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Number of common roots and resultant of two tropical univariate polynomials

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Abstract

It is well known that for two univariate polynomials over complex number field the number of their common roots is equal to the order of their resultant. In this paper, we show that this fundamental relationship still holds for the tropical polynomials under suitable adaptation of the notion of order, if the roots are simple and non-zero.

Keywords. tropical semifield; tropical resultant; common roots

MSC. 15A15, 15A80, 12K10.

1 Introduction

The resultant plays a crucial role in algebra and algebraic geometry [27, 23, 17, 5, 1, 9]. Let m, n be fixed. Let

$$\mathbf{A} = \mathbf{a}_0\mathbf{x}^m + \mathbf{a}_1\mathbf{x}^{m-1} + \cdots + \mathbf{a}_m \in \mathbb{C}[\mathbf{a}, \mathbf{x}]$$
$$\mathbf{B} = \mathbf{b}_0\mathbf{x}^n + \mathbf{b}_1\mathbf{x}^{n-1} + \cdots + \mathbf{b}_n \in \mathbb{C}[\mathbf{b}, \mathbf{x}]$$

Then the *resultant* $R \in \mathbb{C}[\mathbf{a}, \mathbf{b}]$ is defined as the smallest monic polynomial (w.r.t. a given order) such that, for every $a \in \mathbb{C}^{m+1}$ and $b \in \mathbb{C}^{n+1}$, if the two polynomials $\mathbf{A}(a, \mathbf{x}), \mathbf{B}(b, \mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ have a common complex root then $R(a, b) = 0$.[1] We recall the following two well known fundamental properties of resultants. Let $a \in \mathbb{C}^{m+1}$ and $b \in \mathbb{C}^{n+1}$ such that $a_0, b_0 \neq 0$. Then we have

P1. The point (a, b) is a root of R if and only if the polynomials $\mathbf{A}(a, \mathbf{x})$ and $\mathbf{B}(b, \mathbf{x})$ have a common complex root. (Of course, the 'if' part is immediate from the definition and thus the interesting part is the 'only if').

P2. The order of the point (a, b) at R is equal to the number of common complex roots of the polynomials $\mathbf{A}(a, \mathbf{x})$ and $\mathbf{B}(b, \mathbf{x})$. (See the appendix)

[1] It is well-known that the resultant R can be defined in various other ways: for instance, in terms of Sylvester matrix, Bezout matrix, Barnett matrix, Hankel matrix (see [7] for a nice summary). Those definitions are more useful for computational purposes. However they are also more complicated when deducing theoretical or structural properties. Since the main interest of this paper is not computational but structural, we chose the more structural definition.

A natural question arises: *whether/how these properties can be adapted to polynomials over the tropical semifield.* Recall that the tropical semifield is the set $\mathbb{R}\cup\{-\infty\}$ where the addition operation is defined as the usual maximum, the multiplication operation is defined as the usual addition. As a result, it does not allow subtraction (due to lack of additive inverse; hence the name semifield). It has been intensively investigated due to numerous interesting applications [24, 22, 21, 12, 2, 10, 25, 13, 4, 26, 16, 3, 18, 11].

Note that, unlike polynomials over $\mathbb{C}$, it is easy to compute the roots of polynomials over tropical semifield, and thus, counting number of common roots is also easy. Hence the motivation for asking the above question is not for finding an efficient algorithm, but for gaining structural understanding on the relation between roots and the resultant.

There have been several adaptations of resultant over $\mathbb{C}$ to the tropical semifield [19, 8, 2, 28, 20, 15]. In particular, Tabera [28] and Odagiri [20] showed that the above property P1 holds over the tropical semifield, if one redefines the notions of roots and resultant as follows: (1) a root is redefined as a point where the graph of the polynomial is not smooth (2) the resultant is redefined as the tropicalization of resultant over $\mathbb{C}$ ([28]) or as the permanent of the Sylvester matrix ([20]). In this paper, we will follow the definition of [28].

The main contribution of this paper is to show that the property P2 also holds over the tropical semifield, if one puts a slight restriction on (a, b) and if one makes a suitable adaptation of the notion of order, as follows: (1) we restrict (a, b) such that the polynomials $\mathbf{A}(a, \mathbf{x})$ and $\mathbf{B}(b, \mathbf{x})$ have only simple and non-zero roots. (2) the notion of the order of a point p at a multivariate polynomial C is replaced by the new concept of "order", which is the $\log_2$ of the numbers of terms, say t, in C such that $t(p) = C(p)$.

The paper is structured as follows. In Section 2, we state formally the main result of the paper. In Section 3, we provide a proof. In Section 4, we summarize the main result and discuss some potential generalizations and associated difficulties. In Appendix, we include a simple proof of P2 over $\mathbb{C}$, provided by Laurent Busé.

2 Main Result

In this section, we present the main result of this paper. For this, we need to recall some basic notions on the tropical semi-field and the tropical resultant. The *tropical semi-field* is the tuple $(\mathbb{T}, +, \times, /)$, where $\mathbb{T} = \mathbb{R}\cup\{-\infty\}$, $+$ is the usual maximum, $\times$ is the usual addition, and $/$ is the usual subtraction. It is easy to see that the additive identity (tropical zero) is $-\infty$ and that the multiplicative identity (tropical one) is 0.

Let $\mathbb{T}[\mathbf{x}]$ be the set of all polynomials in the indeterminate $\mathbf{x}$. The polynomial $C \in \mathbb{T}[\mathbf{x}]$ represents a function: $\mathbb{T} \to \mathbb{T}$. We say that $\alpha \in \mathbb{T}$ is a *root* of C if the graph of C has a corner over α. The *multiplicity* of α is the change in the slopes of the graph across α. When the multiplicity of α is one, we say that α is *simple.*

Finally, we recall the notion of tropical resultant (see [28] for further details). Let $R \in \mathbb{C}[\mathbf{a}, \mathbf{b}]$ be the resultant w.r.t. the fixed degrees m, n. Then the *tropical resultant* $\mathfrak{R} \in \mathbb{T}[\mathbf{a}, \mathbf{b}]$ is defined as the tropicalization of R, that is, the tropical sum of the supports of R.[2]

Now we adapt the notion of the order to the tropical semifield.

Definition 1 (Order)**.** *Let $C \in \mathbb{T}[\mathbf{z}_1, \ldots, \mathbf{z}_l]$ be a tropical non-zero polynomial. Let $C = t_1 + \cdots + t_r$ where t_i's are terms in $\mathbf{z}$ with tropical non-zero coefficients. Let $p \in \mathbb{T}^l$. Let $E_C(p) := \{t_i : C(p) = t_i(p)\}$. The* order[3] *of p in C, written as $O_C(p)$, is defined by*

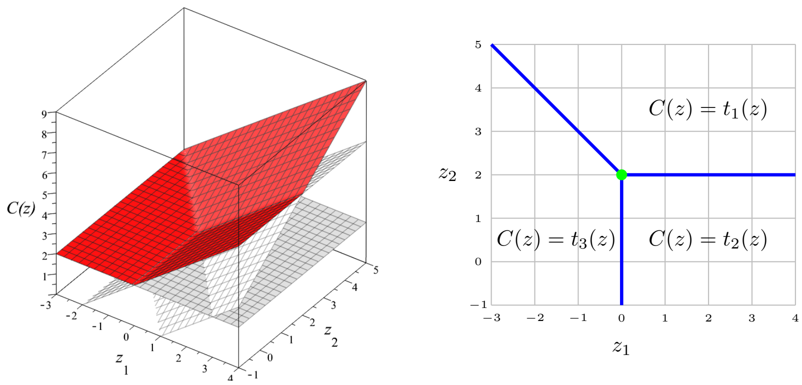
$$O_C(p) := \log_2 \#E_C(p)$$

Remark 2. *The definition of the order given above may look a bit strange, especially because it could be non-integer when $\#E_C(p)$ is not a power of 2. The motivation behind the definition is that it allows us to state our main theorem in a similar way as in the field case. Compare P2 in the introduction and Theorem 4. In particular, when the roots are simple and tropical non-zero, then $\#E_C(p)$ is a power of 2.*

Example 3. *Let $C(z) = \mathbf{z}_1\mathbf{z}_2 + 2\mathbf{z}_1 + 2 \in \mathbb{T}[\mathbf{z}]$. The plots in Figure 1 describe the graph of C.*

[2] A motivation behind this definition comes from the observation that if one carries out the $\log_t$-coordinate transform where $t \to \infty$, then R becomes the tropical sum of the support of R.

[3] Over $\mathbb{T}$, the notions of multiplicity and order are *not* the same, unlike over $\mathbb{C}$, already in the univariate case.



$(p_1, p_2) \in \mathbb{T}^2$	color	$E_C(p)$	$\#E_C(p)$	$O_C(p)$
$t_1 = t_2 = t_3$	green	$\{t_1, t_2, t_3\}$	3	$\log_2 3$
$t_1 = t_2 > t_3$	blue	$\{t_1, t_2\}$	2	1
$t_1 = t_3 > t_2$	blue	$\{t_1, t_3\}$	2	1
$t_2 = t_3 > t_1$	blue	$\{t_2, t_3\}$	2	1
$t_1 > t_2, t_3$	white	$\{t_1\}$	1	0
$t_2 > t_1, t_3$	white	$\{t_2\}$	1	0
$t_3 > t_1, t_2$	white	$\{t_3\}$	1	0

Figure 1: Polynomial C in Example 3

The left plot shows the function represented by C. The red piecewise plane is the graph of $C(\mathbf{z})$ and the three gray planes are the graphs of the terms in C. Let $t_1 = \mathbf{z}_1\mathbf{z}_2$, $t_2 = 2\mathbf{z}_1$, and $t_3 = 2$ be the terms in C. The right plot explicitly shows the partition of the $\mathbf{z}$ plane into pieces and the corresponding term for each piece. The bottom table shows the values of the order on each piece.

Theorem 4 (Main Result)**.** *Let*

$$\begin{aligned} A &= a_0\mathbf{x}^m + a_1\mathbf{x}^{m-1} + \cdots + a_m \in \mathbb{T}[\mathbf{x}], \quad a_0 \neq -\infty \\ B &= b_0\mathbf{x}^n + b_1\mathbf{x}^{n-1} + \cdots + b_n \quad \in \mathbb{T}[\mathbf{x}], \quad b_0 \neq -\infty \end{aligned}$$

be with simple tropical non-zero roots. Then the following two are equivalent:

1. *A and B have exactly k common roots.*
2. *$O_{\mathfrak{R}}(a, b) = k$.*

Example 5. *We will illustrate the main result on a simple example. Let*

$$A = a_0\mathbf{x}^3 + a_1\mathbf{x}^2 + a_2\mathbf{x} + a_3 \in \mathbb{T}[\mathbf{x}]$$

be a monic polynomial with roots $\alpha_1 > \alpha_2 > \alpha_3 \neq -\infty$ and let

$$B = b_0\mathbf{x}^2 + b_1\mathbf{x} + b_2 \in \mathbb{T}[\mathbf{x}]$$

be a monic polynomial with the roots $\beta_1 > \beta_2 \neq -\infty$. Assume that

$$\begin{array}{ccccc} \alpha_1 & & \alpha_3 & \\ \| & > \alpha_2 > & \| & \neq -\infty \\ \beta_1 & & \beta_2 & \end{array}$$

Now we will verify the main result on this example. Note

1. *A and B have exactly 2 common roots, namely $\alpha_1 = \beta_1$ and $\alpha_3 = \beta_2$.*

2. *Table 1 shows the value of each term in $\mathfrak{R}$.*

Term	Value in roots	Value simplified
$\mathbf{a}_0^2\mathbf{b}_2^3$	$(0)^2(\beta_1\beta_2)^3$	$\alpha_1^3\alpha_3^3$
$\mathbf{a}_0\mathbf{a}_1\mathbf{b}_1\mathbf{b}_2^2$	$(0)(\alpha_1)(\beta_1)(\beta_1\beta_2)^2$	$\alpha_1^4\alpha_3^2$
$\mathbf{a}_0\mathbf{a}_2\mathbf{b}_0\mathbf{b}_2^2$	$(0)(\alpha_1\alpha_2)(0)(\beta_1\beta_2)^2$	$\alpha_1^3\alpha_2\alpha_3^2$
$\mathbf{a}_0\mathbf{a}_2\mathbf{b}_1^2\mathbf{b}_2$	$(0)(\alpha_1\alpha_2)(\beta_1)^2(\beta_1\beta_2)$	$\alpha_1^4\alpha_2\alpha_3$
$\mathbf{a}_0\mathbf{a}_3\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$	$(0)(\alpha_1\alpha_2\alpha_3)(0)(\beta_1)(\beta_1\beta_2)$	$\alpha_1^3\alpha_2\alpha_3^2$
$\mathbf{a}_0\mathbf{a}_3\mathbf{b}_1^3$	$(0)(\alpha_1\alpha_2\alpha_3)(\beta_1)^3$	$\alpha_1^4\alpha_2\alpha_3$
$\mathbf{a}_1^2\mathbf{b}_0\mathbf{b}_2^2$	$(\alpha_1)^2(0)(\beta_1\beta_2)^2$	$\alpha_1^4\alpha_3^2$
$\mathbf{a}_1\mathbf{a}_2\mathbf{b}_0\mathbf{b}_1\mathbf{b}_2$	$(\alpha_1)(\alpha_1\alpha_2)(0)(\beta_1)(\beta_1\beta_2)$	$\alpha_1^4\alpha_2\alpha_3$
$\mathbf{a}_1\mathbf{a}_3\mathbf{b}_0\mathbf{b}_1^2$	$(\alpha_1)(\alpha_1\alpha_2\alpha_3)(0)(\beta_1)^2$	$\alpha_1^4\alpha_2\alpha_3$
$\mathbf{a}_1\mathbf{a}_3\mathbf{b}_0^2\mathbf{b}_2$	$(\alpha_1)(\alpha_1\alpha_2\alpha_3)(0)^2(\beta_1\beta_2)$	$\alpha_1^3\alpha_2\alpha_3^2$
$\mathbf{a}_2^2\mathbf{b}_0^2\mathbf{b}_2$	$(\alpha_1\alpha_2)^2(0)^2(\beta_1\beta_2)$	$\alpha_1^3\alpha_2^2\alpha_3$
$\mathbf{a}_2\mathbf{a}_3\mathbf{b}_0^2\mathbf{b}_1$	$(\alpha_1\alpha_2)(\alpha_1\alpha_2\alpha_3)(0)^2(\beta_1)$	$\alpha_1^3\alpha_2^2\alpha_3$
$\mathbf{a}_3^2\mathbf{b}_0^3$	$(\alpha_1\alpha_2\alpha_3)^2(0)^3$	$\alpha_1^2\alpha_2^2\alpha_3^2$

Table 1: Value of each term of $\mathfrak{R}$, in Example 5, at the roots

In the second column, we used the obvious relation

$$\begin{array}{llll} a_0 = 0 & a_1 = \alpha_1 & a_2 = \alpha_1\alpha_2 & a_3 = \alpha_1\alpha_2\alpha_3 \\ b_0 = 0 & b_1 = \beta_1 & b_2 = \beta_1\beta_2 & \end{array}$$

In the last column, we simplified the value using the fact that $\alpha_1 = \beta_1$ and $\alpha_3 = \beta_2$, for the sake of easier comparison among the values. One can straightforwardly verify that $\alpha_1^4\alpha_2\alpha_3$ is the maximum among the values. Thus $\mathfrak{R}(a,b) = \alpha_1^4\alpha_2\alpha_3$. Hence the corresponding terms are given by

$$E_{\mathfrak{R}}(a,b) = \{\mathbf{a}_0\mathbf{a}_2\mathbf{b}_1^2\mathbf{b}_2,\ \mathbf{a_0 a_3 b_1^3},\ \mathbf{a_1 a_2 b_0 b_1 b_2},\ \mathbf{a}_1\mathbf{a}_3\mathbf{b}_0\mathbf{b}_1^2\}$$

Thus $\#E_{\mathfrak{R}}(a,b) = 2^2$ and $O_{\mathfrak{R}}(a,b) = 2$.

We have verified the main result on this example.

3 Proof

In this section, we provide a proof of the main result (Theorem 4). One naturally wonders whether a proof for the main result can be obtained by suitably 'translating" a proof for the field case (such as the one given in Appendix). We have tried the approach, without success. Thus we developed a completely different proof strategy.

Before plunging into a technically detailed proof, we first provide an informal overview of the proof strategy. Note that the tropical resultant $\mathfrak{R}$ is defined as the tropicalization of (i.e., the tropical sum of all the terms appearing in) the resultant over $\mathbb{C}$. Recall that the resultant over $\mathbb{C}$ is same as the determinant of the Sylvester matrix. Hence the tropical resultant $\mathfrak{R}$ is the tropicalization of the determinant of the Sylvester matrix. Recalling that terms in the determinant correspond to permutations of column indices, we observe that each term in $\mathfrak{R}$ comes from one or more permutations of $(1,\ldots,n+m)$. Thus we focus our attention on the permutations. Let S be the set of all the permutations. Then, the main steps of the proof consist of the followings:

1. Lemma 14: We "prune" the set S, obtaining S^*, by removing all the permutations that never yields $\mathfrak{R}(a,b)$, no matter what a and b are.

2. Lemma 16: We show that each permutation in S^* provides a different term in the tropical resultant polynomial $\mathfrak{R}(\mathbf{a},\mathbf{b})$.

3. Lemma 18: Using the above two lemmas, we show that the following three elements of $\mathbb{T}$ are the same: $\mathfrak{R}(a,b)$, P and P^* that are the value of the determinant of the Sylvester matrix by considering all the permutations in S (the case of P) or only the permutations in S^* (for the case of P^*).

4. Lemma 24: We "characterize" the permutations in S^* that yield $\mathfrak{R}(a,b)$, in terms of the ordering among the roots of A and B.

5. Lemma 28: We show that the number of permutation in the previous step is exactly 2^k, where k is the number of common roots.

6. The main result is immediate from the above lemmas.

Now we plunge into the details of the proof. From now on, we fix $a = (a_0,\ldots,a_m) \in \mathbb{T}^{m+1}$ and $b = (b_0,\ldots,b_n) \in \mathbb{T}^{n+1}$ such that the polynomials

$$\begin{aligned} A &= a_0\mathbf{x}^m + a_1\mathbf{x}^{m-1} + \cdots + a_m \in \mathbb{T}[\mathbf{x}],\ a_0 \neq -\infty \\ B &= b_0\mathbf{x}^n + b_1\mathbf{x}^{n-1} + \cdots + b_n \quad \in \mathbb{T}[\mathbf{x}],\ b_0 \neq -\infty \end{aligned}$$

are with simple tropical non-zero roots.

The next two lemmas will be used to reduced the proof to the monic polynomial case.

Lemma 6. *Let $C = c_0\mathbf{x}^d + c_1\mathbf{x}^{d-1} + \cdots + c_d \in \mathbb{T}[\mathbf{x}]$ where $c_0 \neq -\infty$. The roots of C and $\frac{1}{c_0}C$ are the same.*

Proof. Obvious. □

Lemma 7. $O_{\mathfrak{R}}(a,b) = O_{\mathfrak{R}}\left(\frac{1}{a_0}a, \frac{1}{b_0}b\right)$.

Proof. Let $\mathfrak{R}$ be expressed as $\mathfrak{R} = t_1 + \cdots + t_\rho$ where t_i are terms in $\mathbf{a},\mathbf{b}$ with tropical non-zero coefficients. We observe that $\mathfrak{R}$ is bi-homogeneous of degrees n and m in the variables $\mathbf{a}$ and $\mathbf{b}$, respectively. Thus for each term t_i we have

$$a_0^m b_0^n \ \ t_i\left(\frac{1}{a_0}a, \frac{1}{b_0}b\right) \quad = \quad t_i(a,b).$$

Now, the lemma follows immediately from the definition of order. □

Therefore, taking into account of Lemma 6 and Lemma 7, we can restrict the proof of the main result to the monic case, without losing generality. Thus, in the following, we assume that A, B are monic; that is $a_0 = 0 = b_0$. In addition, let $\alpha_1 > \cdots > \alpha_m$ be the roots of A and $\beta_1 > \cdots > \beta_n$ be the roots of B. We obviously have

$$\begin{aligned} a_i &= \alpha_1 \cdots \alpha_i \\ b_i &= \beta_1 \cdots \beta_i \end{aligned}$$

Note that, since $\alpha_i \neq -\infty$ and $\beta_j \neq -\infty$, we see that $a_i \neq -\infty$ and $b_j \neq -\infty$. We will set $a_i = -\infty$ if $i > m$ or $i < 0$ and set $b_i = -\infty$ if $i > n$ or $i < 0$. Let

$$[\mathbf{a}_{j-i}] = \begin{bmatrix} \mathbf{a}_0 & \mathbf{a}_1 & \cdots & \cdots & \mathbf{a}_m & & & \\ & \ddots & \ddots & & & & \ddots & \\ & & \mathbf{a}_0 & \mathbf{a}_1 & \cdots & \cdots & & \mathbf{a}_m \end{bmatrix} \in \mathbb{T}[\mathbf{a}]^{n\times(n+m)}$$

$$[\mathbf{b}_{j-i}] = \begin{bmatrix} \mathbf{b}_0 & \mathbf{b}_1 & \cdots & \mathbf{b}_n & & & \\ & \ddots & \ddots & & \ddots & & \\ & & \ddots & \ddots & & \ddots & \\ & & & \mathbf{b}_0 & \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix} \in \mathbb{T}[\mathbf{b}]^{m\times(n+m)}$$

$$\mathbf{M} := \begin{bmatrix} [\mathbf{a}_{j-i}] \\ [\mathbf{b}_{j-i}] \end{bmatrix} \in \mathbb{T}[\mathbf{a},\mathbf{b}]^{(n+m)\times(n+m)}$$

$$M := \mathbf{M}(a,b) \in \mathbb{T}^{(n+m)\times(n+m)}$$

Notation 8. *Let S stand for the set of all permutations of $(1,\ldots,n+m)$. Furthermore, for $\pi \in S$, let*

$$\mathbf{M}_\pi := \mathbf{M}_{1,\pi_1} \cdots \mathbf{M}_{n+m,\pi_{n+m}}$$
$$M_\pi := \mathbf{M}_\pi(a,b)$$

Moreover, let S^* stand for the set of all $(\nu_1,\ldots,\nu_n,\mu_1,\ldots,\mu_m) \in S$ such that $\nu_1 < \cdots < \nu_n$ and $\mu_1 < \cdots < \mu_m$.

Remark 9. *From the structure of M it follows that, if $\pi = (\nu_1,\ldots,\nu_n,\mu_1,\ldots,\mu_m) \in S$, then*

$$\mathbf{M}_\pi = \mathbf{a}_{\nu_1-1}\cdots\mathbf{a}_{\nu_n-n}\ \mathbf{b}_{\mu_1-1}\cdots\mathbf{b}_{\mu_m-m}$$

In the next example we see that S^* is much smaller that S. Later, we will see that the relevant information for our problem lies in S^*

Example 10. *Let $m = 3$ and $n = 2$. Then*

$$\begin{aligned} S^* = \{\ & (1,2,\ \ 3,4,5), \quad (1,3,\ \ 2,4,5), \quad (1,4,\ \ 2,3,5), \quad (1,5,\ \ 2,3,4), \\ & (2,3,\ \ 1,4,5), \quad (2,4,\ \ 1,3,5), \quad (2,5,\ \ 1,3,4), \\ & (3,4,\ \ 1,2,5), \quad (3,5,\ \ 1,2,4), \\ & (4,5,\ \ 1,2,3)\ \} \end{aligned}$$

Note

$$\#S^* = \frac{(n+m)!}{n!m!} = \frac{(2+3)!}{2!3!} = 10, \ \ \#S = 120.$$

Next, we introduce the following notations.

$$\mathbf{P} = \sum_{\pi\in S} \mathbf{M}_\pi, \text{ and } \mathbf{P}^* = \sum_{\pi\in S^*} \mathbf{M}_\pi.$$
$$P = \mathbf{P}(a,b), \text{ and } P^* = \mathbf{P}^*(a,b).$$

The next lemmas will be used to conclude that $P = P^* = \mathfrak{R}(a,b)$ (see Lemma 18).

Lemma 11. $\mathfrak{R}(a,b) \le P$.

Proof. Note that the terms of $\mathfrak{R}(\mathbf{a},\mathbf{b})$ come from some of the permutations in the definition of $\mathbf{P}$. Thus $\mathfrak{R}(a,b) \le P$. □

Lemma 12 (Odagiri 2008, [20])**.** *Let $\pi = (\nu_1,\ldots,\nu_n,\mu_1,\ldots,\mu_m) \in S$ be such that $M_\pi \ne -\infty$. Then we have the followings.*

1. *Suppose that $\nu_k > \nu_{k+1}$ for some k. Let $\pi' := (\nu_1,\ldots,\nu_{k+1},\nu_k,\ldots,\nu_n,\mu_1,\ldots,\mu_m)$, that is, obtained from π by swapping ν_k and ν_{k+1}. Then $M_{\pi'} > M_\pi$.*

2. *Suppose that $\mu_k > \mu_{k+1}$ for some k. Let $\pi' := (\nu_1, \ldots, \nu_n, \mu_1, \ldots, \mu_{k+1}, \mu_k, \ldots, \mu_m)$, that is, obtained from π by swapping μ_k and μ_{k+1}. Then $M_{\pi'} > M_\pi$.*

Example 13. *Let $m = 3$ and $n = 3$. Let*

$$\pi = (1, 4, 3, 2, 5) \quad \text{and} \quad \pi' = (1, 4, 2, 3, 5)$$

They represent the following choices of elements ("path") encircled

$$\pi : \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \\ & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & & \\ & b_0 & b_1 & b_2 & \\ & & b_0 & b_1 & b_2 \end{bmatrix} \quad \text{and} \quad \pi' : \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \\ & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & & \\ & b_0 & b_1 & b_2 & \\ & & b_0 & b_1 & b_2 \end{bmatrix}$$

Note

$$\frac{M_{\pi'}}{M_\pi} = \frac{a_0 a_2 b_1 b_1 b_2}{a_0 a_2 b_2 b_0 b_2} = \frac{b_1 b_1}{b_2 b_0} = \frac{(\beta_1)(\beta_1)}{(\beta_1 \beta_2)(0)} = \frac{\beta_1}{\beta_2} > 0$$

Thus

$$M_{\pi'} > M_\pi$$

verifying the lemma on π and π'. Observe that π has a "zigzag" in the bottom part, while π' does not have a zigzag. The lemma says that a zigzag makes the value of a path smaller.

Proof of Lemma 12. The proof was given in [20]. However for the sake of reader's convenience and the notational consistency, we provide a complete proof here. We will show the proof of the claim 1 only. The proof for the claim 2 is essentially the same. Note

$$\begin{aligned} \frac{M_{\pi'}}{M_\pi} &= \frac{a_{\nu_1-1} \cdots a_{\nu_{k+1}-k} \; a_{\nu_k-(k+1)} \cdots a_{\nu_n-n} \; b_{\mu_1-1} \cdots b_{\mu_m-m}}{a_{\nu_1-1} \cdots a_{\nu_k-k} \; a_{\nu_{k+1}-(k+1)} \cdots a_{\nu_n-n} \; b_{\mu_1-1} \cdots b_{\mu_m-m}} \\ &= \frac{a_{\nu_{k+1}-k} \quad a_{\nu_k-(k+1)}}{a_{\nu_k-k} \quad a_{\nu_{k+1}-(k+1)}} \\ &= \frac{a_{\nu_{k+1}-(k+1)} \; \alpha_{\nu_{k+1}-k} \quad a_{\nu_k-(k+1)}}{a_{\nu_k-(k+1)} \; \alpha_{\nu_k-k} \quad a_{\nu_{k+1}-(k+1)}} \\ &= \frac{\alpha_{\nu_{k+1}-k}}{\alpha_{\nu_k-k}} \\ &> 0 \end{aligned}$$

Thus $M_{\pi'} > M_\pi$. ☐

Lemma 14. *If $\pi \in S \setminus S^*$, then $P > M_\pi$.*

Proof. Let $\pi \in S \setminus S^*$. From [20, Main Theorem], we have $P = \prod_{ij} (\alpha_i + \beta_j)$. Since $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ are tropically non-zero, we have $P \neq -\infty$ Suppose that $M_\pi = -\infty$. Then we obviously have $P > M_\pi$. Thus from now on, assume that $M_\pi \neq -\infty$. Note

$$P = \sum_{\rho \in S} M_\rho = \sum_{\rho \in S \setminus \{\pi\}} M_\rho + M_\pi$$

From Lemma 12, we have $\pi' \in S \setminus \{\pi\}$ such that $M_{\pi'} > M_\pi$. Hence $P > M_\pi$. ☐

Lemma 15. $P = P^*$.

Proof. Immediate from the definition of P, P^*. and Lemma 14. $\square$

Lemma 16. *Let $\pi, \pi' \in S^*$. The following two are equivalent.*

1. $\pi = \pi'$
2. $\mathbf{M}_\pi = \mathbf{M}_{\pi'}$

Proof. Let $\pi = (\nu, \mu)$, $\pi' = (\nu', \mu') \in S^*$. It is obvious that $1 \Longrightarrow 2$. It remains to show $2 \Longrightarrow 1$. Assume $\mathbf{M}_\pi = \mathbf{M}_{\pi'}$. We will show that $\pi = \pi'$. Note

$$\mathbf{M}_\pi = \mathbf{a}_{\bar{\nu}_1} \cdots \mathbf{a}_{\bar{\nu}_n} \mathbf{b}_{\bar{\mu}_1} \cdots \mathbf{b}_{\bar{\mu}_m}$$
$$\mathbf{M}_{\pi'} = \mathbf{a}_{\bar{\nu}'_1} \cdots \mathbf{a}_{\bar{\nu}'_n} \mathbf{b}_{\bar{\mu}'_1} \cdots \mathbf{b}_{\bar{\mu}'_m}$$

where $\bar{\mu}_i = \mu_i - i$, $\bar{\nu}_i = \nu_i - i$, $\bar{\mu}'_i = \mu'_i - i$ and $\bar{\nu}'_i = \nu'_i - i$. Since $\pi \in S^*$ we have $0 \le \bar{\nu}_1 \le \cdots \le \bar{\nu}_n \le m$ and $0 \le \bar{\mu}_1 \le \ldots \le \bar{\mu}_m \le n$. The same with $(\bar{\nu}', \bar{\mu}')$. Since $\mathbf{M}_\pi = \mathbf{M}_{\pi'}$, we have $\bar{\nu} = \bar{\nu}'$ and $\bar{\mu} = \bar{\mu}'$, in turn, $\nu = \nu'$ and $\mu = \mu'$. Thus $\pi = \pi'$. $\square$

Remark 17. *The above lemma implies that the terms generated by the permutations in S^*, in the determinant of the Sylvester matrix, do not cancel since they appear only once. Thus all the terms in $\mathbf{P}^*$ appear in $\mathfrak{R}(\mathbf{a}, \mathbf{b})$.*

Lemma 18. $\mathfrak{R}(a, b) = P = P^*$.

Proof. From Lemmas 15, we have $P = P^*$. From Lemma 16 and Remark 17, we have $P^* \le \mathfrak{R}(a, b)$. Thus $P \le \mathfrak{R}(a, b)$. By Lemma 11 we know that $\mathfrak{R}(a, b) \le P$. So, the statement holds. $\square$

Remark 19. *We emphasize that the above lemma states that $\mathfrak{R}(\mathbf{a}, \mathbf{b}), \mathbf{P}$ and $\mathbf{P}^*$ are the same as* functions, *but not necessary as tropical* polynomials.

Lemma 20. *Let $\pi = (\nu_1, \ldots, \nu_n, \mu_1, \ldots, \mu_m) \in S^*$. Let $p_i = n - (\mu_i - i)$ and $q_i = m - (\nu_i - i)$. Then we have*

$$M_\pi = \alpha^p \beta^q$$

Example 21. *Let us $m = 3$ and $n = 2$. Let $\pi = (1, 4,\ 2, 3, 5) \in S^*$. Then*

$$\begin{aligned}
M_\pi &= a_{1-1}\ a_{4-2}\ \ b_{2-1}\ b_{3-2}\ b_{5-3} \\
&= a_0 a_2 b_1 b_1 b_2 \\
&= (0)(\alpha_1\alpha_2)(\beta_1)(\beta_1)(\beta_1\beta_2) \\
&= \alpha_1^1 \alpha_2^1 \alpha_3^0 \beta_1^3 \beta_2^1 \\
\alpha^p \beta^q &= \alpha_1^{2-(2-1)} \alpha_2^{2-(3-2)} \alpha_3^{2-(5-3)} \beta_1^{3-(1-1)} \beta_2^{3-(4-2)} \\
&= \alpha_1^1 \alpha_2^1 \alpha_3^0 \beta_1^3 \beta_2^1
\end{aligned}$$

Hence $M_\pi = \alpha^p \beta^q$, verifying the lemma on the particular π.

Proof of Lemma 20. There are two cases: $\mu_1 = 1$ or $\nu_1 = 1$. We will show a proof of the lemma only for the case $\mu_1 = 1$. The proof for the case $\nu_1 = 1$ is essentially the same. We will divide the proof into two steps.

1. Let $s_1, s_2, \ldots$ be the lengths of the consecutive blocks in μ. Likewise let $t_1, t_2, \ldots$ be the lengths of the consecutive blocks in ν. Then

$$\mu = (1, \ldots, s_1, \quad s_1 + t_1 + 1, \ldots, s_1 + t_1 + s_2, \quad s_1 + t_1 + s_2 + t_2 + 1, \ldots, s_1 + t_1 + s_2 + t_2 + s_3, \quad \ldots)$$
$$\nu = (s_1 + 1, \ldots, s_1 + t_1, \quad s_1 + t_1 + s_2 + 1, \ldots, s_1 + t_1 + s_2 + t_2, \quad \ldots)$$

Let $\overline{\mu}_j = \mu_j - j$ and $\overline{\nu}_i = \nu_i - i$. Then

$$\begin{aligned}\bar{\mu} &= (\underbrace{0,\ldots,0}_{s_1}, \quad \underbrace{t_1,\ldots,t_1}_{s_2}, \quad \underbrace{t_1+t_2,\ldots,t_1+t_2}_{s_3}, \quad \ldots) \\ \bar{\nu} &= (\underbrace{s_1,\ldots,s_1}_{t_1}, \quad \underbrace{s_1+s_2,\ldots,s_1+s_2}_{t_2}, \quad \ldots)\end{aligned} \tag{1}$$

2. Note

$$\begin{aligned}M_\pi &= a_{\overline{\nu}_1} a_{\overline{\nu}_2} \cdots b_{\overline{\mu}_1} b_{\overline{\mu}_2} \cdots \\ &= a_{s_1}^{t_1} a_{s_1+s_2}^{t_2} \cdots b_{t_1}^{s_2} b_{t_1+t_2}^{s_3} \cdots \qquad \text{from (1) and the fact that } b_0 = 1 \\ &= \prod_{k=1}^{s_1} \alpha_k^{t_1} \prod_{k=1}^{s_1+s_2} \alpha_k^{t_2} \cdots \prod_{k=1}^{t_1} \beta_k^{s_2} \prod_{k=1}^{t_1+t_2} \beta_k^{s_3} \cdots \\ &= \prod_{k=1}^{s_1} \alpha_k^{t_1+t_2+\cdots} \prod_{k=s_1+1}^{s_1+s_2} \alpha_k^{t_2+t_3+\cdots} \cdots \prod_{k=1}^{t_1} \beta_k^{s_2+s_3+\cdots} \prod_{k=t_1+1}^{t_1+t_2} \beta_k^{s_3+s_4+\cdots} \cdots \\ &= \prod_{k=1}^{s_1} \alpha_k^{n-0} \prod_{k=s_1+1}^{s_1+s_2} \alpha_k^{n-t_1} \cdots \prod_{k=1}^{t_1} \beta_k^{m-s_1} \prod_{k=t_1+1}^{t_1+t_2} \beta_k^{m-(s_1+s_2)} \cdots \\ &= \prod_{k=1}^{s_1} \alpha_k^{n-\bar{\mu}_k} \prod_{k=s_1+1}^{s_1+s_2} \alpha_k^{n-\bar{\mu}_k} \cdots \prod_{k=1}^{t_1} \beta_k^{m-\bar{\nu}_k} \prod_{k=t_1+1}^{t_1+t_2} \beta_k^{m-\bar{\nu}_k} \cdots \\ &= \prod_{k=1}^{m} \alpha_k^{n-\bar{\mu}_k} \prod_{k=1}^{n} \beta_k^{m-\bar{\nu}_k} \\ &= \prod_{k=1}^{m} \alpha_k^{p_k} \prod_{k=1}^{n} \beta_k^{q_k} \\ &= \alpha^p \beta^q\end{aligned}$$

□

Notation 22. *Let $\pi \in S$ and L be a list of length $n+m$. Then $\pi(L)$ is the list obtained from L by permuting the element according to π, that is, by moving the i-th element of L to the π_i-th position.*

Example 23. *Let $m = 3$ and $n = 2$. Let $\pi = (1, 4, 2, 3, 5) \in S^*$. Then*

$$\pi(\beta_1, \beta_2, \alpha_1, \alpha_2, \alpha_3) = (\beta_1, \alpha_1, \alpha_2, \beta_2, \alpha_3)$$

Lemma 24. *Let $\pi \in S$. The followings are equivalent*

1. *The elements of $\pi(\beta, \alpha)$ are non-increasing*
2. *$P = M_\pi$.*

Example 25. *We illustrate the lemma using Example 5, that is, $m = 3$ and $n = 2$ and*

$$\begin{array}{c} \alpha_1 \\ \| \\ \beta_1 \end{array} > \alpha_2 > \begin{array}{c} \alpha_3 \\ \| \\ \beta_2 \end{array} \neq -\infty$$

For $\pi \in S \backslash S^*$, it is easy to check that both 1 and 2 are false and thus the lemma is verified. For $\pi \in S^*$, Table 2 verifies the lemma. In the table, γ stands for $\pi(\beta, \alpha)$. In the 3rd and the 6th columns, we simplified the previous columns using the fact that $\alpha_1 = \beta_1$ and $\alpha_3 = \beta_2$, for the sake of easier checks in the next columns.

$\pi \in S^*$	γ	γ simplified	$\gamma_1 \geq \cdots \geq \gamma_5$	M_π	$M_\pi$simplified	$P = M_\pi$
$(1,2, \quad 3,4,5)$	$(\beta_1,\beta_2,\alpha_1,\alpha_2,\alpha_3)$	$(\alpha_1,\alpha_3,\alpha_1,\alpha_2,\alpha_3)$	false	$\beta_1^3\beta_2^3$	$\alpha_1^3\alpha_3^3$	false
$(1,3, \quad 2,4,5)$	$(\beta_1,\alpha_1,\beta_2,\alpha_2,\alpha_3)$	$(\alpha_1,\alpha_1,\alpha_3,\alpha_2,\alpha_3)$	false	$\alpha_1\beta_1^3\beta_2^2$	$\alpha_1^4\alpha_3^2$	false
$(1,4, \quad 2,3,5)$	$(\beta_1,\alpha_1,\alpha_2,\beta_2,\alpha_3)$	$(\alpha_1,\alpha_1,\alpha_2,\alpha_3,\alpha_3)$	true	$\alpha_1\alpha_2\beta_1^3\beta_2$	$\alpha_1^4\alpha_2\alpha_3$	true
$(1,5, \quad 2,3,4)$	$(\beta_1,\alpha_1,\alpha_2,\alpha_3,\beta_2)$	$(\alpha_1,\alpha_1,\alpha_2,\alpha_3,\alpha_3)$	true	$\alpha_1\alpha_2\alpha_3\beta_1^3$	$\alpha_1^4\alpha_2\alpha_3$	true
$(2,3, \quad 1,4,5)$	$(\alpha_1,\beta_1,\beta_2.\alpha_2,\alpha_3)$	$(\alpha_1,\alpha_1,\alpha_3.\alpha_2,\alpha_3)$	false	$\alpha_1^2\beta_1^2\beta_2^2$	$\alpha_1^4\alpha_3^2$	false
$(2,4, \quad 1,3,5)$	$(\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3)$	$(\alpha_1,\alpha_1,\alpha_2,\alpha_3,\alpha_3)$	true	$\alpha_1^2\alpha_2\beta_1^2\beta_2$	$\alpha_1^4\alpha_2\alpha_3$	true
$(2,5, \quad 1,3,4)$	$(\alpha_1,\beta_1,\alpha_2,\alpha_3,\beta_2)$	$(\alpha_1,\alpha_1,\alpha_2,\alpha_3,\alpha_3)$	true	$\alpha_1^2\alpha_2\beta_1^2\alpha_3$	$\alpha_1^4\alpha_2\alpha_3$	true
$(3,4, \quad 1,2,5)$	$(\alpha_1,\alpha_2,\beta_1,\beta_2,\alpha_3)$	$(\alpha_1,\alpha_2,\alpha_1,\alpha_3,\alpha_3)$	false	$\alpha_1^2\alpha_2^2\beta_1\beta_2$	$\alpha_1^3\alpha_2^2\alpha_3$	false
$(3,5, \quad 1,2,4)$	$(\alpha_1,\alpha_2,\beta_1,\alpha_3.\beta_2)$	$(\alpha_1,\alpha_2,\alpha_1,\alpha_3.\alpha_3)$	false	$\alpha_1^2\alpha_2^2\beta_1\alpha_3$	$\alpha_1^3\alpha_2^2\alpha_3$	false
$(4,5, \quad 1,2,3)$	$(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2)$	$(\alpha_1,\alpha_2,\alpha_3,\alpha_1,\alpha_3)$	false	$\alpha_1^2\alpha_2^2\alpha_3^2$	$\alpha_1^2\alpha_2^2\alpha_3^2$	false

Table 2: Verification of Lemma 24 in Example 25, where $\gamma = \pi(\beta, \alpha)$

Proof of Lemma 24. Let $\gamma = \pi(\beta, \alpha)$. We divide the proof into two cases: $\pi \in S \setminus S^*$ and $\pi \in S^*$.

Case 1: $\pi \in S \setminus S^*$. For some $i < j$, we have $\nu_i > \nu_j$ or $\mu_i > \mu_j$. Recall that $\gamma_{\nu_i} = \beta_i$, $\gamma_{\nu_j} = \beta_j$, $\gamma_{\mu_i} = \alpha_i$ and $\gamma_{\mu_j} = \alpha_j$. Since $\beta_i > \beta_j$ and $\alpha_i > \alpha_j$, we have $\gamma_{\nu_i} > \gamma_{\nu_j}$ or $\gamma_{\mu_i} > \gamma_{\mu_j}$. Thus the statement $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{n+m}$ is false. From Lemma 14, the statement $P = M_\pi$ is also false. Thus the lemma is vacuously true.

Case 2: $\pi \in S^*$. We prove each direction of implication one at a time:

1. If $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{n+m}$ then $P = M_\pi$.

   Assume that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{n+m}$. We need to show $P = M_\pi$. By Lemma 15

   $$P = \sum_{\pi' \in S^*} M_{\pi'}$$

   Let $\pi' = (\nu'_1, \ldots, \nu'_n, \mu'_1, \ldots, \mu'_m) \in S^*$ be such that $\pi' \neq \pi$. It suffices to show that $M_\pi \geq M_{\pi'}$. If $M_{\pi'} = -\infty$ then it is obvious true. Thus from now on, assume that $M_{\pi'} \neq -\infty$. Note, by Lemma 20,

   $$\begin{aligned}
   \frac{M_\pi}{M_{\pi'}} &= \frac{\prod_{i=1}^m \alpha_i^{n-(\mu_i - i)} \prod_{i=1}^n \beta_i^{m-(\nu_i - i)}}{\prod_{i=1}^m \alpha_i^{n-\left(\mu'_i - i\right)} \prod_{i=1}^n \beta_i^{m-\left(\nu'_i - i\right)}} \\
   &= \frac{\prod_{i=1}^m \alpha_i^{\mu_i{}'} \prod_{i=1}^n \beta_i^{\nu'_i}}{\prod_{i=1}^m \alpha_i^{\mu_i} \prod_{i=1}^n \beta_i^{\nu_i}} \\
   &= \frac{\prod_{i=1}^m \gamma_{\mu_i}^{\mu'_i} \prod_{i=1}^n \gamma_{\nu_i}^{\nu'_i}}{\prod_{i=1}^m \gamma_{\mu_i}^{\mu_i} \prod_{i=1}^n \gamma_{\nu_i}^{\nu_i}} \\
   &= \frac{\prod_{j=1}^{n+m} \gamma_j^{\lambda_j}}{\prod_{j=1}^{n+m} \gamma_j^{j}} \qquad \text{where } \lambda \in S \text{ such that } \lambda_{\mu_i} := \mu'_i \text{ and } \lambda_{\nu_i} := \nu'_i \\
   &\geq 0
   \end{aligned}$$

   Hence $M_\pi \geq M_{\pi'}$.

2. If $P = M_\pi$ then $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{n+m}$.

   We will prove the contrapositive. Assume that it is false that $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{n+m}$. We need to show that $P > M_\pi$. Let k be such that $\gamma_k < \gamma_{k+1}$. Let i and j such that $k = \pi_i$ and $k+1 = \pi_j$. We consider the following four potential cases.

(a) $i \le n$ and $j \le n$
Note that π has the following form

$$\pi = (\nu, \mu) = (\underbrace{\dots, k, k+1, \dots}_{n} \underbrace{\dots \dots}_{m})$$

where k appears at the i-th position and $k+1$ appears at the j-th position (in fact, j must be $i+1$, since $\pi \in S^*$) in the ν block. Thus $\gamma_k = \beta_i$ and $\gamma_{k+1} = \beta_j$. Since $\gamma_k < \gamma_{k+1}$, we should have $\beta_i < \beta_j$. However this is not possible due to the global assumption $\beta_1 > \beta_2 > \cdots > \beta_n$. Thus this case cannot occur.

(b) $i > n$ and $j > n$
This case cannot occur, due to the essentially same reason as above.

(c) $i \le n$ and $j > n$
We divide the proof into several steps.

i. Note that π has the following form

$$\pi = (\nu, \mu) = (\underbrace{\dots, k, \dots}_{n} \underbrace{\dots, k+1, \dots}_{m})$$

where k appears on the i-th position in the ν block and $k+1$ appears on the $(j-n)$-th position in the μ block.

ii. Note $\gamma_k = \beta_i$ and $\gamma_{k+1} = \alpha_{j-n}$. Since $\gamma_k < \gamma_{k+1}$, we have $\beta_i < \alpha_{j-n}$.

iii. Let π' be obtained from π by swapping π_i and π_j. Then π' has the following form

$$\pi' = (\nu', \mu') = (\underbrace{\dots, k+1, \dots}_{n} \underbrace{\dots, k, \dots}_{m})$$

where $k+1$ appears on the i-th position in the ν' block and k appears on the $(j-n)$-th position in the μ' block.

iv. We will show that $\pi' \in S^*$. Since $\pi \in S^*$, we have that ν and μ are strictly increasing. By inspecting the form of π shown above, we see that, in the ν block, everything to the left of k is less then k and everything to the right of k is greater than $k+1$ and that, in the μ block, everything to the left of $k+1$ is less than k and everything to the right of $k+1$ is greater than $k+1$. By inspecting the form of π' shown above, we see that ν' and μ' are strictly increasing. Thus $\pi' \in S^*$.

v. From Lemma 20, we have

$$\frac{M_{\pi'}}{M_\pi} = \frac{\cdots \alpha_{j-n}^{n-(k-(j-n))} \quad \cdots \qquad \cdots \beta_i^{m-(k+1-i)} \cdots}{\cdots \alpha_{j-n}^{n-(k+1-(j-n))} \cdots \qquad \cdots \beta_i^{m-(k-i)} \qquad \cdots} = \frac{\alpha_{j-n}}{\beta_i} > 0$$

Thus $M_{\pi'} > M_\pi$. Thus $P > M_\pi$.

(d) $i > n$ and $j \le n$
We can show that $P > M_\pi$, using the essentially same argument as above.

□

Notation 26. *Let* $\Delta = \{\pi \in S : P = M_\pi\}$, *and* $\Theta = \{\pi \in S^* : P^* = M_\pi\}$. *Let* $\delta = \#\Delta$, *i.e. the number of "maximum permutations". Similarly, let* $\theta = \#\Theta$.

Lemma 27. $\#E_{\mathfrak{R}}(a, b) = \delta = \theta$.

Proof. By Lemma 18 we have that $P = P^*$. Thus, $\Theta \subset \Delta$. Moreover, by Lemma 14, we get that $\Delta \subset \Theta$. Therefore, $\Delta = \Theta$ and in turn $\delta = \theta$. On the other hand, we have that $E_{\mathfrak{R}}(a,b) \subset \Delta$, and by Lemma 16 we have $\Theta \subset E_{\mathfrak{R}}(a,b)$. Thus $\Theta \subset E_{\mathfrak{R}}(a,b) \subset \Delta$ in turn $\theta \leq \#E_{\mathfrak{R}}(a,b) \leq \delta$. Now, the lemma follows directly. $\square$

Lemma 28. *The following two are equivalent:*

1. *A and B have exactly k common roots.*
2. *$\delta = 2^k$.*

Proof. Let us prove that $1 \Longrightarrow 2$. Let A and B be of degrees m and n with exactly k common roots, say $\alpha_{i_1} = \beta_{j_1}, \ldots, \alpha_{i_k} = \beta_{j_k}$ where the roots α's and β's are ordered as follows.

$$\cdots > \begin{array}{c}\alpha_{i_1}\\ \| \\ \beta_{j_1}\end{array} > \cdots > \begin{array}{c}\alpha_{i_2}\\ \| \\ \beta_{j_2}\end{array} > \cdots\cdots > \begin{array}{c}\alpha_{i_k}\\ \| \\ \beta_{j_k}\end{array} > \cdots \tag{2}$$

where $\cdots$ represent strict orderings among the other (non-common) roots. Note

$$\begin{aligned}
\delta &= \#\left\{\pi \in S : P = M_\pi\right\} \\
&= \#\left\{\pi \in S : \text{the elements in } \pi\left(\beta,\alpha\right) \text{ are non-increasing}\right\} \quad \text{from Lemma 24} \\
&= \#\left\{\pi \in S : \pi\left(\beta,\alpha\right) = \left(\cdots, \begin{array}{c}\alpha_{i_1},\beta_{j_1}\\ \beta_{j_1},\alpha_{i_1}\end{array}, \cdots, \begin{array}{c}\alpha_{i_2},\beta_{j_2}\\ \beta_{j_2},\alpha_{i_2}\end{array}, \cdots\cdots, \begin{array}{c}\alpha_{i_k},\beta_{j_k}\\ \beta_{j_k},\alpha_{i_k}\end{array}, \cdots\right)\right\} \quad \text{from (2)} \\
&\text{(where } \begin{array}{c}\square\\ \triangle\end{array} \text{ means "either } \square \text{ or } \triangle\text{")} \\
&= 2^k
\end{aligned}$$

Let us show that $2 \Longrightarrow 1$. Assume that $\delta = 2^k$. Let λ be the number of common roots of A and B. Then, since $1 \Longrightarrow 2$, we have $\delta = 2^\lambda$. Thus $2^\lambda = 2^k$, and hence $\lambda = k$. Thus A and B have exactly k common roots. $\square$

Proof of Main Result (Theorem 4). It is a direct consequence of Lemmas 27 and 28. $\square$

4 Conclusion and Discussion

The goal of this paper was to adapt the following well known property of univariate resultant over $\mathbb{C}$ to the tropical semifield: the number of common roots of the two polynomials is the same as the order of the resultant at the tuple of the coefficients. We have shown that the same property holds if we adapt the notion of order, and we restrict the roots of the polynomials to be tropical non-zeros and simple.

In the following, we will briefly and informally discuss a few questions naturally raised by the results given in this paper.

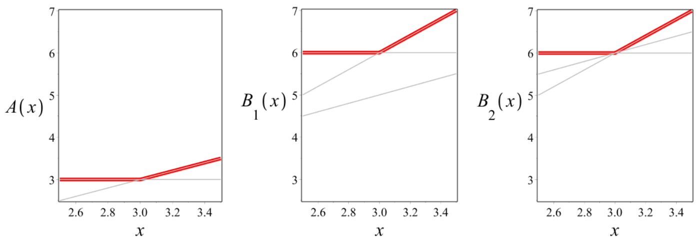
1. *Conditions on the root.* Note that we treated only the case when all the roots are tropical non-zeros and simple. We discuss what happens in the other cases.

   (a) *tropical zero* root: Suppose that the two polynomials A and B have the tropical zero ($-\infty$) as a root. The main result does not hold. The reason is as follows: each term of the polynomial $\mathfrak{R}$ always contains, at least, one of the indeterminates $\mathbf{a}_m$ and $\mathbf{b}_n$. On the other hand, the fact that $-\infty$ is a common root of A, B implies that $a_m = b_n = -\infty$. Thus, for every term t in $\mathfrak{R}$, we have $\mathfrak{R}(a,b) = t_i(a,b) = -\infty$. Therefore the order can be different from (in fact *not* related to) the number of common roots. Hence, in order to cover tropical zero roots, one will need to come up with a different notion of order. We leave it as an open challenge.

(b) *multiple* root: Suppose that a polynomial has a multiple (not simple) root. The main result does not hold. The reason is as follows: a function with a multiple root admits infinitely many polynomial representations. Furthermore the order of the resultant depends on which representation is chosen. Therefore the order can be different from the number of common roots. For example, consider the following three polynomials:

$$\begin{aligned} A &= 0\mathbf{x} + 3 \\ B_1 &= 0\mathbf{x}^2 + 2\mathbf{x} + 6 \\ B_2 &= 0\mathbf{x}^2 + 3\mathbf{x} + 6 \end{aligned}$$

representing the following functions:



- It is obvious that A has a simple root, namely 3. It is also obvious that B_1 and B_2 represent the same function, with one double root, namely 3.
- Thus the number of common roots of A and B_1 is 1. Likewise the number of common roots of A and B_2 is also 1.
- Direct computation show that

$$\mathfrak{R}(\mathbf{a}, \mathbf{b}) = \mathbf{a}_1^2\mathbf{b}_0 + \mathbf{a}_0\mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_0^2\mathbf{b}_2$$

  Thus

$$\begin{aligned} \mathfrak{R}(((0,3),(0,2,6))) &= \max\{6,5,6\} \\ \mathfrak{R}(((0,3),(0,3,6))) &= \max\{6,6,6\} \end{aligned}$$

- Hence

$$O_{\mathfrak{R}}((0,3),(0,2,6)) = \log_2 2 = 1$$

  but

$$O_{\mathfrak{R}}((0,3),(0,3,6)) = \log_2 3 \neq 1$$

Hence, in order to cover multiple roots, one will need to come up with a different notion of order. One natural and potential approach might be to look at the variety (polyhedral fan complex) of the resultant and investigate the co-dimension of the cone where the coefficients vector of two polynomials A and B belongs to. We leave it as an open challenge.

2. *Rank deficiency.* Over $\mathbb{C}$, it is well known that the number of the common roots is the same as the rank deficiency of the Sylvester matrix. Thus, one wonders whether the relation can be adapted to the tropical semifield. We divide the discussion into two cases.

   (a) *All roots are tropical non-zeros and simple*: Through numerous experiments on computer, we conjecture that the relation holds in this case if the rank is taken as the tropical rank [6, 14]. It is relatively easy to prove that the number of the common roots is bounded *below* by the rank deficiency (exploiting some of the proof techniques developed in this paper). However, it seems challenging to prove/disprove that the number of the common roots is bounded *above* by the rank deficiency. We leave it as open challenge.

(b) *The other cases*: The relation does not hold in general, as illustrated by the example

$$A = \mathbf{x}^3 + 2\mathbf{x}^2 + 2\mathbf{x} + 6, \qquad B = \mathbf{x}^2 - 2\mathbf{x} + 4$$

It is easy to verify that the roots of A and B are respectively $(2, 2, 2)$ and $(2, 2)$. Thus, the number of common roots is 2. However, a direct computation shows that the tropical rank deficiency of the Sylvester matrix is 1. In [6], two other different notions of ranks are considered, namely Kapranov and Barvinok. Hence one wonders whether any of those might make the relation hold. However, according to Theorem 1.4 in the paper, the tropical rank is not greater than the other two, and hence the tropical rank deficiency based on the other two can never be greater than 1. Thus, the relation does not hold for Kapranov and Barvinok ranks either.

Hence, in order to cover the tropical zero roots or multiple roots, one will need to come up with a different notion of rank. We leave it as an open challenge.

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5 Appendix: number of common roots and resultants over $\mathbb{C}$ (by Laurent Busé)

Recall the following property mentioned in the introduction: the order of the point at the resultant is equal to the number of common complex roots of the two polynomials over $\mathbb{C}$. This property is definitely part of the folklore but we were not able to find it in the existing literature. In the following, we communicate a simple proof kindly provided by Laurent Busé (laurent.buse@inria.fr). The proof is presented in a bit more general context of unique factorization domain. Furthermore, the number of common roots is seen as the degree of gcd.

Let k be a unique factorization domain. Given two positive integers m, n, consider the homogeneous polynomials

$$f(x,y) = a_0x^m + a_1x^{m-1}y + \cdots + a_my^m$$
$$g(x,y) = b_0x^n + b_1x^{n-1}y + \cdots + b_ny^n$$

in the variables x, y with coefficients in the ring $\mathbb{A} := k[a_0, \ldots, a_m, b_0, \ldots, b_n]$. The Sylvester resultant $R := \mathrm{Res}(f,g)$ of $f(x,y)$ and $g(x,y)$ is a polynomial in $\mathbb{A}$. Given a point

$$s := (p_0, \ldots, p_m, q_0, \ldots, q_n) \in k^{m+n+2},$$

the question is to determine the order of the resultant at s.

Proposition 29. *The order of the resultant polynomial R at the point s is equal to the degree of the gcd of the polynomials*

$$p(x,y) = \sum_{i=0}^{m} p_i x^{m-i} y^i, \;\; q(x,y) = \sum_{i=0}^{n} q_i x^{n-i} y^i,$$

unless $s = (0, \ldots, 0)$, i.e. $(p,q) = (0,0)$, in which case the order is equal to $m+n$.

Proof. The order of the resultant R at s is nothing but the t-valuation of the polynomial

$$R(p_0 + ta_0, \ldots, p_m + ta_m, q_0 + tb_0 + \ldots, q_n + tb_n) = \mathrm{Res}(p(x,y) + tf(x,y), q(x,y) + tg(x,y)) \in \mathbb{A}[t].$$

Denote by $h(x,y)$ the gcd of $p(x,y)$ and $q(x,y)$ and by δ its degree; there exist two polynomials $\tilde{p}$ and $\tilde{q}$ such that $p = \tilde{p}h$ and $q = \tilde{q}h$.

If $p = q = 0$ then the claimed property is clear. If $q = 0$ and $p \neq 0$, then

$$\mathrm{Res}(p + tf, q + tg) = \mathrm{Res}(p + tf, tg) = t^m \,\mathrm{Res}(p + tf, g).$$

Since $\mathrm{Res}(p + tf, g)_{|t=0} = \mathrm{Res}(p, g) \neq 0$, for $p \neq 0$ and g is the generic homogeneous polynomial of degree n, the claimed property is proved.

Now, assume that $q \neq 0$. By applying some classical properties of the resultant, we have that

$$\begin{aligned}\mathrm{Res}(\tilde{q}, tg)\,\mathrm{Res}(\tilde{p}h + tf, \tilde{q}h + tg) &= \mathrm{Res}(\tilde{q}, \tilde{q}h + tg)\,\mathrm{Res}(\tilde{p}h + tf, \tilde{q}h + tg)\\ &= \mathrm{Res}(\tilde{q}\tilde{p}h + t\tilde{q}f, \tilde{q}h + tg) = \mathrm{Res}(t(\tilde{q}f - \tilde{p}g), \tilde{q}h + tg).\end{aligned}$$

It follows that

$$t^{n-\delta}\,\mathrm{Res}(\tilde{q}, g)\,\mathrm{Res}(p + tf, q + tg) = t^n\,\mathrm{Res}(\tilde{q}f - \tilde{p}g, q + tg).$$

From here, the claimed property follows since $\mathrm{Res}(\tilde{q}, g) \neq 0$ and $\mathrm{Res}(\tilde{q}f - \tilde{p}g, q) \neq 0$, for $\tilde{p}$ and $\tilde{q}$ are coprime polynomials. $\square$