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DETERMINATION AND (RE)PARAMETRIZATION OF RATIONAL DEVELOPABLE SURFACES*

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Abstract The developable surface is an important surface in computer aided design, geometric modeling and industrial manufactory. It is often given in the standard parametric form, but it can also be in the implicit form which is commonly used in algebraic geometry. Not all algebraic developable surfaces have rational parametrizations. In this paper, we focus on the rational developable surfaces. For a given algebraic surface, we first determine whether it is developable by geometric inspection, and we give a rational proper parametrization for the affirmative case. For a rational parametric surface, we can also determine the developability and give a proper reparametrization for the developable surface.

Keywords Parametrization, proper, rational developable surface, reparametrization.

1 Introduction

A developable surface can be constructed by bending a planar region at every point. It is a commonly used surface in computer aided design and geometric modeling [1–4]. Developable surfaces have zero Gaussian curvature and they are a subset of ruled surfaces. In general design, the developable surface is often proposed as a parametric form. In recent years, people challenge to geometrically design with algebraic surfaces since they have more geometric features and topologies than those of the parametric surfaces (see [5, 6]). In this situation, a natural problem is to determine the type of algebraic modeling surfaces. For expected cases, some surfaces can be commonly used surfaces, for example, developable surfaces. As a successive problem, we need to find a rational parametrization of the determined surface, since the parametrizations are better representations for manufactural control and computer display.

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The two problems are both difficult for general surfaces, particularly in computation. Since the algebraic surfaces are basic objects in algebraic geometry, there were some classical results associated to these two problems. Let $\mathcal{S}$ be an algebraic surface. If $\mathcal{S}$ is a rational surface then $P_n = q = 0$ for all n, and conversely, any surface with $q = P_2 = 0$ is a rational surface, where P_n and q are the *plurigenus* and the *irregularity* of $\mathcal{S}$, respectively. This is called Castelnuovo's rationality criterion (see [7], V.1). If $\mathcal{S}$ is a ruled surface then $P_n = 0$ for all n, and conversely, any surface with $P_4 = P_6 = 0$ (or $P_{12} = 0$) is a ruled surface. This is called the criterion of ruled surfaces (see [7], VI.18).

The above results gave important theoretical effects, but there were lack of practical method for real computation. In fact, the plurigenus and irregularity are difficult to compute. Therefore, for a general implicit curve and surface, to propose a parametrization algorithm is still an open problem [6]. To meet the practical needs, people tried to design some parametrization algorithm for some special surfaces which are commonly used. Sederberg and Snively [8] proposed four methods of parametrization of cubic algebraic surfaces. Sederberg [9] and Bajaj et al. [10] expanded this method. In [11], a method to parameterize a quadric is given using a stereographic projection; Berry et al. [12] tried to unify the implicitization and parametrization of a nonsingular cubic surface with Hilbert-Burch matrices. Recently, Chen et al. [13] presented a method to deal with the implicitization and parametrization of quadratic and cubic surfaces by the μ-basis which is a developing method. In [14], Shen and Pérez-Díaz characterized the rational ruled surfaces using the reduced standard form. These methods were designed for some special surfaces. For a general given surface, Schicho [15] gave well analysis in parametrization problem. He provided more contributions on theoretical analysis than practical computation, since the problem is quite difficult for general situations.

There were also some numerical mesh parametrization methods designed for the industrial manufactory [16]. A difficult problem in the numerical methods is to set the values of parameter for the points in an implicit surface. This is the main reason that people can only get an approximate parametrization using the numerical methods. Since the numerical approximate method may lose some intrinsic properties of the surfaces, we prefer to find the symbolic parametrization method in this paper for a typical surface named as the developable surface.

As mentioned above, the developable surface is an important modeling surface. But there are few papers that discussed the parametrization of an algebraic developable surface. This motives us to focus on the problem in this paper. In the geometric investigation, a developable surface must be either a cylindrical surface, a conical surface, or a tangential surface of a space curve. We then reduce the problem to determine and parameterize these three special surfaces.

The paper is organized as follows. Some necessary notations and preliminary results are proposed in Section 2. The algebraic rational developable surfaces are characterized in Section 3, and a rational proper parametrization is computed. Examples are given for some typical surfaces. In Section 4, we focus on the parametric surface and the reparametrization problem. The examples are also presented. Finally, we concluded the paper in Section 5.

2 Preliminaries

Let $\mathbb{L}[t]$ be the polynomial ring over the subfield $\mathbb{L}$ of an algebraically closed field of characteristic zero $\mathbb{K}$, and let $\mathbb{L}(t)$ be the field of rational functions over $\mathbb{L}$.

A *ruled surface* is defined by one parameter family of straight lines moving along a curve. The curve is called the *directrix* and the straight lines are called *rulings* or *generators*. A *developable surface* is a ruled surface with zero Gaussian curvature (see [17, 18]). If the rulings all pass through one point which called *apex*, the surface is a *conical surface*. If the rulings of a developable surface are parallel to the same straight line, the surface is a *cylindrical surface*. In the remaining cases the developable surface is the *tangential surface* which is defined by the tangents to a certain space curve. We also call it the tangential surface of the space curve. The space curve is called *cuspidal edge* of the tangential surface.

Although the developable surfaces are often given in parametric form, not all developable surfaces have rational parametrizations.

A proper parametrization of a rational ruled surface in standard form is given by

$$\mathbf{P}(s,t) = \mathbf{P}_0(t) + s\mathbf{P}_1(t). \tag{1}$$

where $\mathbf{P}_i(t) \in \mathbb{L}(t)^3$, $i = 1, 2$, and $\mathbf{P}_1 \neq (0, 0, 0)$. The rational developable surface has three forms by the following lemma presented in [18].

Lemma 2.1 *A ruled surface of the form* (1) *is a developable surface if and only if* $\mathbf{P}_0' \times \mathbf{P}_1' \cdot \mathbf{P}_1 = 0$. *In addition, a rational developable surface can only be one of the following cases:*

- *If* $\mathbf{P}_0(t)$ *is a constant vector, then* $\mathbf{P}(s,t)$ *defines a conical surface.*
- *If* $\mathbf{P}_1(t)$ *is a constant vector, then* $\mathbf{P}(s,t)$ *defines a cylindrical surface.*
- *If* $\mathbf{P}_1(t) = \mathbf{P}_0'(t)$*, then* $\mathbf{P}(s,t)$ *defines a tangential surface.*

Remark 2.2 Note that for a tangential surface $\mathcal{S}$ defined by the parametrization $\mathbf{P}(s,t) = \mathbf{P}_0(t) + s\mathbf{P}_0'(t)$, the cuspidal edge $\mathbf{P}_0(t)$ defines a rational curve and it is a singular curve of $\mathcal{S}$ (one has that $\mathbf{P}(s,t)$ is singular at $(0,t)$).

The parametric form is widely used in computer aided geometric design and in geometric modeling. An algebraic surface implicitly defined by a polynomial $F(x,y,z)$ may not have a rational parametrization. If an algebraic developable surface has a rational parametrization, we call it a rational developable surface. In the following of this paper, we focus on finding the rational parametrization of a given rational developable surface.

3 Implicitly rational developable surface

In this section, we consider a surface $\mathcal{S}$ implicitly defined by a polynomial

$$F(x,y,z) \in \mathbb{L}[x,y,z].$$

We provide necessary and sufficient conditions so that $\mathcal{S}$ represents a developable surface. In the affirmative case, we compute a rational parametrization of the form given in 1.

For this purpose, we start with a theorem that determines the developability of $\mathcal{S}$. The referenced discussion can be found in [17–19].

Theorem 3.1 *Let $\mathcal{S}$ be an algebraic surface implicitly defined by the polynomial $F(x,y,z)$. $\mathcal{S}$ is a developable surface if and only if $K(x,y,z)=0$ on $\mathcal{S}$, where*

$$K(x,y,z)=\begin{vmatrix} F_{xx} & F_{xy} & F_{xz} & F_x \\ F_{yx} & F_{yy} & F_{yz} & F_y \\ F_{zx} & F_{zy} & F_{zz} & F_z \\ F_x & F_y & F_z & 0 \end{vmatrix}. \tag{2}$$

In some papers, it is also said that $\mathcal{S}$ is developable if and only if its Gaussian curvature $\kappa(F)=0$ on $\mathcal{S}$, since we have the formula $\kappa(F)=K(x,y,z)/|\nabla F|^4$, where ∇ means the gradient [18]. Goldman [20] gave a proof for this formula.

Since there are three types of developable surfaces (see Lemma 2.1), in the following we discuss the rationality for each of them.

Lemma 3.2 *Let $\mathcal{S}$ be a conical surface with the apex $\mathbf{P}_0 \in \mathbb{L}^3$. Let $\mathcal{L}$ be a plane not passing through $\mathbf{P}_0$, and let $\mathcal{C}$ be the intersection curve of $\mathcal{S}$ and $\mathcal{L}$. $\mathcal{S}$ has a rational proper parametrization of the form $\mathbf{P}_0 + s\mathbf{P}_1(t) \in \mathbb{L}(s,t)^3$ if and only if $\mathcal{C}$ is rational over $\mathbb{L}$.*

Proof For the necessity, let $\mathcal{S}$ be a conical developable surface and let

$$\mathbf{P}(s,t)=(p_{01},p_{02},p_{03})+s(p_{11}(t),p_{12}(t),p_{13}(t)) \in \mathbb{L}(s,t)^3$$

be a rational proper parametrization of $\mathcal{S}$. Let $\mathcal{L}$ be a plane not passing through the apex $\mathbf{P}_0$ of $\mathcal{S}$, and we assume its implicit equation is given as $L(x,y,z)=0$. Substituting $\mathbf{P}(s,t)$ into $L(x,y,z)=0$, one can solve $s=q(t)\in\mathbb{L}(t)$ because $(p_{11}(t),p_{12}(t),p_{13}(t)) \neq (0,0,0)$ and s is linear in the equation. Then, the curve $\mathcal{C}$, given by the intersection of $\mathcal{L}$ and $\mathcal{S}$, has a proper rational parametrization defined as $\widetilde{\mathbf{P}}(t)=\mathbf{P}(q(t),t)\in\mathbb{L}(t)^3$ (note that $\mathbf{P}(s,t)$ is proper and $\mathcal{L}$ does not pass through the apex).

For the sufficiency, we first consider $\widetilde{\mathbf{P}}(t)\in\mathbb{L}(t)^3$ a rational proper parametrization of $\mathcal{C}$. Note that according to the arguments, the apex $\mathbf{P}_0$ is not on $\mathcal{C}$. Thus, $\mathbf{P}(s,t) := (1-s)\mathbf{P}_0 + s\widetilde{\mathbf{P}}(t)\in\mathbb{L}(s,t)^3$ is a rational proper parametrization of $\mathcal{S}$, since $\mathbf{P}(s,t)$ defines a conical surface covering $\mathcal{S}$. ∎

For the cylindrical surface, we have an equivalent property.

Lemma 3.3 *Let $\mathcal{S}$ be a cylindrical surface with the ruling direction $\mathbf{P}_1 \in \mathbb{L}^3$. Let $\mathcal{L}$ be a plane not parallel to $\mathbf{P}_1$, and let $\mathcal{C}$ be the intersection curve of $\mathcal{S}$ and $\mathcal{L}$. $\mathcal{S}$ has a rational proper parametrization of the form $\mathbf{P}_0(t)+s\mathbf{P}_1 \in \mathbb{L}(s,t)^3$ if and only if $\mathcal{C}$ is rational over $\mathbb{L}$.*

Proof For the necessity, let $\mathcal{S}$ be a cylindrical developable surface, and let

$$\mathbf{P}(s,t) = (p_{01}(t), p_{02}(t), p_{03}(t)) + s(p_{11}, p_{12}, p_{13}) \in \mathbb{L}(s,t)^3$$

be a rational proper parametrization of $\mathcal{S}$. Let $\mathcal{L}$ be a plane not parallel to $\mathbf{P}_1$, and we assume its implicit equation is given by $L(x,y,z) = 0$. Substituting $\mathbf{P}(s,t)$ into $L(x,y,z) = 0$, one can solve $s = q(t) \in \mathbb{L}(t)$ because $(p_{11}(t), p_{12}(t), p_{13}(t)) \neq (0,0,0)$, and s is linear in the equation. Then, the intersection curve $\mathcal{C}$ has a proper rational parametrization defined by $\widetilde{\mathbf{P}}(t) = \mathbf{P}(q(t), t) \in \mathbb{L}(t)^3$ (note that $\mathbf{P}(s,t)$ is proper and $\mathcal{L}$ is not parallel to the rulings).

For the sufficiency, we first note that $\mathcal{C}$ is not a ruling since $\mathcal{L}$ is not parallel to the ruling direction. If $\mathcal{C}$ has a rational proper parametrization $\widetilde{\mathbf{P}}(t) \in \mathbb{L}(t)^3$, then $\mathbf{P}(s,t) := \widetilde{\mathbf{P}}(t) + s\mathbf{P}_1 \in \mathbb{L}(s,t)^3$ is a rational proper parametrization of $\mathcal{S}$, since $\mathbf{P}(s,t)$ defines a cylindrical surface covering $\mathcal{S}$. ∎

A tangential developable surface is generated by the tangent lines of a space curve. The intersection of a tangent developable with the normal plane at a point P of the curve generally has a cusp at that point. Thus the tangential developable surface of a space curve has a cuspidal edge along the curve, and the cuspidal edge is a singular curve of the tangential developable surface (see [21]). In this paper, the space curve is assumed not to be a planar curve in $\mathbb{L}^3$ (note that the tangential surface with a planar cuspidal edge is just a plane).

Lemma 3.4 *Let $\mathcal{S}$ be a tangential surface with the cuspidal edge $\mathcal{C}$. $\mathcal{S}$ has a rational proper parametrization of the form $\mathbf{P}_0(t) + s\mathbf{P}_1 \in \mathbb{L}(s,t)^3$ if and only if $\mathcal{C}$ is rational over $\mathbb{L}$.*

Proof For the necessity, let $\mathcal{S}$ be a tangential surface with a rational proper parametrization

$$\mathbf{P}(s,t) = (p_{01}(t), p_{02}(t), p_{03}(t)) + s(p'_{01}(t), p'_{02}(t), p'_{03}(t)) \in \mathbb{L}(s,t)^3.$$

Then, the cuspidal edge $\mathcal{C} \subset \mathcal{S}$ is defined by the parametrization $\mathbf{P}_0(t) = (p_{01}(t), p_{02}(t), p_{03}(t)) \in \mathbb{L}(t)^3$. Hence, it is a rational curve. In addition, since $(s, 0)$ is always singular, we have that $\mathcal{C}$ is a singular curve of $\mathcal{S}$.

For the sufficiency, if $\mathcal{C}$ has a rational proper parametrization $\widetilde{\mathbf{P}}(t) \in \mathbb{L}(t)^3$, then $\mathbf{P}(s,t) := \widetilde{\mathbf{P}}(t) + s\widetilde{\mathbf{P}}'(t) \in \mathbb{L}(s,t)^3$ is a rational proper parametrization of $\mathcal{S}$, since $\mathbf{P}(s,t)$ defines a tangential surface sharing the common cuspidal edge of $\mathcal{S}$. ∎

We observe that if $\mathcal{S}$ is a tangent developable surface, then the cuspidal edge is a singular curve. Therefore, it is included in the singular set defined by the algebraic system $S := \{F = 0, F_x = 0, F_y = 0, F_z = 0\}$.

In the following, we summarize Theorem 3.1, and Lemmas 3.2, 3.3, 3.4, and we get the following theorem.

Theorem 3.5 *Let $\mathcal{S}$ be an algebraic surface implicitly defined by the polynomial $F(x,y,z) \in \mathbb{L}[x,y,z]$. $\mathcal{S}$ is a rational developable surface if and only if the following statements hold:*

1. *$K(x,y,z) = 0$, for all points (x,y,z) of $\mathcal{S}$.*

2. *One of the following statements holds:*

   2.1. *$\mathcal{S}$ is a conical surface with apex $\mathbf{P}_0 \in \mathbb{L}^3$, and there exists a planar curve $\mathcal{C} \subset \mathcal{S}$ not passing through $\mathbf{P}_0$ and having a proper rational parametrization $\widetilde{\mathbf{P}}(t) \in \mathbb{L}(t)^3$. Furthermore, $(1-s)\mathbf{P}_0 + s\widetilde{\mathbf{P}}(t) \in \mathbb{L}(s,t)^3$ is a proper parametrization of $\mathcal{S}$.*

   2.2. *$\mathcal{S}$ is a cylindrical surface with ruling direction $\mathbf{P}_1 \in \mathbb{L}^3$, and there exists a planar curve $\mathcal{C} \subset \mathcal{S}$ not parallel to $\mathbf{P}_1$ and having a proper rational parametrization $\widetilde{\mathbf{P}}(t) \in \mathbb{L}(t)^3$. Furthermore, $\widetilde{\mathbf{P}}(t) + s\mathbf{P}_1(t) \in \mathbb{L}(s,t)^3$ is a proper parametrization of $\mathcal{S}$.*

   2.3. *$\mathcal{S}$ is a tangential surface, and there exists a space singular curve $\mathcal{C} \subset \mathcal{S}$ having a rational proper parametrization $\mathbf{P}_0(t) \in \mathbb{L}(t)^3$. Furthermore, $\mathbf{P}_0(t) + s\mathbf{P}_0'(t) \in \mathbb{L}(s,t)^3$ is a proper parametrization of $\mathcal{S}$.*

Parameterize the developable surfaces

By Theorem 3.5, before parametrizing the rational developable surface $\mathcal{S}$ implicitly defined by $F(x,y,z) \in \mathbb{L}[x,y,z]$, we need to determine the type of the surface: conical, cylindrical or tangential surface. The normal vector of $\mathcal{S}$ at (x,y,z) is $\mathbf{N}(x,y,z) = (F_x, F_y, F_z)$, where F_{var} is the partial derivative of F with respect to the variable var. Then, the tangent surface of $\mathcal{S}$ at the point (u,v,w) is given by the equation $T(x,y,z) = F_u(x-u) + F_v(y-v) + F_w(z-w) = 0$, where $(F_u, F_v, F_w) = (F_x, F_y, F_z)|_{(x=u, y=v, z=w)}$. Under these conditions, we have the following proposition.

Proposition 3.6 *Let (F_x, F_y, F_z) and $T(x,y,z) = 0$ be the normal vector and tangent surface of the developable surface $\mathcal{S}$, respectively. It holds that:*

1. *If $T(x,y,z) = 0$ passes through a fixed point $\mathbf{P}_0 \in \mathbb{L}^3$, then $\mathcal{S}$ is a conical surface with the apex $\mathbf{P}_0$.*
2. *If there exists $(0,0,0) \neq \mathbf{P}_1 = (p_{11}, p_{12}, p_{13}) \in \mathbb{L}^3$ such that $p_{11}F_x + p_{12}F_y + p_{13}F_z = 0$, then $\mathcal{S}$ is a cylindrical surface with the ruling direction $\mathbf{P}_1$.*
3. *Otherwise, $\mathcal{S}$ is a tangential surface, and its cuspidal edge is included in $\{F = 0, F_x = 0, F_y = 0, F_z = 0\}$.*

Proof We know that $\mathcal{S}$ is a developable surface, and there are three different types of these surfaces (see Lemma 2.1). According to the definitions, the conical surface is the only that has an apex such that any tangent surface $T(x,y,z) = 0$ passes through it (see Lemma 3.2). The situation of the normal direction orthogonal with a constant vector can only happen with the cylindrical surface (see Lemma 3.3). The remain developable surfaces are the tangential surfaces (see Lemma 3.4). ▮

Algorithm 1 INPUT: An algebraic surface $\mathcal{S}$ implicitly defined by the polynomial $F(x,y,z) \in \mathbb{L}[x,y,z]$.

OUTPUT: A proper rational parametrization over $\mathbb{L}$ of the rational developable surface $\mathcal{S}$ or a message for $\mathcal{S}$.

1. Compute $K(x, y, z)$ of the form (2). If it is zero on $\mathcal{S}$ go to Step 2. Otherwise, return "$\mathcal{S}$ is not a developable surface."
2. If the tangent surface defined by the equation $T(x, y, z) = 0$ passes through a fixed point $\mathbf{P}_0 \in \mathbb{L}^3$, compute a plane $\mathcal{L}$ not passing through $\mathbf{P}_0$, and the intersection curve, $\mathcal{C}$, of $\mathcal{S}$ and $\mathcal{L}$.
   2.1. If $\mathcal{C}$ has a rational proper parametrization $\widetilde{\mathbf{P}}(t) \in \mathbb{L}(t)^3$, return $(1-s)\mathbf{P}_0 + s\widetilde{\mathbf{P}}(t) \in \mathbb{L}(s,t)^3$ is a rational proper parametrization of the conical surface $\mathcal{S}$.
   2.2. Otherwise, return "$\mathcal{S}$ is a conical developable surface but not rational."
3. If there exists $(0, 0, 0) \neq \mathbf{P}_1 = (p_{11}, p_{12}, p_{13}) \in \mathbb{L}^3$ such that $p_{11}F_x + p_{12}F_y + p_{13}F_z = 0$, compute a plane $\mathcal{L}$ not parallel to $\mathbf{P}_1$, and the intersection curve, $\mathcal{C}$, of $\mathcal{S}$ and $\mathcal{L}$.
   3.1. If $\mathcal{C}$ has a rational proper parametrization $\widetilde{\mathbf{P}}(t) \in \mathbb{L}(t)^3$, return $\widetilde{\mathbf{P}}(t) + s\mathbf{P}_1 \in \mathbb{L}(s,t)^3$ is a rational proper parametrization of the cylindrical surface $\mathcal{S}$.
   3.2. Otherwise, return "$\mathcal{S}$ is a cylindrical developable surface but not rational."
4. Solve the algebraic system $S := \{F = F_x = F_y = F_z = 0\}$ by applying for instance Wu's zero decomposition (see [22]). Compute a rational proper parametrization $\widetilde{\mathbf{P}}(t) \in \mathbb{L}(t)^3$ of a curve $\mathcal{C} \subset S$ applying for instance the resolvent method in [23].
   4.1. If $\widetilde{\mathbf{P}}(t) + s\widetilde{\mathbf{P}}'(t) \in \mathbb{L}(s,t)^3$ parametrizes $\mathcal{S}$, return $\widetilde{\mathbf{P}}(t) + s\widetilde{\mathbf{P}}'(t)$ is a rational proper parametrization of the tangential surface $\mathcal{S}$.
   4.2. If there does not exist any curve $\mathcal{C} \subset S$ satisfying the condition of Step 4.1, return "$\mathcal{S}$ is a tangential developable surface but not rational."

Remark 3.7 We here give some details for computation.

a. To find the fixed point $\mathbf{P}_0$ in Step 2, we need to solve $S := \{T(x, y, z; u, v, w) = 0, F(u, v, w) = 0\}$ with respect to $\{x, y, z\}$. In order to simplify the computation, we can solve the linear system $T(x, y, z; u_i, v_i, w_i) = 0$ for some random selected points $(u_i, v_i, w_i) \in \mathcal{S}$, and check the solutions lying in S. Observe that one also may use the arithmetic in the quotient field of rational functions $\mathbb{K}(\mathcal{S})$, and to compute remainders with the polynomial $F(u, v, w)$.

b. To find the ruling direction $\mathbf{P}_1$ in Step 3, we only need to consider the coefficient vectors of F_x, F_y and F_z. It holds that for a cylindrical surface, $\mathbf{P}_1$ is linearly dependent with the vector (p_{11}, p_{12}, p_{13}).

c. In step 4, S may has different curves since it is the collection of singular curves. To determine cuspidal edge $\mathcal{C}$ from S, one needs to check two conditions: $\mathcal{C}$ is rational and $\widetilde{\mathbf{P}}(t) + s\widetilde{\mathbf{P}}'(t) \in \mathbb{L}(s,t)^3$ parametrizes $\mathcal{S}$, where $\widetilde{\mathbf{P}}(t)$ is a rational proper parametrization of $\mathcal{C}$.

Examples of Algorithm 1

Example 3.8 Let $\mathcal{S}$ be the algebraic surface defined by the polynomial

$$F(x,y,z)=4x^2+9y^2-4x-6y-z^2+2\in\mathbb{R}[x,y,z].$$

In Step 1, we compute $K(x,y,z)=576F(x,y,z)$, which implies that $\mathcal{S}$ is a developable surface. In Step 2, we get that the tangent plane at (u,v,w) is

$$(8u-4)(x-u)+(18v-6)y-v)-2w(z-w)=0,\quad \text{where}\quad F(u,v,w)=0.$$

One has that some random tangent planes pass through the point $\mathbf{P}_0=(x_0,y_0,z_0)=(1/2,1/3,0)\in\mathbb{R}^3$. We check that $\mathbf{P}_0$ is a common point of $\{T(x,y,z;u,v,w)=0,F(u,v,w)=0\}$. Then, $\mathcal{S}$ is a conical surface.

Let $L(x,y,z)=x-z=0$ be the plane $\mathcal{L}$ not passing through $\mathbf{P}_0$. Then, the intersection curve, $\mathcal{C}$, of the surface $\mathcal{S}$ and the plane $\mathcal{L}$ is defined by the equation $3x^2+9y^2-4x-6y+2=0$ (we eliminate the variable z). The curve $\mathcal{C}$ can be regarded as a planar curve, and it has a rational proper parametrization given by

$$(q_1(t),q_2(t)):=\left(\frac{9+t^2}{27+t^2},\frac{t^2-6t+27}{3t^2+81}\right)\in\mathbb{R}(t)^2.$$

We substitute (q_1,q_2) into the plane $x-z=0$, and solving the variable z, we get the rational proper parametrization of the space curve $\mathcal{C}$

$$\widetilde{\mathbf{P}}(t)=\left(\frac{9+t^2}{27+t^2},\frac{t^2-6t+27}{3t^2+81},\frac{9+t^2}{27+t^2}\right)\in\mathbb{R}(t)^3.$$

Therefore, a parametrization of $\mathcal{S}$ is given by

$$(1-s)\mathbf{P}_0+s\widetilde{\mathbf{P}}(t)=(1-s)(1/2,1/3,0)+s\left(\frac{9+t^2}{27+t^2},\frac{t^2-6t+27}{3t^2+81},\frac{9+t^2}{27+t^2}\right)\in\mathbb{R}(s,t)^3.$$

Example 3.9 Let $\mathcal{S}$ be the algebraic surface defined by the polynomial
$F(x,y,z)=x^4+4\,x^3y+6\,x^2y^2+4\,xy^3+y^4-10\,x^3-27\,x^2y-3\,x^2z-18\,xy^2-18\,xyz+6\,xz^2-2\,y^3-12\,y^2z+3\,yz^2+z^3+16\,x^2+8\,xy+24\,xz+16\,y^2-24\,yz+24\,z^2+64\,x-32\,y+96\,z.$

We have that $K(x,y,z)=0$ on $\mathcal{S}$ which implies that $\mathcal{S}$ is a developable surface. There is not a fixed point that belongs to all the tangent planes. Then, we go to Step 3.

The normal vector $\mathbf{N}(x,y,z)$ of $\mathcal{S}$ is orthogonal to $\mathbf{P}_1=(1,-1,-1)\in\mathbb{R}^3$, i.e., $F_x-F_y-F_z=0$, which implies that $\mathcal{S}$ is a cylindrical surface.

Let $L(x,y,z)=x+y+z=0$ be the plane not parallel to $\mathbf{P}_1$. The intersection curve, $\mathcal{C}$, of $\mathcal{S}$ and the plane has a rational proper parametrization
$\widetilde{\mathbf{P}}(t)=(-554752\,t^2+439520\,t+311296\,t^3-65536\,t^4-130606,65536\,t^4-307200\,t^3+540160\,t^2-422240\,t+123804,14592\,t^2-17280\,t-4096\,t^3+6802)\in\mathbb{R}(t)^3.$
Then, a parametrization of the surface $\mathcal{S}$ is given by $\widetilde{\mathbf{P}}(t)+s(1,-1,-1)\in\mathbb{R}(s,t)^3$.

Example 3.10 Let $\mathcal{S}$ be the algebraic surface defined by the polynomial $F(x,y,z) = 11+16\,z-12\,y-36\,x-4\,z^2-48\,yz+12\,y^2-36\,xz+36\,xy+42\,x^2+48\,y^2z+72\,xyz-24\,xy^2+24\,x^2z-36\,x^2y-20\,x^3-32\,zy^3-48\,y^2zx-24\,zyx^2+12\,x^2y^2-4\,zx^3+12\,x^3y+3\,x^4 \in \mathbb{R}[x,y,z]$.

We follow Steps 1,2 and 3, and we get that $\mathcal{S}$ is neither a conical surface nor a cylindrical surface. Thus, $\mathcal{S}$ is a tangential developable surface. We go to Step 4, and we solve the algebraic system $S := \{F_x = 0, F_y = 0, F_z = 0, F = 0\}$. For this purpose, we use `WSOLVE` (http://www.mmrc.iss.ac.cn/ dwang/wsolve.htm) which is a maple package to solve the characteristic set. We get

$$S = \left\{ \begin{array}{l} \{2\,xy+2\,yz-2\,y-3\,z+xz, 4\,y^3-2\,z+6\,yz+z^2\}; \\ \{3+x^2, y-1, z+2\}; \\ \{2\,x-3, 4\,y-1, 4\,z-1\}; \\ \{x-3, y, z-2\}; \\ \{x-1, y, z\}. \end{array} \right\}$$

Only the first component has dimension one, and then it should be the cuspidal edge. Thus, the cuspidal edge is the algebraic curve $\mathcal{C}$ defined by two surfaces of equations

$$\{2\,xy+2\,yz-2\,y-3\,z+xz=0; 4\,y^3-2\,z+6\,yz+z^2=0\}.$$

We can determine its rationality and we parameterize it by applying for instance the resolvent method in [23]. Actually, in this example, one has that the cylindrical surface $4y^3-2\,z+6\,yz+z^2=0$ can be regarded as a planar curve (in $\mathbb{R}^2$), and it has a rational parametrization given by

$$(q_1(t), q_2(t)) = \left(\frac{(t+2)(t-4)}{4(t-1)^2}, \frac{(t+3)^3}{4(t-1)^3} \right) \in \mathbb{R}(t)^2.$$

We substitute (q_1, q_2) into the surface defined by the equation $2\,xy+2\,yz-2\,y-3\,z+xz=0$, and we solve the variable x. Then, we get a parametrization of the cuspidal edge $\mathcal{C}$ given by

$$\widetilde{\mathbf{P}}(t) = \left(\frac{3(t^2+2)}{2(t-1)^2}, \frac{(t+2)(t-4)}{4(t-1)^2}, \frac{(t+3)^3}{4(t-1)^3} \right) \in \mathbb{R}(t)^3.$$

Therefore, a rational parameterization of $\mathcal{S}$ is $\widetilde{\mathbf{P}}(t) + s\widetilde{\mathbf{P}}'(t) \in \mathbb{R}(s,t)^3$.

Refine the parameterizations

Reparametrizing a rational surface such that it does not contain any base point is usually a cumbersome task, even for a ruled surface. An affine *base point* of a rational surface parameterized by $\mathbf{P}(s,t)$ is a parameter pair (s_0, t_0) such that the numerator and denominator of each component of $\mathbf{P}(s,t)$ at (s_0, t_0) are zero. The μ-basis technique in [24] provides a simple and elegant way to reparameterize a rational ruled surface such that it does not contain any non-generic base point. Furthermore, the directrices of the reparameterized surface have the lowest possible degree. Thus there are both geometrical and computational advantages to be gained from such a reparametrization. Here we refine the parametrization using the μ-basis method in [24]. The more efficient algorithm to compute μ-basis can be found in [25]. Continue to Example 3.10 of the tangential surface, one can get the refined rational reparametrization $\mathbf{P}(u,v)=\mathbf{P}_0(u)+v\mathbf{P}_1(u)\in\mathbb{R}(u,v)^3$ where

$$\mathbf{P}_0(u)=\left(-\frac{5\,u-8}{2(u^2-2\,u+1)},\frac{u^2+7\,u-11}{4(u^2-2\,u+1)},-\frac{7(u^2+4\,u+4)}{8(u^2-2\,u+1)}\right)\in\mathbb{R}(u)^3$$

and

$$\mathbf{P}_1(u)=(2\,u^2+2\,u-4,-3\,u+3,3\,u^2/2+6\,u+6)\in\mathbb{R}(u)^3.$$

4 Parametrically developable surfaces

In this section, we consider a surface $\mathcal{S}$ defined by a parametrization (not necessarily proper),

$$\mathbf{P}(s,t)=(p_1(s,t),p_2(s,t),p_3(s,t))\in\mathbb{L}(s,t)^3. \tag{3}$$

We give the necessary and sufficient condition so that $\mathcal{S}$ represents a developable surface. The following theorem can be deduced from Theorem 3.1, and the details also was proposed in [19].

Theorem 4.1 *A given parametric surface $\mathcal{S}$ defined by $\mathbf{P}(s,t)$ of the form* (3)*, is developable if and only if $K(s,t)=0$, where*

$$K(s,t)=\begin{vmatrix} l_s & l_t & l\\ m_s & m_t & m\\ n_s & n_t & n\end{vmatrix}, \tag{4}$$

and $l=p_{2s}p_{3t}-p_{3s}p_{2t}$, $m=p_{3s}p_{1t}-p_{1s}p_{3t}$ and $n=p_{1s}p_{2t}-p_{2s}p_{1t}$.

Since we can tell whether $\mathcal{S}$ is a developable surface, we compute a proper reparametrization in standard form for the affirmative case. For this purpose, let $\mathbf{N}(s,t)=(n_1,n_2,n_3)=\mathbf{P}_s\times\mathbf{P}_t$ be the normal vector of the parametric surface $\mathcal{S}$, and we denote $\mathbf{X}=(x,y,z)$. Then, the tangent plane of $\mathcal{S}$ is $T(x,y,z)=\mathbf{N}(s,t)\cdot(\mathbf{X}-\mathbf{P}(s,t))=0$. According to the three types of developable surface (see Theorem 3.5), we have the following algorithm.

Algorithm 2 Input: An algebraic surface $\mathcal{S}$ defined by the parametrization $\mathbf{P}(s,t)\in\mathbb{L}(s,t)^3$.
Output: A proper rational parametrization over $\mathbb{L}$ of $\mathcal{S}$ or the message of "$\mathcal{S}$ is not a developable surface".

1. Compute $K(s,t)$ of the form (4). If it is zero go to Step 2. Otherwise, return "$\mathcal{S}$ is not a developable surface."

2. If $T(x,y,z)=0$ passes through a fixed point $\mathbf{P}_0\in\mathbb{L}^3$, compute $\widetilde{\mathbf{P}}(t)\in\mathbb{L}(t)^3$ a rational proper parametrization of the curve $\mathcal{C}$ which is the intersection curve of $\mathcal{S}$ and a plane

not passing through $\mathbf{P}_0$. Then, return $(1-s)\mathbf{P}_0 + s\widetilde{\mathbf{P}}(t) \in \mathbb{L}(s,t)^3$ is a rational proper parametrization of the conical surface $\mathcal{S}$.

3. If there exists $(0,0,0) \neq \mathbf{P}_1 = (p_{11}, p_{12}, p_{13}) \in \mathbb{L}^3$ such that $p_{11}n_1 + p_{12}n_2 + p_{13}n_3 = 0$, compute $\widetilde{\mathbf{P}}(t) \in \mathbb{L}(t)^3$ a rational proper parametrization of the curve $\mathcal{C}$ which is the intersection of $\mathcal{S}$ and a plane not parallel to $\mathbf{P}_1$. Then, return $\widetilde{\mathbf{P}}(t) + s\mathbf{P}_1 \in \mathbb{L}(s,t)^3$ is a rational proper parametrization of the cylindrical surface $\mathcal{S}$.

4. Solve the algebraic system $S := \{\mathbf{X} - \mathbf{P}(s,t) = (0,0,0),\, \mathbf{N}(s,t) = (0,0,0)\}$, and consider a rational curve $\mathcal{C} \subset S$. Compute a rational proper parametrization $\widetilde{\mathbf{P}}(t) \in \mathbb{L}(t)^3$ of $\mathcal{C}$ (apply for instance the resolvent method in [23]), and check whether $\widetilde{\mathbf{P}}(t) + s\widetilde{\mathbf{P}}'(t) \in \mathbb{L}(s,t)^3$ parametrizes $\mathcal{S}$. In the affirmative case, return $\widetilde{\mathbf{P}}(t) + s\widetilde{\mathbf{P}}'(t)$ is a rational proper parametrization of the tangential surface $\mathcal{S}$. Otherwise, consider a new rational curve $\mathcal{C} \subset S$.

Remark 4.2 We give some necessary remarks for **Algorithm** 2.

a. The point $\mathbf{P}_0$ in Step 2 can be obtained from the coefficient set of the tangent plane $T(x,y,z;s,t) = 0$ with respect to $\{s,t\}$. To compute $\mathbf{P}_1$ in Step 3, we just need to find the linearly dependent coefficient vector of n_1, n_2 and n_3.

b. In Step 4, it is known that the cuspidal edge is included in the singular set S of the surface $\mathcal{S}$. The cuspidal edge is a prime set and then, it can be separately solved by the resolvent method in [23].

c. For the intersection curve in Step 2 or Step 3, or the singular curve in Step 4, we may get an improper parameterized curve if the given parametrization $\mathbf{P}(s,t)$ is improper. In this case, we check out the improper case (apply for instance the method in [26]), and we properly reparameterize the curve by some methods such as [27, 28].

d. In Step 4, we should check whether $\widetilde{\mathbf{P}}(t) + s\widetilde{\mathbf{P}}'(t)$ parametrizes $\mathcal{S}$. Here, we recommend to compute the implicit polynomial $F(x,y,z)$ defined by the parametrization $\widetilde{\mathbf{P}}(t) + s\widetilde{\mathbf{P}}'(t)$ (observe that $\widetilde{\mathbf{P}}(t) + s\widetilde{\mathbf{P}}'(t)$ defines a ruled surface and then it has an efficient implicitization method; see for instance [29]). If $F(\mathbf{P}(s,t)) = 0$, one concludes $\widetilde{\mathbf{P}}(t) + s\widetilde{\mathbf{P}}'(t)$ parametrizes $\mathcal{S}$.

Examples of Algorithm 2

Example 4.3 Let $\mathcal{S}$ be a surface defined by the parametrization
$\mathbf{P}(s,t) = (p_1(s,t), p_2(s,t), p_3(s,t)) =$

$$\left(\frac{4\,s^2+t+1-2\,s+t^2+2\,ts}{1-2\,t-2\,s+t^2+2\,ts+s^2}, \frac{6\,ts^2+7\,t^2+6\,s^3+8\,ts-s^2-4\,t+1-2\,s}{1-2\,t-2\,s+t^2+2\,ts+s^2},\right.$$
$$\left.\frac{t^2s^2+2\,ts^3+6\,ts^2+t^3+2\,t^2s+5\,t^2+s^4+5\,s^3+5\,ts}{1-2\,t-2\,s+t^2+2\,ts+s^2}\right) \in \mathbb{R}(s,t)^3.$$

Following Step 1, we have that $\mathcal{S}$ is a developable surface since $K(s,t)=0$. We observe that this parametrization is improper (apply the method in [26]).

In Step 2, solving the coefficient set of $T(x,y,z;s,t)$ with respect to $\{s,t\}$, we get a fixed point $\mathbf{P}_0=(1,1,0)\in\mathbb{R}^3$ of the tangent planes. Therefore, $\mathcal{S}$ is a conical surface with the apex $\mathbf{P}_0$.

Let $L(x,y,z)=0$ be the equation defining a plane $\mathcal{L}$ not passing through $\mathbf{P}_0$. To simplify the computation, one considers the plane defined by the polynomial $L(x,y,z)=z-1\in\mathbb{R}[x,y,z]$, and we compute a proper parametrization of the curve $\mathcal{C}$ that is the intersection curve of $\mathcal{S}$ and $\mathcal{L}$. That is, $\mathcal{C}$ is included in $\{x=p_1(s,t), y=p_2(s,t), z=p_3(s,t), p_3(s,t)-1=0\}$. By the classical properties of the resultant (see for instance [30] or [31]), the implicit equation of the projected curve of $\mathcal{C}$ on the $(x:y)$ plane is a factor of

$$\mathrm{Res}_s(\mathrm{Res}_t(\mathrm{num}(x-p_1),\mathrm{num}(p_3-1)),\mathrm{Res}_t(\mathrm{num}(y-p_2),\mathrm{num}(p_3-1))),$$

where $\mathrm{num}(\cdot)$ returns the numerator of a rational function, and Res_{var} returns the resultant of two polynomials with respect to var. We get that the implicit equation of the projected curve is

$$283-338\,x+64\,x^2-120\,y+102\,xy+9\,y^2=0.$$

We find a rational parametrization, and we lift it to get the parametrization of $\mathcal{C}$. We have that

$$\widetilde{\mathbf{P}}(t)=\left(\frac{-283+120\,t-9\,t^2}{-258+90\,t},\frac{283-507\,t+144\,t^2}{-387+135\,t},1\right)\in\mathbb{R}(t)^3.$$

Finally, we obtain a rational proper reparametrization $(1-s)\mathbf{P}_0+s\widetilde{\mathbf{P}}(t)\in\mathbb{R}(s,t)^3$ for the surface $\mathcal{S}$.

The computation process is similar as above if the surface has a ruling direction.

Example 4.4 Let $\mathcal{S}$ be an algebraic surface defined by the parametrization
$\mathbf{P}(s,t)=((-1+2\,t+2\,s+3\,t^2s^2-2\,ts-ts^2+2\,ts^3+4\,s^5-t^6+4\,s^4t^2-3\,t^4s^2-2\,t^2s^3-2\,t^4s+4\,s^4t-2\,t^3s^2-2\,t^3s-s^3-s^4-2\,t^5-s^2)/\left(t^2+s+t-1\right)^2,(-3\,t^4s-2\,t^2s^2+3\,t^2s+4\,ts^3-5\,t^5-ts^2-t^2+3\,t^3+2\,s^4-s^3-3\,s^4t-6\,t^3s+2\,t^4s^2-6\,t^3s^3+6\,t^3s^2+2\,t^2s^3+6\,t^5s-s^5-2\,s^6-3\,t^4s^4+3\,t^2s^6+3\,s^7+3\,t^6s-3\,t^5s^2-t^6s^2+3\,s^6t-3\,s^4t^3+t^8-3\,t^4s^3+3\,t^7-3\,t^2s^5)/\left(t^2+s+t-1\right)^3,2\,s^4(3\,t^2s^2+3\,s^3+3\,ts^2-2\,s^2-3\,t^4-3\,t^2s-3\,t^3+3\,t^2)/\left(t^2+s+t-1\right)^3)\in\mathbb{R}(s,t)^3$.

$\mathcal{S}$ is a developable surface because $K(s,t)=0$. In addition, $\mathcal{S}$ is a rational tangential surface since it is neither a conical surface nor a cylindrical surface.

In Step 4, we solve the algebraic system $S:=\{\mathbf{X}-\mathbf{P}(s,t)=(0,0,0),\mathbf{N}(s,t)=(0,0,0)\}$ using Maple package WSOLVE. We get a rational space curve defined by the proper parametrization

$$\widetilde{\mathbf{P}}(t)=(2\,t^2-3\,t,-t^3+3/2\,t^2+1/4\,t-3/8,-2\,t^3+3\,t^2-3/2\,t+1/4)\in\mathbb{R}(t)^3.$$

Now, we consider the parametrization $\widetilde{\mathbf{P}}(t)+s\widetilde{\mathbf{P}}'(t)\in\mathbb{R}(s,t)^3$. Using univariate resultant (see [29]), we can get its implicit equation

$F(x,y,z) = 32 + 96\,y + 96\,x - 48\,z + 96\,x^2 + 32\,x^3 + 48\,y^2 - 64\,y^3 - 48\,y^4 - 48\,y^2x^2 + 96\,x^2y + 192\,xy - 96\,xy^3 - 20\,z^2 + 32\,z^3 + 13\,z^4 + 48\,zy + 12\,z^3x - 12\,z^2x^2 - 48\,z^2x - 144\,z^2y + 120\,z^2y^2 - 72\,z^3y - 32\,zy^3 + 192\,zy^2 - 48\,zx^2 - 72\,z^2xy + 144\,y^2xz + 96\,zyx + 48\,zx^2y - 96\,zx = 0.$

It holds that $F(\mathbf{P}(s,t)) = 0$, and hence $\widetilde{\mathbf{P}}(t) + s\widetilde{\mathbf{P}}'(t)$ is a rational proper parametrization of the tangential surface $\mathcal{S}$.

5 Conclusion

Taking into account that the developability of an algebraic surface is associated with the Gaussian curvature, and that there are three different types of developable surfaces, we deal with the (re)parametrization problem for rational developable surfaces. More precisely, given an implicitly algebraic surface $\mathcal{S}$, we analyze whether $\mathcal{S}$ is a rational developable surface and in the affirmative case, we compute a rational proper parametrization of $\mathcal{S}$. For this purpose, we analyze the three types of developable surfaces, and we prove the main theorem constructively. Finally, for a given algebraic surface defined by a parametrization (not necessarily proper), we determine its developability and we find a proper reparametrization.

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