PROBLEMS IN NUMBER THEORY AND HYPERBOLIC GEOMETRY

BY

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DISSERTATION

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Abstract

In the first part of this thesis [S1] we generalize a theorem of Kiming and Olsson concerning the existence of Ramanujan-type congruences for a class of eta quotients. Specifically, we consider a class of generating functions analogous to the generating function of the partition function and establish a bound on the primes ℓ for which their coefficients $c(n)$ obey congruences of the form $c(\ell n + a) \equiv 0 \pmod{\ell}$. We apply this result to obtain a complete characterization of the congruences of the same form that the sequences $c_N(n)$ satisfy, where $c_N(n)$ is defined by $\sum_{n=0}^{\infty} c_N(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-q^{Nn})}$. This last result answers a question of H.-C. Chan.

In the second part of this thesis [S2] we explore a natural analog of D. Calegari's result that there are no hyperbolic once-punctured torus bundles over S^1 with trace field having a real place. We prove a contrasting theorem showing the existence of several infinite families of pairs $(-\chi, p)$ such that there exist hyperbolic surface bundles over S^1 with trace field having a real place and with fiber having p punctures and Euler characteristic χ. This supports our conjecture that with finitely many known exceptions there exist such examples for each pair $(-\chi, p)$.

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During the latter half of my undergraduate years I wanted to know something about low dimensional topology but imagined that it was outside of my grasp on account of my lack of geometric intuition. I thank my advisor Nathan Dunfield for initiating me into the fascinating world of hyperbolic 3-manifolds and explaining the material me to the point that I can comfortably read papers and follow seminars in the area. This has been a wish fulfillment experience which I'll always remember.

Throughout my first three years at UIUC, former postdoc Jeremy Rouse was very generous with his time, spending countless hours discussing math with me and exposing me to new areas of number theory. In particular, he ran a reading course on modular forms for Atul Dixit and me and served as my advisor for my first paper.

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Chapter 1

Background and Preliminaries

1.1 Hyperbolic Geometry

Let $n \geq 2$. A *hyperbolic n-manifold* is a Riemannian n-manifold of constant sectional curvature -1. *Hyperbolic n-space* is the unique simply connected hyperbolic n-manifold and is written as $\mathbb{H}^n$. This space can be identified with the upper half-space $\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_n > 0\}$ together with the Riemannian metric $ds^2 = \frac{dx_1{}^2 + dx_2{}^2 ... + dx_n{}^2}{x_n^2}$. This space has a large group of isometries owing to its homogeneity. When $n = 2$ or $n = 3$ this group has an especially simple description. The group $\mathrm{Isom}^+(\mathbb{H}^2)$ is the same as the group of orientation preserving automorphisms of the upper half plane. This is the group of linear fractional transformations $\{z \rightarrow \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{R}$ and $ad - bc \neq 0\}$ which is in turn isomorphic to $PSL_2(\mathbb{R})$. The group $\mathrm{Isom}^+(\mathbb{H}^3)$, the group of orientation-preserving conformal automorphisms of the Riemann sphere. This is the the group of linear fractional transformations $\{z \rightarrow \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0\}$ which is isomorphic to $PSL_2(\mathbb{C})$. In what follows we write $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ instead of $PSL_2(\mathbb{R})$ and $PSL_2(\mathbb{C})$; while using the former pair of groups introduces redundancy, doing so also simplifies notation. Let Γ be a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$. Then the quotient space $\mathbb{H}^n/\Gamma$ is hyperbolic n-orbifold. If in addition Γ is torsion-free then $\mathbb{H}^n/\Gamma$ is a hyperbolic n-manifold. Conversely, every hyperbolic n-orbifold and n-manifold arises in this fashions. Hyperbolic n-manifolds may be compact, noncompact but of finite volume, or of infinite volume. In this thesis we will be concerned with finite volume hyperbolic n-manifolds for $n = 2, 3$. Almost every topological surface arises as a hyperbolic 2-manifold in the sense that all but finitely many topological surfaces have negative Euler characteristic and every 2-manifold of negative Euler characteristic arises as a hyperbolic 2-manifold. In dimension 3 there are many infinite families of finite volume 3-manifolds which are not hyperbolic but hyperbolic 3-manifolds nevertheless play a prominent role in the study of 3-manifolds in general [Thu1]. There are several important differences between hyperbolic 2-manifolds and hyperbolic 3-manifolds. The hyperbolic structure on a compact orientable hyperbolic 2-manifold M relaxes to a complex structure on M and by the Riemann Roch theorem one can realize this complex 1-manifold as an essentially unique algebraic curve. There is no analog of this fact for hyperbolic 3-manifolds as one can see from the fact that complex algebraic varieties have even real dimension. Another important difference concerns "flexibility" vs. "rigidity" of hyperbolic manifolds. The hyperbolic structure on a surface of negative Euler characteristic other than the 3-punctured sphere can be "deformed" which is to say that are slight perturbations of the corresponding $\mathrm{Isom}^+(\mathbb{H}^2)$ that remain discrete and give the same topological surface, so that there are uncountably many nonisometric hyperbolic surfaces of a given topological type. By way of contrast, the Mostow-Prasad rigidity theorem states that if two finite volume hyperbolic 3-manifolds are topologically equivalent then they are isometric.

There are many differential geometric invariants attached to a finite volume hyperbolic n-manifold. It is interesting to note that by the Mostow-Prasad rigidity theorem these differential geometric invariants are also *topological* invariants of M. In Chapter 3 we consider a differential geometric invariant of $M = \mathbb{H}^3/\Gamma$ called the *trace field* of M defined by $\mathbb{Q}(\mathrm{tr}(\Gamma)) = \mathbb{Q}(\{\mathrm{tr}(\gamma) : \gamma \in \Gamma\})$. Mostow-Prasad rigidity and Weil-Garland rigidity imply ([MR] Chapter 3) that the trace field of M is a *finite* extension of $\mathbb{Q}$.

The trace field of a hyperbolic 3-manifold is not an abstract number field, but a concrete subfield of the complex numbers, that is, a pair (K, σ) where K is an abstract field and $\sigma : K \to \mathbb{C}$ is nonzero ring homomorphism. A short argument shows that $\sigma(K) \not\subset \mathbb{R}$ since otherwise Γ could not have finite covolume [MR]. However, it can happen that there is some other ring homomorphism $\sigma' : K \to \mathbb{C}$ that $\sigma'(K) \subset \mathbb{R}$. If this is so then we say that K

has a *real place*. The condition that K has a real place is the same as the condition that a minimal polynomial for K over $\mathbb{Q}$ has a real root.

A natural question is:

Question 1.1.1. *Is every pair (K, σ) with $\sigma(K) \not\subset \mathbb{R}$ the trace field of some hyperbolic 3-manifold?*

W. Neumann has conjectured that the answer is yes, but the question is still open. An adjacent question is whether there are restrictions on the trace fields that arise from natural families of hyperbolic 3-manifolds. We explore this question for certain classes of hyperbolic 3-manifolds and certain classes of trace fields in Chapter 3.

1.2 Surface Bundles

Here we describe the classes of 3-manifolds that we study in Chapter 3. These classes of 3-manifolds are the surface bundles of type S where S is a surface with negative Euler characteristic other than the thrice punctured sphere.

A simple construction that produces many 3-manifolds is as follows. Let $S = S(-\chi, p)$ be the orientable, connected surface of Euler characteristic $-\chi$ with p punctures. Let $\psi : S \to S$ be an orientation preserving homeomorphism. The *mapping torus* of the pair (S, ψ) is $M = S \times [0, 1]/\sim$, where $\sim$ identifies $S \times \{0\}$ with $S \times \{1\}$ via ψ. A 3-manifold is called a *surface bundle over S^1 with fiber* $S(-\chi, p)$ if it arises from this construction. If ψ and ψ' differ by a homeomorphism isotopic to the identity map, then the associated mapping cylinders are homeomorphic. Hence M depends only on the class that ψ represents in the *mapping class group* $\mathrm{Mod}(S) := \mathrm{Homeo}^+(S)/\mathrm{Homeo}_0(S)$. Here $\mathrm{Homeo}^+(S)$ is the group of orientation preserving homeomorphisms of S and $\mathrm{Homeo}_0(S)$ is the group of homeomorphisms of S that are isotopic to the identity. The mapping class group of a surface is generically an infinite nonabelian group, each element of which gives a different surface bundle, so the construction described does indeed give many 3-manifolds.

In view of the remarks in Section 1.1 about most 3-manifolds being hyperbolic it is natural to ask when these surface bundles are hyperbolic. W. Thurston showed that a mapping torus M is a hyperbolic 3-manifold if and only if ψ is *pseudo-Anosov* [Thu3]. Since empirically almost all elements of a generic mapping class group are pseudo-Anosov, Thurston's theorem is an indication that almost all surface bundles with a given fiber are hyperbolic. Recently J. Maher [Mah] has rigorously proved that if g is the genus of a surface S with Euler characteristic $\chi(S) = 2 - 2g - p < 0$ and $(g, p) \neq (0, 3)$, then almost all mapping classes $\psi \in \mathrm{Mod}(S)$ are pseudo-Anosov (in the sense that there's a natural ordering of elements in the mapping class group by complexity of the element such that there are finitely many elements of a given complexity, the fraction of elements of bounded complexity that are pseudo-Anosov tends toward 1 as the upper bound goes to infinity).

One motivation for studying hyperbolic surface bundles over S^1 is that Thurston has conjectured that every hyperbolic 3-manifold is a finite quotient of such a bundle.

1.3 Modular Forms

A special family of discrete groups of isometries of $\mathbb{H}^2$ is given by the congruence subgroups of $SL_2(\mathbb{Z})$, in particular the subgroups

$\Gamma_0(N) = \{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid \text{and } c \equiv 0 \pmod{N}\}$ and

$\Gamma_1(N) = \{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid c \equiv 0 \pmod{N} \text{ and } a, d \equiv 1 \pmod{N}\}$.

Quotienting out $\mathbb{H}^2$ by these subgroups and compactifying them in the natural way gives surfaces $X_0(N)$ and $X_1(N)$ which are moduli spaces of elliptic curves with extra structure. The hyperbolic structures on these (oriented) surfaces induce complex structures on these surfaces and by the Riemann-Roch theorem these surfaces can be realized as algebraic curves. These algebraic curves have canonical models over number fields owing to their interpretation as moduli spaces and this endows them number theoretic significance. For example, if E is an elliptic curve over $\mathbb{Q}$ with conductor N then the famous modularity theorem by Andrew

Wiles and collaborators is that there is a nonconstant morphism $X_0(N) \to E$; this implies Fermat's Last Theorem and is the starting point for the only known results on the Birch and Swinnerton Dyer conjecture.

In another (not unrelated) direction, analytic functions called modular forms defined on $\mathbb{H}^2$ which transform "nicely" under congruence subgroups of $SL_2(\mathbb{Z})$ and have Fourier expansions and form finite dimensional vector spaces with bases consisting of functions with integer Fourier coefficients. Many arithmetic sequences arise as the Fourier coefficients of such functions. As a consequence, modular forms have been used to study sequences such as $p(n)$ (the number of partitions of n) and $r_4(n)$ (the number of representations of n as a sum of 4 squares).

To be more precise, let $f(z)$ be a holomorphic function on the upper half plane. Given a matrix $M \in SL_2(\mathbb{R})$ and an integer $k > 0$, define the slash operator of weight k corresponding to M by

$$f(z)|_k M = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right). \tag{1.3.1}$$

We say that $f(z)$ is a *modular form of weight k for* $\Gamma_0(N)$ if $f(z)|_k M = f(z)$ for all $M \in \Gamma_0(N)$ and if $f(z)$ is bounded. Since $z \to z+1$ is in $\Gamma_0(N)$, if $f(z)$ is a modular form of weight k for $\Gamma_0(N)$ then the invariance of $f(z)$ under the slash operator gives $f(z) = f(z+1)$ so that $f(z)$ has a Fourier expansion in powers of $q = e^{2\pi i z}$. The requirement that $f(z)$ is bounded guarantees that the Fourier expansion of $f(z)$ has no negative powers of q (because otherwise $f(z)$ would tend toward ∞ exponentially as $\Im z \to \infty$). The definition of modular forms of weight k for $\Gamma_1(N)$ is identical to the above definition save for $\Gamma_0(N)$ being replaced by $\Gamma_1(N)$

As indicated previously, modular forms of weight k for a congruence subgroup of $SL_2(\mathbb{Z}$ form finite dimensional vector spaces over $\mathbb{R}$. If the congruence subgroup in question is $\Gamma_0(N)$ or $\Gamma_1(N)$ one can obtain explicit formulas for the dimensions of these vector spaces as a function of N using the Riemann Roch theorem ([DS] Chapter 2).

There is a theory of (mod ℓ)reductions of modular forms with integer coefficients for $SL_2(\mathbb{Z})$ which was developed by Serre and Swinnerton-Dyer to investigate congruences modulo primes for the coefficients of the modular form of weight 12 for $SL_2(\mathbb{Z})$ given by $\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$. Serre and Swinnerton-Dyer developed this theory by utilizing an explicit characterization of the graded algebra of modular forms for $SL_2(\mathbb{Z})$. However, this sort of explicit characterization seems to be absent for $\Gamma_0(N)$ and $\Gamma_1(N)$ for N sufficiently large. Arithmetic algebraic geometers soon developed a notion of "modular forms (mod ℓ) which coincided with (mod ℓ) reductions of modular forms with integer coefficients but was susceptible to advanced tools from arithmetic algebraic geometry. This allowed facts from the theory of Serre and Swinnerton-Dyer to be extended to modular forms for $\Gamma_1(N)$ [Gr]. We use these facts in Chapter 2.

Chapter 2

Ramanujan congruences for a class of eta quotients

2.1 Introduction

Some of Ramanujan's most influential results are his congruences for the partition function $p(n) \pmod 5$, $\pmod 7$ and $\pmod{11}$. For $n \geq 1$, the function $p(n)$ is defined to be the number of ways of writing n as a sum of positive integers in non-increasing order. By convention, one sets $p(0) = 1$ and $p(n) = 0$ for $n < 0$. Ramanujan discovered that for any $n \in \mathbb{Z}$, we have

$$\begin{cases} p(5n+4) & \equiv 0 \pmod 5 \\ p(7n+5) & \equiv 0 \pmod 7 \\ p(11n+6) & \equiv 0 \pmod{11}. \end{cases} \tag{2.1.1}$$

He proved the congruences in (2.1.1) starting from the fact that $\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n)q^n$. The congruences in (2.1.1) have inspired much research in q-series, combinatorics and modular forms. For a short survey of this work, we refer the reader to [AO].

One noticeable feature of the congruences listed above is that that they all take the form $p(\ell n + a) \equiv 0 \pmod{\ell}$ where ℓ is prime. It is natural to ask whether $p(n)$ satisfies any other congruences of the same form. In [AB], Ahlgren and Boylan showed that Ramanujan's congruences are the only congruences of this form: if ℓ is prime, $0 \leq a \leq \ell - 1$ and $p(\ell n + a) \equiv 0 \pmod{\ell}$, then $(\ell, a) \in \{(5, 4), (7, 5), (11, 6)\}$.

In [HCC], H.-C. Chan defined a sequence $r(n)$ by the formula

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-q^{2n})} = \sum_{n=0}^{\infty} r(n)q^n$$

and proved that $r(3n+2) \equiv 0 \pmod 3$. The form of this last congruence parallels Ramanujan's three congruences listed above: it is of the form $r(\ell n + a) \equiv 0 \pmod \ell$ for ℓ prime. In [HCC2], Chan asked if there are any other congruences of the same form. In this paper we answer his question in the negative as a consequence of Theorem 2.2.1 below. Define a *Ramanujan congruence* for a sequence $c(n)$ to be a congruence of the form $c(\ell n + a) \equiv 0 \pmod \ell$ for all $n \in \mathbb{Z}$ with ℓ prime. Without loss of generality we can take $0 \leq a \leq \ell - 1$.

Theorem 2.1.1. *Let $N > 1$. Define $c_N(n)$ by*

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-q^{Nn})} = \sum_{n=0}^{\infty} c_N(n)q^n.$$

Let ℓ be prime, $0 \leq a \leq \ell - 1$ and suppose that

$$c_N(\ell n + a) \equiv 0 \pmod \ell$$

for all n. Then $2 < \ell \leq 11$. Moreover,

- $\ell = 3$ *if and only if* $N = 2$ *and* $a = 2$,
- $\ell = 5$ *if and only if* $N \equiv 0 \pmod 5$ *and* $a = 4$,
- $\ell = 7$ *if and only if* $N \equiv 0 \pmod 7$ *and* $a = 5$,
- $\ell = 11$ *if and only if* $N \equiv 0 \pmod{11}$ *and* $a = 6$.

Theorem 2.1.1 gives a complete characterization of Ramanujan congruences for the family of sequences $c_N(n)$. The reader should note that when $c_N(n)$ satisfies a sufficient condition for the existence of a Ramanujan congruence (mod ℓ), $\ell = 5, 7$ or 11, the congruence follows trivially from the known congruences for $p(n)$, so that the effect of Theorem 2.1.1 is that the sequences $c_N(n)$ obey no Ramanujan congruences other than Chan's and those that come from the Ramanujan congruences for $p(n)$ in a trivial way.

We prove Theorem 2.1.1 using a more broadly applicable theorem which we now state.

Theorem 2.1.2. *Let $S = (a_1, a_2, \ldots, a_j)$ be a sequence of positive integers with j even and define $c(n)$ by*

$$\prod_{n=1}^{\infty} \prod_{i=1}^{j} \frac{1}{(1 - q^{a_i n})} = \sum_{n=0}^{\infty} c(n) q^n. \tag{2.1.2}$$

Let $N = \operatorname{lcm}(a_1, a_2, \ldots, a_j)$. Then if $c(n)$ obeys a Ramanujan congruence (mod ℓ), then $\ell | N$ or $\ell \leq \max(5, j + 4)$.

It follows that if $c(n)$ obeys a Ramanujan congruence (mod ℓ), then $\ell \leq \max(N, 5, j + 4)$. This finiteness result contrasts with Treneer's result [Tre] that there are infinitely many congruences of the form $c(An + B) \equiv 0 \pmod{M}$ where $A, M \in \mathbb{N}$ are allowed to be arbitrary. Treneer's result is a broad generalization of the celebrated theorem of Ono [Ono1] showing the existence of infinitely many congruences for the partition function $p(n)$ and its extension by Ahlgren [Ahl]. These results are quite a bit sharper than we indicate here; we refer the reader to the original sources for more information.

Upon taking $a_i = 1$ for each i, Theorem 2.1.2 reduces to a result of Kiming and Olsson [KO] that there is an explicit bound on those ℓ for which there is a Ramanujan congruence (mod ℓ) for the coefficients of an even power of the generating function of the partition function. Our method of proof is essentially that of Kiming and Olsson but we do not follow their exposition in detail. Kiming and Olsson used the theory of modular forms (mod ℓ) for $SL_2(\mathbb{Z}) = \Gamma_1(1)$. To generalize their results we use certain facts about the ring of modular forms (mod ℓ) for $\Gamma_1(N)$ which were provided by Gross [Gr].

The upper bound on ℓ implied by Theorem 2.1.2 is very close to being sharp in j and is sharp in N: this follows from the unexceptional congruences for even powers of the generating function for $p(n)$ reported on in [KO], the exceptional Ramanujan congruences (mod ℓ) for coefficients of odd powers of the generating function of $p(n)$ as reported on in [Boy] and a line of elementary algebra to use the latter Ramanujan congruences to produce Ramanujan congruences for $c(n)$ with j even.

The reader may wonder why Theorem 2.1.2 is stated for even j. We suspect that there is an explicit bound on ℓ in for odd j as well, however, rigorously establishing an upper bound on ℓ for odd j appears to be substantially more difficult than doing so for even j. Indeed, even if we take $a_i = 1$ for all i, in contrast to the Kiming and Olsson bound on ℓ for even j, it appears that there is no established bound on ℓ for an arbitrary odd j (but see [Boy] for substantial partial results on this matter). The results of Sections 2.3 and 2.4 hold independent of the parity of j; these results may be of use in establishing a generalization of Theorem 2.1.2 that includes the case with j odd.

In Section 2.2 we state the facts that we need about the ring of modular forms (mod ℓ) for $\Gamma_1(N)$. In Section 2.3 we use Lemma 2.4.1 to determine determine a if $c(\ell n + a) \equiv 0 \pmod{\ell}$ and ℓ is larger than an explicit bound. In Section 4 we prove Lemma 2.4.1 which we use in Section 2.5 to prove Theorem 2.1.2. In Section 2.6 we use Theorem 2.1.2 to prove Theorem 2.1.1. In Section 7 we conclude with comments and open questions.

2.2 Modular Forms (mod ℓ) for $\Gamma_1(N)$, $N \geq 4$

Throughout this section we assume that ℓ does not divide N Before stating the facts that we need about modular forms (mod ℓ) for $\Gamma_1(N)$, we define the filtration, the operator θ and the Eisenstein series for $SL_2(\mathbb{Z})$.

Given an element $f(z) \in M_k(\Gamma_1(N)) \cap \mathbb{Z}[[q]]$ and a prime $\ell \in \mathbb{Z}$, reducing the Fourier expansion of $f(z)$ (mod ℓ) gives an element $\tilde{f} \in \mathbb{F}_\ell[[q]]$. We call such a series a "modular form (mod ℓ) for $\Gamma_1(N)$." We want a notion of "weight" for such a series. At first blush one might attempt to define the weight of such a series as the weight of the preimage under the reduction map, but there are many preimages of any such series and not all have the same weight. This motivates the definition of the filtration of a modular form $f \in M_k(\Gamma_1(N)) \cap \mathbb{Z}[[q]], f \not\equiv 0 \pmod{\ell}$ which is defined as follows:

$$w_\ell(f) := \min\{k' : \tilde{f} \in \widetilde{M}_{k'}(\Gamma_1(N))$$

where

$$\widetilde{M}_{k'}(\Gamma_1(N)) = \{\tilde{f} : f(z) \in M_{k'}(\Gamma_1(N)) \cap \mathbb{Z}[[q]]\}.$$

We mildly abuse notation and given $\tilde{f}$ a modular form $\pmod{\ell}$ with $w_\ell(f) = k$, we also call the preimages of $\tilde{f}$ under the reduction map "modular forms $\pmod{\ell}$ with filtration k."

Given $f(z) = \sum_{n=0}^{\infty} c(n)q^n$ where $q = e^{2\pi i z}$, define

$$\theta f := \frac{1}{2\pi i}\frac{df}{dz} = \sum_{n=0}^{\infty} nc(n)q^n.$$

The Eisenstein series for $SL_2(\mathbb{Z})$ of weight 2k is

$$E_{2k}(z) = 1 - \frac{4k}{B_{2k}}\sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n$$

For $k > 1$, $E_{2k}(z)$ is a modular form for $SL_2(\mathbb{Z})$ of weight $2k$. For $k = 1$, $E_{2k}(z)$ is not a modular form for $SL_2(\mathbb{Z})$ but rather a quasi-modular form. Given a complex analytic function $f(z)$ defined on the upper half plane, an integer $k > 0$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, as usual define the slash operator of weight k by

$$f(z)|_k M = (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right). \tag{2.2.1}$$

Though the slash operator depends on k we often omit the subscript k to avoid cumbersome notation. Returning to our comment about $E_2(z)$, as mentioned on pg. 18 of [DS], if M is as above we have

$$E_2(z)|_2 M = E_2(z) - \frac{6ic}{\pi(cz+d)}. \tag{2.2.2}$$

If f is a modular form of weight k for $\Gamma_1(N)$ then $12\theta f - kE_2 f$ is a modular form of weight $k+2$ for $\Gamma_1(N)$. This is Lemma 3 of [SwD] for $N = 1$ and is proved for arbitrary N in exactly the same way as for $N = 1$: by unpackaging the definitions and using (2.2.2). Theorem 2a) from [SwD] is that $E_{\ell-1} \equiv 1 \pmod{\ell}$ and $E_{\ell+1} \equiv E_2 \pmod{\ell}$. Putting these results together we obtain Lemma 2.2.1.

Lemma 2.2.1. *If $f \in M_k(\Gamma_1(N)) \cap \mathbb{Z}[[q]]$, then defining R to be*

$$R = \left(\theta f - \frac{k}{12}E_2 f\right) E_{\ell-1} + \frac{k}{12}E_{\ell+1} f, \tag{2.2.3}$$

R is a modular form of weight $k + \ell + 1$ such that $R \equiv \theta f \pmod{\ell}$. In particular, θf is a modular form (mod ℓ) for $\Gamma_1(N)$. It follows that if $\tilde{f} \not\equiv 0 \pmod{\ell}$, then $w_\ell(\theta f) \leq w_\ell(f) + \ell + 1$.

With Lemma 2.2.1 and the preceding setup in mind we cite the remaining facts that we need about modular forms (mod ℓ) for $\Gamma_1(N)$.

Lemma 2.2.2. *Let $N \geq 4$, let $f, g \in M(\Gamma_1(N)) \cap \mathbb{Z}[[q]]$, and let $\ell \geq 5$ be prime. Then*
(i) We have $w_\ell(\theta f) = w_\ell(f) + \ell + 1$ if and only if $w_\ell(f) \not\equiv 0 \pmod{\ell}$.
(ii) If f and g have weights k_1 and k_2 respectively and $\tilde{f} \equiv \tilde{g} \not\equiv 0 \pmod{\ell}$, then $k_1 \equiv k_2 \pmod{\ell - 1}$.
(iii) For $i \geq 0$, $w_\ell(f^i) = i \cdot w_\ell(f)$.

For a proof of Lemma 2.2.2, see Section 4 of [Gr].

If $f(z) = \sum_{n=0}^{\infty} c(n)q^n$ where $q = e^{2\pi iz}$, define $f|U_\ell$ by

$$f|U_\ell := \sum_{n=0}^{\infty} c(\ell n)q^n.$$

A crucial elementary fact is that if $f \in M_k(\Gamma_1(N)) \cap \mathbb{Z}[[q]]$, then there is a relationship between θf and $f|U_\ell$:

$$(f|U_\ell)^\ell \equiv f - \theta^{\ell-1} f \pmod{\ell}.$$

It follows that

$$f|U_\ell \equiv 0 \pmod{\ell} \iff \theta^{\ell-1} f \equiv f \pmod{\ell}. \tag{2.2.4}$$

2.3 Determination of a if $c(\ell n + a) \equiv 0 \pmod{\ell}$

Let $\delta_\ell = \frac{\ell^2-1}{24}$ and as usual let $\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$. In this section we prove the following.

Lemma 2.3.1. *Let $c(n)$, j, $a_1, ..., a_j$ and N be as in Theorem 1.2, and let $\ell > \max(5, j+3)$ be a prime such that $\ell \nmid N$.*
(i) Then $c(\ell n + a) \equiv 0 \pmod{\ell}$ if and only if $d(\ell n + b) \equiv 0 \pmod{\ell}$, where $d(n)$ is defined by

$$\sum_{n=0}^{\infty} d(n)q^n = \{\prod_{i=1}^{j} \Delta(a_i z)\}^{\delta_\ell}, \tag{2.3.1}$$

and b is defined by $24a \equiv 24b + (\sum_{i=1}^{j} a_i) \pmod{\ell}$.
(ii) In part (i) we have $b \equiv 0 \pmod{\ell}$ so that $24a \equiv (\sum_{i=1}^{j} a_i) \pmod{\ell}$.

The specific tool that we use is a modified form of Proposition 3 from [KO]. One modification is the addition of an additional hypothesis which is implicitly assumed in the proof of Proposition 3 and not explicitly stated. The other modification is that we replace the space $M_k(\Gamma_1(1))$ in Proposition 3 with $M_k(\Gamma_1(N))$ for $N \geq 4$. This yields a true statement because the proof of Proposition 3 given in [KO] is the same word for word for any N for which Lemma 2.2.1 and Lemma 2.2.2 of Section 2.2 are true.

Proposition 2.3.2. *(After Proposition 3 in [KO]) Let $\ell \geq 5$ be prime and $N \geq 4$, $\ell \nmid N$. Suppose that $f(z) \in M_k(\Gamma_1(N))$ has ℓ-integral Fourier coefficients, $w_\ell(f(z)) \not\equiv 0 \pmod{\ell}$, and $\theta(f(z)) \not\equiv 0 \pmod{\ell}$. Suppose further that $w_\ell(\theta^m f(z)) \geq w_\ell(f(z))$. Then if the Fourier coefficients $d(n)$ of $f(z)$ satisfy $d(\ell n + b) \equiv 0 \pmod{\ell}$, one of the following is true: $b = 0$, $w_\ell(f(z)) \equiv (\ell+1)/2 \pmod{\ell}$ or $w(f(z)) \equiv (\ell+3)/2 \pmod{\ell}$.*

The hypothesis that is implicitly assumed in the proof of Proposition 3 of [KO] is that $w_\ell(\theta^m f(z)) \geq w_\ell(f(z))$.

Proof of Lemma 2.3.1. Since $\ell > 3$, we have $24 | (\ell^2 - 1)$. Write $-1 = -\ell^2 + (\ell^2 - 1)$. Then we have

$$\begin{aligned}
\sum_{n=0}^{\infty} c(n) q^n &= \prod_{i=1}^{j} \prod_{n=1}^{\infty} (1 - q^{a_i n})^{-1} = \prod_{i=1}^{j} \prod_{n=1}^{\infty} (1 - q^{a_i n})^{-\ell^2} (1 - q^{a_i n})^{\ell^2 - 1} \\
&= \prod_{i=1}^{j} \prod_{n=1}^{\infty} \left\{ (1 - q^{a_i n})^{-\ell^2} q^{\frac{-a_i(\ell^2-1)}{24}} ((q^{a_i/24})(1 - q^{a_i n}))^{\ell^2 - 1} \right\} \\
&= \left\{ \prod_{i=1}^{j} \prod_{n=1}^{\infty} (1 - q^{a_i n}) \right\}^{-\ell^2} \left\{ \prod_{i=1}^{j} q^{-a_i \cdot \delta_\ell} \Delta(a_i z)^{\delta_\ell} \right\}.
\end{aligned}$$

It follows that

$$q^{-(\delta_\ell \cdot \sum_{i=1}^{j} a_i)} \left\{ \prod_{i=1}^{j} \Delta(a_i z) \right\}^{\delta_\ell} = \left\{ \prod_{i=1}^{j} \prod_{n=1}^{\infty} (1 - q^{a_i n}) \right\}^{\ell^2} \left\{ \sum_{n=0}^{\infty} c(n) q^n \right\}. \tag{2.3.2}$$

Multiplying (2.3.1) by q^{-a}, applying the operator U_ℓ and recalling the definition of $d(n)$ gives

$$\sum_{n=0}^{\infty} d\left(\ell n + \left(\delta_\ell \cdot \sum_{i=1}^{j} a_i \right) + a \right) q^n = \left\{ \prod_{i=1}^{j} \prod_{n=1}^{\infty} (1 - q^{a_i n}) \right\}^{\ell} \left\{ \sum_{n=0}^{\infty} c(\ell n + a) q^n \right\}.$$

It follows that

$$\sum_{n=0}^{\infty} d\left(\ell n + \left(\delta_\ell \cdot \sum_{i=1}^{j} a_i\right) + a\right) q^n \equiv 0 \pmod{\ell} \iff \sum_{n=0}^{\infty} c(\ell n + a) q^n \equiv 0 \pmod{\ell}.$$

This completes the proof of Lemma 2.3.1(i).

In view of Lemma 2.3.1(i), to prove Lemma 2.3.1(ii), it suffices to show that if $d(\ell n+b) \equiv 0 \pmod{\ell}$, then $b \equiv 0 \pmod{\ell}$. The point is that we can show that $b \equiv 0 \pmod{\ell}$ using the theory of modular forms $\pmod{\ell}$ for $\Gamma_1(N)$. Indeed, it is a standard fact that $\Delta(a_i z)$ is a modular form for $\Gamma_0(a_i)$ so that $\Delta(a_i z)$ is a modular form for $\Gamma_1(a_i)$ and

$$F_\ell(z) = \{\prod_{i=1}^{j} \Delta(a_i z)\}^{\delta_\ell}$$

is a modular form for $\Gamma_1(N)$ where $N = \mathrm{lcm}(a_1, a_2, \ldots, a_j)$.

To apply Proposition 2.3.2, we treat F_ℓ as a modular form on $\Gamma_1(N')$ where $N' = N$ if $N > 3$ and $N' = 6$ if $N \leq 3$. We now verify that $F_\ell(z)$ satisfies the hypotheses of Proposition 2.3.2. Clearly $F_\ell(z)$ has ℓ-integral Fourier coefficients. By Lemma 2.4.1 below, $w_\ell(F_\ell) = j(\ell^2-1)/2 \not\equiv 0 \pmod{\ell}$. Since $F_\ell \not\equiv 0 \pmod{\ell}$, $\theta F_\ell \not\equiv 0 \pmod{\ell}$. Also by Lemma 2.4.1, $w_\ell(\theta^m F_\ell) \geq w_\ell(F_\ell)$.

Applying Proposition 2.3.2, we see that if $b \neq 0$, either $w_\ell(F_\ell) = j(\ell^2-1)/2 \equiv \frac{\ell+1}{2} \pmod{\ell}$ or $w_\ell(F_\ell) = j(\ell^2-1)/2 \equiv \frac{\ell+3}{2} \pmod{\ell}$, but neither possibility occurs since $\ell > j+3$ by hypothesis. So $b = 0$ as claimed. $\square$

2.4 A lemma about $\Theta^m F_\ell \pmod{\ell}$

In this section we prove a lemma which we used in the proof of Lemma 2.3.1 and which we will use further in the proof of Theorem 2.1.2.

Lemma 2.4.1. *If $m \geq 1$ and $\ell > 3$ is a prime, then*

$$w_\ell(\theta^m F_\ell) \geq w_\ell(F_\ell) = \frac{j(\ell^2-1)}{2}.$$

Lemma 2.4.1 appears in [KO] for $N = 1$. The situation is more subtle for a general N than it is for $N = 1$. While to prove Lemma 2.4.1 for $N = 1$ suffices to consider the Fourier expansion of $F_\ell \pmod{\ell}$ at ∞, for a general N, $\Gamma_1(N)$ has multiple cusps and we find it necessary to consider the Fourier expansions of F_ℓ at each cusp of $\Gamma_1(N)$.

Enumerate the cosets of $\Gamma_1(N)$ in $SL_2(\mathbb{Z})$ with $\{i\}_{i=1,\ldots 2d_N}$. Let M_i be a representative of the i^{th} coset. Let α_i be the cusp that M_i sends to ∞. Denote the minimal period of $F_\ell|M_i$ by t_i. Then $F_\ell|M_i$ has a Fourier expansion in powers of $q_{t_i} = e^{\frac{2\pi i z}{t_i}}$. The order of vanishing of F_ℓ at α_i is then defined to be the index of the first nonvanishing Fourier coefficient of F_ℓ in powers of q_{t_i} and is denoted $\mathrm{ord}_{\alpha_i}(f(z))$.

The Fourier expansions of F_ℓ about cusps other than ∞ need not have coefficients in $\mathbb{Z}$, but by the q-expansion principle, for $N' > 4$ if the Fourier expansion of a modular form f for $\Gamma_1(N')$ about ∞ has integer coefficients, then the Fourier coefficients of f about another cusp must lie in $\mathbb{Q}(\zeta_N)$ where ζ_N is a primitive N'th root of unity and have bounded denominators (see section 12.3 of [DI]). To apply this property we view F_ℓ as a modular form for $\Gamma_1(N')$ where $N' = N$ if $N > 4$ and $N' = 12$ if $N \leq 4$.

Before proceeding, we make a remark about the first few paragraphs of Section 2.2. Rather than considering an element f of $M_k(\Gamma_1(N)) \cap \mathbb{Z}[[q]]$ and reducing $f \pmod{\ell}$ for some rational prime ℓ we can consider elements g of $M_k(\Gamma_1(N)) \cap L[[q]]$ where L is an algebraic number field and reduce $g \pmod{v}$ for any prime v such that the v-adic valuation of g is 0. This defines the notion of a "modular form (mod v)" and allows us to define the filtration w_v for nonvanishing modular forms (mod v) in the obvious way. Define the v-adic valuation of a power series with coefficients in L to be the minimum of the v-adic valuations of the coefficients of the power series (this minimum exists by the bounded denominator property).

If we modify the statement of Lemma 2.2.1 by replacing $\mathbb{Z}[[q]]$ by $L[[q]]$ and replace the modulus of reduction by v where v is a prime above ℓ such that the v-adic valuation of f is 0, then the modified Lemma 2.2.1 is true. We use these facts with $L = \mathbb{Q}(\zeta_N)$. Define $\widetilde{\mathrm{ord}}_{\alpha_i}(f(z))$ to be the order of vanishing of $f \pmod{v}$ at the cusp α_i.

As a preliminary to the proof of Lemma 2.4.1 we prove the following.

Lemma 2.4.2. *Let $m \geq 1$ be an integer and let v be a prime in $\mathbb{Z}[\zeta_N]$ such that $v \nmid 2, 3, N$. Let $f(z)$ be a modular form for $\Gamma_1(N)$ such that $f(z)|M_i$ has coefficients in $\mathbb{Q}(\zeta_N)$ and v-adic valuation 0. Let α_i be a cusp of $\Gamma_1(N)$. Then*

$$\widetilde{ord}_{\alpha_i}(\theta^m f) \geq \widetilde{ord}_{\alpha_i}(f).$$

Proof of Lemma 2.4.2. By induction it suffices to prove the claim for $m = 1$. Since R in (2.2.2) satisfies $R \equiv \theta f \pmod{v}$ for $v \nmid 2, 3$ it suffices to show that $\widetilde{\mathrm{ord}}_{\alpha_i}(R) \geq \widetilde{\mathrm{ord}}_{\alpha_i}(f)$. Take M_i to be as in the discussion preceding the statement of Lemma 2.4.1. Let k be the weight of $f(z)$. Applying the slash operator $|M_i$ of weight $k + \ell + 1$ to both sides of (2.2.3), we obtain

$$\begin{aligned} R|M_i &= \left(\left(\theta f - \frac{k}{12}E_2 f\right) E_{\ell-1}\right)|M_i + \left(\frac{k}{12}E_{\ell+1}f\right)|M_i \\ &= \left((\theta f)|M_i - \frac{k}{12}(E_2|M_i)(f|M_i)\right)(E_{\ell-1}|M_i) + \frac{k}{12}\left(E_{\ell+1}|M_i\right)\left(f|M_i\right). \end{aligned} \tag{2.4.1}$$

In the second line of (2.4.1) and in what follows, the slash operators applied to θf, E_2, f, $E_{\ell-1}$ and $E_{\ell+1}$ are of weights $k+2$, 2, k, $\ell - 1$, and $\ell + 1$ respectively. Now since $E_{\ell-1}$ and $E_{\ell+1}$ are modular forms for $SL_2(\mathbb{Z})$, equation (2.4.1) becomes

$$R|M_i = \left((\theta f)|M_i - \frac{k}{12}(E_2|M_i)(f|M_i)\right) E_{\ell-1} + \frac{k}{12}\left(E_{\ell+1}\right)\left(f|M_i\right). \tag{2.4.2}$$

Next we find an alternate expression for $(\theta f)|M_i$. Applying θ to both sides of the equation

(2.1) gives

$$(\theta f)|M_i = \theta(f|M_i) + \frac{kc}{2\pi i}(cz+d)^{-1}(f|M_i). \tag{2.4.3}$$

Replacing $(\theta f)|M_i$ in (2.4.2) by the righthand side of (2.4.3) and replacing $E_2|M_i$ in (2.4.2) by the righthand side of (2.2.2), after simplification (2.4.2) becomes

$$R|M_i = \left(\theta(f|M_i) - \frac{k}{12}E_2 \cdot (f|M_i)\right) E_{\ell-1} + \frac{k}{12}(E_{\ell+1})(f|M_i).$$

Since $v \nmid N$ and $f|M_i$ has v-adic valuation 0, the Fourier expansion of $\theta(f|M_i)$ has v-adic valuation 0. It is clear from the definition of θ that the index of the Fourier first coefficient of $\theta(f|M_i)$ that is nonvanishing (mod v) is no smaller than the first Fourier coefficient of $f|M_i$ that is nonvanishing (mod v). But then the index of the first Fourier coefficient of $R|M_i$ that is nonvanishing (mod v) is no smaller than the first Fourier coefficient of $f|M_i$ that is nonvanishing (mod v). This completes the proof. □

Proof of Lemma 2.4.1. First we prove that $w(F_\ell) = \frac{j(\ell^2-1)}{2}$. Consider the functions $F_\ell|M_i$ for $i \in \{1, ..., 2d_N\}$. Since the v-adic valuation of each $F_\ell|M_i$ is finite, for each i there exists $\beta_i \in \mathbb{Q}$ such that $(\beta_i F_\ell)|M_i$ has v-adic valuation 0. Now consider

$$G(z) := \prod_{i=1}^{2d_N}(\beta_i F_\ell)|M_i.$$

Since F_ℓ is a modular form of weight $\frac{j(\ell^2-1)}{2}$ for $\Gamma_1(N)$ and the M_i's are a complete set of representatives of cosets of $\Gamma_1(N)$ in $SL_2(\mathbb{Z})$, $G(z)$ is a modular form of weight $d_N j(\ell^2-1)$ for $SL_2(\mathbb{Z})$. Let v be a prime above ℓ in $\mathbb{Z}[\zeta_N]$.

Since F_ℓ is zero-free on $\mathbb{H}$, $F_\ell|M_i$ is zero-free on $\mathbb{H}$, so $G(z)$ is zero-free on $\mathbb{H}$ and the zeros of $G(z)$ all occur at ∞. So $G(z)$ must be a nonzero constant multiple of $\Delta(z)^e$ where $e = \frac{d_N j(\ell^2-1)}{12}$. Moreover, by our choice of β_i, this constant must be nonvanishing (mod v).

So $w_v(G(z)) = 12e$. But then $w_v(F_\ell) = \frac{j(\ell^2-1)}{2}$ and since $F_\ell \in \mathbb{Z}[[q]]$, $w_\ell(F_\ell) = \frac{j(\ell^2-1)}{2}$. So we need only show that $w_\ell(\theta^m F_\ell) \geq \frac{j(\ell^2-1)}{2}$. Since G is a nontrivial multiple of Δ^e, it must be that $\widetilde{\mathrm{ord}}_\infty(G) = e$. Since $\widetilde{\mathrm{ord}}_{\alpha_i}$ is defined in terms of powers of q_{t_i} while $\widetilde{\mathrm{ord}}_\infty$ is defined in terms of powers of q_{t_1} we have $\widetilde{\mathrm{ord}}_\infty(\beta_i F_\ell|M_i) = \frac{\widetilde{\mathrm{ord}}_{\alpha_i}(\beta_i F_\ell)}{t_i}$. From the definition of G we see that $\sum_{i=1}^{2d_N} \frac{\widetilde{\mathrm{ord}}_{\alpha_i}(\beta_i F_\ell)}{t_i} = \widetilde{\mathrm{ord}}_\infty(G(z))$. So

$$\sum_{i=1}^{2d_N} \frac{\widetilde{\mathrm{ord}}_{\alpha_i}(\beta_i F_\ell)}{t_i} = e.$$

Now consider

$$H = \prod_{i=1}^{2d_N} (\beta_i \theta^m F_\ell)|M_i.$$

Then H is a modular form (mod v) for $SL_2(\mathbb{Z})$. We have

$$\widetilde{\mathrm{ord}}_\infty(H) = \sum_{i=1}^{2d_N} \frac{\widetilde{\mathrm{ord}}_{\alpha_i}(\beta_i \theta^m F_\ell)}{t_i} \geq \sum_{i=1}^{2d_N} \frac{\widetilde{\mathrm{ord}}_{\alpha_i}(\beta_i F_\ell)}{t_i} = e,$$

where the inequality is a consequence of Lemma 2.4.2.

The definition of the β_i forces $(\beta_i \theta^m F_\ell)|M_i \not\equiv 0 \pmod{v}$, so $H \not\equiv 0 \pmod{v}$. By Sturm's theorem [Stu], $w_v(H) \geq 12e$. But then we see that $w_v(\theta^m F_\ell) \geq \frac{j(\ell^2-1)}{2}$. Since $\theta^m F_\ell \in \mathbb{Z}[[q]]$, we deduce that $w_\ell(\theta^m F_\ell) \geq \frac{j(\ell^2-1)}{2}$, completing the proof. □

2.5 Proof of Theorem 2.1.2

Let $\ell > \max(5, j+4)$, $\ell \nmid N$. By Lemma 2.3.1, to prove Theorem 2.1.2 it suffices to show that $d(\ell n) \equiv 0 \pmod{\ell}$ leads to a contradiction. Note that it follows from (2.2.3) that $d(\ell n) \equiv 0 \pmod{\ell}$ implies $\theta^{\ell-1} F_\ell \equiv F_\ell \pmod{\ell}$. We analyze the consequences that this has for the sequence $w_\ell(\theta^i f)$, $i \in \{1, 2, \ldots, \ell-1\}$. We will see that the congruence $\theta^{\ell-1} f \equiv f$ (mod ℓ) leads to the existence of an m violating the conclusion of Lemma 2.4.1

To proceed, we need information about the possible sequences $w_\ell(\theta^i F_\ell)$, $i \in \{1, 2, \ldots, \ell - 1\}$ in terms of $w_\ell(F_\ell)$. Proposition 2 of [KO] is false as stated in [KO] (see appendix) but is true when an additional hypothesis is added to the statement of Proposition 2 and the proof in [KO] is valid once the hypothesis is added. The proof of this modified Proposition 2 carries through without modification when $SL_2(\mathbb{Z})$ is replaced by $\Gamma_1(N)$ for any N for which Lemma 2.2.1 and Lemma 2.2.2 hold. For context recall Lemma 2.1 and Lemma 2.2.

Proposition 2.5.1. *(After Proposition 2 of [KO]) Let $\ell \geq 5$ be prime and $N \geq 4$, $\ell \nmid N$. Suppose that $f(z) \in M_k(\Gamma_1(N))$ has ℓ-integral Fourier coefficients, $w_\ell(f(z)) \not\equiv 0 \pmod{\ell}$ and $\theta(f(z)) \not\equiv 0 \pmod{\ell}$. Suppose further that $w_\ell(\theta^m f(z)) \geq w_\ell(f(z))$. Let $i_1 < i_2 < \ldots < i_v$ be those i with $0 \leq i \leq \ell - 1$ for which $w_\ell(\theta^i f) \equiv 0 \pmod{l}$. Write $w_\ell(\theta^{i_j+1} f) = w_\ell(\theta^{i_j} f) + (\ell + 1) - s_j(\ell - 1)$. Write $k = w_\ell(f)$ and let $k_0 \in \{1, 2, \ldots, \ell - 1\}$ be such that $k \equiv -k_0 \pmod{\ell}$. Then one of the four cases below holds:*

- *(I) $k \equiv 1 \pmod{\ell}$, $v = 1$, $i_1 = \ell - 1$, and $s_1 = \ell + 1$*
- *(II) $k \equiv 2 \pmod{\ell}$, $v = 1$, $i_1 = \ell - 2$, and $s_1 = \ell + 1$*
- *(III) $k \not\equiv 1 \pmod{\ell}$, $v = 2$, $(i_1, i_2) = (k_0, \ell - 1)$, and $(s_1, s_2) = (k_0 + 1, \ell - k_0)$*
- *(IV) $k \not\equiv 1 \pmod{\ell}$, $v = 2$, $(i_1, i_2) = (k_0, \ell - 2)$, and $(s_1, s_2) = (k_0 + 2, \ell - k_0 - 1)$*

We have $w_\ell(f) = w_\ell(\theta^{\ell-1} f)$ if and only if case (II) or case (IV) holds.

The necessary hypothesis that is missing in the statement of Proposition 2 of [KO] is that $w_\ell(\theta^m f) \geq w_\ell(f)$. For a counterexample to the original statement, let $f(z) = \Delta(z)$ and take $\ell = 5$. By Lemma 2.2.1, there exists a modular form $g(z) \in M_{18}(\Gamma_1(N))$ such that $g \equiv f \pmod{5}$. By the equality in Lemma 2.4.1, $w_5(f) = 12$. Applying Lemma 2.2(i) then gives $w_5(\theta f) = 18$, $w_5(\theta^2 f) = 24$ and $w_5(\theta^3 f) = 30$. Applying w_5 to both sides of $\theta^5 f \equiv \theta f \pmod{\ell}$ forces $w_5(\theta^4 f) = 12$, so that $v = 1$ for $f(z)$ which implies $v = 1$ for g. The function

g satisfies the hypotheses of the original Proposition 2, but $w_5(g) = 18 \equiv 3 \pmod 5$ so that if the conclusion of Proposition 2 is true for $g(z)$ then $v = 2$ for $g(z)$, and as we just saw this is not the case.

Now we prove Theorem 2.1.2. We verified that F_ℓ satisfies the hypotheses of Proposition 2.5.1 in our proof of Lemma 2.3.1. Taking $f = F_\ell$ in Proposition 2.5.1 we see that $\theta^{\ell-1}F_\ell \equiv F_\ell \pmod \ell$ implies that we are in case (II) or case (IV) of Proposition 2.5.1. Actually, we cannot be in case (II) of Proposition 2.5.1 since Lemma 2.4.1 shows that $w_\ell(F_\ell) = j(\ell^2-1)/2$ and $\ell > j+4$, so we are in case (IV) of Proposition 2.5.1. This implies that if we take $k_0 \equiv -(j)(\ell^2-1)/2 \pmod l$, then $w_\ell(\theta^{k_0+1}F_\ell) = w_\ell(F_\ell) + (\ell+1)(k_0+1) - (k_0+2)(\ell-1) = w(F_\ell) + 2k_0 + 3 - \ell$. We can determine k_0 as follows: we have $2k_0 \equiv j \pmod \ell$ so since j is **even** and $\ell > j$, it must be that $k_0 = j/2$. So $w_\ell(\theta^{k_0+1}F_\ell) = w(F_\ell) + j + 3 - \ell$ and since $\ell > j+3$, we have $w(\theta^{k_0+1}F_\ell) < w(F_\ell)$, contradicting Lemma 2.4.1 and proving Theorem 2.1.2. □

2.6 Proof of Theorem 2.1.1

Let $c_N(n)$ be as in the statement of Theorem 2.1.1, and assume that $c_N(\ell n + a) \equiv 0 \pmod \ell$. Then by Theorem 2.1.2, we may assume that $\ell \leq 5$ or $\ell \mid N$. First suppose that $\ell | N$. Write

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \left(\sum_{n=0}^{\infty} c_N(n)q^n\right)\left(\prod_{n=1}^{\infty}(1-q^{Nn})\right).$$

Since $\ell | N$ we can write $\prod_{n=1}^{\infty}(1-q^{Nn}) = \sum_{n=0}^{\infty} y(n)q^{\ell n}$ so that (2.5.1) becomes

$$\sum_{n=0}^{\infty} p(n)q^n = \left(\sum_{n=0}^{\infty} c_N(n)q^n\right)\left(\sum_{n=0}^{\infty} y(n)q^{\ell n}\right). \tag{2.6.1}$$

Multiplying (5.1) by q^{-a} and applying U_ℓ to both sides gives

$$\sum_{n=0}^{\infty} p(\ell n + a) q^n = \left(\sum_{n=0}^{\infty} c_N(\ell n + a) q^n \right) \left(\sum_{n=0}^{\infty} y(n) q^n \right).$$

Since $y(0) = 1$, we have $c_N(\ell n + a) \equiv 0 \pmod{\ell}$ if and only if $p(\ell n + a) \equiv 0 \pmod{\ell}$ from which it follows that $(\ell, a) \in \{(5, 4), (7, 5), (11, 6)\}$ by the result from [AB] quoted in the Section 2.1. This establishes Theorem 2.1.1 assuming that $\ell | N$. So we need only establish Theorem 2.1.1 assuming that $\ell \leq 5$.

If $\ell \leq 5$ then since $c_N(n) = p(n)$ for $n \leq N$ a short computation shows unless N is as in the bulleted portion of the conclusion of Theorem 2.1.1, $N \leq 5$. Another short computation together with Chan's result for $N = 2$ show that Theorem 2.1.1 holds for $\ell \leq 5$. $\square$

2.7 Conclusion

In light of our results it is natural to ask:

Question 2.7.1. *Let $c(n)$ be given by $\prod_{n=1}^{\infty} \prod_{i=1}^{j} \frac{1}{(1-q^{a_i n})} = \sum_{n=0}^{\infty} c(n) q^n$ where j is **odd**. Are there only finitely many ℓ for which there is a Ramanujan congruence (mod ℓ) for $c(n)$? Can one give an explicit bound on ℓ if this is so?*

In [Boy], Boylan treated many cases where j is odd and $a_i = 1$ for all i. Boylan also reported on the existence of several infinite families of pairs $(j,\, \ell)$ such that the coefficients of $\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^j}$ obey a Ramanujan congruence (mod ℓ), but remarks that there are some pairs $(j,\, \ell)$ that do not fit into these families for which there is nevertheless a Ramanujan congruence. A complete characterization of the pairs $(j,\, \ell)$ for which there is a Ramanujan congruence appears to be absent from the literature. So we ask the following:

Question 2.7.2. *Can one give a complete characterization of all tuples $(\ell; a_1, a_2, \ldots, a_j)$ for which $c(n)$ given by $\prod_{n=1}^{\infty} \prod_{i=1}^{j} \frac{1}{(1-q^{a_i n})} = \sum_{n=0}^{\infty} c(n) q^n$ obeys a Ramanujan congruence (mod ℓ)?*

While it seems likely that the answer to both parts of Question 2.7.1 can be answered in the affirmative, the extent of the phenomenon of there being only finitely Ramanujan congruences for the Fourier coefficients of a modular form is quite unclear, motivating:

Question 2.7.3. *Is there a characterization of those weakly holomorphic modular forms $f(z)$ for congruence subgroups of $SL_2(\mathbb{Z})$ with integer Fourier coefficients such that the Fourier coefficients of $f(z)$ obey only finitely many Ramanujan congruences?*

Chapter 3

Real Places and Surface Bundles

3.1 Introduction

One of the few known results relating the topology of a hyperbolic 3-manifold to its trace field is D. Calegari's intriguing theorem [Cal]. Calegari's theorem is

Theorem 3.1.1. *The trace field of a hyperbolic once punctured torus bundle has no real places.*

Here we consider whether Calegari's result can be extended to hyperbolic surface bundles with other fibers. We show that many surfaces occur as fibers of hyperbolic surface bundles with trace field having real place.

Recall from Chapter 1 that we refer to the unique surface with p punctures and $\chi(S) = \chi$ as the *surface of type* $(-\chi, p)$, and a 3-manifold that can be realized as a surface bundle fiber of type $(-\chi, p)$ as a *surface bundle of type* $(-\chi, p)$. We remark that the type of a surface bundle is not in general uniquely determined, since a given manifold can fiber over S^1 in multiple ways.

In this language, Theorem 3.1.1 is that the trace field of a hyperbolic surface bundle of type $(1, 1)$ has no real places. Calegari's proof does not readily generalize to surface bundles of other types. Our goal is to substantiate

Conjecture. *For each pair $(-\chi, p)$ with $-\chi > 1$, there exists a hyperbolic mapping torus of type $(-\chi, p)$ with trace field having a real place.*

Those surfaces with $\chi \geq 0$ and $(-\chi, p) = (1, 3)$ and are excluded since there are no hyperbolic surface bundles with these as fibers, while the surface of type $(1, 1)$ is excluded by Theorem 3.1.1. Throughout the remainder of the paper we restrict ourselves to surfaces with $-\chi > 1$.

We were led to Conjecture 3.1 by utilizing existing software to collect data which led to the following.

Observation 3.1.2. *For each of the following pairs $(-\chi, p)$, there exists a hyperbolic mapping torus of that type with trace field having a real place:*

$-\chi$	p	$-\chi$	p	$-\chi$	p	$-\chi$	p
2	2, 4	7	1, 3, 5	12	0	18	0, 2
3	1, 3, 5	8	0, 2, 4	14	0, 2	19	1
4	0, 2, 4, 6	9	1, 3	15	1	20	0, 2
5	0, 1, 3, 5, 7	10	0, 2, 4	16	0, 2	21	1, 3
6	0, 2, 4	11	1, 3	17	1	22	0, 2

We were not able to prove our conjecture in general, but were able to prove it for several infinite families of surfaces.

Theorem 3.1.3. *Conjecture 3.1 is true for mapping tori of type $(-\chi, p)$ if*

a) $p \geq 4$ *and* $2|(-\chi)$

b) $p = 0$

c) $5p \leq -\chi$

We use a simple fact in the proof of Theorem 3.1.3.

Proposition 3.1.4. *If M is a finite volume hyperbolic 3-manifold with trace field having real place then any finite cover M' of M is a finite volume hyperbolic 3-manifold with trace field having real place.*

The volume of M' is finite since $\mathrm{Vol}(M') = d \cdot \mathrm{Vol}(M)$ where d is the degree of the covering map $M' \to M$. Since the trace field of a finite cover of a hyperbolic 3-manifold is a subfield of the trace field of the base and a subfield of a field $K \subset \mathbb{R}$ is itself contained in $\mathbb{R}$, the truth of the Proposition 3.1.4 is immediate. So to show that Conjecture 3.1 is true for a surface of type $(-\chi, p)$, it suffices to show that a mapping torus of type $(-\chi, p)$ covers a hyperbolic 3-manifold with trace field having real place.

In Section 3.2 we prove a special case of Theorem 3.1.3 (a) by combining J. Hoste and P. Shanahan's explicit determination of the trace fields of hyperbolic twist knots [HS] with the constructions of fibered covers of hyperbolic twist knots due to G. Walsh [Wal]. In Section 3.3 we give a criterion for a surface bundle of type $(-\chi, p)$ to be covered by a surface bundle of type $(-\chi', p')$ and use this to prove parts (a) and (b) of Theorem 3.1.3. In Section 3.4 we use properties of the Thurston norm on the cohomology of a 3-manifold to prove that a particular 3-manifold with trace field having real place is a surface bundle of type $(-\chi, p)$ for many pairs $(-\chi, p)$ and this gives a proof of Theorem 3.1.3 (c). In Section 3.5 we offer empirical data substantiating Observation 3.1.2, discuss patterns that we observed and state some open questions.

3.2 Fibered Covers of Twist Knots

In [HS] Hoste and Shanahan compute the trace fields of an infinite family of 3-manifolds, namely the complements of the twist knots in S^3. The twist knot K_m is defined by Figure 1 for $m > 0$. When $m = 2$, K_m is the trefoil knot and when $m = 3$, K_m is the figure-eight knot. For $m \geq 3$, S^3/K_m is hyperbolic so that the trace field is well-defined. Hoste and Shanahan found that if $m \geq 3$ and m is even then the trace field of S^3/K_m is of odd degree over $\mathbb{Q}$. As a polynomial of odd degree has a real root, the trace field of such a knot complement S^3/K_m has a real place.

Every two-bridge knot is associated to a fraction $\frac{p}{q}$, written in lowest reduced terms and with $q > 0$. In [Wal], Walsh showed that the complement of such a knot is finitely covered by surface bundle of type $(q-3, q-1)$. As Figure 1 shows, K_m is the two-bridge knot associated to the rational number $\dfrac{1}{2+\frac{1}{m-1}} = \frac{m-1}{2m-1}$. Since $\gcd(m-1, 2m-1) = 1$, it follows that S^3/K_m is a finitely covered by a surface bundle of type $(2m-4, 2m-2)$.

By Proposition 3.1.4, the result in [HS] implies that this surface bundle has a real place if m is even. This proves Conjecture 3.1 for surfaces of type $(4n-4, 4n-2)$ for $n \geq 2$. As we will see in Section 3.3, by utilizing another method we can prove Theorem 3.1.3 (b) which is a strictly stronger statement.

We remark that since Leininger [Lein] showed that the twist knot complements are virtually fibered by punctured torus bundles, one can use Leininger's work in conjunction with Hoste and Shanahan's work to exhibit punctured torus bundles with trace field having real place. However, though we have not checked this rigorously, it appears that following this strategy can only prove Conjecture 3.1 for a subset of the family of surfaces in Theorem 3.1.3 (b).

3.3 Covers of Surface Bundles

In this section we prove Theorem 3.1.3 (a) and (b). First we prove

Theorem 3.3.1. *A surface bundle of type $(-\chi, p)$ is finitely covered by a surface bundle of type $(-\chi', p')$ if $\chi'/\chi = d \in \mathbb{N}$ and $p \leq p' \leq dp$.*

After doing this we exhibit surface bundles of types $(2,4)$ and $(2,0)$ with trace field having real place. Parts (a) and (b) of Theorem 3.1.3 will then follow easily from Theorem 3.3.1.

To prove Theorem 3.3.1 we need a theorem of Massey [Mas] characterizing when one punctured surface covers another.

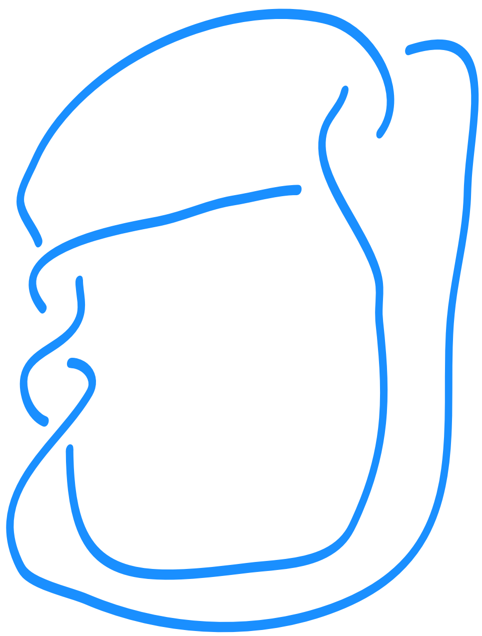

Figure 3.1: A depiction of K_m for $m = 4$. Hoste and Shanahan's K_m is isomorphic to the knot depicted with $m - 1$ crossings on the left hand side of the figure.

Theorem 3.3.2. *For the surface of type $(-\chi', p')$ to be a degree d cover of the surface of type $(-\chi, p)$ it is necessary and sufficient that (i) $\frac{-\chi'}{\chi} = d \in \mathbb{N}$ and (ii) $p \leq p' \leq pd$.*

Massey's proof proceeds by cut and paste arguments. In order to use our knowledge of covers of surfaces of type $(-\chi, p)$ to pass to knowledge about covers of surface bundles of type $(-\chi, p)$ we need another lemma.

Lemma 3.3.3. *Suppose M is a surface bundle with fiber S and that S is finitely covered by S'. Then M has a finite cover $\tilde{M}$ which is a surface bundle with fiber S'.*

Proof of Lemma 3.3.3: Let $\psi : S \to S$ define a mapping torus homeomorphic to M, and let $\rho : S' \to S$ be a covering map. After picking base points for fundamental groups, we have the inclusion $\rho_*(\pi_1(S') \subset \pi_1(S)$. The map $\psi_* : \pi_1(S) \to \pi_1(S)$ is a bijection and so maps index d subgroups to index d subgroups. Since the number m of degree d subgroups of $\pi_1(S)$ is finite, $(\psi_*)^m = (\psi^m)_*$ fixes $\rho_*(\pi_1(S')$. Define M_1 to be the degree m cover of M that is the mapping torus defined by ψ^m, corresponding to "unwinding" M in the direction transverse to the fiber m times. According to the lifting criterion (Prop. 1.33 of [Hat]) there exists a lift $\widetilde{\psi^m}$ of ψ^m so that the following diagram commutes.

$$\begin{array}{ccc} S' & \xrightarrow{\widetilde{\psi^m}} & S \\ \downarrow \rho & & \downarrow \rho \\ S & \xrightarrow{\psi^m} & S \end{array}$$

Then the surface bundle $\widetilde{M}$ with fiber S' and monodromy $\widetilde{\psi^m}$ is a cover of M of degree dm. $\square$

Proof of Theorem 3.3.1: This is immediate from Theorem 3.3.2 and Lemma 3.3.3. $\square$

Proof of Theorem 3.1.3 (a): Let S be the four-punctured sphere, that is, the surface of type $(2, 4)$. Let S' be a surface of type $(-\chi', p')$, where $p' \geq 4$ and $2|(-\chi')$, as in the statement of Theorem 3.1.3 (a). Then since $\chi' = 2-2g'-p'$, $p' \leq -\chi'+2$ so that by Theorem 3.3.1, to prove

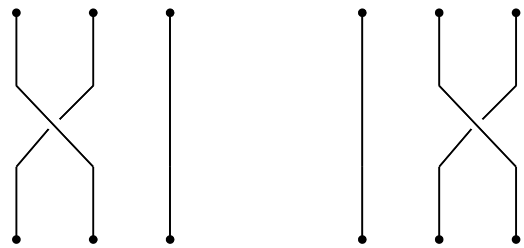

Figure 3.2: A depiction of the braids σ_1 (on the left) and σ_2 (on the right)

Theorem 3.1.3 (a) it suffices to exhibit an S-bundle with trace field having a real place. An S-bundle can be specified by giving the mapping class $[\psi] \in \mathrm{Mod}(S)$. The subgroup of $\mathrm{Mod}(S)$ fixing one of the four punctures is isomorphic to the braid group on three strands. This group is generated by the braids σ_1 and σ_2 in Figure 2. We take M to be the S-bundle given by the mapping class $[\psi]$ corresponding to the braid $\sigma_1{\sigma_2}^4{\sigma_1}^{-2}\sigma_2\sigma_1{\sigma_2}^{-1}\sigma_1\sigma_2$. SnapPy [CDW] computes a hyperbolic structure on the resulting S-bundle, and Snap [CGHN] rigorously proves that the trace field of M is generated by the roots of the polynomial $x^4-x^3-2x^2-x+1$. Since this polynomial has two real roots, the trace field of M has a real place.

Now Theorem 3.1.3 (a) follows from the existence of a surface bundle of type $(2, 4)$ with trace field having a real place and Theorem 3.3.1. $\square$

Proof of Theorem 3.1.3 (b): Let S be the closed surface of genus 2, that is, the surface of type $(2, 0)$. As above, we specify an S-bundle by specifying a mapping class in $[\psi] \in \mathrm{Mod}(S)$. The group $\mathrm{Mod}(S)$ is generated by Dehn twists in 5 standard curves [Bir]. We order these curves according to x value on pg. 169 of [Bir] as a, b, c, d and e. Let τ_α be the mapping class of the Dehn twist about the curve α. Let M be the S-bundle corresponding to $\tau_a(\tau_b\tau_c\tau_d\tau_e\tau_b)^{-1}$. Twister [Hal] generates a triangulation of M, and Snap [CGHN] rigorously proves that the trace field of M is generated by the roots of the polynomial $x^8 - 5x^6 + 12x^4 - 9x^2 + 2$. Since this polynomial has four real roots, the trace field of M has a real place.

Theorem 3.1.3 (b) follows from the existence of a surface bundle of type $(2, 0)$ with trace field having a real place and Theorem 3.3.1. $\square$

3.4 A 3-Manifold Fibering In Many Ways

In this section we prove

Theorem 3.4.1. *There is a hyperbolic 3-manifold with trace field having real place that is a surface bundle of type* $(-\chi, p)$ *for every pair with* $5p \leq -\chi$ *coming from a surface except for* $(5, 1)$.

Since according to Observation 3.1.2, Conjecture 3.1 is true for $(-\chi, p) = (5, 1)$; Theorem 3.4.1 implies Theorem 3.1.3 (c).

If a hyperbolic 3-manifold M fibers over S^1, then the 1-form $\frac{1}{2\pi}d\theta$ on S^1 pulls back to a cohomology class $\phi \in H^1(M;\mathbb{Z})$, and we say that ϕ induces the fibration. In [Thu2], Thurston provides an elegant characterization of all fibrations of M over S^1. First Thurston defines a norm $|| * ||_T$ (now called the *Thurston norm*) on $H^1(M;\mathbb{R})$. We do not need the definition of the Thurston norm here but will use the following consequence of the definition.

Proposition 3.4.2. *If* ϕ *induces a fibration of a hyperbolic 3-manifold* M *over* S^1 *with fiber* S *then* $||\phi||_T = -\chi$ *where* $-\chi$ *is the Euler characteristic of* S.

In [Thu2] Thurston shows that the unit norm ball of $|| * ||_T$ is a convex polyhedron symmetric about the origin and with coordinates of the vertices in $\mathbb{Q}$. We denote its unit norm ball by B_T. Setting $b_1(M) = \dim(H^1(M;\mathbb{R}))$, we say that a face of B_T is *top dimensional* if its dimension is $b_1(M) - 1$. For $\phi \in H^1(M;\mathbb{R})$, the ray from the origin to ϕ intersects the interior of a unique face of B_T, and we say that ϕ *lies above* that face. A face and its reflection through the origin play the same role, so we abuse notation, referring to a *pair* of opposite faces of the unit norm ball as a face. The main theorem of [Thu2] is:

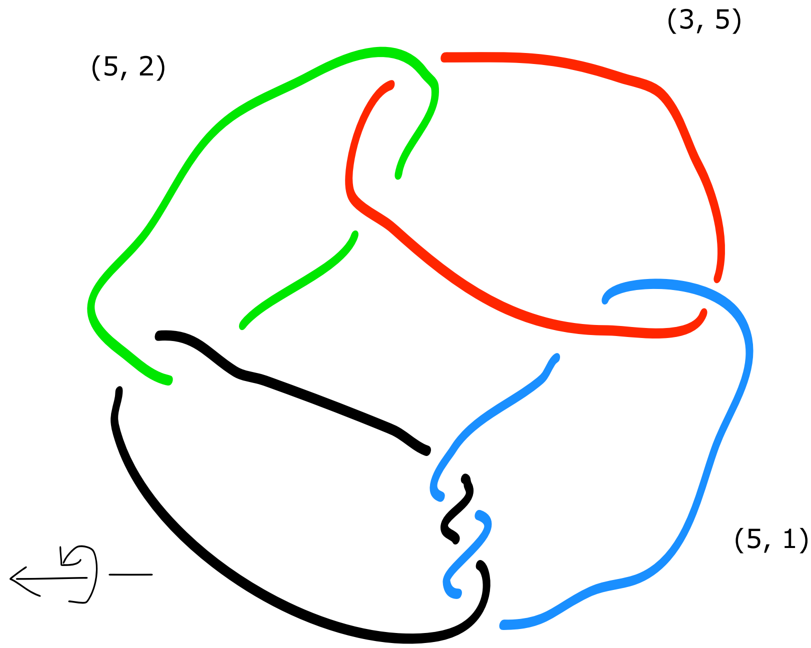

Figure 3.3: The ordered pairs adjacent to each of the colored loops denote Dehn surgery coefficients, with Dehn surgery convention as indicated in the lower lefthand corner.

Theorem 3.4.3. *There is a collection of top dimensional faces of B_T called the fibered faces so that $\phi \in H^1(M;\mathbb{Z})$ induces a fibration of M if and only if ϕ lies over one of the faces in this collection.*

Thurston's theorem is trivial for M if $b_1(M) = 1$, but if $b_1(M) > 1$, it shows that if M is a surface bundle then there are infinitely many pairs $(-\chi, p)$ such that M is a surface bundle of type $(-\chi, p)$. In this section we give an example of a hyperbolic 3-manifold M with trace field having real place and $b_1(M) > 1$, and work out the pairs $(-\chi, p)$ such that M is a surface bundle of type $(-\chi, p)$.

For our manifold, we take M as in Figure 3.3. We found this manifold in Morwen Thistlethwaite's recent census of cusped hyperbolic 3-manifolds that are gluings of 8 ideal tetrahedra. Thistlewaite's data has been incorporated into the most recent version of Snappy [CDW].

Snap [CGHN] computes that a minimal polynomial for the trace field of M is $x^6 - 2x^4 +$

$4x^2 - 1$. Since this polynomial has two real roots, the trace field of M has a real place. To study the surfaces S such that M is a surface bundle with fiber S we need to compute B_T and determine which faces of B_T are fibered. This will allow us to use Proposition 3.4.2 to determine the Euler characteristics of the fibers. First we show:

Theorem 3.4.4. *Let B_T be the unit ball of the Thurston norm on $H^1(M;\mathbb{R})$. Then all top dimensional faces of B_T are fibered.*

Proof of Theorem 3.4.4: Stallings [Sta] gave an algebraic criterion for a class $\phi \in H^1(M;\mathbb{Z})$ to induce a fibration of M over S^1. Under the assumption that $\pi_1(M)$ is specified by a presentation with two-generators and one-relator presentation, Brown [Br] gave an algorithm for determining all $\phi \in H^1(M;\mathbb{Z})$ that satisfy the hypothesis of Stallings' criterion. We use Brown's algorithm to apply Stallings' criterion and determine all $\phi \in H^1(M;\mathbb{Z})$ that induce fibrations of M. For a nice exposition of the perspective on these topics that we use here, see sections 4 and 5 of [DT].

SnapPy [CDW] computes a presentation of $\pi_1(M)$ given by

$$< \alpha, \beta \mid \alpha^2\beta^3\alpha^2\beta^{-2}\alpha^{-3}\beta^{-2}\alpha^2\beta^3 > .$$

Since this presentation is 2-generator and 1-relator, Brown's criterion can be applied. Brown's criterion is stated in terms of a bounded path P in $\mathbb{R}^2$ consisting of horizontal and vertical segments that can be produced from a presentation of a 2-generator 1-relation group G. The path starts at the origin and is formed by reading the relator from left to right, drawing a horizontal unit segment for each appearance of α and a vertical unit segment for each appearance of β. We give a depiction of the path associated to the above presentation of $\pi_1(M)$ in Figure 3.4.

Brown's criterion is that $\phi = (c, d) \in H^1(M;\mathbb{Z})$ induces a fibration of M if and only if as e varies over elements of $\mathbb{R}$, the largest and smallest value of e for which the line L_e given by $cx + dy = e$ intersects P each give a line which intersects P exactly once. Here L_e is

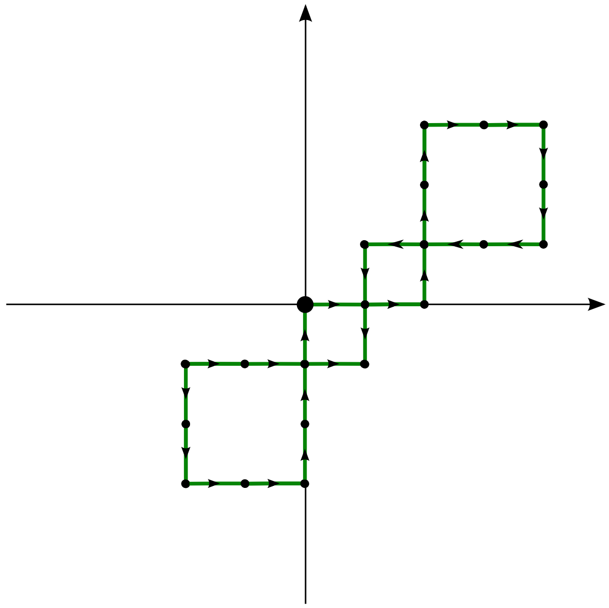

Figure 3.4: Path associated to presentation of $\pi_1(M)$. Defined only up to translation by a lattice point.

considered to intersect P multiple times if P crosses over itself where it meets L_e, but L_e is considered to intersect P only once if L_e is horizonal (resp. vertical) and L_e intersects a single horizontal (resp. vertical) segment. Applying Brown's criterion to Figure 4, it is clear that the only pairs $(c, d) \in H^1(M; \mathbb{Z})$ that do not induce fibrations of M are those lie on the three lines that go through the origin and have slope 0, -1 and ∞. By Theorem 3.4.3, there cannot be any top dimensional faces of B_T that are not fibered, completing the proof of Theorem 3.4.4. $\square$

We can use Theorem 3.4.4 to determine the precise shape of B_T.

Theorem 3.4.5. *The unit ball of the Thurston norm B_T is as pictured in Figure 3.5.*

Proof of Theorem 3.4.5: We use Theorem 3.4.4 together with properties of the Alexander norm on $H^1(M; \mathbb{R})$. The Alexander norm $|| * ||_A$ on $H^1(M; \mathbb{R})$ was introduced by McMullen in [McM]. McMullen showed that if N is a 3-manifold and $\phi \in H^1(N; \mathbb{Z})$ then $||\phi||_A \leq ||\phi||_T$ where $|| * ||_T$ is the Thurston norm, and that equality holds if $\dim(H^1(M; \mathbb{R})) > 1$ and ϕ

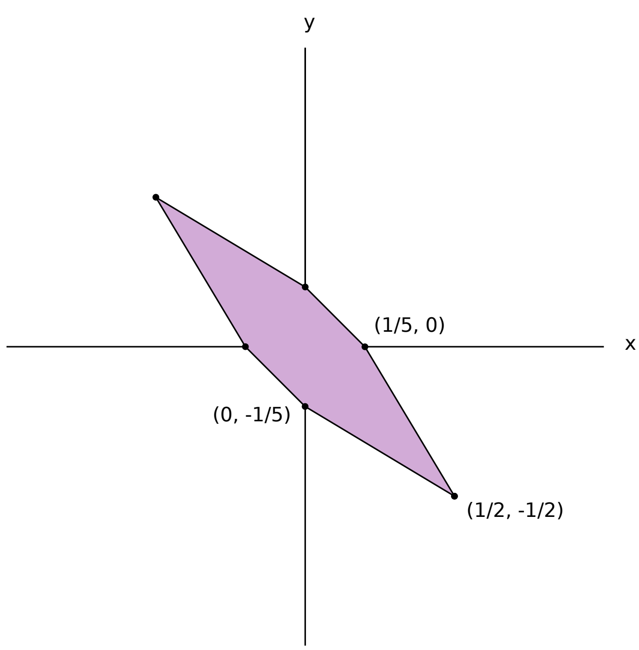

Figure 3.5: The unit ball B_T of the Thurston norm on $H^1(M;\mathbb{R})$

induces a fibration of N. Applying this result to M, $||\phi||_A = ||\phi||_T$ for each ϕ not lying on the lines through the origin with slope 0, -1 and ∞. Since McMullen [McM] and Thurston [Thu2] showed that their respective norms are linear on rays through the origin and continuous on $H^1(M;\mathbb{R})$, we must have $||\phi||_A = ||\phi||_T$ identically. So $B_T = B_A$ where B_A is the unit ball of the Alexander norm on $H^1(M;\mathbb{R})$.

The advantage of working with the Alexander norm is that it is easy to compute. Indeed, the unit norm ball on $H^1(M;\mathbb{R})$ is defined to be the dual of the Newton polytope of the Alexander polynomial of N. The Alexander polynomial of M is

$$a^5b^5 + a^5b^4 + a^4b^5 + a^4b^4 - a^3b^3 + a^2b^2 - ab - a - b - 1$$

Thus, its Newton polytope has vertices $(0,0)$, $(1,0)$, $(0,1)$, $(4,5)$, $(5,4)$ and $(5,5)$. The dual of this polytope has vertices given by $(0,1/5)$, $(1/5,0)$, $(-1/2,1/2)$, $(-1/5,0)$, $(0,-1/5)$, and $(1/2,-1/2)$ (Figure 3.5). This completes the proof of Theorem 3.4.5 . $\Box$

Using our computation of B_T, the fact that the Thurston norm is linear on rays, and Proposition 3.4.2, if ϕ induces a fibration of M then we can compute the Euler characteristic of the corresponding fiber. To completely determine the fiber we need to be able to compute the number of punctures of the fiber. To this end we prove:

Theorem 3.4.6. *If $\psi = (x,y)$ lies over a fibered face of B_T then ψ induces fibration of M of type $(-\chi, p)$, where $p = |x+y|$.*

Proof of Theorem 3.4.6: Since M is a one-cusped hyperbolic 3-manifold, removing a neighborhood of the cusp of M gives a 3-manifold M' with boundary $\partial M'$ consisting of a torus. Pick ϕ inducing a fibration of M. Pick generators μ and λ of $H_1(\partial M';\mathbb{Z})$ with intersection number 1 such that $i(\mu)$ is transverse to the fiber and $i(\lambda)$ is parallel to the fiber in the fibration of M induced by ϕ. Then $\phi(\mu) = p$ and $\phi(\lambda) = 0$, where p is the number of punctures of a fiber. In some sense this gives a formula for p, but this formula depends on a choice of generators of $H_1(\partial M';\mathbb{Z})$ which is special to ϕ. A formula for p that does not depend on generators of ϕ is given by

$$|\gcd(\phi(\mu),\phi(\lambda))| = p. \tag{3.4.1}$$

To see that (4.1) is true, first note that it is true if μ and λ are generators that depend on ϕ as above. Now, any choice of generators of $H_1(\partial M';\mathbb{Z}) \cong \mathbb{Z}^2$ differs from that of μ, λ by an invertible linear change of coordinates, that is, an element of $GL(2,\mathbb{Z})$. Since the gcd of two numbers is invariant under $\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}$, $\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}$, and $\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}$ and these matrices generate $GL(2,\mathbb{Z})$, equation (4.1) holds for any choice of generators of $H_1(\partial M;\mathbb{Z})$.

With its preferred choice of generators of $H_1(\partial M';\mathbb{Z})$, SnapPy [CDW] computes that their images in $H_1(M;\mathbb{Z})/(\text{torsion})$ to be $\mu = (-4,-4)$ and $\lambda = (-5,-5)$. Writing $\phi = (x,y)$, we have $\gcd(\phi(\mu),\phi(\lambda)) = \gcd(-4x-4y,-5x-5y) = x+y$. Thus Theorem 3.4.6 follows from (4.1). $\square$

Proof of Theorem 3.4.1: We fix $p > 0$ and determine those χ such that M is a surface bundle of type $(-\chi, p)$. By Theorem 3.4.6, such χ come from classes (x,y) with $x+y=\pm p$. Since the classes (x,y) and $(-x,-y)$ are the same up to orientation, there is no loss of generality in restricting ourselves to those classes lying on the line $x+y=p$. Since the Thurston norm is symmetric about the lines $y=x$ and $x=0$ and those classes lying on these lines do not induce fibrations of M, we can and do further restrict our attention to those classes (x,y) with $x>0$ and $y>x$. Of these classes, those on the line $y=0$ do not induce fibrations of M, but all others do. Recall that when a class induces a fibration, the Thurston norm computes its Euler characteristic. The formula for the Thurston norm of (x,y) depends on whether $y<0$ or $y>0$. Thus we divide our analysis into two cases.

Case 1: Suppose $y>0$. Then (x,y) lies over the face of B_T determined by the two vertices

$(1/5, 0)$ and $(0, 1/5)$, so in this case

$$||(x, y)||_T = 5x + 5y = x + 5(p - x) = 5p.$$

So every class (x, y) with $y > 0$ such that $y < x$ and $x + y = p$ gives a realization of M as a surface bundle of type $(5p, p)$. The pairs $(-\chi, p)$ such that there is a class (x, y) with $y > 0$ that induces a fibration of M of type $(-\chi, p)$ are those pairs $(5p, p)$ for $p \geq 2$.

Case 2: Suppose $y < 0$. Then (x, y) lies in the cone over the face of the Thurston norm ball determined by $(1/5, 0)$ and $(1/2, -1/2)$, so in this case

$$||(x, y)||_T = 5x + 3y = 5x + 3(p - x) = 2x + 3p.$$

In this case the condition $y < x$ is automatically satisfied since $x + y = p$. The condition $y < 0$ is equivalent to $p < x$ which is in turn equivalent to

$$5p < 2x + 3p = ||(x, y)||_T$$

Recall that for any pair $(-\chi, p)$ that comes from a surface, p and χ have the same parity. So for any pair $(-\chi, p)$ with $5p < -\chi$ we can find a class (x, y) with $y < 0$ that induces a fibration of M of type $(-\chi, p)$.

From our above analysis we see that M is a surface bundle of type $(-\chi, p)$ precisely when $5p \leq \chi$ and $(-\chi, p) \neq (5, 1)$ as claimed. $\square$

3.5 Empirical Data and Open Questions

Here we give a table of cusped census manifolds with trace fields having real places which are surface bundles of type $(-\chi, p)$ for various pairs $(-\chi, p)$. This table together with Theorem 3.1.3 (a) and (b) substantiates Observation 3.1.2. We compiled it using N. Dunfield's data [Dun] and Snap [CGHN]. The manifolds are labelled by their labels in the cusped census [CHW].

$(-\chi, p)$	Manifold
(2, 2)	v2640
(3, 1)	m036
(3, 3)	s493
(3, 5)	m043
(4, 2)	v2677
(4, 4)	s500

$(-\chi, p)$	Manifold
(4, 6)	s147
(5, 1)	m034
(5, 3)	v0163
(5, 5)	v0003
(5, 7)	m172
(6, 2)	v3212

$(-\chi, p)$	Manifold
(6, 4)	s500
(7, 1)	m078
(7, 3)	v3238
(7, 5)	v0022
(8, 2)	m297
(8, 4)	v3193

$(-\chi, p)$	Manifold
(9, 1)	m011
(9, 3)	v0170
(10, 2)	m200
(10, 4)	v1251
(11, 1)	m019
(11, 3)	v1721

$(-\chi, p)$	Manifold
(14, 2)	s156
(15, 1)	m070
(16, 2)	s550
(17, 1)	m044
(18, 2)	s133

$(-\chi, p)$	Manifold
(19, 1)	m055
(20, 2)	s457
(21, 1)	m064
(21, 3)	v1609
(22, 2)	v2603

While investigating the trace fields of the fibered manifolds in cusped census, we found that the trace fields attached to hyperbolic surface bundles of type $(-\chi, p)$ for $(-\chi, p) \neq (1, 1)$ do not appear to exhibit any regularities. We are thus led to ask

Question 3.5.1. *Suppose* $(-\chi, p) \neq (1,1)$. *Is every pair* (K, σ) *with* $\sigma(K) \not\subset \mathbb{R}$ *the trace field of some hyperbolic surface bundle of type* $(-\chi, p)$*?*

Empirically, it seems that the trace field of a small volume cusped hyperbolic surface bundle typically has about 1/3 as many real places as its degree over $\mathbb{Q}$. Since a complex place corresponds to a conjugate pair of nonreal roots of a polynomial, this observation suggests the possibility that a "random" place of a "random" cusped hyperbolic 3-manifold is real with probability 1/2. We therefore ask the following pair of questions:

Question 3.5.2. *Is there a natural notion of "random number field" such that the expected fraction of places of a random number field that are real is* $1/2$*?*

Question 3.5.3. *Assuming that a notion of "random number field" as in the preceding question is found, can one find a natural notion of "random cusped hyperbolic surface bundle" such that the trace fields of such manifolds are modeled well by this notion of "random number field" (with the restriction that such number fields have at least one complex place) and deduce that the fraction of places of a "random cusped hyperbolic surface bundle" that are real is* $1/2$*?*

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