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How To Retrieve The Normal Modes Using The POD

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Abstract

Proper orthogonal decomposition (POD) is an increasingly popular way to analyze data, and to obtain either a low-dimensional approximated description of a high-dimensional process, or useful information for damage assessment. In the case of a dynamic system with *n* degrees of freedom, the purpose of POD is to retrieve the modal properties from the measured response. Until now, POD has been used for systems with a diagonal mass matrix. The aim of this presentation is double: first, to demonstrate that POD can also be used for a non-diagonal symmetric mass matrix; and second, to present sufficient conditions on the response sampling, in order to retrieve the modal characteristics with a prescribed accuracy.
The conditions to obtain this approximation with a given accuracy are first explicitly given for the case without damping. Then the case of proportional damping is addressed and similar conditions are shown. The critical case of two modal frequencies close together is also studied, given that it requires particular conditions.
The obtained conditions show that the expected accuracy is explicitly limited by the damping ratio. Some numerical tests illustrate the accuracy evolution of the approximated normal modes obtained by this method, with respect to the variation of the observation time and the damping ratio. This careful analysis can be useful for finding the cause of poor approximation properties in more complex cases, such as analysis of variation in nonlinear normal modes.

Keywords: dynamic system; POD; normal modes; accuracy of an approximation;

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1. Introduction

Studying the behavior of structures in various common situations such as earthquakes, high wind, etc., is very important for civil engineering. These studies are based on the mode shapes and modal frequencies of the structures, explaining why the development of new efficient mathematical techniques to retrieve this information is so important in the scientific field.

In this paper, we study a mathematically based technique that allows one to find the mode shapes and modal frequencies of a structure depending on the type of initial data. This method uses Proper Orthogonal Decomposition (POD) in order to determine the modal characteristics of a second-order free linear discrete dynamic system with *n* degrees of freedom (DoF). While several previous works have proven the accuracy of this method when the mass

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matrix is proportional to the identity matrix, in this paper we will prove its efficiency for all symmetric, definite and positive mass matrices, and even in the case of light proportional damping.

Two problem types will be studied in this paper: **the direct problem** and **the indirect problem**.

In the direct problem, the mass matrix **M**, the damping matrix **C** and the stiffness matrix **K** are known, and are used to determine the displacement vector **x**, the eigenvectors and eigenvalues of the system and their properties.

In the indirect problem, the displacement matrix **X** is used to retrieve the modal characteristics of a structure. Engineers very often have to deal with this type of problem. Indeed, when working on real structures, the matrices **M**, **C** and **K** remain unknown. Determining the modal characteristics of the system thus cannot be completed with the methods used for a direct problem. Therefore, it is valuable to develop methods that use only the displacement matrix **X**, which can be easily obtained through experimental data.

To find the best approximation of a system's characteristics, one relies on the POD: the eigenvectors of the correlation matrix **R** approach the eigenmodes of the system when **M** is a scalar matrix. When using this method, scientists found severe limitations of the POD. The first problem is that the eigenmodes of the system are orthogonal with respect to the mass matrix **M**, whereas the eigenvectors of the system are orthogonal to each other. In fact, the approximations of the eigenmodes are the eigenvectors of the matrix **R * M**. Studies in [1] and [2] showed that when the mass matrix of a structure is proportional to the identity matrix, the eigenvectors of **R** converge to the modes of vibration. The accuracy of the method has been verified by a large number of numerical tests. It has been proven numerically that, using the matrix **R * M**, the eigenmodes can be retrieved even when the mass matrix **M** is not proportional to the identity, but to a known diagonal matrix [1]. A second problem appears when two frequencies are close to each other [1(p.1-7)], since POD cannot distinguish between them.

The purpose of this work is to extend the use of the POD method in order to retrieve the modal characteristics of a structure for indirect problems, in the case when the mass matrix is symmetric and positive definite, but not necessarily diagonal. Moreover, quantitative conditions on the total observation time are given in order to approximate the eigenmodes with a given accuracy. It is also shown how these conditions are modified when the system has frequencies that are very close together. These conditions are also given for a low proportionally damped system.

Researchers have recently developed this method for different domains. POD was successfully adapted for discrete linear symmetric undamped systems [2] and for light damping systems [3]. This approach also became useful for the case of nonlinear systems [4,5], discretely sampled continuous systems (uniform sampling), [6] and even (with a modified version of POD) for a non-uniform discretization with homogenous structures.

In the literature, one can find three closely related methods that can be considered to be POD methods: the Karhunène-Loève decomposition (KLD) (for continuous time), the principal component analysis (PCA) (for random discrete variables), and the singular value decomposition (SVD) (which is originally a mere algebraic decomposition) [7].

This presentation is divided into five main sections. In section two, the direct problem is studied. It is shown how one can retrieve the modal characteristics of a structure in a rather simple way. In section three, we are interested in solving indirect problems. A POD-based method is proven to be accurate for all symmetric positive definite mass matrices. Conditions on the sampling frequency and the total observation time are developed in order to approximate the eigenmodes and the eigenvectors of the system, with a given accuracy ε. Both damped and undamped systems are studied. Numerical results are presented in section four, and a conclusion is developed in the last section.

2. Direct problem

This dynamic system considered here is governed by the following second-order differential equation:

$$\mathbf{M}\,\ddot{\mathbf{x}}(t)+\mathbf{C}\,\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{0} \qquad (1)$$

with $\mathbf{x}(0)=\mathbf{x}_0$ and $\dot{\mathbf{x}}(0)=\dot{\mathbf{x}}_0$ and with **M** the mass matrix, **C** the damping matrix and **K** the stiffness matrix. The matrices **M**, **C** and **K** are symmetric and positive definite. The purpose of this section is to determine the eigenmodes $\boldsymbol{\phi}$ of the system. Therefore, the system given by (1) is transformed into a system of *n* uncoupled ordinary equations (7). Applying the coordinate transformation, as shown in [1,6] and in [5(p.67)] :

$$\mathbf{x}(t) = \mathbf{M}^{-\frac{1}{2}}\mathbf{q}(t) \tag{2}$$

to the equation (1) yields to:

$$\mathbf{M}^{\frac{1}{2}}\ddot{\mathbf{q}}(t) + \mathbf{C}\mathbf{M}^{-\frac{1}{2}}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{M}^{-\frac{1}{2}}\mathbf{q}(t) = \mathbf{0} \tag{3}$$

Multiplying by $\mathbf{M}^{-\frac{1}{2}}$ recasts the system as:

$$\ddot{\mathbf{q}}(t) + \mathbf{M}^{-\frac{1}{2}}\mathbf{C}\mathbf{M}^{-\frac{1}{2}}\dot{\mathbf{q}}(t) + \mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-\frac{1}{2}}\mathbf{q}(t) = \mathbf{0} \tag{4}$$

Since the matrices **M**, **K** and **C** are symmetric and positive definite, the matrices $\mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-\frac{1}{2}}$ and $\mathbf{M}^{-\frac{1}{2}}\mathbf{C}\,\mathbf{M}^{-\frac{1}{2}}$ are, too. It should be noted that a symmetric matrix is always diagonalizable over an orthogonal basis of eigenvectors. For proportional (Rayleigh) damping, the matrices $\mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-\frac{1}{2}}$ and $\mathbf{M}^{-\frac{1}{2}}\mathbf{C}\,\mathbf{M}^{-\frac{1}{2}}$ have the same eigenvectors $\mathbf{p}_i$, so these two matrices can be diagonalized by the same change of basis. Only the proportionally damped case is considered in the following. Let **P** be the matrix the columns of which are the orthogonal vectors $\mathbf{p}_i$.

By setting,

$$\mathbf{q}(t) = \mathbf{P}\,\mathbf{r}(t) \tag{5}$$

and multiplying by $\mathbf{P}^T$, the system yields:

$$\ddot{\mathbf{r}}(t) + \mathbf{diag}(\mu_i)\dot{\mathbf{r}}(t) + \mathbf{diag}(\lambda_i)\mathbf{r}(t) = \mathbf{0} \tag{6}$$

where μ_i and λ_i are respectively the eigenvalues of $\mathbf{M}^{-\frac{1}{2}}\mathbf{C}\,\mathbf{M}^{-\frac{1}{2}}$ and $\mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-\frac{1}{2}}$.

Since $\mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-\frac{1}{2}}$ and $\mathbf{M}^{-\frac{1}{2}}\mathbf{C}\,\mathbf{M}^{-\frac{1}{2}}$ are positive definite, their eigenvalues are positive and non null. Hence, one can define $\omega_i > 0$ and $\zeta_i > 0$ by :

$$\lambda_i = \omega_i^2 \text{ and } \mu_i = 2\omega_i\zeta_i$$

Thus, the system (6) becomes:

$$\ddot{\mathbf{r}}(t) + \mathbf{diag}(2\omega_i\zeta_i)\dot{\mathbf{r}}(t) + \mathbf{diag}(\omega_i^2)\mathbf{r}(t) = 0 \tag{7}$$

Since this above system is completely uncoupled, it is possible to solve an ordinary differential equation for each $i \in \{1, \ldots, n\}$. The characteristic equation for $r_i(t)$ is given by:

$$s^2 + (2\omega_i\zeta_i)s + \omega_i^2 = 0 \tag{8}$$

If the damping is low, i.e., $\zeta_i < 1$, the equation (8) admits two complex roots defined as follows:

$$s = -\omega_i\;\zeta_i \pm i\omega_i\;\sqrt{1-\zeta_i^2} \tag{9}$$

By setting $\tilde{\omega}_i = \omega_i\sqrt{1-\zeta_i^2}$, the system response due to initial conditions may be expressed as:

$$r_i(t) = e^{-\omega_i\zeta_i t}\left[r_i(0)\cos(\tilde{\omega}_i t) + \frac{1}{\tilde{\omega}_i}\left[\dot{r}_i(0) + \omega_i\zeta_i r_i(0)\right]\sin(\tilde{\omega}_i t)\right] \tag{10}$$

Let a_i be defined by:

$$a_i^2 = \left(r_i^2(0) + \frac{1}{\tilde{\omega}_i^2}\left(\dot{r}_i(0) + \omega_i\zeta_i r_i(0)\right)^2\right) \tag{11}$$

In the undamped case, a_i is twice the energy σ_i^2.

By introducing the angle $0 \le \theta_i < \pi$, i.e.,

$$\tan\theta_i = \frac{-\frac{1}{\tilde{\omega}_i}\left[\dot{r}_i(0) + \omega_i\zeta_i r_i(0)\right]}{r_i(0)} \tag{12}$$

one can express $r_i(t)$ as follows:

$$r_i(t) = e^{-\omega_i \zeta_i t} a_i \cos(\tilde{\omega}_i t - \theta_i) \tag{13}$$

Since $\mathbf{x}(t) = \mathbf{M}^{-\frac{1}{2}}\mathbf{q}(t) = \mathbf{M}^{-\frac{1}{2}}\mathbf{Pr}(t)$, introducing $\boldsymbol{\phi}$, such that:

$$\boldsymbol{\phi} = \mathbf{M}^{-\frac{1}{2}}\mathbf{P} \tag{14}$$

the response $\mathbf{x}(t)$ becomes:

$$\mathbf{x}(t) = \boldsymbol{\phi}\mathrm{r}(t) = \sum_{i=1}^{n} r_i(t)\boldsymbol{\varphi}_i \tag{15}$$

where the φ_i are the columns of $\boldsymbol{\phi}$. These are the eigenmodes of the system. Therefore, as $\boldsymbol{\varphi}_i = \mathbf{M}^{-\frac{1}{2}}\mathbf{p}_i$, and as the $\mathbf{p}_i$ form an orthonormal base of eigenvectors for $\mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-\frac{1}{2}}$ and for $\mathbf{M}^{-\frac{1}{2}}\mathbf{C}\,\mathbf{M}^{-\frac{1}{2}}$ (case of proportional damping), then:

$$\mathbf{M}^{-\frac{1}{2}}\mathbf{K}\mathbf{M}^{-\frac{1}{2}}\mathbf{p}_i = \omega_i^2\mathbf{p}_i \quad \Leftrightarrow \quad \mathbf{M}^{-1}\mathbf{K}\boldsymbol{\varphi}_i = \omega_i^2\boldsymbol{\varphi}_i \text{ and,}$$

$$\mathbf{M}^{-\frac{1}{2}}\mathbf{C}\mathbf{M}^{-\frac{1}{2}}\mathbf{p}_i = 2\omega_i\zeta_i\mathbf{p}_i \quad \Leftrightarrow \quad \mathbf{M}^{-1}\mathbf{C}\boldsymbol{\varphi}_i = 2\omega_i\zeta_i\boldsymbol{\varphi}_i$$

This emphasizes that (φ_i) form a basis of eigenvectors for $\mathbf{M}^{-1}\mathbf{K}$ and also for $\mathbf{M}^{-1}\mathbf{C}$. Although this basis is not orthonormal with respect to the L_2 norm, it is worth pointing out that as seen in [8(p.45)], one has:

$$\boldsymbol{\phi}^T\mathbf{M}\boldsymbol{\phi} = \mathbf{I}\,, \quad \boldsymbol{\phi}^T\mathbf{K}\boldsymbol{\phi} = \mathbf{diag}(\boldsymbol{\omega}_i^2) \quad \text{and} \quad \boldsymbol{\phi}^T\mathbf{C}\boldsymbol{\phi} = \mathbf{diag}(2\boldsymbol{\omega}_i\boldsymbol{\zeta}_i)$$

In addition, from (14), one has $\boldsymbol{\phi}^\mathbf{T} = \mathbf{P}^\mathbf{T}\mathbf{M}^{-\frac{1}{2}}$, and so:

$$\boldsymbol{\phi}^{-1} = \mathbf{P}^\mathbf{T}\mathbf{M}^{\frac{1}{2}} = \boldsymbol{\phi}^\mathbf{T}\mathbf{M} \tag{16}$$

3. Indirect problem

In practice, the mass, stiffness and damping matrices are unknown, the only given information about the structure is the displacements (eventually the accelerations), and the data are usually discretized in time. The main purpose of the POD is to find the eigenmodes of the initial differential system by finding the eigenmodes of a matrix **R** called the correlation matrix.

For *m* samples in time of the n-dimensional vector **x**, the (*n* x *m*) data matrix **X** is given by:

$$\mathbf{X} = \begin{pmatrix} x_1(t_1) & \dots & x_1(t_m) \\ \vdots & \ddots & \vdots \\ x_n(t_1) & \dots & x_n(t_m) \end{pmatrix} \tag{17}$$

and the correlation matrix is defined by:

$$\mathbf{R} = \frac{1}{m}\mathbf{X}\mathbf{X}^\mathbf{T}$$

As seen in equation (15):

$\mathbf{x}(t) = \boldsymbol{\phi}\mathrm{r}(t) = \sum_{i=1}^{n} r_i(t)\boldsymbol{\varphi}_i$ hence,

$\mathbf{x}(t_k) = \sum_{i=1}^{n} r_i(t_k)\boldsymbol{\varphi}_i$, and therefore, the data matrix **X** can be written as:

$$\mathbf{X} = \boldsymbol{\phi}\begin{pmatrix} r_1(t_1) & \dots & r_1(t_m) \\ \vdots & \ddots & \vdots \\ r_n(t_1) & \dots & r_n(t_m) \end{pmatrix} \tag{18}$$

Consequently, $\mathbf{R} = \frac{1}{m}\mathbf{X}\mathbf{X}^\mathbf{T}$ is expressed by:

$$\mathbf{R} = \frac{1}{m}\boldsymbol{\phi}\left[r_{ij}\right]\boldsymbol{\phi}^{\mathbf{T}} \tag{19}$$

with

$$\left[\mathrm{r}_{ij}\right] = \begin{pmatrix} \mathrm{r}_1(\mathrm{t}_1) & \cdots & \mathrm{r}_1(\mathrm{t}_m) \\ \vdots & \ddots & \vdots \\ \mathrm{r}_n(\mathrm{t}_1) & \cdots & \mathrm{r}_n(\mathrm{t}_m) \end{pmatrix} \begin{pmatrix} \mathrm{r}_1(\mathrm{t}_1) & \cdots & \mathrm{r}_n(\mathrm{t}_1) \\ \vdots & \ddots & \vdots \\ \mathrm{r}_1(\mathrm{t}_m) & \cdots & \mathrm{r}_n(\mathrm{t}_m) \end{pmatrix} \tag{20}$$

where, for $1 \le i, j \le n$, the general term r_{ij} of the *i*-th row and *j*-th column has the following expression :

$$r_{ij} = \frac{1}{m}\sum_{k=1}^{m} r_i(t_k) r_j(t_k) \tag{21}$$

When the mass matrix **M** is proportional to the identity, numerous studies [1,2] have numerically shown that the eigenvectors of the correlation matrix $\mathbf{R} = \frac{1}{m}\mathbf{X}\mathbf{X}^{\mathbf{T}}$ are a good approximation of the system's eigenvectors. It will be proved in the following section that this result is also verified even when **M** is no longer proportional to the identity and not even necessarily diagonal.

The main step is to show that under certain conditions, the matrix $[r_{ij}]$ is close to the matrix $\mathbf{diag}(\sigma_i^2)$ with $\sigma_i^2 = \frac{a_i^2}{2}$. Hence, the following approximation can be written:

$\mathbf{R} \approx \boldsymbol{\phi}\ \mathbf{diag}(\sigma_i^2)\ \boldsymbol{\phi}^{\mathbf{T}}$ As $\boldsymbol{\phi}^{-1} = \boldsymbol{\phi}^{\mathbf{T}}\mathbf{M}$, (16) implies that

$$\mathbf{RM} \approx \boldsymbol{\phi}\ \mathbf{diag}(\sigma_i^2)\ \boldsymbol{\phi}^{\mathbf{T}}\ \mathbf{M} = \boldsymbol{\phi}\ \mathbf{diag}(\sigma_i^2)\ \boldsymbol{\phi}^{-1}$$

This shows that the columns of $\boldsymbol{\phi}$ are close to the eigenvectors of the matrix **R * M**.

We will give sufficient conditions to ensure that the matrix $[r_{ij}]$ is close to the matrix $\mathbf{diag}(\sigma_i^2)$.

In the following sections, we first recall the classical condition for the sampling rate $\frac{1}{\Delta t}$, and then we study the sufficient conditions concerning the total observation time $T = m\,\Delta t$ for the case of an undamped system. Finally, we generalize these conditions for the case of a system with light proportional damping.

3.1. Choice of the sampling rate

We recall that the average of a continuous function g on an interval [0, T] is the limit for $m \rightarrow +\infty$ of the average values taken at t_k, with $t_k = k\,\Delta t$ and $m\,\Delta t = T$. This can be expressed as follows:

$$\frac{1}{m\ \Delta t}\left[\sum_{k=1}^{m} g(t_k)\right]\Delta t \xrightarrow[m\to+\infty]{} \frac{1}{T}\int_0^T g(t)\,dt \quad \text{with} \quad m\,\Delta t = T \tag{22}$$

The necessary condition for $\frac{1}{m}\sum_{k=1}^{m} g(t_k)$ to be reasonably approximated by the average value of g on [0, T] is the verification of the Nyquist-Shannon sampling theorem, i.e., the sampling rate $\frac{1}{\Delta t}$ should be higher than twice the maximum frequency of the signal g.

For the discrete dynamic system presented in (3), frequencies are given by $f_i = \frac{\omega_i}{2\pi}$. Therefore, one must choose Δt such that:

$$\frac{1}{\Delta t} > \frac{1}{\pi}\max_{1\leq i\,\text{n}}(\omega_i) \tag{23}$$

Then, for $T = m\,\Delta t$, where T is the observation time and m the number of samples, the approximation is as follows:

$$r_{i\,j} = \frac{1}{m}\sum_{k=1}^{m} r_i(t_k)r_j(t_k) \approx \tilde{r}_{i\,j} = \frac{1}{T}\int_0^T r_i(t)r_j(t)\,dt \tag{24}$$

3.2. Choice of total observation time T

In this section, we study the limit of the matrix $[\tilde{r}_{i\,j}]$ when T increases. We will first consider the undamped case (3.2.1), and then the case with proportional light damping (3.2.2).

3.2.1. Undamped system

Equation (13), used in the undamped case, gives the following expression of the temporal evolution of each mode:

$$r_i(t) = a_i\cos(\omega_i t - \theta_i) \tag{25}$$

We will prove that when certain conditions are verified, the matrix $[\tilde{r}_{i\,j}]$ tends to $\mathbf{diag}\left(\frac{a_i^2}{2}\right)$, when $m \rightarrow +\infty$, with:

$$\tilde{r}_{i\,j} = \frac{1}{T}\int_0^T r_i(t)r_j(t)\,\text{dt}.$$

In greater detail, for when a_i and a_j are nonzero, we will show that:

$$\left|\frac{\tilde{r}_{ii}}{a_i^2} - \frac{1}{2}\right| \leq \frac{\varepsilon}{2} \quad \textit{and} \quad \left|\frac{\tilde{r}_{ij}}{a_i a_j}\right| \leq \frac{\varepsilon}{2} \tag{26}$$

with $\varepsilon = \frac{T_M}{T}$, where the constant T_M is function of the analyzed signal and T is the time of observation. By increasing T, ε might arbitrarily become small. In the following, we will consider the case where $i = j$, and then the case where $i \neq j$.

- The diagonal terms are:

$$\tilde{r}_{i\,i} = \frac{1}{T}\int_0^T a_i^2\cos^2(\omega_i t - \theta_i)\,dt \tag{27}$$

As

$$\frac{1}{T}\int_0^T \cos^2(\omega_i t - \theta_i)\, dt = \frac{1}{2} + \frac{-\sin(2(\omega_i T - \theta_i)) + \sin(2(-\theta_i))}{4\omega_i T}$$

the following estimates hold:

$$\frac{1}{2} - \frac{2}{4\omega_i T} \le \frac{1}{T}\int_0^T \cos^2(\omega_i t - \theta_i)\, dt \le \frac{1}{2} + \frac{2}{4\omega_i T} \tag{28}$$

Therefore,

$$\left|\frac{\tilde{r}_{i\,i}}{a_i^2} - \frac{1}{2}\right| \le \frac{1}{2\omega_i T} \le \frac{1}{2T}\frac{1}{\min \omega_i} \tag{29}$$

By introducing T_i, the time period for each mode and T_{max}

$$T_{max} = \frac{1}{2\min \omega_i} = \frac{\max(T_i)}{4\pi} \tag{30}$$

This shows that the first part of (26) is verified if $\frac{T_{max}}{T} \le \varepsilon$.

- The non-diagonal terms are:

$$\tilde{r}_{i\,j} = \frac{a_i a_j}{T}\int_0^T \cos(\omega_i t - \theta_i)\cos(\omega_j t - \theta_j)\, dt$$

For the sake of simplicity, we assume that the phase angles are zero. Therefore:

$$\int_0^T \cos(\omega_i t)\cos(\omega_j t)\, dt = \frac{1}{2}\left[\frac{1}{\omega_i + \omega_j}\sin(\omega_i + \omega_j)T + \frac{1}{\omega_i - \omega_j}\sin(\omega_i - \omega_j)T\right]$$

which leads to:

$$\left|\int_0^T \cos(\omega_i t)\cos(\omega_j t)\, dt\right| \le \frac{1}{2}\left(\frac{1}{\omega_i + \omega_j} + \frac{1}{|\omega_i - \omega_j|}\right) \tag{31}$$

Since $\omega_i + \omega_j \ge 2\min(\omega_i)$, one has:

$$\frac{1}{(\omega_i + \omega_j)} \le \frac{1}{2\min(\omega_i)} = \frac{\max(T_i)}{4\pi} = T_{max} \tag{32}$$

Defining T_{gap} by:

$$T_{gap} = \frac{1}{\min|\omega_i - \omega_j|} = \frac{1}{2\pi \min|f_i - f_j|} \tag{33}$$

where f_i are the modal frequencies, we obtain:

$$\left|\frac{\tilde{r}_{ij}}{a_i a_j}\right| \le \frac{1}{2}\frac{T_{max} + T_{gap}}{T}$$

When two frequencies are close to each other, T_{gap} can be quite large in respect to T_{max} . Thus, in order to have a good approximation of the matrix $\mathbf{diag}\left(\frac{a_i^2}{2}\right)$, one should make up for this by increasing the observation time *T*.

- <u>To conclude</u>, in the undamped case, the coefficients of the matrix $[\tilde{r}_{i\,j}]$ are equal to those of the matrix $\mathbf{diag}\left(\frac{a_i^2}{2}\right)$ with a relative error of order $\frac{\varepsilon}{2}$ for $\varepsilon = \frac{T_M}{T}$, where

$$T_M = T_{max} + T_{gap} \tag{34}$$

Finally, let us note that if for a certain i_0 , a_{i_0} is null, $\tilde{r}_{i_0\,j}$ and $\tilde{r}_{i_0\,i_0}$ are also null, which means that the $i_0 - th$ mode is not activated, thus it is not necessary to take it into account in the previous inequalities.

3.2.2. *Proportionally damped system*

We recall that for a system with proportional damping, $r_i(t)$ is given by equation (13).

As for the undamped system, we simplify the presentation by assuming that all phase angles are null. Using the same procedure as in section (3.2.1), the aim is to show that, under certain conditions, the matrix $[\tilde{r}_{ij}]$ approaches $\mathbf{diag}\left(\frac{a_i^2}{2}\right)$, when T is large enough. In greater detail, when a_i and a_j are nonzero, conditions are set in order to insure that $\left|\frac{\tilde{r}_{ii}}{a_i^2}-\frac{1}{2}\right|\le\frac{\varepsilon}{2}$ and $\left|\frac{\tilde{r}_{ij}}{a_i a_j}\right|\le\frac{\varepsilon}{2}$ for ε arbitrarily small. The cases $i=j$ and $i\neq j$ will be studied.

- The diagonal terms are:

$$\tilde{r}_{ii}=\frac{a_i^2}{T}\int_0^T e^{-2\omega_i\zeta_i t}\cos^2(\tilde{\omega}_i t)\,dt \tag{35}$$

As the Taylor series of e^{-x} is an alternating series for $x\ge 0$, one has $1-x\le e^{-x}\le 1$. Thus, for $0\le t\le T$ we have:

$$1-2\omega_i\zeta_i T\le 1-2\tilde{\omega}_i\zeta_i t\le e^{-2\tilde{\omega}_i\zeta_i t}\le 1$$

Since the function given by $\cos^2(\tilde{\omega}_i t)$ is positive, we deduce that:

$$(1-2\tilde{\omega}_i\zeta_i T)\frac{1}{T}\int_0^T\cos^2(\tilde{\omega}_i t)\,dt\le\frac{1}{T}\int_0^T e^{-2\tilde{\omega}_i\zeta_i t}\cos^2(\tilde{\omega}_i t)\,dt\le\frac{1}{T}\int_0^T\cos^2(\tilde{\omega}_i t)\,dt$$

Using the inequality (28) one has:

$$(1-2\tilde{\omega}_i\zeta_i T)\left(\frac{1}{2}+\frac{-1}{4\tilde{\omega}_i T}\right)\le\frac{\tilde{r}_{ii}}{a_i^2}\le\frac{1}{2}+\frac{1}{4\tilde{\omega}_i T}$$

which yields to:

$$-\tilde{\omega}_i\zeta_i T-(1-2\tilde{\omega}_i\zeta_i T)\frac{1}{4\tilde{\omega}_i T}\le\frac{\tilde{r}_{ii}}{a_i^2}-\frac{1}{2}\le\frac{1}{4\tilde{\omega}_i \mathrm{T}}$$

$$-\tilde{\omega}_i\zeta_i T-\frac{1}{4\tilde{\omega}_i T}-\zeta_i\frac{1}{2}\le\frac{\tilde{r}_{ii}}{a_i^2}-\frac{1}{2}\le\frac{1}{4\tilde{\omega}_i T}$$

Using the following notations

$$\tilde{T}_{\max}=\frac{\max(\tilde{T}_i)}{4\pi}=\frac{1}{2\min(\tilde{\omega}_i)}\quad,\quad\tilde{T}_{\min}=\frac{1}{\max(\tilde{\omega}_i)}\quad and\quad 0\le\zeta_i\le\eta$$

one has

$$-\eta\frac{T}{2\tilde{T}_{\min}}-\frac{\tilde{T}_{\max}}{2T}-\frac{\eta}{2}\le-\tilde{\omega}_i\zeta_i T-\frac{1}{4\tilde{\omega}_i T}-\zeta_i\frac{1}{2}\le\frac{\tilde{r}_{ii}}{a_i^2}-\frac{1}{2}\le\frac{1}{4\tilde{\omega}_i T}\le\frac{\tilde{T}_{\max}}{2T} \tag{36}$$

If T is large enough and if the damping ratio η is low enough to satisfy the following inequalities:

$$\frac{\tilde{T}_{\max}}{T}\le\varepsilon\quad,\quad\eta\frac{T}{\tilde{T}_{\min}}\le\varepsilon\quad and\quad\eta\le\varepsilon \tag{37}$$

then the estimation (36) allows one to prove that the inequality $\eta\frac{T}{T_{\min}}+\frac{\tilde{T}_{\max}}{T}+\eta\le 3\varepsilon$ is verified, and so:

$$-\frac{3\varepsilon}{2}\le\frac{\tilde{r}_{ii}}{a_i^2}-\frac{1}{2}\le\frac{\varepsilon}{2}$$

- The non-diagonal terms are:

$$\tilde{r}_{ij}=\frac{a_i a_j}{T}\int_0^T e^{(-\omega_i\zeta_i-\omega_j\zeta_j)t}\cos(\tilde{\omega}_i t-\theta_i)\cos(\tilde{\omega}_j t-\theta_j)\,dt$$

The function $g(t) = \cos(\tilde{\omega}_i t - \theta_i)\cos(\tilde{\omega}_j t - \theta_j)$ is continuous but of either sign and $f(t) = e^{(-\omega_i\zeta_i - \omega_j\zeta_j)t}$ is a positive decreasing function. It is thus possible to use the second mean value theorem on [*a*,*b*]:

$$\exists\, c \in]a,\, b[\quad \textit{such that} \quad \int_a^b f(t)g(t)dt = f(a)\int_a^c g(t)\, dt$$

As previously in (31):

$$\left|\int_0^c \cos(\omega_i t - \theta_i)\cos(\omega_j t - \theta_j)\, dt\right| \;\le\; \frac{1}{2}\left(\frac{1}{\omega_i + \omega_j} + \frac{1}{|\omega_i - \omega_j|}\right)$$

then:

$$\left|\frac{\tilde{r}_{ij}}{a_i a_j}\right| \le \frac{e^0}{2T}\left(\frac{1}{\tilde{\omega}_i + \tilde{\omega}_j} + \frac{1}{|\tilde{\omega}_i - \tilde{\omega}_j|}\right)$$

which is the same result found in the undamped case. In order to have $\left|\frac{\tilde{r}_{ij}}{a_i a_j}\right| \;\le\; \frac{\varepsilon}{2}$, with ε set in advance, it is sufficient to choose:

$$\frac{T_M}{T} \;\le \varepsilon \qquad \textit{with} \qquad T_M = T_{max} + T_{gap} \tag{38}$$

In conclusion, the conditions (37) and (38) are sufficient for the coefficients of the matrix $[\tilde{r}_{ij}]$ to be approximations, of order $\frac{3\varepsilon}{2}$, of those of the matrix $\mathbf{diag}\left(\frac{a_i^2}{2}\right)$. It is important to note that for a given damping ratio η , one cannot expect to have an approximation of degree ε smaller than $\sqrt{\eta\frac{\tilde{T}_{max}}{\tilde{T}_{min}}}$.

4. **Numerical results and conclusion**

To illustrate this theoretical study of the normal modes approximation accuracy by the POD, some numerical tests are presented in the following.

The relative error is calculated for different elementary initial data and for different observation times. The figures present the evolution of the mean relative errors with respect to T/T_{max}.

To begin, we consider a diagonal matrix **M**. The example presented was previously chosen by Feeny and Kappagantu in [1 section(2.2)]. They used a total observation time *T* equal to four fundamental periods:

$$T = \max(T_i) = 4 * 4\pi * \frac{\max(T_i)}{4\pi} = 16\pi * T_{max} \approx 50 T_{max}$$

Figure (1) illustrates the mean relative error evolution of the approximated normal modes, all the way to $T = 100\ T_{max}$. One can notice that the error is less than 10% when *T* is greater than $10\ T_{max}$ (which is about the same *T* considered in [1 section(2.2)], and that the accuracy is equal to 0.01589 and is coherent with Feeny's results. Figure (1) proves that, for systems with distinct modal frequencies ($f_1 = 0.06,\ f_2 = 0.15$ and $f_3 = 0.26$), where $T_{max} = 1.188\ s$ and $T_{gap} = 1.4703\ s$, the error is acceptable (lower than 10%) when we vary T/T_M and do not consider T_{gap} (since the gap between the frequencies is large).

The next example concerns a non-diagonal, symmetric and positive definite mass matrix **M**, and two different matrices K, such that the modal pulsations of the system are respectively $\omega = [1, 2, 3]$ and $\omega = [1, 2, 2.05]$:

$$M = \begin{pmatrix} 3 & 2\sqrt{3} & 0 \\ 2\sqrt{3} & 7 & 0 \\ 0 & 0 & 16 \end{pmatrix}$$

In both cases $min(w_i) = 1$, hence $T_{max} = 0.5\ s$. However, in the last one, $min(w_i - w_j) = 0.05$, which is much smaller than $min(w_i) = 1$. Figure (2) shows the poor accuracy obtained in the second case for the same T/T_{max}. In the first case, $T_{max} = 0.5\ s$ and $T_{gap} = 1\ s$, but in the second one, since it has two similar modal frequencies $T_{max} = 0.5\ s$ and $T_{gap} = 20\ s$, $T_M = T_{max} + T_{gap} = 20.5\ s$. One can easily deduce from figure (2) that: when $min(w_i - w_j)$ are of the same order as $min(w_i)$, in order to have a relative error of order epsilon, it is sufficient to have an observation time *T* less than $\varepsilon * T_{max}$. But when two frequencies are too close to each other, in order to obtain an accuracy of order epsilon, it is necessary to take into account T_{gap} and to have *T* less than $\varepsilon * (T_{max} + T_{gap})$.

Finally, figure (3) considers the evolution of the relative error when T/T_M increases for different damping ratios ζ. When $T \geq T_M$, one should examine the values of the relative error for three different values of proportional damping ζ. The comparison leads one to deduce that the proportional damping ζ must be low. Otherwise, the relative error becomes large and thus unacceptable.

To conclude, it is now numerically and theoretically proven that the POD, under the sufficient conditions discussed in this paper, can be used for dynamical low damping systems with non-diagonal **M**, even when two modal frequencies are close together. This careful analysis can be useful for finding the cause of poor approximation properties in more complex cases, such as analysis of nonlinear normal modes in vibrating systems.

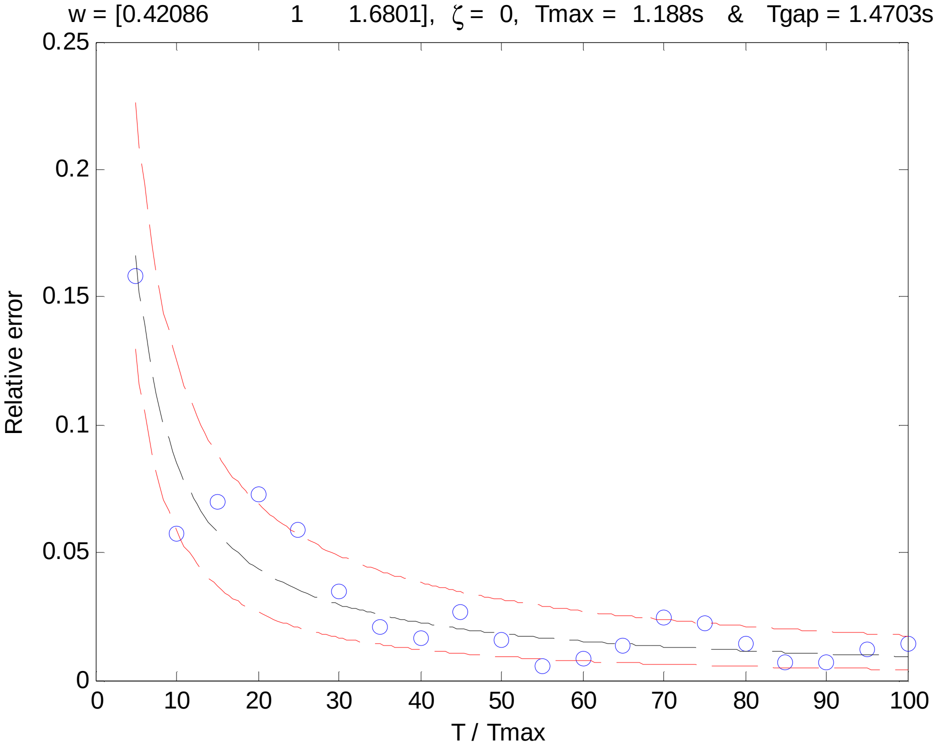

Fig. 1.The evolution of the relative error as a function of $T\,/\,\mathrm{T}_{\max}$ for system with a diagonal **M**

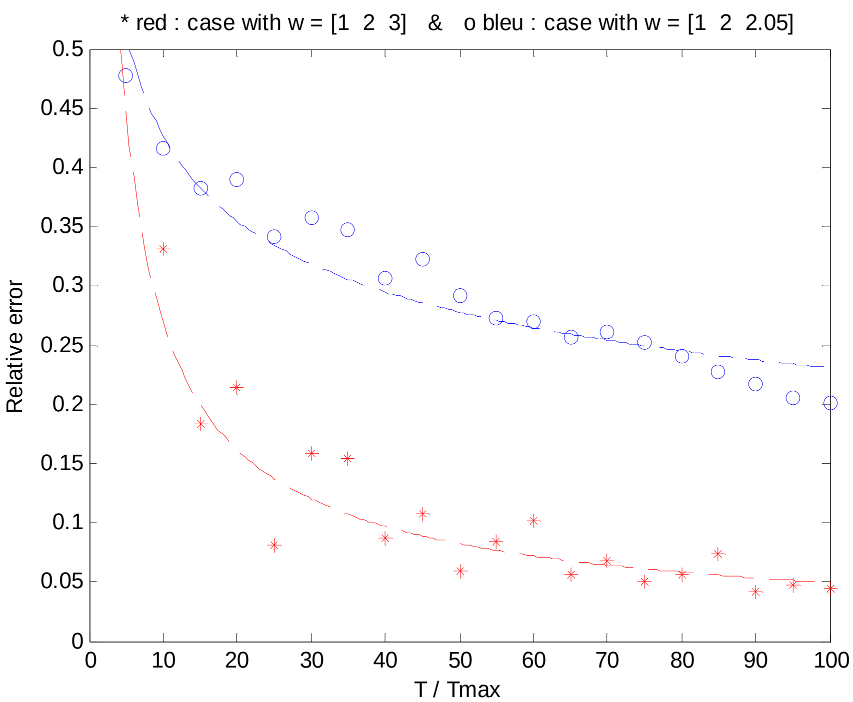

Fig. 2.The evolution of the relative error as a function of $T\,/\,\mathrm{T}_{\max}$ for system with a non-diagonal **M**

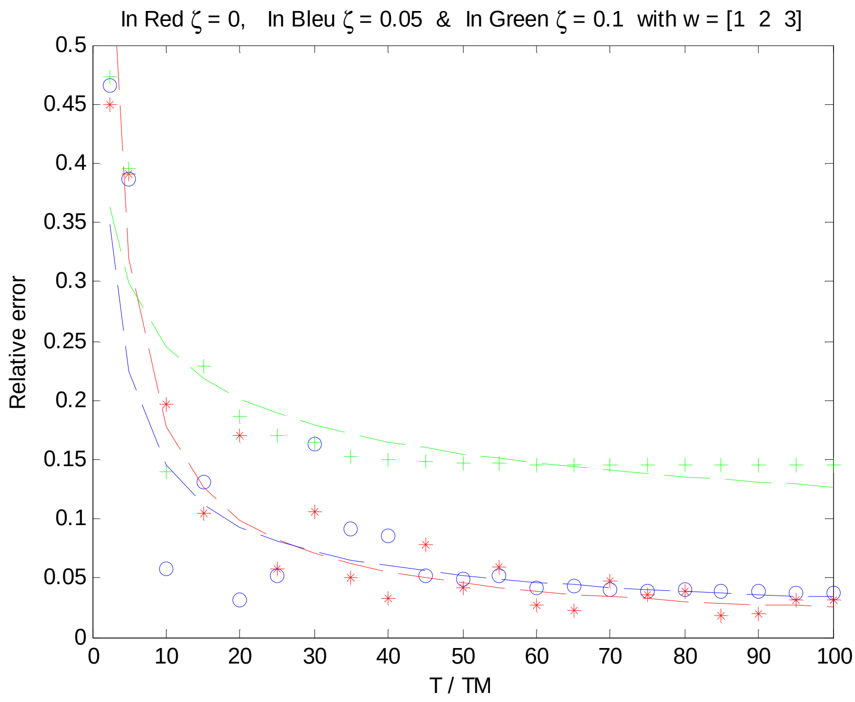

Fig. 3.The evolution of the relative error as a function of T / T_{max} for system with a non-diagonal **M**

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