Uniqueness of equilibrium on rings

Frédéric Meunier, Thomas Pradeau

UNIQUENESS OF EQUILIBRIUM ON RINGS

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Abstract. We consider congestion games on networks with nonatomic users and user-specific costs. We are interested in the uniqueness property defined by Milchtaich [Milchtaich, I. 2005. Topological conditions for uniqueness of equilibrium in networks. Math. Oper. Res. 30 225-244] as the uniqueness of equilibrium flows for all assignments of strictly increasing cost functions. He settled the case with two-terminal networks.

In the present work we characterize completely bidirectional rings for which the uniqueness property holds. The main result is that it holds precisely for nine networks and those obtained from them by elementary operations. For other bidirectional rings, we exhibit affine cost functions yielding to two distinct equilibrium flows. Related results are also proven.

1. Introduction

In many areas, different users share a common network to travel or to exchange informations or goods. Each user wishes to select a path connecting a certain origin to a certain destination. However, the selection of paths in the network by the users induces congestion on the arcs, leading to an increase of the costs. Taking into account the choices of the other users, each user looks for a path of minimum cost. We expect therefore to reach a Nash equilibrium: each user makes the best reply to the actions chosen by the other users.

This kind of games is studied since the 50's, with the seminal works by Wardrop [War52] or Beckman [BMW56]. When the users are assumed to be nonatomic – the effect of a single user is negligible – equilibrium is known to exist [Mil00]. Moreover, when the users are affected equally by the congestion on the arcs, the costs supported by the users are the same in all equilibria [AM81]. In the present paper, we are interested by the case when the users may be affected differently by the congestion. In such a case, examples are known for which these costs are not unique. Various conditions have been found that ensure nevertheless uniqueness. For instance, if the user's cost functions attached to the arcs are continuous, strictly increasing and identical up to additive constants, then we have uniqueness of the equilibrium flows, and thus of the equilibrium costs [AK01]. In 2005, continuing a work initiated by Milchtaich [Mil00] and Konishi [Kon04] for networks with parallel routes, Milchtaich [Mil05] found a topological characterization of two-terminal networks for which, given any assignment of stricly increasing and continuous cost functions, the flows are the same in all equilibria. Such networks are said to enjoy the *uniqueness property*.

The purpose of this paper is to find similar characterization for networks with more than two terminals. We are able to characterize completely the ring networks having the uniqueness property, whatever the number of terminals is. The main result is that it holds precisely for nine networks and those obtained from them by elementary operations. For other rings, we exhibit affine cost functions yielding to two distinct equilibrium flows. Related results are also proven. Following Milchtaich, we show that the uniqueness property for these networks coincides generically with the equivalence of all equilibria: if the users are split into a finite number of *classes* of users with the

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same features and if the cost functions are not linked by special relations, the contribution of each class to the flow on each arc is the same in all equilibria.

2. Preliminaries on graphs

An *undirected graph* is a pair $G = (V, E)$ where V is a finite set of *vertices* and E is a family of unordered pairs of vertices called *edges*. A *directed graph*, or *digraph* for short, is a pair $D = (V, A)$ where V is a finite set of *vertices* and A is a family of ordered pairs of vertices called *arcs*. A *mixed graph* is a graph having edges and arcs. More formally, it is a triple $M = (V, E, A)$ where V is a finite set of vertices, E is a family of unordered pairs of vertices (edges) and A is a family of ordered pairs of vertices (arcs). Given an undirected graph $G = (V, E)$, we define the *directed version* of G as the digraph $D = (V, A)$ obtained by replacing each (undirected) edge in E by two (directed) arcs, one in each direction. An arc of G is then understood as an arc of its directed version. In these graphs, *loops* – edge or arc having identical endpoints – are not allowed, but pairs of vertices occuring more than once – *parallel edges* or *parallel arcs* – are allowed.

A *walk* in a directed graph D is a sequence

$$P = (v_0, a_1, v_1, \dots, a_k, v_k)$$

where $k \geq 0$, $v_0, v_1, \dots, v_k \in V$, $a_1, \dots, a_k \in A$ and $a_i = (v_{i-1}, v_i)$ for $i = 1, \dots, k$. If all v_i are distinct, the walk is called a *path*. If no confusion may arise, we identify sometimes a path P with the set of its vertices or with the set of its arcs, allowing to use the notation $v \in P$ (resp. $a \in P$) if a vertex v (resp. an arc a) occurs in P.

An undirected graph $G' = (V', E')$ is a *subgraph* of an undirected graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. An undirected graph G' is a *minor* of an undirected graph G if G' is obtained by contracting edges (possibly none) of a subgraph of G. *Contracting* an edge uv means deleting it and identifying both endpoints u and v. Two undirected graphs are *homeomorphic* if they arise from the same undirected graph by subdivision of edges, where a *subdivision* of an edge uv consists in introducing a new vertex w and in replacing the edge uv by two new edges uw and wv.

The same notions hold for directed graphs and for mixed graphs.

Finally, let $G = (V, E)$ be an undirected graph, and $H = (T, L)$ be a directed graph with $T \subseteq V$, then $G + H$ denotes the mixed graph (V, E, L).

3. Model

Similarly, as in the multiflow theory (see for instance [Sch03] or [KV00]), we are given a *supply graph* $G = (V, E)$ and a *demand digraph* $H = (T, L)$ with $T \subseteq V$. The graph G models the (transportation) *network*. The arcs of H model the origin-destination pairs, also called in the sequel the *OD-pairs*. H is therefore assumed to be simple. A *route* is an (o, d)-path of the directed version of G with $(o, d) \in L$ and is called an (o, d)-route. The set of all routes (resp. (o, d)-routes) is denoted by $\mathcal{R}$ (resp. $\mathcal{R}_{(o,d)}$).

The population of *users* is modelled as a bounded real interval I endowed with the Lebesgue measure λ, the *population measure*. I is partitioned into measurable subsets $I_{(o,d)}$ with $(o, d) \in L$, modelling the users wishing to select an (o, d)-route.

For a given pair of supply graph and demand digraph, we define the *strategy profile* by a measurable mapping $\sigma : I \to \mathcal{R}$ such that $\sigma(i) \in \mathcal{R}_{(o,d)}$ for all $i \in I_{(o,d)}$ and $(o, d) \in L$. For each arc $a \in A$ of the directed version of G, the measure of the set of all users i such that a is in $\sigma(i)$ is the *flow* on a in σ and is denoted f_a:

$$f_a = \lambda\{i \in I : a \in \sigma(i)\}.$$

The cost of each arc $a \in A$ for each user $i \in I$ is given by a nonnegative, continuous and strictly increasing *cost function* $c_a^i : \mathbb{R}_+ \to \mathbb{R}_+$, such that $i \mapsto c_a^i(x)$ is measurable for any $a \in A$ and

$x \in \mathbb{R}_+$. When the flow on a is f_a, the cost for user i of traversing a is $c_a^i(f_a)$. For user i, the cost of a route r is defined as the sum of the costs of the arcs contained in r. A *class* is a set of users having the same cost functions on all arcs.

The game we are interested in is defined by the supply graph G, the demand digraph H, the population user set I and the cost functions c_a^i for $a \in A$ and $i \in I$. A strategy profile is a (pure) Nash equilibrium if each route is only chosen by users for whom it is a minimal-cost route. In other words, a strategy profile is a Nash equilibrium if for each pair $(o, d) \in L$ and each user $i \in I_{(o,d)}$ we have

$$\sum_{a \in \sigma(i)} c_a^i(f_a) = \min_{r \in \mathcal{R}_{(o,d)}} \sum_{a \in r} c_a^i(f_a). \tag{1}$$

Under the conditions stated above on the cost functions, a Nash equilibrium is always known to exist. It can be proven similarly as Theorem 3.1 in [Mil00], or, as noted by Milchtaich [Mil05], it can be deduced from more general results (Theorem 1 of [Sch70] or Theorems 1 and 2 of [Rat70]). However, such an equilibrium is not necessarily unique, and even the equilibrium flows are not necessarily unique.

4. Results

Milchtaich [Mil05] raised the question whether it is possible to characterize networks having the *uniqueness property*, i.e. networks for which flows at equilibrium are unique. A pair (G, H) defined as in Section 3 is said to have the *uniqueness property* if for any partition of I into measurable subsets $I_{(o,d)}$ with $(o, d) \in L$, and for any assignment of (strictly increasing) cost functions, the flow on each arc is the same in all equilibria.

Milchtaich found a positive answer for the two-terminal networks, i.e. when $|L| = 1$. More precisely, he gave a (polynomial) characterization of a family of two-terminal undirected graphs such that, for the directed versions of this family and for any assignment of (strictly increasing) cost functions, the flow on each arc is the same in all equilibria. For two-terminal undirected graphs outside this family, he gave explicit cost functions for which equilibria with different flows on some arcs exist.

The objective of this paper is to address the uniqueness property for some networks having more than two terminals, especially for ring networks. In a ring network, each user has exactly two possible strategies. See Figure 1 for an illustration of this kind of supply graph G, demand digraph H and mixed graph $G + H$. We prove the following theorem.

Theorem 1. *Assume that the supply graph G is a cycle. Then, for any demand digraph H, the pair (G, H) has the uniqueness property if and only if each arc of G is contained in at most 2 routes.*

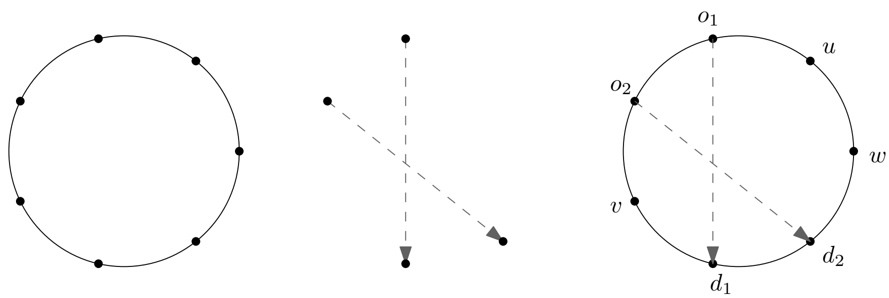
Whether such a pair (G, H) of supply graph and demand digraph is such that each arc in contained in at most 2 routes is obviously polynomially checkable, since we can test each arc one after the other. We will show that it can actually be tested by making only one round trip, in any direction.

Furthermore, we have the following corollaries.

Corollary 1. *If the supply graph is a cycle and if there are at most two OD-pairs, i.e. $|L| \leq 2$, then the uniqueness property holds.*

Corollary 2. *If the supply graph is a cycle and if the uniqueness property holds, then the number of OD-pairs is at most 4, i.e. $|L| \leq 4$.*

We are also able to describe such pairs (G, H) having the uniqueness property as the pairs such that $G + H$ is homeomorphic to a minor of one of nine mixed graphs, see Figures 2–5. Except for



undirected graph G digraph H mixed graph $G + H$

FIGURE 1. Example of a supply graph G, a demand digraph H, and the mixed graph $G + H$.

the smallest one, none of the uniqueness property of these graphs can be derived from the result by Milchtaich, even by adding fictitious vertices as suggested p.235 of his article [Mil05].

The organization of the remaining goes as follows. Section 5 is devoted to the proof of Theorem 1. Section 6 explores the nature of ring networks having each of their arcs in at most 2 routes. It provides the whole list of such pairs (G, H). In this section, we also prove Corollaries 1 and 2 and explain how to compute the maximal number of routes containing an arc with only one round trip. Section 7 contains complementary results. For instance, in Section 7.1, it is proven that when the uniqueness property holds for such pairs, then we have a stronger result, namely that the contribution of each class to the flows are generically unique (equivalence of the equilibria). Section 7.3 focus on a *strong uniqueness property* that may hold for general graphs independently of the demand digraph, i.e. of the OD-pairs.

5. PROOF OF THE MAIN RESULT

5.1. **Proof strategy and some preliminary results.** The proof works in two steps. The first step, Section 5.2, consists in proving Proposition 1 stating that when each arc is contained in at most 2 routes, then the uniqueness property holds. The second step, Section 5.3, consists in exhibiting cost functions for which flows at equilibrium are non-unique for any pair (G, H) with an arc in at least 3 routes.

From now on, we assume that the cycle G is embedded in the plane. It allows to use an orientation for G. Each route is now either clockwise, or counterclockwise. The same holds for arcs: we have counterclockwise arcs and clockwise arcs.

Claim 1. *For any $(o, d) \in L$, if a and a^{-1} are the two arcs stemming from a given edge $e \in E$, then exactly one of a and a^{-1} is in an (o, d)-route.*

Proof. Indeed, given an $(o, d) \in L$ and an edge $e \in E$, exactly one of the counterclockwise and clockwise (o, d)-routes contains e. $\square$

The following notations will be useful for the proof of Theorem 1. For each user i, we define r_i^+ (resp. r_i^-) to be the unique counterclockwise (resp. clockwise) route connecting the origin of i to its destination. For any subset $J \subseteq L$, consider all counterclockwise (resp. clockwise) (o, d)-routes with $(o, d) \in J$. Then A_J^+ (resp. A_J^-) is the set of counterclockwise (resp. clockwise) arcs contained in all of these routes and no other. We define moreover $A_J = A_J^+ \cup A_J^-$. The sets A_J form a partition of the set A of arcs of G.

On the example of Figure 1, we have for instance

$$A^+_{\{(o_1,d_1),(o_2,d_2)\}} = \{(o_2, v), (v, d_1)\}$$

$$A^-_{\{(o_2,d_2)\}} = \{(o_2, o_1)\}$$
$$A_\emptyset = \{(d_1, v), (v, o_2), (d_2, w), (w, u), (u, o_1)\}.$$

The sets A^ε_J enjoy three useful properties.

Claim 2. *For any $\varepsilon \in \{-,+\}$ and any $(o,d) \in L$, there is at least one $J \subseteq L$ such that A^ε_J is nonempty and $(o,d) \in J$.*

Proof. Indeed, there is at least one arc of G on the (o,d)-route oriented according to ε. □

Claim 3. *For any $\varepsilon \in \{-,+\}$ and any $J \subseteq L$, we have*

$$A^\varepsilon_J \neq \emptyset \Longleftrightarrow A^{-\varepsilon}_{\overline{J}} \neq \emptyset.$$

Proof. It is a consequence of Claim 1: if $a \in A^\varepsilon_J$, then $a^{-1} \in A^{-\varepsilon}_{\overline{J}}$. □

Claim 4. *For any distinct ℓ and ℓ' in L, there is at least one J such that $|\{\ell, \ell'\} \cap J| = 1$ and $A_J \neq \emptyset$.*

Proof. Indeed, let $\ell = (o,d)$ and $\ell' = (o', d')$ be two distinct OD-pairs of H. Since H is simple, $o \neq o'$ or $d \neq d'$. It means that there is at least one arc of G which is in exactly one of the four (o,d)- and (o',d')-routes. □

5.2. **If each arc of G is contained in at most 2 routes, the uniqueness property holds.** For a strategy profile σ and a subset $J \subseteq L$, we define f^+_J and f^-_J to be:

$$f^+_J = \int_{i \in \bigcup_{\ell \in J} I_\ell} 1_{\{\sigma(i) = r^+_i\}} d\lambda \ \text{ and } \ f^-_J = \int_{i \in \bigcup_{\ell \in J} I_\ell} 1_{\{\sigma(i) = r^-_i\}} d\lambda.$$

The quantity f^+_J (resp. f^-_J) is thus the number of users i in a I_ℓ with $\ell \in J$ choosing a counter-clockwise (resp. clockwise) route. Note that the quantity $f^+_J + f^-_J = \sum_{\ell \in J} \lambda(I_\ell)$ does not depend on the strategy σ.

Assume that we have two distinct equilibria σ and $\hat{\sigma}$. The flows induced by $\hat{\sigma}$ are denoted with a hat: $\hat{f}$. We define for any subset $J \subseteq L$:

$$\Delta_J = f^+_J - \hat{f}^+_J = \hat{f}^-_J - f^-_J. \tag{2}$$

By a slight abuse of notation, we let $\Delta_\ell := \Delta_{\{\ell\}}$ for $\ell \in L$. Note that for every subset J, $J' \subseteq L$ such that $J \cap J' = \emptyset$, we have $\Delta_J + \Delta_{J'} = \Delta_{J \cup J'}$.

We have for each user i:

$$\sum_{J \subseteq L} \delta(i) \left(\sum_{a \in A^+_J \cap r^+_i} (c^i_a(f^+_J) - c^i_a(\hat{f}^+_J)) - \sum_{a \in A^-_J \cap r^-_i} (c^i_a(f^-_J) - c^i_a(\hat{f}^-_J)) \right) \leq 0. \tag{3}$$

where $\delta(i) = 1_{\{\sigma(i) = r^+_i\}} - 1_{\{\hat{\sigma}(i) = r^+_i\}}$.

Indeed, as σ is an equilibrium we have

$$\sum_{a \in A} c^i_a(f_a) \left(1_{\{a \in \sigma(i)\}} - 1_{\{a \in \hat{\sigma}(i)\}} \right) \leq 0. \tag{4}$$

For $a \in A^+_J$, we have $f_a = f^+_J$ and $1_{\{a \in \sigma(i)\}} = 1_{\{\sigma(i) = r^+_i\}} 1_{\{a \in r^+_i\}}$, and the same holds for $a \in A^-_J$. By decomposing the sum (4), we obtain that

$$\sum_{J \subseteq L} \left(\sum_{a \in A^+_J \cap r^+_i} c^i_a(f^+_J) \delta(i) - \sum_{a \in A^-_J \cap r^-_i} c^i_a(f^-_J) \delta(i) \right) \leq 0.$$

We can write a similar equation for the equilibrium $\hat{\sigma}$. Equation (3) is then obtained by summing them.

As a consequence we get the following lemma.

Lemma 1. *Let $\ell \in L$ and $i \in I_\ell$ such that $\delta(i) \neq 0$. Then exactly one of the following alternatives holds.*

- *There is a $J \subseteq L$ with $\ell \in J$, $A_J \neq \emptyset$ and $\delta(i)\Delta_J < 0$.*
- *For all $J \subseteq L$ with $\ell \in J$ and $A_J \neq \emptyset$, we have $\Delta_J = 0$.*

Proof. In Equation (3), if all terms of the sum are equal to 0, the second point of the statement of the lemma holds: according to Equation (2) and using the fact that the maps c_a^i are strictly increasing, $\sum_{a \in A_J^+ \cap r_i^+}(c_a^i(f_J^+) - c_a^i(\hat{f}_J^+))$ and $-\sum_{a \in A_J^- \cap r_i^-}(c_a^i(f_J^-) - c_a^i(\hat{f}_J^-)$ have same sign and therefore are both equal to 0.

If at least one term of the sum is < 0, we get the first point in the statement of the lemma. $\square$

With the help of this lemma, we get one direction of Theorem 1.

Proposition 1. *If each arc of G is contained in at most 2 routes, the uniqueness property holds.*

Proof. Note that the assumption of the proposition assures that $A_J = \emptyset$ if $|J| \geq 3$.

Assume for a contradiction that there is a J_0 such that $\Delta_{J_0} \neq 0$ and $A_{J_0} \neq \emptyset$. Then there is a $\ell_0 \in J_0$ such that $\Delta_{\ell_0} \neq 0$. At least one $i_0 \in I_{\ell_0}$ is such that $\delta(i_0)\Delta_{\ell_0} > 0$.

Suppose that the first case of Lemma 1 occurs. Then, there exists $\ell_1 \in L$, $\ell_1 \neq \ell_0$ with $A_{\{\ell_0,\ell_1\}} \neq \emptyset$ and $\delta(i_0)\Delta_{\{\ell_0,\ell_1\}} < 0$. Then, $\delta(i_0)(\Delta_{\{\ell_0,\ell_1\}}) = \delta(i_0)(\Delta_{\ell_0} + \Delta_{\ell_1}) < 0$, which implies that $|\Delta_{\ell_0}| < |\Delta_{\ell_1}|$. It follows that $\Delta_{\ell_1} \neq 0$, and taking $i_1 \in I_{\ell_1}$ with $\delta(i_1)\Delta_{\ell_1} > 0$, only the first case of Lemma 1 can occur for $i = i_1$ and $\ell = \ell_1$. Indeed, the second case would imply that $\Delta_{\{\ell_0,\ell_1\}} = 0$ since $A_{\{\ell_0,\ell_1\}} \neq \emptyset$. Repeating the same argument, we build an infinite sequence $(\ell_0, \ell_1, \ldots)$ of elements of L such that for each $k \geq 0$, $A_{\{\ell_k,\ell_{k+1}\}} \neq \emptyset$ and $|\Delta_{\ell_k}| < |\Delta_{\ell_{k+1}}|$. This last condition implies that the ℓ_k are distincts, which is impossible since $|L|$ is finite.

Thus, the second case of Lemma 1 occurs for ℓ_0, and then for any $\ell' \in L$, $\ell' \neq \ell_0$, if $A_{\{\ell_0,\ell'\}} \neq \emptyset$, then $\Delta_{\{\ell_0,\ell'\}} = 0$. Since $\Delta_{\ell_0} \neq 0$, the lemma also gives that $A_{\{\ell_0\}} = \emptyset$, and thus $J_0 \neq \{\ell_0\}$. We get that $J_0 = \{\ell_0, \ell'\}$ for some $\ell' \in L$, and hence $\Delta_{J_0} = 0$, which is in contradiction with the starting assumption. On any arc, we have a total flow that remains the same when changing from σ to $\hat{\sigma}$. $\square$

5.3. **If an arc of G is contained in at least 3 routes, a counterexample exists.** We give an explicit construction of multiple equilibria when an arc is contained in at least 3 routes.

5.3.1. *If $|L| = 3$.* In order to ease the notation, we use 1, 2 and 3 to denote the three OD-pairs of H. We denote accordingly by I_1, I_2 and I_3 the three sets of users associated to each of these OD-pairs.

We can assume without loss of generality that $A^+_{\{1,2,3\}} \neq \emptyset$, $A_{\{1,2\}} \neq \emptyset$ and $A_{\{1,3\}} \neq \emptyset$. The first assumption can be done since there is an arc in three routes. For the other ones: with the help of Claim 4, and if necessary of Claim 3, we get that there is at least a J of cardinality 2 such that $A_J \neq \emptyset$. Again, using Claim 4, this time with the two elements of J, and if necessary Claim 3, we get another J' of cardinality 2 such that $A_{J'} \neq \emptyset$.

We define three classes of users. Each of these classes will be attached to one of the OD-pairs. For a class $k \in \{1, 2, 3\}$, we define also the following cost functions $c_J^{k,\varepsilon}(x)$ for each arc according to the set A_J^ε to which it belongs, with $J \subseteq \{1, 2, 3\}$ and $\varepsilon \in \{-, +\}$. If the set does not exist, the definition of $c_J^{k,\varepsilon}(x)$ is simply discarded.

Class 1: We define this class to be the users of the set I_1. We set $\lambda(I_1) = 1.5$, and choose $J_1 \subseteq \{1,2,3\}$ with $1 \in J_1$ such that $A^-_{J_1} \neq \emptyset$ (with the help of Claim 2).

$$\begin{cases} c^{1,+}_{\{1,2,3\}}(x) = \dfrac{24x+7}{|A^+_{\{1,2,3\}}|} \\ c^{1,+}_J(x) = \dfrac{x}{|A^+_J|} \quad \text{for any } J \neq \{1,2,3\} \text{ with } 1 \in J \\ c^{1,-}_{J_1}(x) = \dfrac{x+48}{|A^-_{J_1}|} \\ c^{1,-}_J(x) = \dfrac{x}{|A^-_J|} \quad \text{for any } J \neq J_1 \text{ with } 1 \in J. \end{cases}$$

Class 2: We define this class to be the users of the set I_2. We set $\lambda(I_2) = 1$. We have assumed that $A_{\{1,2\}} \neq \emptyset$. We distinguish hereafter the cases $A^+_{\{1,2\}} \neq \emptyset$ and $A^-_{\{1,2\}} \neq \emptyset$ (which may hold simultaneously, in which case we make an arbitrary choice).

If $A^+_{\{1,2\}} \neq \emptyset$: We choose $J_2 \subseteq \{1,2,3\}$ with $2 \in J_2$ such that $A^-_{J_2} \neq \emptyset$ (with the help of Claim 2).

$$\begin{cases} c^{2,+}_{\{1,2\}}(x) = \dfrac{25x}{|A^+_{\{1,2\}}|} \\ c^{2,+}_J(x) = \dfrac{x}{|A^+_J|} \quad \text{for any } J \neq \{1,2\} \text{ with } 2 \in J \\ c^{2,-}_{J_2}(x) = \dfrac{x+31}{|A^-_{J_2}|} \\ c^{2,-}_J(x) = \dfrac{x}{|A^-_J|} \quad \text{for any } J \neq J_2 \text{ with } 2 \in J. \end{cases}$$

If $A^-_{\{1,2\}} \neq \emptyset$:

$$\begin{cases} c^{2,+}_{\{1,2,3\}}(x) = \dfrac{x+26}{|A^+_{\{1,2,3\}}|} \\ c^{2,+}_J(x) = \dfrac{x}{|A^+_J|} \quad \text{for any } J \neq \{1,2,3\} \text{ with } 2 \in J \\ c^{2,-}_{\{1,2\}}(x) = \dfrac{22x}{|A^-_{\{1,2\}}|} \\ c^{2,-}_J(x) = \dfrac{x}{|A^-_J|} \quad \text{for any } J \neq \{1,2\} \text{ with } 2 \in J. \end{cases}$$

Class 3: We define this class to be the users of the set I_3. We set $\lambda(I_3) = 1$. We have assumed that $A_{\{1,3\}} \neq \emptyset$. We distinguish hereafter the cases $A^+_{\{1,3\}} \neq \emptyset$ and $A^-_{\{1,3\}} \neq \emptyset$ (which may hold simultaneously, in which case we make an arbitrary choice).

If $A^+_{\{1,3\}} \neq \emptyset$: We choose $J_3 \subseteq \{1,2,3\}$ with $3 \in J_3$ such that $A^-_{J_3} \neq \emptyset$ (with the help of Claim 2).

$$\begin{cases} c^{3,+}_{\{1,3\}}(x) = \dfrac{25x}{|A^+_{\{1,3\}}|} \\ c^{3,+}_J(x) = \dfrac{x}{|A^+_J|} \quad \text{for any } J \neq \{1,3\} \text{ with } 3 \in J \\ c^{3,-}_{J_3}(x) = \dfrac{x+31}{|A^-_{J_3}|} \\ c^{3,-}_J(x) = \dfrac{x}{|A^-_J|} \quad \text{for any } J \neq J_3 \text{ with } 3 \in J. \end{cases}$$

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We define now two strategy profiles σ and $\hat{\sigma}$, inducing distinct flows on some arcs and we check that each of them is an equilibrium.

σ**:** For all $i \in I_1$, we set $\sigma(i) = r^+_i$ and for all $i \in I_2 \cup I_3$, we set $\sigma(i) = r^-_i$.

$\hat{\sigma}$**:** For all $i \in I_1$, we set $\hat{\sigma}(i) = r^-_i$ and for all $i \in I_2 \cup I_3$, we set $\hat{\sigma}(i) = r^+_i$.

For a class $k \in \{1,2,3\}$, we denote with a slight abuse of notation the common counterclockwise (resp. clockwise) route of the class k users by r^+_k (resp. r^-_k).

Then for each strategy, the flows are the following:

For σ:

J	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
f^+_J	1.5	0	0	1.5	1.5	0	1.5
f^-_J	0	1	1	1	1	2	2

For $\hat{\sigma}$:

J	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\hat{f}^+_J$	0	1	1	1	1	2	2
$\hat{f}^-_J$	1.5	0	0	1.5	1.5	0	1.5

We check now that σ and $\hat{\sigma}$ are equilibria, by computing the cost of each of the two possible routes for each class. Since some A^ε_J may be empty, we get intervals, and those known to be nonempty are used in the computation.

Class 1: We put in the Tables 1 and 2, the costs experienced by the class 1 users on the various arcs of G for each of σ and $\hat{\sigma}$. For a given $J \subseteq \{1,2,3\}$ with $1 \in J$, we indicate the cost experienced by any class 1 user on the whole collection of arcs in A^ε_J, with $\varepsilon \in \{-,+\}$ according to the direction. For instance in σ, consider $J = \{1,2,3\}$, then $f^+_J = 1.5$, and the cost of all arcs together in A^+_J is $|A^+_J| c^{1,+}_J(1.5) = 43$.

	$\varepsilon = +$		$\varepsilon = -$	
J with $1 \in J$	{1,2,3}	other	J_1	other
f_J^ε	1.5	1.5	0 or 1 or 2	0 or 1 or 2
Cost on A_J^ε	43	1.5	48 or 49 or 50	0 or 1 or 2

TABLE 1. σ: flows and costs on the arcs of G for a class 1 user

Using the fact that $A^+_{\{1,2,3\}} \neq \emptyset$, the total cost of r_1^+ in σ for a class 1 user is equal to

$$43 + 1.5 \times (\text{number of } J \neq \{1,2,3\} \text{ such that } A_J^+ \neq \emptyset \text{ and } 1 \in J).$$

Since there are at most 3 sets $J \neq \{1,2,3\}$ such that $A_J^+ \neq \emptyset$, we get that the total cost of r_1^+ lies in $[43; 47.5]$. Similarly, we get that the total cost of r_1^- for a class 1 user lies in $[48; 54]$. Therefore the users of class 1 are not incitated to change their choice in σ.

	$\varepsilon = +$		$\varepsilon = -$	
J with $1 \in J$	{1,2,3}	other	J_1	other
$\hat{f}_J^\varepsilon$	2	0 or 1	1.5	1.5
Cost on A_J^ε	55	0 or 1	49.5	1.5

TABLE 2. $\hat{\sigma}$: flows and costs on the arcs of G for a class 1 user

In $\hat{\sigma}$, the total cost of r_1^+ for a class 1 user lies in $[55; 58]$ and the total cost of r_1^- for a class 1 user lies in $[49.5; 54]$. Therefore the users of class 1 are not incitated to change their choice in $\hat{\sigma}$.

Class 2: **If $A^+_{\{1,2\}} \neq \emptyset$:** We put in the Tables 3 and 4, the costs experienced by the class 2 users on the various arcs of G for each of σ and $\hat{\sigma}$.

	$\varepsilon = +$		$\varepsilon = -$	
J with $2 \in J$	{1,2}	other	J_2	other
f_J^ε	1.5	0 or 1.5	1 or 2	1 or 2
Cost on A_J^ε	37.5	1.5	32 or 33	1 or 2

TABLE 3. σ: flows and costs on the arcs of G for a class 2 user

In σ, the total cost of r_2^+ for a class 2 user is precisely 39 (we use the fact that $A^+_{\{1,2,3\}} \neq \emptyset$) and the total cost of r_2^- lies in $[32; 38]$. The users of class 2 are not incitated to change their choice in σ.

In $\hat{\sigma}$, the total cost of r_2^+ for a class 2 user lies in $[27; 30]$ and the total cost of r_2^- lies in $[31; 35.5]$. The users of class 2 are not incitated to change their choice in $\hat{\sigma}$.

	$\varepsilon = +$		$\varepsilon = -$	
J with $2 \in J$	{1,2}	other	J_2	other
$\hat{f}_J^\varepsilon$	1	1 or 2	0 or 1.5	0 or 1.5
Cost on A_J^ε	25	1 or 2	31 or 32.5	0 or 1.5

TABLE 4. $\hat{\sigma}$: flows and costs on the arcs of G for a class 2 user

If $A^-_{\{1,2\}} \neq \emptyset$: We put in the Tables 5 and 6, the costs experienced by the class 2 users on the various arcs of G for each of σ and $\hat{\sigma}$.
In σ, the total cost of r_2^+ for a class 2 user lies in $[27.5; 29]$ and the total cost of r_2^- lies in $[22; 27]$. The users of class 2 are not incitated to change their choice in σ.

	$\varepsilon = +$		$\varepsilon = -$	
J with $2 \in J$	{1,2,3}	other	$\{1, 2\}$	other
f_J^ε	1.5	0 or 1.5	1	1 or 2
Cost on A_J^ε	27.5	0 or 1.5	22	1 or 2

TABLE 5. σ: flows and costs on the arcs of G for a class 2 user

	$\varepsilon = +$		$\varepsilon = -$	
J with $2 \in J$	{1,2,3}	other	$\{1, 2\}$	other
$\hat{f}_J^\varepsilon$	2	1 or 2	1.5	0 or 1.5
Cost on A_J^ε	28	1 or 2	33	0 or 1.5

TABLE 6. $\hat{\sigma}$: flows and costs on the arcs of G for a class 2 user

In $\hat{\sigma}$, the total cost of r_2^+ for a class 2 user lies in $[28; 32]$ and the total cost of r_2^- lies in $[33; 34.5]$. The users of class 2 are not incitated to change their choice in $\hat{\sigma}$.

Class 3: The symmetry of the cost functions for classes 2 and 3 gives the same tables for class 3 as for class 2, by substituting $\{1, 3\}$ to $\{1, 2\}$. Therefore, we get the same conclusions: neither in σ, nor in $\hat{\sigma}$, the class 3 users are incitated to change their choice.

We have proven that if there exists an arc in at least 3 routes and if $|L| = 3$, then the uniqueness property does not hold. It remains to check the case when $|L| > 3$.

Remark. A classical question when there are several equilibria is whether one of them dominates the others. An equilibrium is said to *dominate* another one if it is preferable for all users. In this construction, no equilibrium dominates the other, except when $A^+_{\{1,2,3\}} \neq \emptyset$, $A^-_{\{1,2\}} \neq \emptyset$ and $A^-_{\{1,3\}} \neq \emptyset$.

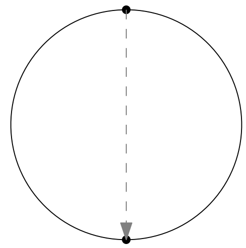

Figure 2. All rings with $|L| = 1$ (i.e. 1 OD-pair) having the uniqueness property are homeomorphic to this graph

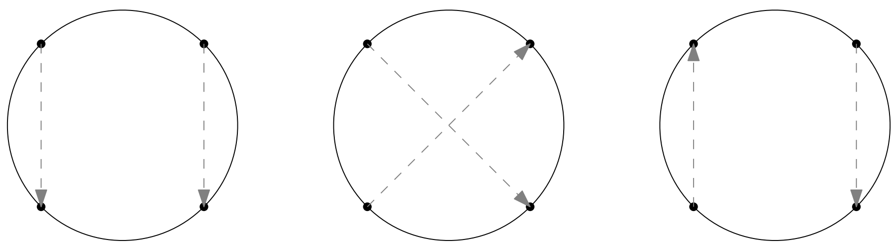

Figure 3. All rings with $|L| = 2$ (i.e. 2 OD-pairs) having the uniqueness property are homeomorphic to one or to minors of these graphs

5.3.2. *If* $|L| > 3$. Denote 1, 2, and 3 the three OD-pairs of H giving the three routes containing the same arc of G. For these three arcs of H, we make the same construction as above, in the case $|L| = 3$. For the other arcs of H, i.e. the other OD-pairs, we use a fourth class, whose costs are very small on all counterclockwise arcs of G and very large on all clockwise arcs of G, and whose measure is a small positive quantity δ. Each user of this class chooses always a counterclockwise route, whatever the other users do. For δ small enough, the users of this class have no impact on the choices of the users of the classes 1, 2, and 3, as the difference of cost between the routes is always bounded below by 0.5.

6. Each arc of G in at most two routes

In this subsection, we provide a further combinatorial analysis of the characterization of the uniqueness property stated in Theorem 1.

6.1. **Corollaries 1 and 2.** We first note that Corollary 1 is straightforward, and that Corollary 2 is a direct consequence of Claim 1: if $|L| \geq 5$, then there is necessarily an arc of G in three routes.

6.2. **How to compute in one round trip the maximal number of routes containing an arc of G.** In an arc (u, v), vertex u is called the *tail* and vertex v is called the *head*. We select any tail of an arc in H. We make a round trip in an arbitrary direction, while maintaining in a list all tails of arcs in H whose heads have not yet be encountered and while maintaining the minimum and maximum sizes m and M reached by the list during the round trip. To take into account the arcs of H whose tails are encoutered after their heads during the trip we add 1 to the current m and M when such a tail is encountered.

At the end, $\max(|L| - m, M)$ is the maximal number of routes containing an arc of G (Claim 1).

6.3. **Explicit description of the networks having the uniqueness property when the supply graph is a cycle.**

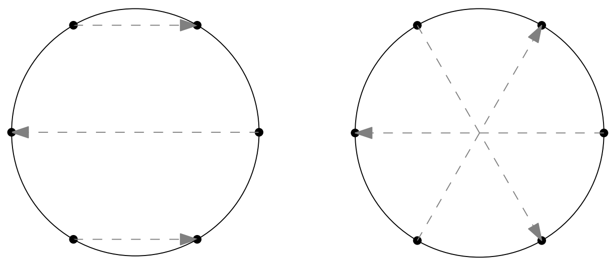

Figure 4. All rings with $|L| = 3$ (i.e. 3 OD-pairs) having the uniqueness property are homeomorphic to one or to minors of these graphs

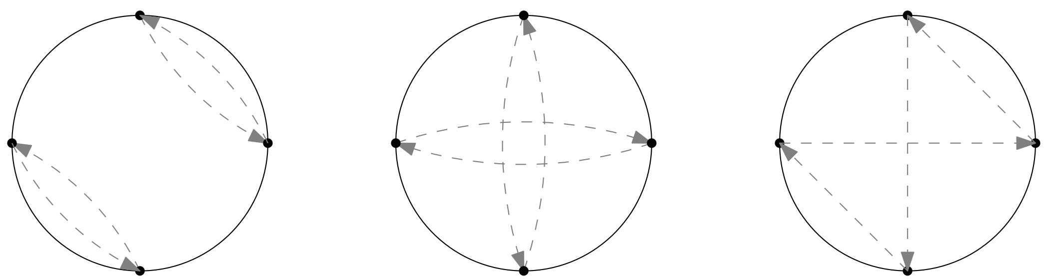

Figure 5. All rings with $|L| = 4$ (i.e. 4 OD-pairs) having the uniqueness property are homeomorphic to one or to minors of these graphs

Proposition 2. *Let the supply graph G be a cycle. Then, for any demand digraph H, the pair (G, H) is such that each arc of G is in at most 2 routes if and only if the mixed graph $G + H$ is homeomorphic to a minor of one of the nine mixed graphs of Figures 2–5.*

Combined with Theorem 1, this proposition allows to describe explicitely all pairs (G, H) having the uniqueness property, when G is a cycle.

Proof of Proposition 2. We can assume that $V = T$. As we have already noted, we can also assume that $|L| \in \{1, 2, 3, 4\}$.

If $|L| \in \{1, 2\}$, there is nothing to prove.

If $|L| = 3$, we can first assume that L contains two disjoint arcs that are crossing in the plane embedding. By trying all possibilities for the third arc, we get that the only possible configuration is the right one on Figure 4 and the ones obtained from it by edge contraction. Second, we assume that there are no "crossing" arcs. The three heads of the arcs can not be consecutive on the cycle otherwise we would have an arc of G in three routes. Again by enumerating all possibilities, we get that the only possible configuration is the left one on Figure 4 and the ones obtained from it by edge contraction.

If $|L| = 4$, Claim 1 shows that each arc of G belongs to exactly two routes. It implies that, in H, the indegree of any $\ell \in L$ is at least 1 and the outdegree as well. There are therefore circuits in H. It is straigtworward to check that it is impossible to have a length 3 circuit. It remains to enumerate the possible cases for length 2 and length 4 circuits to get that the only possible configurations are the ones of Figure 5 and the common one obtained from them by edge contraction. $\square$

7. Discussion

7.1. **Equivalence of equilibria.** Let us assume that we have a finite set K of classes. We denote by I_ℓ^k the class k users of I_ℓ and we assume that all I_ℓ^k are measurable.

Let σ and $\hat{\sigma}$ be two Nash equilibria. We define for an $\ell \in L$, a class k and an arc a the quantity

$$f_{\ell,a}^k = \lambda\{i \in I_\ell^k : a \in \sigma(i)\},$$

and

$$\hat{f}_{\ell,a}^k = \lambda\{i \in I_\ell^k : a \in \hat{\sigma}(i)\}.$$

Following Milchtaich [Mil05], we say that the two equilibria are *equivalent* if not only the flow on each arc is the same but the contribution of each pair and each class to the flow on each arc is the same i.e. $f_{\ell,a}^k = \hat{f}_{\ell,a}^k$ for any arc a, OD-pair ℓ and class k. Milchtaich proved that a two-terminal network has the uniqueness property if and only if every two Nash equilibria are equivalent for generically all cost functions (Theorem 5.1 in [Mil05]). A property is considered *generic* if it holds on an open dense set. “Open” and “dense” are understood according to the following metric on the cost functions.

Define the set $\mathcal{G}$ of feasible assignment of cost functions: $\mathcal{G}$ contains all collections of continuous and strictly increasing cost functions $(c_a^i)_{a\in A, i\in I}$ with $c_a^i : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $\ell \in L$ and for all class k, whenever $i, i' \in I_\ell^k$, then $c_a^i = c_a^{i'}$.

Given a particular element of $\mathcal{G}$, the function $i \mapsto c_a^i(x)$ is measurable for all $a \in A$ and $x \in \mathbb{R}_+$. Every element of $\mathcal{G}$ has therefore a nonempty set of Nash equilibria. Note that the set $\mathcal{G}$ depends on the partition of the population. We can define the distance between two elements $(c_a^i)_{a\in A, i\in I}$ and $(\hat{c}_a^i)_{a\in A, i\in I}$ of $\mathcal{G}$ by $\max |c_a^i(x) - \hat{c}_a^i(x)|$, where the maximum is taken over all $a \in A$, $i \in I$ and $x \in \mathbb{R}_+$. This defines a metric for $\mathcal{G}$.

Theorem 2. *Assume that the supply graph G is a cycle. Then, for any demand digraph H, the following assertions are equivalent:*

(i) *(G, H) has the uniqueness property.*
(ii) *For every partition of the population, there is an open dense set in $\mathcal{G}$ such that for any assignment of cost functions that belongs to this set, every two equilibria are equivalent.*

Sketch of proof. Up to slight adaptations, the proof is the same as the one of Theorem 5.1 in [Mil05].

If (i) does not hold, we can use the construction of Section 5.3 to build two strict equilibria for an assignment in $\mathcal{G}$. In a ball centered on this assignment with radius $\rho > 0$ small enough, we still have two equilibria with distinct flows, which can not be equivalent. Therefore (ii) does not hold either.

If (i) holds, three claims (Claims 1, 2 and 4 of [Mil05]) lead to the desired conclusion, namely that (ii) holds. These three claims are now sketched. Their original proof does not need to be adapted, except for the second one, which is the only moment where the topology of the network is used. Its proof in our case is in fact simpler.

For an assignment in $\mathcal{G}$, we denote ϕ_ℓ^k the number of minimal-cost routes for users in I_ℓ^k, which is in our case 1 or 2. Since the uniqueness property is assumed to hold, this number is fully determined by the assignment in $\mathcal{G}$. Define the mean number of minimal-cost routes

$$\phi = \sum_{k\in K, \ell\in L} \lambda(I_\ell^k)\phi_\ell^k.$$

The first claim states that the map $\phi : \mathcal{G} \to \mathbb{R}$ is upper semicontinuous and has finite range.

The second claim states that for every assignment of cost functions in $\mathcal{G}$ that is a point of continuity of ϕ, all Nash equilibria are equivalent. To prove this second claim, we consider two Nash equilibria assumed to be nonequivalent σ and $\hat{\sigma}$. Using these two equilibria, a new one is built, $\tilde{\sigma}$, such that $f_{\ell,r_1}^k > 0$, $\hat{f}_{\ell,r_1}^k > 0$ and $\tilde{f}_{\ell,r_1}^k = 0$ for some $\ell \in L$, some class k and some ℓ-route r_1. As the two ℓ-routes do not share any arc (Claim 1), for any a in r_1, we have $f_{\ell,a}^k > 0$, $\hat{f}_{\ell,a}^k > 0$, and $\tilde{f}_{\ell,a}^k = 0$.

The second claim is achieved by choosing any a_1 in r_1 and by adding any small value $\delta > 0$ to the cost function $c_{a_1}^i$ for $i \in I_\ell^k$, while keeping the others unchanged. It can be checked that the set of minimal-cost routes is the same as for $\delta = 0$, minus the route r_1 for users in I_ℓ^k. The map ϕ has therefore a discontinuity of at least $\lambda(I_\ell^k)$ at the original assignment of cost functions.

Finally, the third claim allows to conclude: in every metric space, the set of all points of continuity of a real-valued upper semicontinuous function with finite range is open and dense. □

7.2. **Results for more general graphs.** The following proposition formalizes the fact that for general graphs, the uniqueness property is kept whenever one "glues" together two supply graphs on a vertex. This latter operation is called a 1-*sum* in the usual terminology of graphs. Since it is an elementary result, we state it without proof.

Proposition 3. *Let $G = (V, E)$ and $G' = (V', E')$ be two graphs, and let $H = (T, L)$ and $H' = (T', L')$ two directed graphs with $T \subseteq V$ and $T' \subseteq V'$, such that (G, H) and (G', H') have the uniqueness property. Assume that $V \cap V' = T \cap T'$ is reduced to a unique vertex v and define $G'' = (V \cup V', E \cup E')$ and $H'' = (T \cup T', L \cup L' \cup L'')$ with $L'' := \{(u, w) : (u, v) \in L$ and $(v, w) \in L'\}$.*

Then (G'', H'') as the uniqueness property.

7.3. **The strong uniqueness property.** A supply graph is said to have the *strong uniqueness property* if for any choice of the OD-pairs, the uniqueness property holds. In other words, $G = (V, E)$ has the strong uniqueness property if, for any digraph $H = (T, L)$ with $T \subseteq V$, the pair (G, H) has the uniqueness property.

Theorem 3. *A graph has the strong uniqueness property if and only if no cycle is of length* 3 *or more.*

A graph has the strong uniqueness property if and only it is obtained by taking a forest (a graph without cycles) and by replacing some edges by parallel edges.

Proof of Theorem 3. If there is a cycle of length 3, there exists a H such that an arc of G is in three routes, see Figure 6. With the help of Theorem 1, we see that multiple equilibria with distinct flows on some arcs exist for some cost functions.

If not, Proposition 3 applied on graphs reduced to parallel edges leads to the desired conclusion. We use here the fact that graphs reduced to parallel edges have the strong uniqueness property, see [Kon04] or [Mil05]. □

7.4. **When there are only two classes.** When exhibiting multiple equilibrium flows in the proof of Theorem 1, the number of classes is always at least three. The same remark holds for the characterization of the two-terminal networks having the uniqueness property in the article by Milchtaich [Mil05]: all cases of non-uniqueness are built with three classes. We may wonder whether there are also multiple equilibrium flows with only two classes of users. The answer is yes as shown by the following examples. The first example is in the framework of the ring network; according to Theorem 1, such an example requires at least 3 OD-pairs. Since it will contain exactly 3 OD-pairs, it is in a sense a minimum example for ring network. The second example involves a two-terminal network. It is inspired by the work by Bashkar et al. [BFHH09].

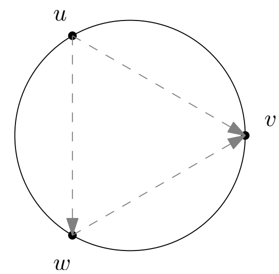

FIGURE 6. A cycle of length 3 without the uniqueness property: arc (u, v) is in three routes

7.4.1. *Multiple equilibrium flows on the ring with only two classes.* Consider the network of Figure 6, and use the construction as in Section 5.3.

There are two classes 1 and 2, with the following population measures.

$\ell \in L$	(u,w)	(u,v)	(w,v)
$\lambda(I_\ell^1)$	1	0	1
$\lambda(I_\ell^2)$	0	1.5	0

Cost functions are

Arcs of G	(u,w)	(w,v)	(v,u)	(w,u)	(u,v)	(v,w)
Class 1	$22x$	$22x$		x	$x+26$	x
Class 2	x	$x+48$			$24x+7$	

We define the strategy profile σ (resp. $\hat{\sigma}$) such that all users of the class 1 select a counterclockwise (resp. clockwise) route and all users of the class 2 select a clockwise (resp. counterclockwise) route. We get the following (distinct) flows.

Arcs a of G	(u,w)	(w,v)	(v,u)	(w,u)	(u,v)	(v,w)
f_a	1	1	0	0	1.5	0
$\hat{f}_a$	1.5	1.5	0	1	2	1

We check that σ is an equilibrium.

For users in $I^1_{(u,w)}$ and in $I^1_{(w,v)}$, the cost of the counterclockwise route is 22 and of the clockwise 27.5. For users in $I^2_{(u,v)}$, the cost of the counterclockwise route is 50 and of the clockwise 43.

No user is incitated to change its route choice.

We check that $\hat{\sigma}$ is an equilibrium.

For users in $I^1_{(u,w)}$ and in $I^1_{(w,v)}$, the cost of the counterclockwise route is 33 and of the clockwise 29. For users in $I^2_{(u,v)}$, the cost of the counterclockwise route is 51 and of the clockwise 55.

No user is incitated to change its route choice.

7.4.2. *Multiple equilibrium flows for a two-terminal network with only two classes.* Consider the two-terminal network K_4 of Figure 7.

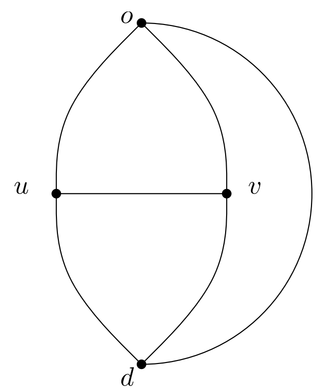

FIGURE 7. A two-terminal network for which multiple equilibrium flows exist with only two classes

Suppose that we have two classes of nonatomic users I^1 and I^2, with $\lambda(I^1) = 3$ and $\lambda(I^2) = 4$, with the following cost functions on each arc, where "∞" means a prohibitive high cost function.

Arcs of G	(o,u)	(o,v)	(u,v)	(v,u)	(u,d)	(v,d)	(o,d)
Class 1	x	"∞"	$x+18$	"∞"	"∞"	x	$7x$
Class 2	$5x$	x	"∞"	"∞"	x	$5x$	$x+10$

Users of class 1 have only the choice between the two routes *ouvd* and *od*, while users of class 2 can choose between the three routes *oud*, *ovd* and *od*.

The strategy profile σ is defined such that all class 1 users select the route *ouvd* and all class 2 users select the route *od*.

The strategy profile $\hat{\sigma}$ is defined such that all class 1 users select the route *od*, half of class 2 users select the route *oud* and the other half select the route *ovd*. We get the following (distinct) flows.

Arcs a of G	(o,u)	(o,v)	(u,v)	(v,u)	(u,d)	(v,d)	(o,d)
f_a	3	0	3	0	0	3	4
$\hat{f}_a$	2	2	0	0	2	2	3

We check that σ is an equilibrium.

For users of the class 1, the cost of *ouvd* is 27 and the cost of *od* is 28. For users of the class 2, the cost of *oud* is 15, the cost of *ovd* is 15 and the cost of *od* is 14.

No user is incitated to change its route choice.

We check that $\hat{\sigma}$ is an equilibrium.

For users of the class 1, the cost of *ouvd* is 22 and the cost of *od* is 21. For users of the class 2, the cost of *oud* is 12, the cost of *ovd* is 12 and the cost of *od* is 13.

No user is incitated to change its route choice.

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