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# On the Maximal Sum of Exponents of Runs in a String 

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#### Abstract

A run is an inclusion maximal occurrence in a string (as a subinterval) of a repetition $v$ with a period $p$ such that $2 p \leq|v|$. The exponent of a run is defined as $|v| / p$ and is $\geq 2$. We show new bounds on the maximal sum of exponents of runs in a string of length $n$. Our upper bound of $4.1 n$ is better than the best previously known proven bound of $5.6 n$ by Crochemore \& Ilie (2008). The lower bound of $2.035 n$, obtained using a family of binary words, contradicts the conjecture of Kolpakov \& Kucherov (1999) that the maximal sum of exponents of runs in a string of length $n$ is smaller than $2 n$.


## 1 Introduction

Repetitions and periodicities in strings are one of the fundamental topics in combinatorics on words $[1,14]$. They are also important in other areas: lossless compression, word representation, computational biology, etc. In this paper we consider bounds on the sum of exponents of repetitions that a string of a given length may contain. In general, repetitions are studied also from other points of view, like: the classification of words (both finite and infinite) not containing repetitions of a given exponent, efficient identification of factors being repetitions of different types and computing the bounds on the number of various types of repetitions occurring in a string. The known results in the topic and a deeper description of the motivation can be found in a survey by Crochemore et al. [4].

The concept of runs (also called maximal repetitions) has been introduced to represent all repetitions in a string in a succinct manner. The crucial property of runs is that their maximal number in a string of length $n$ (denoted as $\rho(n))$ is $O(n)$, see Kolpakov \& Kucherov [10]. This fact is the cornerstone of any

[^0]algorithm computing all repetitions in strings of length $n$ in $O(n)$ time. Due to the work of many people, much better bounds on $\rho(n)$ have been obtained. The lower bound $0.927 n$ was first proved by Franek \& Yang [7]. Afterwards, it was improved by Kusano et al. [13] to $0.944565 n$ employing computer experiments, and very recently by Simpson [18] to $0.944575712 n$. On the other hand, the first explicit upper bound $5 n$ was settled by Rytter [16], afterwards it was systematically improved to $3.48 n$ by Puglisi et al. [15], $3.44 n$ by Rytter [17], $1.6 n$ by Crochemore \& Ilie [2,3] and $1.52 n$ by Giraud [8]. The best known result $\rho(n) \leq 1.029 n$ is due to Crochemore et al. [5], but it is conjectured [10] that $\rho(n)<n$. Some results are known also for repetitions of exponent higher than 2. For instance, the maximal number of cubic runs (maximal repetitions with exponent at least 3 ) in a string of length $n$ (denoted $\rho_{\text {cubic }}(n)$ ) is known to be between $0.406 n$ and $0.5 n$, see Crochemore et al. [6].

A stronger property of runs is that the maximal sum of their exponents in a string of length $n$ (notation: $\sigma(n)$ ) is linear in terms of $n$, see Kolpakov \& Kucherov [12]. It has applications to the analysis of various algorithms, such as computing branching tandem repeats: the linearity of the sum of exponents solves a conjecture of [9] concerning the linearity of the number of maximal tandem repeats and implies that all can be found in linear time. For other applications, we refer to [12]. The proof that $\sigma(n)<c n$ in Kolpakov and Kucherov's paper [12] is very complex and does not provide any particular value for the constant $c$. A bound can be derived from the proof of Rytter [16] but he mentioned only that the bound that he obtains is "unsatisfactory" (it seems to be $25 n$ ). The first explicit bound $5.6 n$ for $\sigma(n)$ was provided by Crochemore and Ilie [3], who claim that it could be improved to $2.9 n$ employing computer experiments. As for the lower bound on $\sigma(n)$, no exact values were previously known and it was conjectured $[11,12]$ that $\sigma(n)<2 n$.

In this paper we provide an upper bound of $4.1 n$ on the maximal sum of exponents of runs in a string of length $n$ and also a stronger upper bound of $2.5 n$ for the maximal sum of exponents of cubic runs in a string of length $n$. As for the lower bound, we bring down the conjecture $\sigma(n)<2 n$ by providing an infinite family of binary strings for which the sum of exponents of runs is greater than $2.035 n$.

## 2 Preliminaries

We consider words (strings) u over a finite alphabet $\Sigma, u \in \Sigma^{*}$; the empty word is denoted by $\varepsilon$; the positions in $u$ are numbered from 1 to $|u|$. For $u=$ $u_{1} u_{2} \ldots u_{m}$, let us denote by $u[i \ldots j]$ a factor of $u$ equal to $u_{i} \ldots u_{j}$ (in particular $u[i]=u[i \ldots i]$ ). Words $u[1 \ldots i]$ are called prefixes of $u$, and words $u[i \ldots|u|]$ suffixes of $u$.

We say that an integer $p$ is the (shortest) period of a word $u=u_{1} \ldots u_{m}$ (notation: $p=\operatorname{per}(u)$ ) if $p$ is the smallest positive integer such that $u_{i}=u_{i+p}$ holds for all $1 \leq i \leq m-p$. We say that words $u$ and $v$ are cyclically equivalent
(or that one of them is a cyclic rotation of the other) if $u=x y$ and $v=y x$ for some $x, y \in \Sigma^{*}$.

A run (also called a maximal repetition) in a string $u$ is an interval $[i, j]$ such that:

- the period $p$ of the associated factor $u[i \ldots j]$ satisfies $2 p \leq j-i+1$,
- the interval cannot be extended to the right nor to the left, without violating the above property, that is, $u[i-1] \neq u[i+p-1]$ and $u[j-p+1] \neq u[j+1]$.
A cubic run is a run $[i . . j]$ for which the shortest period $p$ satisfies $3 p \leq j-i+1$. For simplicity, in the rest of the text we sometimes refer to runs and cubic runs as to occurrences of the corresponding factors of $u$. The (fractional) exponent of a run is defined as $(j-i+1) / p$.

For a given word $u \in \Sigma^{*}$, we introduce the following notation:
$-\rho(u)$ and $\rho_{\text {cubic }}(u)$ are the numbers of runs and cubic runs in $u$ resp.
$-\sigma(u)$ and $\sigma_{\text {cubic }}(u)$ are the sums of exponents of runs and cubic runs in $u$ resp.
For a non-negative integer $n$, we use the same notations $\rho(n), \rho_{c u b i c}(n), \sigma(n)$ and $\sigma_{\text {cubic }}(n)$ to denote the maximal value of the respective function for a word of length $n$.

## 3 Lower bound for $\sigma(n)$

Tables 1 and 2 list the sums of exponents of runs for several words of two known families that contain very large number of runs: the words $x_{i}$ defined by Franek and Yang [7] (giving the lower bound $\rho(n) \geq 0.927 n$, conjectured for some time to be optimal) and the modified Padovan words $y_{i}$ defined by Simpson [18] (giving the best known lower bound $\rho(n) \geq 0.944575712 n$ ). These values have been computed experimentally. They suggest that for the families of words $x_{i}$ and $y_{i}$ the maximal sum of exponents could be less than $2 n$.

We show, however, a lower bound for $\sigma(n)$ that is greater than $2 n$.
Theorem 1. There are infinitely many binary strings $w$ such that

$$
\frac{\sigma(w)}{|w|}>2.035
$$

Proof. Let us define two morphisms $\phi:\{a, b, c\} \mapsto\{a, b, c\}$ and $\psi:\{a, b, c\} \mapsto$ $\{0,1\}$ as follows:

$$
\begin{array}{cl}
\phi(a)=b a a b a, & \phi(b)=c a, \quad \phi(c)=b c a \\
\psi(a)=01011, & \psi(b)=\psi(c)=01001011
\end{array}
$$

We define $w_{i}=\psi\left(\phi^{i}(a)\right)$. Table 3 shows the sums of exponents of runs in words $w_{i}$, computed experimentally.

Clearly, for any word $w=\left(w_{8}\right)^{k}, k \geq 1$, we have

$$
\frac{\sigma(w)}{|w|}>2.035
$$

| $i$ | $\left\|x_{i}\right\|$ | $\rho\left(x_{i}\right) /\left\|x_{i}\right\|$ | $\sigma\left(x_{i}\right)$ | $\sigma\left(x_{i}\right) /\left\|x_{i}\right\|$ |
| ---: | ---: | ---: | ---: | ---: |
| 1 | 6 | 0.3333 | 4.00 | 0.6667 |
| 2 | 27 | 0.7037 | 39.18 | 1.4510 |
| 3 | 116 | 0.8534 | 209.70 | 1.8078 |
| 4 | 493 | 0.9047 | 954.27 | 1.9356 |
| 5 | 2090 | 0.9206 | 4130.66 | 1.9764 |
| 6 | 8855 | 0.9252 | 17608.48 | 1.9885 |
| 7 | 37512 | 0.9266 | 74723.85 | 1.9920 |
| 8 | 158905 | 0.9269 | 316690.85 | 1.9930 |
| 9 | 673134 | 0.9270 | 1341701.95 | 1.9932 |

Table 1. Number of runs and sum of exponents of runs in Franek \& Yang's [7] words $x_{i}$.

| $i$ | $\left\|y_{i}\right\|$ | $\rho\left(y_{i}\right) /\left\|y_{i}\right\|$ | $\sigma\left(y_{i}\right)$ | $\sigma\left(y_{i}\right) /\left\|y_{i}\right\|$ |
| ---: | ---: | ---: | ---: | ---: |
| 4 | 37 | 0.7568 | 57.98 | 1.5671 |
| 8 | 125 | 0.8640 | 225.75 | 1.8060 |
| 12 | 380 | 0.9079 | 726.66 | 1.9123 |
| 16 | 1172 | 0.9309 | 2303.21 | 1.9652 |
| 20 | 3609 | 0.9396 | 7165.93 | 1.9856 |
| 24 | 11114 | 0.9427 | 22148.78 | 1.9929 |
| 28 | 34227 | 0.9439 | 68307.62 | 1.9957 |
| 32 | 105405 | 0.9443 | 210467.18 | 1.9967 |
| 36 | 324605 | 0.9445 | 648270.74 | 1.9971 |
| 40 | 999652 | 0.9445 | 1996544.30 | 1.9972 |

Table 2. Number of runs and sum of exponents of runs in Simpson's [18] modified Padovan words $y_{i}$.

| $i$ | $\left\|w_{i}\right\|$ | $\sigma\left(w_{i}\right)$ | $\sigma\left(w_{i}\right) /\left\|w_{i}\right\|$ |
| ---: | ---: | ---: | ---: |
| 1 | 31 | 47.10 | 1.5194 |
| 2 | 119 | 222.26 | 1.8677 |
| 3 | 461 | 911.68 | 1.9776 |
| 4 | 1751 | 3533.34 | 2.0179 |
| 5 | 6647 | 13498.20 | 2.0307 |
| 6 | 25205 | 51264.37 | 2.0339 |
| 7 | 95567 | 194470.30 | 2.0349 |
| 8 | 362327 | 737393.11 | 2.0352 |
| 9 | 1373693 | 2795792.39 | 2.0352 |
| 10 | 5208071 | 10599765.15 | 2.0353 |

Table 3. Sums of exponents of runs in words $w_{i}$.

## 4 Upper bounds for $\sigma(n)$ and $\sigma_{\text {cubic }}(n)$

In this section we utilize the concept of handles of runs as defined in [6]. The original definition refers only to cubic runs, but here we extend it also to ordinary runs.

Let $u \in \Sigma^{*}$ be a word of length $n$. Let us denote by $P=\left\{p_{1}, p_{2}, \ldots, p_{n-1}\right\}$ the set of inter-positions in $u$ that are located between pairs of consecutive letters of $u$. We define a function $H$ assigning to each run $v$ in $u$ a set of some interpositions within $v$ (called later on handles) - $H$ is a mapping from the set of runs occurring in $u$ to the set $2^{P}$ of subsets of $P$. Let $v$ be a run with period $p$ and let $w$ be the prefix of $v$ of length $p$. Let $w_{\min }$ and $w_{\max }$ be the minimal and maximal words (in lexicographical order) cyclically equivalent to $w . H(v)$ is defined as follows:
a) if $w_{\min }=w_{\max }$ then $H(v)$ contains all inter-positions within $v$,
b) if $w_{\min } \neq w_{\max }$ then $H(v)$ contains inter-positions between consecutive occurrences of $w_{\min }$ in $v$ and between consecutive occurrences of $w_{\max }$ in $v$.

Note that $H(v)$ can be empty for a non-cubic-run $v$.


Fig. 1. An example of a word with two highlighted runs $v_{1}$ and $v_{2}$. For $v_{1}$ we have $w_{\min 1} \neq w_{\max 1}$ and for $v_{2}$ the corresponding words are equal to $b$ (a one-letter word). The inter-positions belonging to the sets $H\left(v_{1}\right)$ and $H\left(v_{2}\right)$ are pointed by arrows

Proofs of the following properties of handles of runs can be found in [6]:

1. Case (a) in the definition of $H(v)$ implies that $\left|w_{\min }\right|=1$.
2. $H\left(v_{1}\right) \cap H\left(v_{2}\right)=\emptyset$ for any two distinct runs $v_{1}$ and $v_{2}$ in $u$.

To prove the upper bound for $\sigma(n)$, we need to state an additional property of handles of runs. Let $\mathcal{R}(u)$ be the set of all runs in a word $u$, and let $\mathcal{R}_{1}(u)$ and $\mathcal{R}_{\geq 2}(u)$ be the sets of runs with period 1 and at least 2 respectively.

## Lemma 1.

If $v \in \mathcal{R}_{1}(u)$ then $\sigma(v)=|H(v)|+1$.
If $v \in \mathcal{R}_{\geq 2}(u)$ then $\lceil\sigma(v)\rceil \leq \frac{|H(v)|}{2}+3$.

Proof. For the case of $v \in \mathcal{R}_{1}(u)$, the proof is straightforward from the definition of handles. In the opposite case, it is sufficient to note that both words $w_{\min }^{k}$ and $w_{\max }^{k}$ for $k=\lfloor\sigma(v)\rfloor-1$ are factors of $v$, and thus

$$
|H(v)| \geq 2 \cdot(\lfloor\sigma(v)\rfloor-2)
$$

Now we are ready to prove the upper bound for $\sigma(n)$. In the proof we use the bound $\rho(n) \leq 1.029 n$ on the number of runs from [5].

Theorem 2. The sum of the exponents of runs in a string of length $n$ is less than $4.1 n$.

Proof. Let $u$ be a word of length $n$. Using Lemma 1, we obtain:

$$
\begin{align*}
\sum_{v \in \mathcal{R}(u)} \sigma(v) & =\sum_{v \in \mathcal{R}_{1}(u)} \sigma(v)+\sum_{v \in \mathcal{R}_{\geq 2}(u)} \sigma(v) \\
& \leq \sum_{v \in \mathcal{R}_{1}(u)}(|H(v)|+1)+\sum_{v \in \mathcal{R}_{\geq 2}(u)}\left(\frac{|H(v)|}{2}+3\right) \\
& =\sum_{v \in \mathcal{R}_{1}(u)}|H(v)|+\left|\mathcal{R}_{1}(u)\right|+\sum_{v \in \mathcal{R}_{\geq 2}(u)} \frac{|H(v)|}{2}+3 \cdot\left|\mathcal{R}_{\geq 2}(u)\right| \\
& \leq 3 \cdot|\mathcal{R}(u)|+A+B / 2, \tag{1}
\end{align*}
$$

where $A=\sum_{v \in \mathcal{R}_{1}(u)}|H(v)|$ and $B=\sum_{v \in \mathcal{R}_{\geq 2}(u)}|H(v)|$. Due to the disjointness of handles of runs (the second property of handles), $A+B<n$, and thus, $A+B / 2<n$. Combining this with (1), we obtain:

$$
\sum_{v \in \mathcal{R}(u)} \sigma(v)<3 \cdot|\mathcal{R}(u)|+n \leq 3 \cdot \rho(n)+n \leq 3 \cdot 1.029 n+n<4.1 n
$$

A similar approach for cubic runs, this time using the bound of $0.5 n$ for $\rho_{\text {cubic }}(n)$ from [6], enables us to immediately provide a stronger upper bound for the function $\sigma_{\text {cubic }}(n)$.

Theorem 3. The sum of the exponents of cubic runs in a string of length $n$ is less than $2.5 n$.

Proof. Let $u$ be a word of length $n$. Using same inequalities as in the proof of Theorem 2, we obtain:
$\sum_{v \in \mathcal{R}_{c u b i c}(u)} \sigma(v)<3 \cdot\left|\mathcal{R}_{\text {cubic }}(u)\right|+n \leq 3 \cdot \rho_{\text {cubic }}(n)+n \leq 3 \cdot 0.5 n+n=2.5 n$,
where $\mathcal{R}_{\text {cubic }}(u)$ denotes the set of all cubic runs of $u$.

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[^0]:    * Some parts of this paper were written during the author's Erasmus exchange at King's College London

