}} \mathrm{d}M_t.$$

The couple (U_0, H), where $U_0 \in \mathcal{H}_0$ and $H^{\mathcal{H}}$ is an $\mathbb{H}$-predictable process in $L^2(\Omega, \mathrm{d}\langle M \rangle \otimes \mathrm{d}\mathbb{P})$, uniquely identifies the projection, that exists and it is well-defined since $\mathcal{L}_T^{\mathcal{H}}$ is closed. We set

$$A_T := \xi - P_{\mathcal{L}_T^{\mathcal{H}}}(\xi) \in (\mathcal{L}_T^{\mathcal{H}})^{\perp},$$

where $(\mathcal{L}_T^{\mathcal{H}})^{\perp}$ denotes the orthogonal subspace of $\mathcal{L}_T^{\mathcal{H}}$. Here A_T corresponds to the final value of a square-integrable $\mathbb{F}$-martingale A with zero initial value, that implies $\mathbb{E}\left[\xi - U_0 \middle| \mathcal{F}_0\right] = 0$. Clearly, we have

$$\mathcal{L}_T^{\mathcal{H}} \oplus (\mathcal{L}_T^{\mathcal{H}})^{\perp} = L^2(\Omega, \mathcal{F}_T, \mathbb{P}).$$

Now we define the process Y as follows:

$$\begin{aligned} Y_t &:= \mathbb{E}\left[\xi \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[U_0 + \int_0^T H_t^{\mathcal{H}} \mathrm{d}M_t + A_T \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[U_0 \middle| \mathcal{F}_0\right] + \int_0^t H_s^{\mathcal{H}} \mathrm{d}M_s + A_t \\ &= Y_0 + \int_0^t H_s^{\mathcal{H}} \mathrm{d}M_s + A_t, \quad 0 \le t \le T, \end{aligned}$$

and we set $Z_t := H_t^{\mathcal{H}}$ and $O_t := A_t$, for every $t \in [0, T]$. Then we get

$$Y_t = \xi - \int_t^T Z_s \mathrm{d}M_s - (O_T - O_t), \quad 0 \le t \le T.$$

Uniqueness. Let (Y, Z), (Y', Z') be two solutions to the BSDE (2.1) under partial information associated to the terminal condition $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. We set $(\bar{Y}, \bar{Z}) = (Y - Y', Z - Z')$. Then $(\bar{Y}, \bar{Z})$ satisfies the BSDE

$$\bar{Y}_t = -\int_t^T \bar{Z}_s \mathrm{d}M_s - (\bar{O}_T - \bar{O}_t), \quad 0 \le t \le T, \tag{3.3}$$

with final condition $\bar{Y}_T = 0$. In addition, we have set $\bar{O} := O - O'$ in (3.3), where O and O' denote the square-integrable $\mathbb{F}$-martingales with $O_0 = O_0^{'} = 0$ satisfying the orthogonality condition

$$\mathbb{E}\left[O_T \int_0^T \varphi_t \mathrm{d}M_t\right] = \mathbb{E}\left[O_T^{'} \int_0^T \varphi_t \mathrm{d}M_t\right] = 0,$$

for all $\mathbb{H}$-predictable processes φ such that $\mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$. Since $(\bar{Y}, \bar{Z})$ is a solution of (3.3), then

$$\bar{Y}_t = \bar{Y}_0 + \int_0^t \bar{Z}_s \mathrm{d}M_s + \bar{O}_t, \quad 0 \le t \le T. \tag{3.4}$$

Since the process $\bar{Y}$ is an $\mathbb{F}$-martingale such that $\bar{Y}_T = 0$, we have

$$\bar{Y}_t = \mathbb{E}\left[\bar{Y}_T \middle| \mathcal{F}_t\right] = 0, \quad \text{for all } t \in [0,T].$$

Thus $Y_t = Y_t^{'}$ $\mathbb{P}$-a.s., for every $t \in [0,T]$. Then we can rewrite (3.4) as follows

$$0 = \int_0^t \bar{Z}_s \mathrm{d}M_s + \bar{O}_t, \quad 0 \le t \le T.$$

By computing the predictable covariation of $\int_0^\cdot \bar{Z}_s \mathrm{d}M_s + \bar{O}$ and $\bar{O}$ and by taking the expectation of both sides in the equality, for each $t \in [0,T]$, we obtain

$$\begin{aligned} 0 &= \int_0^t \bar{Z}_s \mathrm{d}\langle M, \bar{O}\rangle_s + \langle \bar{O}\rangle_t \\ &= \mathbb{E}\left[\int_0^t \bar{Z}_s \mathrm{d}\langle M, \bar{O}\rangle_s\right] + \mathbb{E}\left[\langle \bar{O}\rangle_t\right]. \end{aligned}$$

Since $\bar{Z}$ and $\bar{O}$ are differences of solutions to the BSDE (2.1), then $\mathbb{E}\left[\int_0^t \bar{Z}_s \mathrm{d}\langle M, \bar{O}\rangle_s\right] = 0$ for $t \in [0,T]$, and it follows

$$\mathbb{E}\left[\langle \bar{O}\rangle_t\right] = 0, \quad 0 \le t \le T. \tag{3.5}$$

By Theorem 4.2 of [11], since $\bar{O}$ is a square-integrable $\mathbb{F}$-martingale null at zero, we have that $\bar{O}^2 - \langle \bar{O}\rangle$ is an $\mathbb{F}$-martingale null at zero. Then by (3.5)

$$\mathbb{E}\left[\bar{O}_t^2\right] = \mathbb{E}\left[\langle \bar{O}\rangle_t\right] = 0, \quad 0 \le t \le T,$$

that implies $\bar{O}_t^2 = 0$ $\mathbb{P}$-a.s. for every $t \in [0,T]$ and then $O_t = O_t^{'}$ $\mathbb{P}$-a.s. for every $t \in [0,T]$. Now, let Y be the unique solution of (2.1) for a certain $\mathbb{H}$-predictable Z such that $\mathbb{E}\left[\int_0^T |Z_t|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$, i.e.

$$Y_t = Y_0 + \int_0^t Z_s \mathrm{d}M_s + O_t, \quad 0 \le t \le T. \tag{3.6}$$

It only remains to prove that Z is unique. For $t = T$ equation (3.6) becomes

$$Y_T = \xi = Y_0 + \int_0^T Z_s \mathrm{d}M_s + O_T.$$

By Proposition 3.2, $Z_t = H_t^{\mathcal{H}}$ $\mathbb{P}$-a.s., for each $t \in [0,T]$ and then Z is univocally determined. This concludes the proof. □

4 Galtchouk-Kunita-Watanabe representation under partial information

We now wish to provide an explicit characterization of the integrand process $H^{\mathcal{H}}$ appearing in the representation (3.2) in terms of the one given in the classical Galtchouk-Kunita-Watanabe decomposition, by means in particular of the concept of $\mathbb{H}$-predictable dual projection.
Let $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. We consider the well-known Galtchouk-Kunita-Watanabe decomposition of ξ with respect to M:

$$\xi = \tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} \mathrm{d}M_t + \tilde{O}_T, \quad \mathbb{P} - \text{a.s.}, \tag{4.1}$$

where $\tilde{U}_0 \in \mathcal{F}_0$, the integrand $H^{\mathcal{F}} = (H_t^{\mathcal{F}})_{0 \le t \le T}$ is an $\mathbb{F}$-predictable process such that $\mathbb{E}\left[\int_0^T |H_t^{\mathcal{F}}|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$ and $\tilde{O} = (\tilde{O}_t)_{0 \le t \le T}$ is a square-integrable $\mathbb{F}$-martingale with $\tilde{O}_0 = 0$ such that $\langle \tilde{O}, M\rangle_t = 0$, for every $t \in [0,T]$. Moreover, let us observe that $\tilde{U}_0 = \mathbb{E}\left[\xi \middle| \mathcal{F}_0\right]$.

In the sequel we will denote by ${}^pX$ the $\mathbb{H}$-predictable projection of a (generic) integrable process $X = (X_t)_{0 \le t \le T}$, defined as the unique $\mathbb{H}$-predictable process such that

$$\mathbb{E}\left[X_\tau \mathbf{1}_{\{\tau < \infty\}} \middle| \mathcal{H}_{\tau -}\right] = {}^pX_\tau \mathbf{1}_{\{\tau < \infty\}} \quad \mathbb{P} - \text{a.s.}$$

for every $\mathbb{H}$-predictable stopping time τ.
First we give a preliminary result under the additional assumption that the predictable quadratic variation $\langle M\rangle$ of the $\mathbb{F}$-martingale M is an $\mathbb{H}$-predictable process. In Theorem 4.7 we extend such result to the general case.

Proposition 4.1. *Let $(\tilde{U}_0, H^{\mathcal{F}}, \tilde{O}_T)$ be the triplet corresponding to decomposition* (4.1) *of $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. Suppose that the predictable quadratic variation $\langle M\rangle$ of the $\mathbb{F}$-martingale M is an $\mathbb{H}$-predictable process. Then*

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} \mathrm{d}M_t + O_T, \quad \mathbb{P} - \text{a.s.},$$

with

$$U_0 = \mathbb{E}\left[\tilde{U}_0 \middle| \mathcal{H}_0\right], \tag{4.2}$$

$$H_t^{\mathcal{H}} = {}^p(H_t^{\mathcal{F}}), \quad 0 \le t \le T, \tag{4.3}$$

and O is a square-integrable $\mathbb{F}$-martingale with $O_0 = 0$ such that $\mathbb{E}[O_T \cdot \eta] = 0$, for every $\eta \in \mathcal{L}_T^{\mathcal{H}}$.

Proof. Let

$$\xi = \tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} \mathrm{d}M_t + \tilde{O}_T, \quad \mathbb{P} - \text{a.s.}$$

be the classical Galtchouk-Kunita-Watanabe decomposition of $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. By taking the expectation of ξ with respect to $\mathcal{H}_0$, we have:

$$\mathbb{E}\left[\xi \middle| \mathcal{H}_0\right] = \mathbb{E}\left[\tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} \mathrm{d}M_t \middle| \mathcal{H}_0\right] + \mathbb{E}\left[\tilde{O}_T \middle| \mathcal{H}_0\right]. \tag{4.4}$$

Since $(\tilde{U}_0 + \int_0^\cdot H_t^{\mathcal{F}} \mathrm{d}M_t)$ is an $\mathbb{F}$-martingale, it follows:

$$\begin{aligned}\mathbb{E}\left[\tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} \mathrm{d}M_t \middle| \mathcal{H}_0\right] &= \mathbb{E}\left[\mathbb{E}\left[\tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} \mathrm{d}M_t \middle| \mathcal{F}_0\right] \middle| \mathcal{H}_0\right] \\ &= \mathbb{E}\left[\tilde{U}_0 \middle| \mathcal{H}_0\right],\end{aligned}$$

so that we can rewrite (4.4) as follows:

$$\mathbb{E}\left[\xi \middle| \mathcal{H}_0\right] = \mathbb{E}\left[\tilde{U}_0 \middle| \mathcal{H}_0\right] + \mathbb{E}\left[\tilde{O}_T \middle| \mathcal{H}_0\right].$$

Moreover, since $\tilde{O}$ is an $\mathbb{F}$-martingale null at zero, we have

$$\mathbb{E}\left[\tilde{O}_T \middle| \mathcal{H}_0\right] = \mathbb{E}\left[\mathbb{E}\left[\tilde{O}_T \middle| \mathcal{F}_0\right] \middle| \mathcal{H}_0\right] = 0.$$

This implies equality (4.2). To prove equality (4.3), we need to calculate the orthogonal projection of ξ onto the space $\mathcal{L}_T^{\mathcal{H}}$, see (3.1). For the sake of brevity, we suppose that $\tilde{U}_0 = 0$. Thanks to Proposition 3.2, this means we need to check the following condition:

$$\mathbb{E}\left[\xi \int_0^T \varphi_t \mathrm{d}M_t\right] = \mathbb{E}\left[\int_0^T {}^p(H_t^{\mathcal{F}}) \mathrm{d}M_t \int_0^T \varphi_t \mathrm{d}M_t\right],$$

for every $\mathbb{H}$-predictable process φ such that $\mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M \rangle_t\right] < \infty$. Taking decomposition (4.1) into account, this corresponds to the following equality:

$$\mathbb{E}\left[\int_0^T H_t^{\mathcal{F}} \varphi_t \mathrm{d}\langle M \rangle_t\right] = \mathbb{E}\left[\int_0^T {}^p(H_t^{\mathcal{F}}) \varphi_t \mathrm{d}\langle M \rangle_t\right], \tag{4.5}$$

for every $\mathbb{H}$-predictable process φ such that $\mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M \rangle_t\right] < \infty$. If we write the process φ as follows

$$\varphi = \varphi^+ - \varphi^-,$$

where φ^+ and φ^- denote the positive and the negative part of φ respectively, and define the $\mathbb{F}$-martingales

$$R_t^+ = \int_0^t \sqrt{\varphi_s^+} \mathrm{d}M_s, \quad R_t^- = \int_0^t \sqrt{\varphi_s^-} \mathrm{d}M_s, \quad 0 \le t \le T,$$

equality (4.5) is equivalent to the following relationships:

$$\mathbb{E}\left[\int_0^T H_t^{\mathcal{F}} \mathrm{d}\langle R^+\rangle_t\right] = \mathbb{E}\left[\int_0^T {}^p(H_t^{\mathcal{F}}) \mathrm{d}\langle R^+\rangle_t\right]$$
$$\mathbb{E}\left[\int_0^T H_t^{\mathcal{F}} \mathrm{d}\langle R^-\rangle_t\right] = \mathbb{E}\left[\int_0^T {}^p(H_t^{\mathcal{F}}) \mathrm{d}\langle R^-\rangle_t\right].$$

Hence, we can reduce the problem by assuming directly $\varphi_t = 1$ in (4.5), for each $t \in [0, T]$. Then, it is enough to prove the equality

$$\mathbb{E}\left[\int_0^T H_t^{\mathcal{F}} \mathrm{d}\langle M\rangle_t\right] = \mathbb{E}\left[\int_0^T {}^p(H_t^{\mathcal{F}}) \mathrm{d}\langle M\rangle_t\right]. \tag{4.6}$$

Since $\langle M\rangle$ is $\mathbb{H}$-predictable, Theorem VI.57 in [4] guarantees that equality (4.6) holds, once we have the positivity of the process $H^{\mathcal{F}}$. By writing

$$H^{\mathcal{F}} = (H^{\mathcal{F}})^+ - (H^{\mathcal{F}})^-,$$

and applying the above theorem to the positive and negative parts of $H^{\mathcal{F}}$, $(H^{\mathcal{F}})^+$ and $(H^{\mathcal{F}})^-$ respectively, and to the associated $\mathbb{H}$-predictable projections, we can get the result by setting

$$H^{\mathcal{H}} := {}^p(H^{\mathcal{F}}) = {}^p((H^{\mathcal{F}})^+) - {}^p((H^{\mathcal{F}})^-).$$

□

Example 4.2. *Let us consider the particular case where M is a square-integrable $\mathbb{F}$-martingale that is in addition a Lévy process, $\mathcal{F}_t = \mathcal{F}_t^M$ and $\mathcal{H}_t = \mathcal{F}_{(t-\tau)^+}^M$, with $\tau \in (0, T)$ a fixed delay. We assume $\xi = h(M_T) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, for some measurable function $h : \mathbb{R} \longrightarrow \mathbb{R}$.*
In this framework, by Lemma A.1 (see Appendix), we know that the integrand appearing in the Galtchouk-Kunita-Watanabe decomposition (4.1) *can be written as*

$$H_t^{\mathcal{F}} = F(t, M_{t-}), \quad t \in [0, T],$$

where the function F is such that the condition $\mathbb{E}\left[\int_0^T |F(t, M_{t-})|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$ is satisfied. Since in this case $\langle M\rangle$ is a deterministic process, we can apply Proposition 4.1 and get

$$H_t^{\mathcal{H}} = {}^pF(t, M_{t-}) = \mathbb{E}\left[F(t, M_{t-})|\mathcal{H}_{t-}\right], \quad t \in [0, T].$$

Then, it is easy to derive the following:

$$H_t^{\mathcal{H}} = \begin{cases} c(t, M_{(t-\tau)^-}) & \text{if t} > \tau \\ c(t, M_0) & \text{if t} \le \tau, \end{cases}$$

where the function c is given by

$$c(t, y) = \int_{\mathbb{R}} F(t, y + z) \mathrm{d}\rho_{t \wedge \tau}(z),$$

with ρ_t denoting the law of M_t, for every $t \in [0, T]$.

4.1 The $\mathbb{H}$-predictable dual projection

It is possible to extend the result of Proposition 4.1 by using the concept of $\mathbb{H}$-predictable dual projection. For reasons of clarity, we provide a self-contained discussion about this kind of projection in presence of more than one filtration. Let $G = (G_t)_{0 \le t \le T}$ be a càdlàg $\mathbb{F}$-adapted process of integrable variation, that is, $\mathbb{E}\left[\|G\|_T\right] < \infty$. Here the process $\|G\| = (\|G\|)_{0 \le t \le T}$ defined, for each $t \in [0, T]$, by

$$\|G\|_t(\omega) = \sup_{\Delta} \sum_{i=0}^{n(\Delta)-1} |G_{t_i+1}(\omega) - G_{t_i}(\omega)|,$$

where $\Delta = \{t_0 = 0 < t_1 < \ldots < t_{n(\Delta)} = t\}$ is a partition of $[0, t]$, denotes the total variation of the function $t \rightsquigarrow G_t(\omega)$.

Proposition 4.3. *Let $G = (G_t)_{0 \le t \le T}$ be a càdlàg $\mathbb{F}$-adapted process of integrable variation. Then there exists a unique $\mathbb{H}$-predictable process $G^{\mathbb{H}} = (G_t^{\mathbb{H}})_{0 \le t \le T}$ of integrable variation, such that*

$$\mathbb{E}\left[\int_0^T \varphi_t \mathrm{d}G_t^{\mathbb{H}}\right] = \mathbb{E}\left[\int_0^T \varphi_t \mathrm{d}G_t\right],$$

for every $\mathbb{H}$-predictable (bounded) process φ. The process $G^{\mathbb{H}}$ is called the $\mathbb{H}$-predictable dual projection *of G.*

Proof. Without loss of generality, we can restrict our attention to the case where G is an increasing process and prove the statement on the generators φ of the form $\varphi_u = \mathbf{1}_{(s,t]}(u)\mathbf{1}_B$, with $B \in \mathcal{H}_s$ and $s, t \in [0, T]$ with $s < t$. Indeed, decomposing the process G as $G = G^+ - G^-$, where both the positive and negative parts of G are assumed to be increasing integrable processes, we can suppose G to be increasing such that

$$\mathbb{E}\left[G_T\right] = \mathbb{E}\left[\|G\|_T\right] < \infty.$$

If G is a càdlàg, increasing, integrable $\mathbb{F}$-adapted process, we will prove that there exists a unique increasing, integrable $\mathbb{H}$-predictable process $G^{\mathbb{H}}$ such that for every $s, t \in [0, T]$ with $s < t$ and $B \in \mathcal{H}_s$, the following relationship holds

$$\mathbb{E}\left[\mathbf{1}_B(G_t - G_s)\right] = \mathbb{E}\left[\mathbf{1}_B(G_t^{\mathbb{H}} - G_s^{\mathbb{H}})\right].$$

Let $\tilde{G} = (\tilde{G}_t)_{0 \le t \le T}$ be the $\mathbb{H}$-optional projection of G, such that for fixed times $t \in (0, T]$

$$\tilde{G}_t = \mathbb{E}\left[G_t | \mathcal{H}_t\right] \; \mathbb{P} - \text{a.s.}.$$

We observe that for every $s, t \in [0, T]$ with $s < t$ and $B \in \mathcal{H}_s$, we have

$$\mathbb{E}\left[\mathbf{1}_B(G_t - G_s)\right] = \mathbb{E}\left[\mathbf{1}_B(\tilde{G}_t - \tilde{G}_s)\right].$$

Indeed,

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_B(G_t - G_s)\right] &= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_B(G_t - G_s)\middle|\, \mathcal{H}_s\right]\right] = \mathbb{E}\left[\mathbf{1}_B\left(\mathbb{E}\left[G_t\middle|\,\mathcal{H}_s\right] - \tilde{G}_s\right)\right] \\
&= \mathbb{E}\left[\mathbf{1}_B\left(\mathbb{E}\left[\tilde{G}_t\middle|\,\mathcal{H}_s\right] - \tilde{G}_s\right)\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_B(\tilde{G}_t - \tilde{G}_s)\middle|\,\mathcal{H}_s\right]\right] \\
&= \mathbb{E}\left[\mathbf{1}_B(\tilde{G}_t - \tilde{G}_s)\right].
\end{aligned}$$

Furthermore, since G is increasing, then $\tilde{G}$ is an $\mathbb{H}$-submartingale, that is

$$\mathbb{E}\left[\tilde{G}_t\middle|\,\mathcal{H}_s\right] = \mathbb{E}\left[\mathbb{E}\left[G_t\middle|\,\mathcal{H}_t\right]\middle|\,\mathcal{H}_s\right] = \mathbb{E}\left[G_t\middle|\,\mathcal{H}_s\right] \geq \mathbb{E}\left[G_s\middle|\,\mathcal{H}_s\right] = \tilde{G}_s, \quad 0 \leq s \leq t \leq T.$$

Thanks to Doob-Meyer Theorem on decomposition of submartingales, see e.g. Theorem 3.15 of [11], there exists a unique increasing, integrable $\mathbb{H}$-predictable process $G^{\mathbb{H}}$ such that $\tilde{G} - G^{\mathbb{H}}$ is an $\mathbb{H}$-martingale, that is, for every $s, t \in [0, T]$ with $s < t$ and $B \in \mathcal{H}_s$, we have

$$\mathbb{E}\left[\mathbf{1}_B(\tilde{G}_t - \tilde{G}_s)\right] = \mathbb{E}\left[\mathbf{1}_B(G^{\mathbb{H}}_t - G^{\mathbb{H}}_s)\right].$$

□

Remark 4.4. *If G is an $\mathbb{H}$-predictable process of integrable variation and X is an $\mathbb{F}$-adapted process satisfying $\mathbb{E}\left[\int_0^T X_t \mathrm{d}G_t\right] < \infty$, then*

$$(X_t \mathrm{d}G_t)^{\mathbb{H}} = {}^p X_t \mathrm{d}G_t, \quad \mathbb{P} - \text{a.s.}, \ \textit{for every } t \in [0, T].$$

Indeed, by Theorem VI.57 in [4], for any $\mathbb{H}$-predictable (bounded) process φ we can prove that

$$\mathbb{E}\left[\int_0^T \varphi_t X_t \mathrm{d}G_t\right] = \mathbb{E}\left[\int_0^T \varphi_t {}^p X_t \mathrm{d}G_t\right].$$

4.2 Explicit representation results

We now can apply the results of Subsection 4.1 to extend Proposition 4.1. Let $\mathcal{P}^{\mathcal{H}}$ and $\mathcal{P}$ be the $\mathbb{H}$-predictable and $\mathbb{F}$-predictable σ-field respectively. We consider the measures $\mu^{\mathcal{H}}$ (respectively μ) defined on $\mathcal{P}^{\mathcal{H}}$ (respectively $\mathcal{P}$) such that

$$\mu^{\mathcal{H}}((s, t] \times B) = \mathbb{E}\left[\mathbf{1}_B(A^{\mathbb{H}}_t - A^{\mathbb{H}}_s)\right], \quad B \in \mathcal{H}_s, \ \ s, t \in [0, T], \ s < t, \tag{4.7}$$

where $A^{\mathbb{H}}$ is the $\mathbb{H}$-predictable dual projection of $A := (\int_0^t H^{\mathcal{F}}_s \mathrm{d}\langle M \rangle_s)_{0 \leq t \leq T}$, that exists thanks to Theorem 4.3, and

$$\mu((u, v] \times F) = \mathbb{E}\left[\mathbf{1}_F(\langle M \rangle^{\mathbb{H}}_v - \langle M \rangle^{\mathbb{H}}_u)\right], \quad F \in \mathcal{F}_u, \ u, v \in [0, T], \ u < v. \tag{4.8}$$

Here $H^{\mathcal{F}}$ is the integrand appearing in the Galtchouk-Kunita-Watanabe decomposition (4.1).

Lemma 4.5. *Let $\mu^{\mathcal{H}}$ and μ measures satisfying conditions* (4.7) *and* (4.8) *respectively. Then $\mu^{\mathcal{H}} \ll \mu$ on $\mathcal{P}^{\mathcal{H}}$, that is, $\mu^{\mathcal{H}}$ is absolutely continuous with respect to the restriction of μ on $\mathcal{P}^{\mathcal{H}}$.*

Proof. By using the definition of absolute continuity, we wish to show that if whenever $\mu(E) = 0$ for $E \in \mathcal{P}^{\mathcal{H}}$, then $\mu^{\mathcal{H}}(E) = 0$. Let $\psi = (\psi_t)_{0 \le t \le T}$ be a nonnegative $\mathbb{H}$-predictable process such that

$$\mathbb{E}\left[\int_0^T \psi_t \mathrm{d}\langle M\rangle_t^{\mathbb{H}}\right] = 0.$$

Then

$$\mathbb{E}\left[\int_0^T \psi_t \mathrm{d}\langle M\rangle_t\right] = 0,$$

that implies that $\psi = 0$ $\mathrm{d}\langle M\rangle \otimes \mathrm{d}\mathbb{P}$ a.e., since ψ is nonnegative. Finally

$$\mathbb{E}\left[\int_0^T \psi_t \mathrm{d}A_t^{\mathbb{H}}\right] = \mathbb{E}\left[\int_0^T \psi_t \mathrm{d}A_t\right] = \mathbb{E}\left[\int_0^T \psi_t H_t^{\mathcal{F}} \mathrm{d}\langle M\rangle_t\right] = 0.$$

□

Since $\mu^{\mathcal{H}} \ll \mu$ on $\mathcal{P}^{\mathcal{H}}$, thanks to Lemma 4.5, by the Radon-Nikodym theorem there exists a $\mathcal{P}^{\mathcal{H}}$-measurable function g on $[0,T] \times \Omega$ such that

$$\mu^{\mathcal{H}}(E) = \int_E g(t,\omega)\mathrm{d}\mu(t,\omega), \quad \forall E \in \mathcal{P}^{\mathcal{H}}.$$

This allows to identify the process $H^{\mathcal{H}}$ as the Radon-Nikodym derivative:

$$H_t^{\mathcal{H}}(\omega) := \left.\frac{\mathrm{d}\mu^{\mathcal{H}}(t,\omega)}{\mathrm{d}\mu(t,\omega)}\right|_{\mathcal{P}^{\mathcal{H}}}, \quad (t,\omega) \in [0,T] \times \Omega. \tag{4.9}$$

Finally, we are ready to state the following theorem.

Theorem 4.6. *For any nonnegative $\mathbb{F}$-measurable process $H^{\mathcal{F}}$, the following equality holds*

$$\mathbb{E}\left[\int_0^T \varphi_t H_t^{\mathcal{F}} \mathrm{d}\langle M\rangle_t\right] = \mathbb{E}\left[\int_0^T \varphi_t H_t^{\mathcal{H}} \mathrm{d}\langle M\rangle_t\right], \tag{4.10}$$

for every $\mathbb{H}$-predictable process φ such that $\mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$. Here $H^{\mathcal{H}}$ is given by (4.9).

Proof. By relationship (4.9) and definition of the measure μ, see (4.8), we have for every $s,t \in [0,T]$ with $s < t$ and $B \in \mathcal{H}_s$

$$\mu^{\mathcal{H}}((s,t]\times B) = \int_s^t \int_B H_u^{\mathcal{H}}(\omega)\mathrm{d}\mu(u,\omega) = \mathbb{E}\left[\mathbf{1}_B \int_s^t H_u^{\mathcal{H}} \mathrm{d}\langle M\rangle_u^{\mathbb{H}}\right] = \mathbb{E}\left[\mathbf{1}_B \int_s^t H_u^{\mathcal{H}} \mathrm{d}\langle M\rangle_u\right].$$

On the other hand, by (4.7)

$$\mu^{\mathcal{H}}((s,t]\times B)=\mathbb{E}\left[\mathbf{1}_B(A^{\mathbb{H}}_t-A^{\mathbb{H}}_s)\right]=\mathbb{E}\left[\mathbf{1}_B\int_s^t H^{\mathcal{F}}_u\mathrm{d}\langle M\rangle_u\right].$$

If φ is of the form $\varphi_u=\mathbf{1}_{(s,t]}(u)\mathbf{1}_B$, with $B\in\mathcal{H}_s$ and $s,t\in[0,T]$ with $s<t$, then the statement is proved since relationship (4.10) is verified on the generators of $\mathcal{P}^{\mathcal{H}}$. □

We now give the analogous of Proposition 4.1, without assuming that the process $\langle M\rangle$ is $\mathbb{H}$-predictable.

Theorem 4.7. *Let $(\tilde{U}_0,H^{\mathcal{F}},\tilde{O}_T)$ be the triplet corresponding to decomposition* (4.1) *of $\xi\in L^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R})$. Then*

$$\xi=U_0+\int_0^T H^{\mathcal{H}}_t\mathrm{d}M_t+O_T,\quad \mathbb{P}-\text{a.s.},\tag{4.11}$$

with

$$U_0=\mathbb{E}\left[\tilde{U}_0\middle|\mathcal{H}_0\right],$$

$$H^{\mathcal{H}}_t=\frac{\mathrm{d}\mu^{\mathcal{H}}(t,\omega)}{\mathrm{d}\mu(t,\omega)}\bigg|_{\mathcal{P}^{\mathcal{H}}},\quad (t,\omega)\in[0,T]\times\Omega,$$

where $\mu^{\mathcal{H}}$ and μ are given in (4.7) *and* (4.8) *respectively, and O is a square-integrable $\mathbb{F}$-martingale with $O_0=0$ such that $\mathbb{E}[O_T\cdot\eta]=0$, for every $\eta\in\mathcal{L}^{\mathcal{H}}_T$.*

Proof. We proceed as in the proof of Proposition 4.1 by observing that condition (4.10) plays the same role of condition (4.5). □

In the next proposition we give a useful result which allows us to compute $H^{\mathcal{H}}$ as the Radon-Nikodym derivative of the $\mathbb{H}$-predictable dual projection $A^{\mathbb{H}}$ of the process $A=(\int_0^t H^{\mathcal{F}}_s\mathrm{d}\langle M\rangle_s)_{0\le t\le T}$ with respect to the $\mathbb{H}$-predictable dual projection $\langle M\rangle^{\mathbb{H}}$ of the $\mathbb{F}$-predictable quadratic variation $\langle M\rangle$.

Proposition 4.8. *The process $A^{\mathbb{H}}=(\int_0^\cdot H^{\mathcal{F}}_s\mathrm{d}\langle M\rangle_s)^{\mathbb{H}}$ is absolutely continuous with respect to $\langle M\rangle^{\mathbb{H}}$ and it is given by*

$$A^{\mathbb{H}}_t=\int_0^t H^{\mathcal{H}}_s\mathrm{d}\langle M\rangle^{\mathbb{H}}_s,\quad 0\le t\le T.$$

As a consequence

$$H^{\mathcal{H}}_t=\frac{\mathrm{d}A^{\mathbb{H}}_t}{\mathrm{d}\langle M\rangle^{\mathbb{H}}_t},\quad 0\le t\le T.\tag{4.12}$$

Proof. Set $\widetilde{A}_t:=\int_0^t H^{\mathcal{H}}_s\mathrm{d}\langle M\rangle^{\mathbb{H}}_s$, for each $t\in[0,T]$. It is sufficient to prove that

$$\mathbb{E}\left[\int_0^T\varphi_u\mathrm{d}A_u\right]=\mathbb{E}\left[\int_0^T\varphi_u\mathrm{d}\widetilde{A}_u\right]$$

for every $\mathbb{H}$-predictable (bounded) process φ. As before, we can consider φ of the form $\varphi_u = \mathbf{1}_{(s,t]}(u)\mathbf{1}_B$, with $B \in \mathcal{H}_s$ and $s < t \in [0, T]$.
Then by the definitions of the measure μ and $\mu^{\mathcal{H}}$, see (4.8) and (4.7), and recalling (4.9) we get

$$\mathbb{E}\left[\int_0^T \varphi_u \mathrm{d}\widetilde{A}_u\right] = \mathbb{E}\left[\mathbf{1}_B \int_s^t H_u^{\mathcal{H}} \mathrm{d}\langle M\rangle_u^{\mathbb{H}}\right] = \int_s^t \int_B H_u^{\mathcal{H}}(\omega)\mathrm{d}\mu(u,\omega) = \mu^{\mathcal{H}}((s,t] \times B) =$$

$$\mathbb{E}\left[\mathbf{1}_B(A_t^{\mathbb{H}} - A_s^{\mathbb{H}})\right] = \mathbb{E}\left[\mathbf{1}_B \int_s^t H_u^{\mathcal{F}} \mathrm{d}\langle M\rangle_u\right] = \mathbb{E}\left[\int_0^T \varphi_u \mathrm{d}A_u\right]$$

which concludes the proof. □

Example 4.9. *Suppose that the process $\langle M\rangle$ is of the form*

$$\langle M\rangle_t = \int_0^t a_s \mathrm{d}G_s, \quad 0 \le t \le T$$

for some $\mathbb{F}$-predictable process $a = (a_t)_{0\le t\le T}$ and an increasing deterministic function G. Then by Remark 4.4

$$\langle M\rangle_t^{\mathbb{H}} = \int_0^t {}^p a_s \mathrm{d}G_s, \quad A_t^{\mathbb{H}} = \int_0^t {}^p(H_s^{\mathcal{F}} a_s)\mathrm{d}G_s, \quad 0 \le t \le T,$$

and as a consequence of Proposition 4.8 we get

$$H_t^{\mathcal{H}} = \frac{{}^p(H_t^{\mathcal{F}} a_t)}{{}^p a_t}, \quad 0 \le t \le T.$$

Remark 4.10. *Let us observe that if the process $\langle M\rangle$ is $\mathbb{H}$-predictable, then again by Remark 4.4*

$$\langle M\rangle_t^{\mathbb{H}} = \langle M\rangle_t, \quad A_t^{\mathbb{H}} = \int_0^t {}^p(H_s^{\mathcal{F}})\mathrm{d}\langle M\rangle_s, \quad 0 \le t \le T,$$

and by applying Proposition 4.8 we obtain that

$$H_t^{\mathcal{H}} = {}^p(H_t^{\mathcal{F}}), \quad 0 \le t \le T.$$

5 Risk-minimization under restricted information

In relation to the connection between risk-minimization under full and partial information respectively, we now show how our result obtained in Proposition 4.8 fits in the approach of [13] of risk-minimization under restricted information.
On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we consider a financial market with one riskless asset with (discounted) price 1 and one risky asset whose (discounted) price is given by a square-integrable (càdlàg) martingale $M = (M_t)_{0\le t\le T}$ adapted to a (large) filtration $\mathbb{F} := (\mathcal{F}_t)_{0\le t\le T}$.
We will study the problem of hedging a contingent claim, whose final payoff is given by

a random variable $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, in the case where investors acting in the market can access only to the information flow $\mathbb{H} := (\mathcal{H}_t)_{0 \le t \le T}$ with $\mathcal{H}_t \subseteq \mathcal{F}_t$, for each $t \in [0,T]$. We choose the risk-minimization approach to solve the above hedging problem. In the case of full information, in [6] the authors proved that there exists a unique $\mathbb{F}$-risk-minimizing hedging strategy $\phi^* = (\theta^*, \eta^*)$, where $\theta^* = (\theta_t^*)_{0 \le t \le T}$ is given by the integrand with respect to M in the classical Galtchouk-Kunita-Watanabe decomposition of ξ, i.e. $\theta^* = H^{\mathcal{F}}$ (see equation (4.1)).
In this section we extend this result to the case where there are restrictions on the available information, by using the Galtchouk-Kunita-Watanabe decomposition under partial information (see equation (3.2)). More precisely, we prove that the $\mathbb{H}$-risk-minimizing hedging strategy $\phi^{\mathcal{H}} = (\theta^{\mathcal{H}}, \eta^{\mathcal{H}})$ (see Definition 5.2 below) is such that $\theta^{\mathcal{H}} = H^{\mathcal{H}}$.

Risk-minimization under restricted information was studied in [13] by using a different approach. We obtain the same explicit representation given in Theorem 3.1 of [13] by applying Proposition 4.8. About risk-minimization under partial information for defaultable markets via nonlinear filtering, we refer to [8]. In particular, they consider the case where the contingent claim ξ is $\mathcal{H}_T$-measurable, in which we can solve the risk-minimization problem by using the classical Galtchouk-Kunita-Watanabe decomposition.
We now assume that the agent has at her/his disposal the information flow $\mathbb{H}$ about trading in stocks while a complete information about trading in the riskless asset.

Definition 5.1. *An $\mathbb{H}$-strategy is a pair $\phi = (\theta, \eta)$ (θ_t is the number of shares of the risky asset to be held at time t, while η_t is the amount invested in the riskless asset at time t) where θ is $\mathbb{H}$-predictable and η is $\mathbb{F}$-adapted and such that*

$$\mathbb{E}\left[\int_0^T \theta_s^2 \mathrm{d}\langle M \rangle_s\right] < \infty$$

and the value process $V(\phi) := \theta M + \eta$ satisfies

$$\mathbb{E}\left[\left(\sup_{t \in [0,T]} |V_t(\phi)|\right)^2\right] < \infty.$$

For any $\mathbb{H}$-strategy ϕ, the associated cost process $C(\phi)$ is given by

$$C_t(\phi) := V_t(\phi) - \int_0^t \theta_r \mathrm{d}M_r, \quad 0 \le t \le T.$$

Finally the $\mathbb{H}$-risk process of ϕ is defined by

$$R_t(\phi) := \mathbb{E}\left[(C_T(\phi) - C_t(\phi))^2 \middle| \mathcal{H}_t\right], \quad 0 \le t \le T.$$

Definition 5.2. *An $\mathbb{H}$-strategy $\phi = (\theta, \eta)$ is called $\mathbb{H}$-risk-minimizing if $V_T(\phi) = \xi$ $\mathbb{P}$-a.s. and if for any $\mathbb{H}$-strategy ψ such that $V_T(\psi) = \xi$ $\mathbb{P}$-a.s., we have $R_t(\phi) \le R_t(\psi)$ $\mathbb{P}$-a.s. for every $t \in [0,T]$.*

Remark 5.3. *By Corollary 3.1 in [13] we have that if $\phi = (\theta, \eta)$ is an $\mathbb{H}$-risk-minimizing strategy then ϕ is mean-self-financing, i.e. the cost process $C(\phi)$ is an $\mathbb{F}$-martingale. Moreover, if $\phi = (\theta, \eta)$ is a mean-self-financing $\mathbb{H}$-strategy, then $V(\phi)$ is an $\mathbb{F}$-martingale, hence for a given $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, we have that $V_t(\phi) = \mathbb{E}\left[\xi | \mathcal{F}_t\right]$, for every $t \in [0,T]$.*

To prove the main result of this section, we need the following Lemma.

Lemma 5.4. *Let $O = (O_t)_{0 \le t \le T}$ be a square-integrable $\mathbb{F}$-martingale with $O_0 = 0$, satisfying the orthogonality condition*

$$\mathbb{E}\left[O_T \int_0^T \varphi_t \mathrm{d}M_t\right] = 0$$

for all $\mathbb{H}$-predictable processes $\varphi = (\varphi_t)_{0 \le t \le T}$ such that $\mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M \rangle_t\right] < \infty$. Then for any $t \in [0,T]$

$$\mathbb{E}\left[(O_T - O_t) \int_t^T \varphi_s \mathrm{d}M_s \middle| \mathcal{H}_t\right] = 0 \quad \mathbb{P}-\text{a.s.}.$$

Proof. Since for any $\mathbb{H}$-predictable process φ

$$\mathbf{1}_{(t,T]}(s)\mathbf{1}_B \varphi_s, \quad B \in \mathcal{H}_t,\ t \in [0,T),$$

is $\mathbb{H}$-predictable, we get

$$\mathbb{E}\left[O_T \mathbf{1}_B \int_t^T \varphi_s \mathrm{d}M_s\right] = \mathbb{E}\left[\mathbf{1}_B \mathbb{E}\left[O_T \int_t^T \varphi_s \mathrm{d}M_s \middle| \mathcal{H}_t\right]\right] = 0, \quad \forall B \in \mathcal{H}_t,$$

and then

$$\mathbb{E}\left[O_T \int_t^T \varphi_s \mathrm{d}M_s \middle| \mathcal{H}_t\right] = 0 \quad \mathbb{P}-\text{a.s.}.$$

Finally, let us observe that

$$\mathbb{E}\left[O_t \int_t^T \varphi_s \mathrm{d}M_s \middle| \mathcal{H}_t\right] = \mathbb{E}\left[O_t \int_0^T \varphi_s \mathrm{d}M_s \middle| \mathcal{H}_t\right] - \mathbb{E}\left[O_t \int_0^t \varphi_s \mathrm{d}M_s \middle| \mathcal{H}_t\right] = 0$$

since

$$\mathbb{E}\left[O_t \int_0^T \varphi_s \mathrm{d}M_s \middle| \mathcal{H}_t\right] = \mathbb{E}\left[\mathbb{E}\left[O_t \int_0^T \varphi_s \mathrm{d}M_s \middle| \mathcal{F}_t\right] \middle| \mathcal{H}_t\right] = \mathbb{E}\left[O_t \int_0^t \varphi_s \mathrm{d}M_s \middle| \mathcal{H}_t\right],$$

and this concludes the proof. □

We are now in the position to provide an alternative proof to that given in [13], concerning the explicit representation for an $\mathbb{H}$-risk-minimizing strategy, by applying the Galtchouk-Kunita-Watanabe decomposition under partial information and the representation result given in Proposition 4.8.

Theorem 5.5. *For every $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, there exists a unique $\mathbb{H}$-risk-minimizing strategy $\phi^{\mathcal{H}} = (\theta^{\mathcal{H}}, \eta^{\mathcal{H}})$ such that $\theta^{\mathcal{H}} = H^{\mathcal{H}}$, where $H^{\mathcal{H}}$ is given by (4.12) and $\eta^{\mathcal{H}}_t = \mathbb{E}\left[\xi | \mathcal{F}_t\right] - \theta^{\mathcal{H}}_t M_t$, for every $t \in [0,T]$.*

Proof. The proof is similar to that of Theorem 2.4 of [14] performed in the full information case. Let $\phi = (\theta, \eta)$ be a mean-self-financing $\mathbb{H}$-strategy such that $V_T(\phi) = \xi$ $\mathbb{P}$-a.s.. Hence, by computing the Galtchouk-Kunita-Watanabe decomposition under partial information, see (4.11), we have

$$\begin{aligned} C_T(\phi) - C_t(\phi) &= V_T(\phi) - V_t(\phi) - \int_t^T \theta_s \mathrm{d}M_s = \xi - V_t(\phi) - \int_t^T \theta_s \mathrm{d}M_s \\ &= U_0 + \int_0^T H^{\mathcal{H}}_t \mathrm{d}M_t + O_T - \int_t^T \theta_s \mathrm{d}M_s - V_t(\phi), \end{aligned}$$

where $H^{\mathcal{H}}$ is given by (4.12). Since $V_t(\phi) = E[\xi|\mathcal{F}_t]$, for every $t \in [0,T]$, see Remark 5.3, we get that

$$V_t(\phi) = U_0 + \int_0^t H^{\mathcal{H}}_s \mathrm{d}M_s + O_t$$

and

$$C_T(\phi) - C_t(\phi) = \int_t^T \{H^{\mathcal{H}}_s - \ \theta_s\} \mathrm{d}M_s + O_T - O_t.$$

By similar computation we get that

$$C_T(\phi^{\mathcal{H}}) - C_t(\phi^{\mathcal{H}}) = O_T - O_t.$$

Finally

$$\begin{aligned} (C_T(\phi) - C_t(\phi))^2 = \Big(C_T(\phi^{\mathcal{H}}) - C_t(\phi^{\mathcal{H}})\Big)^2 + \Big(\int_t^T \{H^{\mathcal{H}}_s - \theta_s\} \mathrm{d}M_s\Big)^2 \\ + 2(O_T - O_t) \int_t^T \{H^{\mathcal{H}}_s - \ \theta_s\} \mathrm{d}M_s \end{aligned}$$

and by Lemma 5.4 we obtain that

$$R_t(\phi) = R_t(\phi^{\mathcal{H}}) + \mathbb{E}\left[\left(\int_t^T \{H^{\mathcal{H}}_s - \theta_s\} \mathrm{d}M_s\right)^2 \middle| \mathcal{H}_t\right] \geq R_t(\phi^{\mathcal{H}}).$$

Hence $\phi^{\mathcal{H}}$ is $\mathbb{H}$-risk-minimizing. If some other ϕ is also $\mathbb{H}$-risk-minimizing then

$$\mathbb{E}\left[\left(\int_0^T \{H^{\mathcal{H}}_s - \theta_s\} \mathrm{d}M_s\right)^2 \middle| \mathcal{H}_0\right] = \mathbb{E}\left[\int_0^T \{H^{\mathcal{H}}_s - \theta_s\}^2 \mathrm{d}\langle M\rangle_s \middle| \mathcal{H}_0\right] = 0,$$

which implies $H^{\mathcal{H}} = \theta$. Since $V_t(\phi) = V_t(\phi^{\mathcal{H}}) = \mathbb{E}\left[\xi | \mathcal{F}_t\right]$ for each $t \in [0,T]$, we also obtain $\phi = \phi^{\mathcal{H}}$. □

In the rest of the section we investigate the case where there is a relationship between the information flow $\mathbb{H}$ and the filtration generated by the stock price M, that we denote by $\mathbb{F}^M$. A possible choice is the assumption that investors acting in the market have access only to the information contained in past asset prices, that is $\mathbb{H} = \mathbb{F}^M$. Such a situation has been studied for instance in [7] and [3] for stock price dynamics with jumps. In the sequel we will assume $\mathbb{H} \subseteq \mathbb{F}^M$, which takes also into account, for instance, the case where the asset price is only observed at discrete times or with a fixed delay $\tau \in (0,T)$, i.e. $\mathcal{H}_t = \mathcal{F}^M_{(t-\tau)^+}$, for every $t \in (0,T)$.

In a such particular case, when in addition $\xi \in L^2(\Omega, \mathcal{F}^M_T, \mathbb{P}; \mathbb{R})$, we can find an $\mathbb{H}$-risk minimizing strategy, $\phi^{\mathcal{H}} = (\theta^{\mathcal{H}}, \eta^{\mathcal{H}})$, where θ^H is $\mathbb{H}$-predictable and η^H is $\mathbb{F}^M$-adapted, while in the general case $\eta^{\mathcal{H}}$ has been taken $\mathbb{F}$-adapted. This means that we study the situation where the agent has at her/his disposal the information flow $\mathbb{H} \subseteq \mathbb{F}^M$ about trading in stocks and the filtration $\mathbb{F}^M$ about trading in the riskless asset, and when $\mathbb{H} = \mathbb{F}^M$ the same information flow.

More precisely, from now on we restrict ourself to consider $\mathbb{H}$-strategies $\phi = (\theta, \eta)$ as in Definition 5.1 where η is chosen $\mathbb{F}^M$-adapted.

Remark 5.6. *Let us observe that given an $\mathbb{H}$-strategy $\phi = (\theta, \eta)$, the associated value process $V(\phi) := \theta M + \eta$ turns out to be $\mathbb{F}^M$-adapted. By Corollary 3.1 in [13], we have that if $\phi = (\theta, \eta)$ is an $\mathbb{H}$-risk-minimizing strategy according to this new definition, then ϕ is mean-self-financing, i.e. the cost process $C(\phi)$ is an $\mathbb{F}^M$-martingale. Moreover, if $\phi = (\theta, \eta)$ is a mean-self-financing $\mathbb{H}$-strategy, then $V(\phi)$ is an $\mathbb{F}^M$-martingale, hence $V_t(\phi) = E[\xi|\mathcal{F}^M_t]$, for every $t \in [0,T]$.*

We are now ready to give the following result.

Theorem 5.7. *For every $\xi \in L^2(\Omega, \mathcal{F}^M_T, \mathbb{P}; \mathbb{R})$, there exists a unique $\mathbb{H}$-risk-minimizing strategy $\phi^{\mathcal{H}} = (\theta^{\mathcal{H}}, \eta^{\mathcal{H}})$ such that $\theta^{\mathcal{H}} = H^{\mathcal{H}}$, where $H^{\mathcal{H}}$ is given by (4.12) and $\eta^{\mathcal{H}} = E[\xi|\mathcal{F}^M_t] - \theta^{\mathcal{H}}_t M_t$, for every $t \in [0,T]$.*

Proof. Since ξ is $\mathcal{F}^M_T$-measurable, by decomposition (4.11) we obtain that

$$\xi = \mathbb{E}[\xi|\mathcal{F}^M_T] = U_0 + \int_0^T H^{\mathcal{H}}_t \mathrm{d}M_t + \mathbb{E}[O_T|\mathcal{F}^M_T],$$

where $H^{\mathcal{H}}$ is given by (4.12). Set $\hat{O}_t := \mathbb{E}[O_t|\mathcal{F}^M_t]$, for each $t \in [0,T]$. It is known that $\hat{O}$ is an $\mathbb{F}^M$-martingale and

$$\mathbb{E}\left[\hat{O}_T \int_0^T H^{\mathcal{H}}_t \mathrm{d}M_t\right] = \mathbb{E}\left[\mathbb{E}\left[\hat{O}_T|\mathcal{F}^M_T\right] \int_0^T H^{\mathcal{H}}_t \mathrm{d}M_t\right] = \mathbb{E}\left[O_T \int_0^T H^{\mathcal{H}}_t \mathrm{d}M_t\right] = 0.$$

Therefore we obtain the Galtchouk-Kunita-Watanabe decomposition of ξ under restricted information with respect to the filtration $\mathbb{F}^M$, that is

$$\xi = U_0 + \int_0^T H^{\mathcal{H}}_t \mathrm{d}M_t + \hat{O}_T, \quad \mathbb{P} - \text{a.s.}.$$

The rest of the proof follows from Theorem 5.5 by replacing the filtration $\mathbb{F}$ by $\mathbb{F}^M$ and the $\mathbb{F}$-martingale O by the $\mathbb{F}^M$-martingale $\hat{O}$. □

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APPENDIX

A Technical results

Recall that $M = (M_t)_{0\leq t\leq T}$ is a square-integrable $\mathbb{F}$-martingale and assume that $\mathbb{F} = \mathbb{F}^M := (\mathcal{F}_t^M)_{0\leq t\leq T}$, i.e. the information flow $\mathbb{F}$ coincides with the canonical filtration $\mathbb{F}^M$ of M.

Lemma A.1. *Let M be a Lévy process and $\xi = h(M_T) \in L^2(\Omega, \mathcal{F}_T^M, \mathbb{P}; \mathbb{R})$ for some measurable function $h : \mathbb{R} \to \mathbb{R}$. Then, there exists a measurable function $F : [0,T]\times\mathbb{R} \to \mathbb{R}$ such that*

$$\xi = \mathbb{E}[\xi] + \int_0^T F(s, M_{s-})\mathrm{d}M_s + \tilde{O}_T, \quad \mathbb{P} - a.s.,$$

where $\tilde{O} = (\tilde{O}_t)_{0\leq t\leq T}$ is a square-integrable $\mathbb{F}^M$-martingale null at zero such that $\langle \tilde{O}, M\rangle_t = 0$, for every $t \in [0,T]$. Moreover, the following integrability condition is satisfied

$$\mathbb{E}\left[\int_0^T |F(s, M_{s-})|^2 \mathrm{d}\langle M\rangle_s\right] < \infty.$$

Proof. If ξ is given as a Fourier transform of M_T, that is, the function h is of the form

$$h(x) = \int_{\mathbb{R}} e^{iax}\mathrm{d}\nu(a), \quad \text{for all } x \in \mathbb{R}, \tag{A.1}$$

where ν is a finite measure, the result is contained in Proposition 4.3 of [9], which was an adaptation of [10].
As a consequence, the thesis follows once we show the existence of a sequence $(h_n)_{n\in\mathbb{N}}$ of functions of the kind (A.1) such that

$$\mathbb{E}\left[|h_n(M_T) - h(M_T)|^2\right] \xrightarrow[n\to\infty]{} 0. \tag{A.2}$$

To see that, denoting by $F_n(t, M_{t-})$ the integrand in the Galtchouk-Kunita-Watanabe decomposition of $h_n(M_T)$, $n \in \mathbb{N}$, we can proceed as in the proof of Lemma 3.1 and we get that the sequence $(F_n(t, M_{t-}))_{n\in\mathbb{N}}$ converges in $L^2(\Omega, \mathrm{d}\langle M\rangle \otimes \mathrm{d}\mathbb{P})$ to the integrand $H^{\mathcal{F}}$ in the Galtchouk-Kunita-Watanabe decomposition of $h(M_T)$. Now, there is a subsequence converging $\mathrm{d}\langle M\rangle \otimes \mathrm{d}\mathbb{P}$-a.e. to the $\mathbb{F}$-predictable process $(H_t^{\mathcal{F}})_{0\leq t\leq T}$ and for almost all $t \in [0, T]$, $H_t^{\mathcal{F}}$ is $\sigma(M_{t-})$-measurable. Finally this implies the existence of a measurable function $F : [0, T] \times \mathbb{R} \to \mathbb{R}$ such that $H_t^{\mathcal{F}} = F(t, M_{t-})$.
It remains to show the existence of a sequence $(h_n)_{n\in\mathbb{N}}$ of functions of type (A.1) verifying (A.2). If ρ_T is the law of M_T, (A.2), translates into

$$\int_{\mathbb{R}} (h_n(y) - h(y))^2 \mathrm{d}\rho_T(y) \xrightarrow[n\to\infty]{} 0. \tag{A.3}$$

Since ρ_T is a finite non-negative measure, it is well-known that the space of smooth functions with compact support is dense in $L^2(\rho_T)$. This implies that the Schwartz space $\mathcal{S}(\mathbb{R})$ of the fast decreasing functions is dense in $L^2(\rho_T)$. Let $(h_n)_{n\in\mathbb{N}}$ belong to $\mathcal{S}(\mathbb{R})$ such that (A.3), and consequently, (A.2) holds. Since the inverse Fourier transform $\mathcal{F}^{-1}$ maps $\mathcal{S}(\mathbb{R})$ into itself, then we observe that h_n are of the type (A.1) with $\nu(\mathrm{d}a) = \mathcal{F}^{-1}h_n(a)\mathrm{d}a$.
□

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