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The μ-Basis of Improper Rational Parametric Surface and Its Application

Sonia Pérez-Díaz [1,*,†] and Li-Yong Shen [2,3,†]

[1] Dpto de Física y Matemáticas, Universidad de Alcalá, E-28871 Madrid, Spain
[2] School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China; lyshen@ucas.ac.cn
[3] Key Laboratory of Big Data Mining and Knowledge Management, CAS, Beijing 100190, China
* Correspondence: sonia.perez@uah.es
† These authors contributed equally to this work.

Abstract: The μ-basis is a newly developed algebraic tool in curve and surface representations and it is used to analyze some essential geometric properties of curves and surfaces. However, the theoretical frame of μ-bases is still developing, especially of surfaces. We study the μ-basis of a rational surface $\mathcal{V}$ defined parametrically by $\mathcal{P}(\bar{t})$, $\bar{t} = (t_1, t_2)$ not being necessarily proper (or invertible). For applications using the μ-basis, an inversion formula for a given proper parametrization $\mathcal{P}(\bar{t})$ is obtained. In addition, the degree of the rational map $\phi_{\mathcal{P}}$ associated with any $\mathcal{P}(\bar{t})$ is computed. If $\mathcal{P}(\bar{t})$ is improper, we give some partial results in finding a proper reparametrization of $\mathcal{V}$. Finally, the implicitization formula is derived from $\mathcal{P}$ (not being necessarily proper). The discussions only need to compute the greatest common divisors and univariate resultants of polynomials constructed from the μ-basis. Examples are given to illustrate the computational processes of the presented results.

Keywords: μ-basis; rational surfaces; inversion; improper; reparametrization; implicitization; resultant

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1. Introduction

The study of representations of rational curves and surfaces is a fundamental task in computer aided geometric design (CAGD) and computer algebra. There exist two typical problems in the study of representations.

- Implicitization: for a rational parametric curve or surface, implicitization is to find an algebraic expression of the curve or surface.
- Proper Reparametrization: for an improper rational parametric curve or surface, proper reparametrization is to find a proper parametric expression of the curve or surface.

The parametric expression of a curve or surface is widely used in geometric modeling, such as NURBS representations. The algebraic equation, which is also called implicit equation, is another important representation, and this is much better than the parametric expression in determining whether or not a point is on the curve or surface. Hence the implicitization problem is classical in CAGD and there are implicitization algorithms for rational curves and surfaces proposed over the past several decades [1–14]. Among all of these techniques, the Gröbner bases [2] is well-known, since it is theoretically complete. However, this method has exponential computational complexity and, thus, it is inefficient. This is the reason that people can not apply the Gröbner basis method for practical implicitization in application. Alternatively, in computational application, people prefer to find the implicit equations from certain implicit matrices. The implicit matrices can be constructed as resultant matrices or matrices of moving curves/surfaces. The implicit matrix of the curve is much simpler than that of the surface, since the curve only introduces one variable. Actually, the construction of bivariate resultant is not uniform and it is still a developing technique in computer algebra [15–21]. The implicit matrix of moving curves/surfaces was introduced in [10] and developed by more researchers [5,11,22–25]. The μ-basis of a curve or surface was later defined by moving lines or moving planes with certain properties [5,23].

The implicit equation of the surface is generally included as a factor in its constructed resultants, but a constructed resultant may have extraneous factors that are not easy to identify and remove. For implicit matrices, some works attempt to construct the matrix whose determinant is exactly the implicit equation [10,24,26,27], but the ways to construct such implicit matrices are not complete for general surfaces. In the implicit matrix, there is more information than the implicit equation, such as singularities with multiplicity counting. Accordingly, people sometimes construct the implicit matrix by simple way and then the determinant of this matrix may have extraneous factors other than the implicit equation [22]. For the matrix constructed from the μ-basis of arbitrary three linearly independent syzygies, the extraneous factors are identified completely based on the analysis of base points or to parameters at infinity of tensor product surfaces [11,25], but some computations need the Gröbner bases of a zero dimensional algebraic variety.

When considering the proper reparametrization problem, an essential question is to decide whether a rational parametrization is proper. If a given rational parametrization is not proper, also called improper, a generic point lying on the variety corresponds to more than one parameter. On the other side, if a rational parametrization is improper, we ask whether it can be reparameterized, such that we can get a new proper parametrization. For algebraic curves, is well-known that the existence of a proper reparametrization for a given improper rational parametrization is certified by Lüroth's Theorem [28]. One can have a look, for instance, at a previous bibliography, as, for instance, [29–31], where some efficient methods are proposed to find a proper reparametrization for an improper parametrization of an algebraic curve. For a given algebraic surface, Castelnuovo's Theorem states that unirationality and rationality are equivalent over algebraically closed fields, but only some partial algorithmic methods approaching the problem are known (see [30,32]).

The μ-basis was first used in [17]. Here, the authors provided a representation for the implicit equation of a given curve defined parametrically. The μ-basis developing as a new algebraic tool can be used to obtain the parametric equation of a rational curve or a rational surface, in order to compute the implicit equation defining these varieties, and to study singularities and intersections [33]. There are several methods to compute the μ-basis for rational curves by computing two moving lines that satisfy the required properties [17], based on Gröbner basis [34] or based on vector elimination [23]. The μ-basis has also been generalized to rational surfaces [5], although the case of rational surfaces is different; for instance, the degrees of the μ-basis elements can be different. An algorithm to compute a μ-basis of a rational surface is designed that is based on polynomial matrix factorization [35]. Another possible way is to compute a basis of the syzygy module of the surface with the application of Quillen–Suslin Theorem [36,37]. In order to avoid the extraneous factor in implicitization, people tried to find the strong μ-bases of surfaces that have the very similar properties of the μ-bases of curves. However, the surfaces with strong μ-bases are relatively rare [25,38].

The μ-basis has shown different advantages by assuming that the given parametrization is always proper. A recent result attempts to find inverse formula, proper reparametrization, and algebraic equation for an improper parametrization of an algebraic curve by using μ-basis [39]. In this paper, we pay attention to the μ-basis of an improper parametrization of an algebraic surface and then apply the μ-basis in the problems of proper reparametrization and implicitization. There are intrinsical differences in the discussions between the surface and the curve; hence, some results of the curve can not be extended to the surface straightforward. After we give the definition of μ-basis of a rational parametric surface defined parametrically by $\mathcal{P}(\bar{t})$, we find the inversion formula (if $\mathcal{P}$ is proper) and the degree of the rational map that is induced by $\mathcal{P}$ while using the μ-basis. Although the proper reparametrization problem is still opening, we address the problem of proper reparametrization partially based on the latest results and the properties of the μ-basis. As an important application, we derive the implicitization formula from the μ-basis from a given parametrization not being necessarily proper. Starting from the μ-basis, the computations only involve greatest common divisors (gcds) and univariate resultants of some polynomials constructed from the μ-basis. While the surface implicitization form μ-basis involved the computation of Gröbner bases in [5].

We have structured the article, as indicated below. In Section 2, we present some important definitions and properties for the μ-basis of rational surfaces. In Section 3, given a rational parametrization $\mathcal{P}$, we study the inversion computation (if $\mathcal{P}$ is proper) and the degree of the induced rational map. In Section 4, we focus on the proper reparametrization problem while using μ-basis. In Section 5, we come to the implicitization problem from a given rational parametrization not necessarily proper using μ-basis. We finish the paper with Section 6, where we present a brief summary of our work.

2. μ-Basis for Rational Surfaces: Definition and Previous Results

Let R denote the polynomial ring $\mathbb{K}[t_1, t_2]$ over an algebraically closed field $\mathbb{K}$ of characteristic zero and R^m denote the set of $m-$dimensional row vectors with entries in the polynomial ring R. A submodule M of R^m is a subset of R^m, for which this condition holds: for any $\mathbf{f}_1, \mathbf{f}_2 \in M$ and $h_1, h_2 \in R$, we have $h_1\mathbf{f}_1 + h_2\mathbf{f}_2 \in M$. A set of elements $\mathbf{f}_i \in M$, for $i = 1, \ldots, k$, is called a generating set of M if for any $\mathbf{m} \in M$, there exist $h_i \in R$, for $i = 1, \ldots, k$ satisfying that

$$\mathbf{m} = h_1\mathbf{f}_1 + \ldots + h_k\mathbf{f}_k.$$

The Hilbert Basis Theorem states that every submodule $M \subset R^m$ has a finite generating set. If, for any $m \in M$, the above expression is unique, then $\{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$ is called a basis of the module M. If a module has a basis, then it is called a free module. For any $(\mathbf{f}_1, \ldots, \mathbf{f}_k) \in R^k$, the set

$$\mathrm{syz}(\mathbf{f}_1, \ldots, \mathbf{f}_k) := \{(h_1, \ldots, h_k) \in R^k | h_1\mathbf{f}_1 + \ldots + h_k\mathbf{f}_k = 0\}$$

is a module over R, called a syzygy module [40]. An important result regarding syzygy modules is that if $a, b, c, d \in \mathbb{R}[t_1, t_2]$ are four relatively prime polynomials then, the syzygy module $\mathrm{syz}(a, b, c, d)$ is a free module of rank 3 (see [5]).

Let $\mathcal{V}_a$ be a rational affine irreducible surface, and let

$$\mathcal{P}_a(\bar{t}) = \left(\frac{\wp_1(\bar{t})}{\wp_4(\bar{t})}, \frac{\wp_2(\bar{t})}{\wp_4(\bar{t})}, \frac{\wp_3(\bar{t})}{\wp_4(\bar{t})}\right) \in \mathbb{K}(\bar{t})^3, \;\; \bar{t} = (t_1, t_2)$$

be a rational affine parametrization of $\mathcal{V}_a$, where $\gcd(\wp_1, \wp_2, \wp_3, \wp_4) = 1$. Sometimes, we write the parametrization in the homogenous coordinate form $\mathcal{P}(\bar{t}) = (\wp_1(\bar{t}) : \wp_2(\bar{t}) : \wp_3(\bar{t}) : \wp_4(\bar{t}))$ and, in this case, we denote the surface in the projective space as $\mathcal{V}$.

A moving surface of degree l is a family of algebraic surfaces with parameter pairs (t_1, t_2)

$$S(\bar{x}, \bar{t}) = \sum_{i=1}^{\sigma} f_i(\bar{x}) b_i(\bar{t})$$

, where $f_i(\bar{x}), i = 1, \ldots, \sigma$ are polynomials of degree l, and $b_i(\bar{t}) \in \mathbb{R}[\bar{t}], i = 1, \ldots, \sigma$ (called blending functions) are linearly independent. We say that a moving surface follows the rational surface $\mathcal{P}$ if

$$\wp_4(\bar{t})^l S(\mathcal{P}_a(\bar{t}), \bar{t}) = 0.$$

We observe that the implicit equation of a given rational surface $\mathcal{V}$ is a moving surface of $\mathcal{P}$. A moving plane is a moving surface of degree one. We denote the next moving plane

$$A(\bar{t})x_1 + B(\bar{t})x_2 + C(\bar{t})x_3 + D(\bar{t})x_4$$

by $\mathbf{L}(\bar{t}) := (A(\bar{t}), B(\bar{t}), C(\bar{t}), D(\bar{t})) \in \mathbb{R}[\bar{t}]^4$. In the following, we denote, by $\mathbf{L}_{\bar{t}}$, the set of the moving planes that follow the rational surface that is parametrized by $\mathcal{P}$. Thus, $\mathbf{L}_{\bar{t}}$

is exactly the syzygy module $\mathrm{syz}(\wp_1, \wp_2, \wp_3, \wp_4)$. Now, we define the μ-basis of the rational surface $\mathcal{P}$.

Definition 1. *Let* $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbf{L}_{\bar{t}}$ *be three moving planes satisfying that* $[\mathbf{p}, \mathbf{q}, \mathbf{r}] = k\,\mathcal{P}(\bar{t})$*, where* k *is a nonzero constant. Subsequently, it is said that* $\mathbf{p}, \mathbf{q}, \mathbf{r}$ *form a* μ*-basis of the rational surface* $\mathcal{P}$*. In the following,* $[\mathbf{p}, \mathbf{q}, \mathbf{r}]$ *denotes the outer product of* $\mathbf{p}, \mathbf{q}, \mathbf{r}$*.*

Geometrically, the above definition means that the point of rational surface $\mathcal{P}$ can be represented as the intersection of three moving planes $\mathbf{p}, \mathbf{q}, \mathbf{r}$. This definition is generalized from the moving lines of a rational curves [23]. Notice that the result in the curve case was proposed twenty years ago, but the surface case has been a mystery for a long time. The μ-bases surfaces that have the very similar properties of the μ-bases of curves is called strong μ-basis, but the strong μ-bases are relatively rare [25]. Therefore, we have to study the μ-basis of the surface from initial definition and, here, we review some basic theorems in [5].

Theorem 1. *For any rational surface* $\mathcal{P}$*, there always exist three moving planes* $\mathbf{p}, \mathbf{q}, \mathbf{r}$*, such that* $[\mathbf{p}, \mathbf{q}, \mathbf{r}] = k \cdot \mathcal{P}(\bar{t})$ *holds. In fact, any basis* $\mathbf{p}, \mathbf{q}, \mathbf{r}$ *of* $\mathrm{syz}(\wp_1(\bar{t}), \wp_2(\bar{t}), \wp_3(\bar{t}), \wp_4(\bar{t}))$ *satisfies the above equality.*

Theorem 2. *Let* $\mathbf{p}, \mathbf{q}, \mathbf{r}$ *be a* μ*-basis of the rational surface* $\mathcal{P}$*. Thus,* $\mathbf{p}, \mathbf{q}, \mathbf{r}$ *provide a basis for the module* $\mathbf{L}_{\bar{t}}$ *(hence,* $\mathbf{L}_{\bar{t}}$ *is a free module). That is, for any* $\mathbf{l}(\bar{t}) \in \mathbf{L}_{\bar{t}}$*, there exist some polynomials* $h_i(\bar{t}), i = 1, 2, 3$*, satisfying that*

$$\mathbf{l}(\bar{t}) = h_1\mathbf{p} + h_2\mathbf{q} + h_3\mathbf{r}.$$

In addition, the above expression is unique.

An immediate consequence of the above theorems is that if $\mathbf{p}, \mathbf{q}, \mathbf{r}$ form a μ-basis if and only if $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are a basis of $\mathrm{syz}(\wp_1(\bar{t}), \wp_2(\bar{t}), \wp_3(\bar{t}), \wp_4(\bar{t}))$

3. Inversion and Degree Using μ-Basis

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. We denote, by $f(x_1, x_2, x_3) \in \mathbb{K}[x_1, x_2, x_3]$, the defining polynomial of a rational affine irreducible surface $\mathcal{V}_a$ defined by the rational affine parametrization

$$\mathcal{P}_a(\bar{t}) = \left(\frac{\wp_1(\bar{t})}{\wp_4(\bar{t})}, \frac{\wp_2(\bar{t})}{\wp_4(\bar{t})}, \frac{\wp_3(\bar{t})}{\wp_4(\bar{t})}\right) \in \mathbb{K}(\bar{t})^3, \quad \bar{t} = (t_1, t_2).$$

The homogeneous implicit polynomial defining the corresponding the projective rational surface $\mathcal{V}$ will be denoted as $F(x_1, x_2, x_3, x_4) \in \mathbb{K}[x_1, x_2, x_3, x_4]$, where $F(x_1, x_2, x_3, x_4) = x_4^{\deg(f)} f(x_1/x_4, x_2/x_4, x_3/x_4)$, and the parametrization in the homogenous coordinate form is given as $\mathcal{P}(\bar{t}) = (\wp_1(\bar{t}) : \wp_2(\bar{t}) : \wp_3(\bar{t}) : \wp_4(\bar{t}))$.

Besides implicitization, other applications of μ-basis include, as in the case of algebraic curves, point inversion and, in general, the computation of the fiber. The point inversion problem can be stated, as follows: given a point Q on the space, decide whether the point is on a rational surface $\mathcal{V}$ defined parametrically by $\mathcal{P}(\bar{t})$ or not, and, in the affirmative case, compute the corresponding parameter values t_1, t_2, such that $\mathcal{P}(t_1, t_2) = Q$. In this section, we recall some efficient algorithms that allow for computing the point inversion and, in general, the computation of the degree of the rational map that is induced by $\mathcal{P}$. For this purpose, we will use μ-basis.

In order to deal with these problems, we first recall that, associated with the parametrization $\mathcal{P}_a(\bar{t})$, we consider the induced rational map $\phi_{\mathcal{P}} : \mathbb{K} \longrightarrow \mathcal{V}_a \subset \mathbb{K}^3; \bar{t} \longmapsto \mathcal{P}_a(\bar{t})$. We denote, by $\deg(\phi_{\mathcal{P}})$, the degree of the induced rational map $\phi_{\mathcal{P}}$ (see [41] p. 143, and [42] p. 80). Observe that the birationality of $\phi_{\mathcal{P}}$, which is the properness of the input parametrization, is characterized by $\deg(\phi_{\mathcal{P}}) = 1$ (see [41,42]). We additionally remind that $\deg(\phi_{\mathcal{P}})$ determines the cardinality of the fiber of a generic element (see Theorem 7, p. 76 in [41]). The degree measures the number of times the parametrization traces the curve when the parameter takes values in $\mathbb{K}^2$. Finally, let $\mathcal{F}_{\mathcal{P}}(Q)$ be the fiber of a point $Q \in \mathcal{V}_a$; that is $\mathcal{F}_{\mathcal{P}}(Q) = \mathcal{P}_a^{-1}(Q) = \{\bar{t} \in \mathbb{K}^2 \mid \mathcal{P}_a(\bar{t}) = Q\}$.

In the following, given the projective parametrization $\mathcal{P}(\bar{t})$ of a surface $\mathcal{V}$ and

$$\mathbf{p}(\bar{t}) = (p_1, p_2, p_3, p_4),\ \mathbf{q}(\bar{t}) = (q_1, q_2, q_3, q_4),\ \mathbf{r}(\bar{t}) = (r_1, r_2, r_3, r_4)$$

a μ-basis for $\mathcal{P}(\bar{t})$, we consider a generic point $Q = (x_1, x_2, x_3, x_4)$ on the surface, and the polynomials

$$p^{\mathcal{P}}(\bar{t}, \bar{x}) = \mathbf{p}(\bar{t}) \cdot Q, q^{\mathcal{P}}(\bar{t}, \bar{x}) = \mathbf{q}(\bar{t}) \cdot Q, r^{\mathcal{P}}(\bar{t}, \bar{x}) = \mathbf{r}(\bar{t}) \cdot Q,\ \bar{x} = (x_1, x_2, x_3, x_4).$$

Remind that $p^{\mathcal{P}}(\bar{t}, \mathcal{P}(\bar{t})) = q^{\mathcal{P}}(\bar{t}, \mathcal{P}(\bar{t})) = r^{\mathcal{P}}(\bar{t}, \mathcal{P}(\bar{t})) = 0$ ($\mathbf{p}, \mathbf{q}, \mathbf{r}$ is a μ-basis for $\mathcal{P}(\bar{t})$). We denote, by V_1, V_2, V_3, the auxiliary curves over $\mathbb{K}(\mathcal{V})$, defined, respectively, by the polynomials $p^{\mathcal{P}}(\bar{t}, \bar{x}), q^{\mathcal{P}}(\bar{t}, \bar{x}), r^{\mathcal{P}}(\bar{t}, \bar{x}) \in \mathbb{K}(\mathcal{V})[\bar{t}]$, where $\mathbb{K}(\mathcal{V})$ is the field of rational functions of the given surface.

Finally, let

$$S^{\mathcal{P}}(t_1, \bar{x}) = \mathrm{pp}_{\bar{x}}(\mathrm{Cont}_Z(\mathrm{Res}_{t_2}(p^{\mathcal{P}}, q^{\mathcal{P}} + Zr^{\mathcal{P}}))) \in \mathbb{K}(\mathcal{V})[t_1],$$

where $\mathrm{Cont}_x(h)$ returns the content of a polynomial h w.r.t. some variable x, $\mathrm{pp}_x(h)$ returns the primitive part of a polynomial h with respect to a variable x and $\mathrm{Res}_x(h_1, h_2)$ returns the resultant of two polynomials h_1 and h_2 w.r.t. some variable x. Similarly, one also considers the polynomial

$$T^{\mathcal{P}}(t_2, \bar{x}) = \mathrm{pp}_{\bar{x}}(\mathrm{Cont}_Z(\mathrm{Res}_{t_1}(p^{\mathcal{P}}, q^{\mathcal{P}} + Zr^{\mathcal{P}}))) \in \mathbb{K}(\mathcal{V})[t_2].$$

The computation of $S^{\mathcal{P}}, T^{\mathcal{P}}$ can be done in two different ways. First, we consider that the implicit equation defining the input surface is known. In this case, we carry out the arithmetic over $\mathbb{K}(\mathcal{V})$ while using this implicit equation. We observe that, since $I(\mathcal{V}) = \langle F \rangle$ ($I(\mathcal{V})$ represents the ideal of $\mathcal{V}$), the basic arithmetic on $\mathbb{K}[\mathcal{V}]$ can be carried out by computing polynomial remainders. Thus, we conclude that the quotient field $\mathbb{K}(\mathcal{V})$ is computable. Furthermore, we note that we calculate the resultants of polynomials in $\mathbb{K}(\mathcal{V})[\bar{t}]$, which is a unique factorization domain, and we compute gcds of univariate polynomials over $\mathbb{K}(\mathcal{V})$ and, thus, in an Euclidean domain. In the second way, we avoid the requirement on the implicit equation. More precisely, the elements are represented (not uniquely) as function of polynomials in the variables x_1, x_2, x_3, x_4. We check the zero equality while using the input rational parametrization. This way could be too time consuming. In order to avoid this problem, one may test zero–equality by substituting a random point on the surface. The result of this test is correct with probability almost one. Additionally, one may also test the correctness of the computation of the inverse by checking it on a randomly chosen point on the given surface. In this way, we avoid the computation of the implicit polynomial.

In the following theorem, we provide the technique for computing the components of the inverse of a given rational proper parametrization $\mathcal{P}(\bar{t})$. Additionally, we characterize the properness of $\mathcal{P}(\bar{t})$. We illustrate this result in Example 1.

Theorem 3. *The rational parametrization $\mathcal{P}(\bar{t})$ is proper if and only if for a generic point $Q = (x_1, x_2, x_3, x_4)$ on the surface, it holds that $\deg_{t_1}(S^{\mathcal{P}}) = 1$. In this case, the t_1-coordinate of the inverse of $\mathcal{P}(\bar{t})$ is given by solving $S^{\mathcal{P}}(t_1, \bar{x}) = 0$ w.r.t the variable t_1.*

Proof. Using the results shown in [43] (see Proposition 1), we deduce that the non-constant t_1-coordinates of the intersections points in V_i, $i = 1, 2, 3$ are given by the roots of the polynomial $S^{\mathcal{P}}(t_1, \bar{x})$. Thus, we only have to prove that $\mathcal{P}(\bar{t})$ is proper if and only there exists one unique point $A = (A_1, A_2) \in (V_1 \cap V_2 \cap V_3) \cap (K \setminus \mathbb{K})^2$ (K denotes the algebraic closure of the field $\mathbb{K}(\mathcal{V})$; i.e $K = \overline{\mathbb{K}(\mathcal{V})}$). Indeed: first, let $M = (M_1(\overline{x}), M_2(\overline{x}))$ be the inverse of the rational proper parametrization $\mathcal{P}(\overline{t})$. Subsequently, $M(\mathcal{P}) = \overline{t}$ and, thus, $p^{\mathcal{P}}(M(\mathcal{P}), \mathcal{P}) = q^{\mathcal{P}}(M(\mathcal{P}), \mathcal{P}) = r^{\mathcal{P}}(M(\mathcal{P}), \mathcal{P}) = 0$, which implies that

$$M \in V_1 \cap V_2 \cap V_3 \cap K^2.$$

In addition, since M is the inverse of $\mathcal{P}(\overline{t})$, one has that $M \in (K \setminus \mathbb{K})^2$. Hence, $M \in (V_1 \cap V_2 \cap V_3) \cap (K \setminus \mathbb{K})^2$. Now, let us see that M is unique. Let $M^* \in (V_1 \cap V_2 \cap V_3) \cap (K \setminus \mathbb{K})^2$. The equalities $p^{\mathcal{P}}(M^*, \overline{x}) = q^{\mathcal{P}}(M^*, \overline{x}) = r^{\mathcal{P}}(M^*, \overline{x}) = 0$ imply that

$$p^{\mathcal{P}}(R(\overline{t}), \mathcal{P}) = q^{\mathcal{P}}(R(\overline{t}), \mathcal{P}) = r^{\mathcal{P}}(R(\overline{t}), \mathcal{P}) = 0, \quad R(\overline{t}) = M^*(\mathcal{P}).$$

Afterwards, by the properties of resultants and by Lemma 1, we get that $\mathcal{P}(\overline{t}) = k\,\mathcal{P}(R(\overline{t}))$ and since $\mathcal{P}$ is proper we deduce that $R(\overline{t}) = M^*(\mathcal{P}(\overline{t})) = \overline{t}$. Thus, left composing by $\mathcal{P}^{-1}$, we get that $M^* = \mathcal{P}^{-1} = M$.

Reciprocally, because there exists a unique point in $(V_1 \cap V_2 \cap V_3) \cap (K \setminus \mathbb{K})^2$, A is fixed under the action of the Galois group and, thus, $A \in \mathbb{K}(\mathcal{V})^2$. Reasoning similarly, as we did for the uniqueness in the above implication, one gets that $A \circ \mathcal{P} = \overline{t}$ and, then, we conclude that A is the inverse of $\mathcal{P}$. $\square$

Remark 1. *Theorem 3 can be stated similarly for $T^{\mathcal{P}}(t_2, \overline{x})$. More precisely, $\mathcal{P}(\overline{t})$ is proper if and only if, for a generic point $Q = (x_1, x_2, x_3, x_4)$ on the surface, it holds that $\deg_{t_2}(T^{\mathcal{P}}) = 1$. In this case, the t_2-coordinate of the inverse of $\mathcal{P}(\overline{t})$ is given by solving $T^{\mathcal{P}}(t_2, \overline{x}) = 0$ w.r.t the variable t_2.*

We also note that we may work over the affine space (i.e., $x_4 = 1$) and the obtained results are the same, but over the affine space. If $x_4 = 1$, then the computations are more efficient.

Example 1. *Let $\mathcal{V}$ be the rational surface that is defined by the parametrization*

$$\mathcal{P}(\overline{t}) = (t_2 t_1 : t_2 + t_1 : t_2 - t_1 : t_2^2 + t_1^2 + 2).$$

First, we compute the polynomials

$$\begin{aligned} p^{\mathcal{P}}(\overline{t}, \overline{x}) &= \mathbf{p}(\overline{t}) \cdot \overline{x} = -2x_1 + t_1 x_2 + t_1 x_3, \\ q^{\mathcal{P}}(\overline{t}, \overline{x}) &= \mathbf{q}(\overline{t}) \cdot \overline{x} = -2t_1 x_1 + (t_1^2 + 1 + t_2 t_1)x_2 - x_3 - t_1 x_4, \\ r^{\mathcal{P}}(\overline{t}, \overline{x}) &= \mathbf{r}(\overline{t}) \cdot \overline{x} = (-2t_2 + 2t_1)x_1 + (t_2^2 - t_1^2)x_2 + 2x_3 + (-t_2 + t_1)x_4, \end{aligned}$$

where the μ-basis is given as

$$\begin{aligned} \mathbf{p}(\overline{t}) &= (-2, t_1, t_1, 0), \\ \mathbf{q}(\overline{t}) &= (-2t_1, t_1^2 + 1 + t_2 t_1, -1, -t_1), \\ \mathbf{r}(\overline{t}) &= (-2t_2 + 2t_1, t_2^2 - t_1^2, 2, -t_2 + t_1). \end{aligned}$$

Now, we determine $S^{\mathcal{P}}(t_1, \overline{x})$ and $T^{\mathcal{P}}(t_2, \overline{x})$. We obtain

$$S^{\mathcal{P}}(t_1, \overline{x}) = -2x_1 + t_1 x_2 + t_1 x_3,$$

$$T^{\mathcal{P}}(t_2, \overline{x}) = -2x_1 x_4 x_2 - 2x_1 x_4 x_3 + 2x_1 x_2^2 t_2 + 2x_1 x_2 t_2 x_3 - 4x_1^2 x_3 - x_2 x_3^2 - x_3^3 + x_2^3 + x_3 x_2^2.$$

Because $\deg_{t_1}(S^{\mathcal{P}}) = 1$, we conclude that $\mathcal{P}$ is proper and the first coordinate of the inverse is given as

$$I_1 = \frac{2x_1}{x_2 + x_3}.$$

Reasoning similarly with $T^{\mathcal{P}}$, we obtain the second coordinate of the inverse, which is given as

$$I_2 = \frac{2x_1x_4x_2 + 2x_1x_4x_3 + x_3^3 - x_2^3 + 4x_1^2x_3 + x_2x_3^2 - x_3x_2^2}{2x_1x_2(x_2 + x_3)}.$$

Based upon the above theorem, we may compute $\mathcal{F}_{\mathcal{P}}(Q)$ for a generic point Q and, thus, to obtain the degree of the rational map induced by $\deg(\phi_{\mathcal{P}})$. For this purpose, we consider $Q = \mathcal{P}_a(\bar{s})$, where $\bar{s} = (s_1, s_2)$ are new variables, and the polynomials

$$S^{\mathcal{P}}(t_1, \bar{s}) = \mathrm{pp}_{\bar{s}}(\mathrm{Cont}_Z(\mathrm{Res}_{t_2}(p^{\mathcal{P}}, q^{\mathcal{P}} + Zr^{\mathcal{P}}))) \in \mathbb{K}[t_1, \bar{s}],$$

$$T^{\mathcal{P}}(t_2, \bar{s}) = \mathrm{pp}_{\bar{s}}(\mathrm{Cont}_Z(\mathrm{Res}_{t_1}(p^{\mathcal{P}}, q^{\mathcal{P}} + Zr^{\mathcal{P}}))) \in \mathbb{K}[t_2, \bar{s}],$$

where

$$p^{\mathcal{P}}(\bar{t}, \bar{s}) = \mathbf{p}(\bar{t}) \cdot \mathcal{P}(\bar{s}), \quad q^{\mathcal{P}}(\bar{t}, \bar{s}) = \mathbf{q}(\bar{t}) \cdot \mathcal{P}(\bar{s}), \quad r^{\mathcal{P}}(\bar{t}, \bar{s}) = \mathbf{r}(\bar{t}) \cdot \mathcal{P}(\bar{s}).$$

Remind that $p^{\mathcal{P}}(\bar{t}, \bar{t}) = q^{\mathcal{P}}(\bar{t}, \bar{t}) = r^{\mathcal{P}}(\bar{t}, \bar{t}) = 0$ ($\mathbf{p}, \mathbf{q}, \mathbf{r}$ is a μ-basis for $\mathcal{P}(\bar{t})$). We denote, by V_1, V_2, V_3, the auxiliary curves over $\mathbb{K}(\bar{s})$ defined, respectively, by the polynomials $p^{\mathcal{P}}(\bar{t}, \bar{s}), q^{\mathcal{P}}(\bar{t}, \bar{s}), r^{\mathcal{P}}(\bar{t}, \bar{s}) \in \mathbb{K}[\bar{t}, \bar{s}]$. Thus, one obtains the following proposition.

Theorem 4. *For a generic point $Q = \mathcal{P}_a(\bar{s})$, it holds that*

$$\deg(\phi_{\mathcal{P}}) = \mathrm{Card}(\mathcal{F}_{\mathcal{P}}(Q)) = \deg_{t_1}(S^{\mathcal{P}}(t_1, \bar{s})) = \deg_{t_2}(T^{\mathcal{P}}(t_2, \bar{s})).$$

Proof. First, we use Proposition 1 in [43], and we deduce that the non-constant t_1-coordinates of the intersections points in V_i, $i = 1, 2, 3$ are given by the roots of the polynomial $S^{\mathcal{P}}(t_1, \bar{s})$. Thus, we only have to prove that $M \in \mathcal{F}_{\mathcal{P}}(Q)$ if and only if $M \in V_1 \cap V_2 \cap V_3 \cap (K \setminus \mathbb{K})^2$, where $K = \overline{\mathbb{K}(\bar{s})}$ is the algebraic closure of the field. Indeed, if $M \in V_1 \cap V_2 \cap V_3 \cap (K \setminus \mathbb{K})^2$ thus $p^{\mathcal{P}}(M, \bar{s}) = q^{\mathcal{P}}(M, \bar{s}) = r^{\mathcal{P}}(M, \bar{s}) = 0$, which implies that

$$\mathbf{p}(M) \cdot \mathcal{P}(\bar{s}) = \mathbf{q}(M) \cdot \mathcal{P}(\bar{s}) = \mathbf{r}(M) \cdot \mathcal{P}(\bar{s}) = 0.$$

Because

$$\mathbf{p}(M) \cdot \mathcal{P}(M) = \mathbf{q}(M) \cdot \mathcal{P}(M) = \mathbf{r}(M) \cdot \mathcal{P}(M) = 0$$

, we get that $\mathcal{P}(M) = k\,\mathcal{P}(\bar{s})$ with $k \neq 0$ (since $M \notin \mathbb{K}^2$), which implies that $\mathcal{P}_a(M) = \mathcal{P}_a(\bar{s})$. Hence, $M \in \mathcal{F}_{\mathcal{P}}(Q)$.

Reciprocally, let $M \in \mathcal{F}_{\mathcal{P}}(Q)$. Subsequently, $\mathcal{P}_a(M) = \mathcal{P}_a(\bar{s})$ which implies that $\mathcal{P}(M) = k\,\mathcal{P}(\bar{s})$ with $k \neq 0$. Because

$$\mathbf{p}(M) \cdot \mathcal{P}(M) = \mathbf{q}(M) \cdot \mathcal{P}(M) = \mathbf{r}(M) \cdot \mathcal{P}(M) = 0,$$

we get that

$$\mathbf{p}(M) \cdot \mathcal{P}(\bar{s}) = \mathbf{q}(M) \cdot \mathcal{P}(\bar{s}) = \mathbf{r}(M) \cdot \mathcal{P}(\bar{s}) = 0.$$

Thus, $p^{\mathcal{P}}(M, \bar{s}) = q^{\mathcal{P}}(M, \bar{s}) = r^{\mathcal{P}}(M, \bar{s}) = 0$ and, hence, $M \in V_1 \cap V_2 \cap V_3$. Furthermore, since $M \in \mathcal{F}_{\mathcal{P}}(Q)$, we also get that $M \in (K \setminus \mathbb{K})^2$. $\square$

Clearly, Theorem 4 can be also stated for a generic point $Q = (x_1, x_2, x_3, x_4)$ on the surface, and the polynomials

$$p^{\mathcal{P}}(\bar{t}, \bar{x}) = \mathbf{p}(\bar{t}) \cdot Q,\ q^{\mathcal{P}}(\bar{t}, \bar{x}) = \mathbf{q}(\bar{t}) \cdot Q,\ r^{\mathcal{P}}(\bar{t}, \bar{x}) = \mathbf{r}(\bar{t}) \cdot Q,\ \bar{x} = (x_1, x_2, x_3, x_4).$$

(remind that $p^{\mathcal{P}}(\bar{t}, \mathcal{P}(\bar{t})) = q^{\mathcal{P}}(\bar{t}, \mathcal{P}(\bar{t})) = r^{\mathcal{P}}(\bar{t}, \mathcal{P}(\bar{t})) = 0$ ($[\mathbf{p}, \mathbf{q}, \mathbf{r}] = k\,\mathcal{P}(\bar{t})$). For this purpose, one considers the polynomials

$$S^{\mathcal{P}}(t_1, \bar{x}) = \mathrm{pp}_{\bar{x}}(\mathrm{Cont}_Z(\mathrm{Res}_{t_2}(p^{\mathcal{P}}, q^{\mathcal{P}} + Zr^{\mathcal{P}}))) \in \mathbb{K}(\mathcal{V})[t_1],$$

and

$$T^{\mathcal{P}}(t_2, \overline{x}) = \mathrm{pp}_{\overline{x}}(\mathrm{Cont}_Z(\mathrm{Res}_{t_1}(p^{\mathcal{P}}, q^{\mathcal{P}} + Zr^{\mathcal{P}}))) \in \mathbb{K}(\mathcal{V})[t_2].$$

where computation can be done, as we described in the paragraph before Theorem 3, i.e., over the field of rational functions $\mathbb{K}(\mathcal{V})$. Thus, one has the following corollary.

Corollary 1. *For a generic point $Q = (x_1, x_2, x_3, x_4) \in \mathcal{V}$, it holds that*

$$\deg(\phi_{\mathcal{P}}) = \mathrm{Card}(\mathcal{F}_{\mathcal{P}}(Q)) = \deg_{t_1}(S^{\mathcal{P}}(t_1, \overline{x})) = \deg_{t_2}(T^{\mathcal{P}}(t_2, \overline{x})).$$

Remark 2. *From the proof of Theorem 4, we deduce that $\mathcal{F}_{\mathcal{P}}(Q) = V_1 \cap V_2 \cap V_3 \cap (K \setminus \mathbb{K})^2$.*

Example 2. *Let $\mathcal{V}$ be the rational surface that is defined by the parametrization*

$$\mathcal{P}(\overline{t}) = (t_2^2t_1^2 - t_1^4 : -t_2 + t_2^3 + t_2t_1^4 : -t_1^2 + t_2t_1^2 + t_2^2t_1^2 - t_1^4 : -t_1^2 + t_2^2t_1^2 + t_1^4).$$

We determine the polynomials $p^{\mathcal{P}}(\overline{t}, \overline{s}) = \mathbf{p}(\overline{t}) \cdot \mathcal{P}(\overline{s})$, $q^{\mathcal{P}}(\overline{t}, \overline{s}) = \mathbf{q}(\overline{t}) \cdot \mathcal{P}(\overline{s})$, $r^{\mathcal{P}}(\overline{t}, \overline{s}) = \mathbf{r}(\overline{t}) \cdot \mathcal{P}(\overline{s})$, where the μ-basis is given by

$$\begin{array}{rcl} \mathbf{p}(\overline{t}) & = & (t_2t_1^2 + 2t_1^2 + 4t_2^4 + 6t_2^3 - 4t_2 - 4, -2t_1^2, \\ & & -2t_2t_1^2 - t_1^2 - 4t_2^4 - 4t_2^3 + 4t_2 + 2, t_2t_1^2 + t_1^2 + 2t_2^3 - 2) \\ \mathbf{q}(\overline{t}) & = & (-2t_2t_1^2 - 3t_1^2 + t_2 + 1, 0, 2t_2t_1^2 + 2t_1^2 - t_2 - 1, -t_1^2 + 1) \\ \mathbf{r}(\overline{t}) & = & (-2t_2^2 - t_2 + 2, 0, 2t_2^2 - 1, -t_2 + 1). \end{array}$$

Now, we determine $S^{\mathcal{P}}(t_1, \overline{s})$ (similarly, if we compute $T^{\mathcal{P}}(t_2, \overline{s})$), and we obtain

$$S^{\mathcal{P}}(t_1, \overline{s}) = \mathrm{pp}_{\overline{s}}(\mathrm{Cont}_Z(\mathrm{Res}_{t_2}(p^{\mathcal{P}}, q^{\mathcal{P}} + Zr^{\mathcal{P}}))) = t_1^2 - s_1^2 \in \mathbb{K}[t_1, \overline{s}].$$

Therefore, applying Theorem 4, we conclude that $\mathcal{P}$ is not proper and in fact $\deg(\phi_{\mathcal{P}}) = 2$. Furthermore,

$$\mathcal{F}_{\mathcal{P}}(\mathcal{P}_a(\overline{s})) = \{(s_1, s_2), (-s_1, s_2)\}.$$

4. On the Problem of the Reparametrization Using μ-Basis

In this section, we consider the problem of computing a rational proper reparametrization of a given algebraic surface defined by an improper parametrization. That is, *given an algebraically closed field $\mathbb{K}$, and $\mathcal{P}(\overline{t})$, $\overline{t} = (t_1, t_2)$, a rational parametrization of a surface $\mathcal{V}$ over $\mathbb{K}$, we want to compute a proper parametrization of $\mathcal{V}$, $\mathcal{Q}(\overline{t})$, and $R(\overline{t}) \in (\mathbb{K}(\overline{t}) \setminus \mathbb{K})^2$, such that*

$$\mathcal{P}(\overline{t}) = \mathcal{Q}(R(\overline{t})).$$

Notice that we consider $\mathcal{Q}(R(\overline{t}))$, with $R(\overline{t}) = (r_1(\overline{t})/r(\overline{t}), r_2(\overline{t})/r(\overline{t})) \in (\mathbb{K}(\overline{t}) \setminus \mathbb{K})^2$, in homogenous form, i.e., $\mathcal{P}(\overline{t}) = \mathcal{Q}(R(\overline{t}))$ means that $\mathcal{P}(\overline{t}) = \mathcal{Q}\Big(\frac{r_1(\overline{t})}{r(\overline{t})}, \frac{r_2(\overline{t})}{r(\overline{t})}\Big) r(\overline{t})^{\deg(\mathcal{Q})}$, which is a polynomial vector in homogenous form.

In this section, although we do not provide a solution to the general reparametrization problem, we show how the μ-basis can be used to provide some information concerning $R(\overline{t}) \in (\mathbb{K}(\overline{t}) \setminus \mathbb{K})^2$. We address the problem partially and the idea presented is based in the results in [32], but we trust that we could develop deeply these new approaches in future works, and more results concerning this topic allow us to get more advances.

The approach that is presented in this section is based on the computation of polynomial gcds and univariate resultants. These techniques always work and the time performance is very effective. The algorithm presented follows directly from the algorithm that was developed in [39], which solves the problem for the case of curves. Accordingly, we first outline this approach and illustrate it with an example.

Example 3. *Let $\mathcal{C}$ be the rational curve that is defined by the parametrization*

$$\mathcal{P}(t) = (-t^4 + t^3 + 2t^2 + 2t + 1 : (t+1)(3t^2 + 2t + 2) : 2t^4 + 3t^3 + 5t^2 + 4t + 2).$$

In Step 2 of the algorithm, we determine the polynomials $p^{\mathcal{P}}(t,s)$ and $q^{\mathcal{P}}(t,s)$, where the μ-basis is
$\mathbf{p}(t) = \left(40t^2 + 40t + 40, -30t^2 - 35t - 35, 20t^2 + 15t + 15 \right)^T$
$\mathbf{q}(t) = \left(30t^2 + 20t + 20, -25t^2 - 20t - 20, 15t^2 + 10t + 10 \right)^T.$

Now, we compute $G^{\mathcal{P}}(t,s)$,

$$G^{\mathcal{P}}(t,s) = C_0(t) + C_1(t)s + C_2(t)s^2,$$

where $C_0(t) = -10t^2$, $C_1(t) = -10t^2$, and $C_2(t) = 10 + 10t$. Because $m := \deg_t(G^{\mathcal{P}}) > 1$, we go to Step 5 of Algorithm 1, and we consider

$$R(t) = \frac{C_2(t)}{C_0(t)} = \frac{-1-t}{t^2}.$$

Note that $\gcd(C_0, C_2) = 1$. Now, we determine the polynomials

$$L_1(s, x_1) = \mathrm{Res}_t(x_1\wp_3(t) - \wp_1(t), sC_0(t) - C_2(t)) = (x_3 + sx_3 + 2x_1 - s^2x_3 - 3sx_1 + 2s^2x_1)^2,$$

$$L_2(s, x_2) = \mathrm{Res}_t(x_2\wp_3(t) - \wp_2(t), sC_0(t) - C_2(t)) = (3sx_3 + 2x_2 - 3sx_2 + 2s^2x_2 - 2s^2x_3)^2.$$

Finally, the algorithm outputs the proper parametrization $\mathcal{Q}(t)$, and the rational function $R(t)$ (see Step 7)

$$\mathcal{Q}(t) = \left(t^2 - 1 - t : t(-3+2t) : 2 - 3t + 2t^2\right), \qquad R(t) = \frac{-1-t}{t^2}.$$

Algorithm 1 Proper Reparametrization for Curves using μ-Basis

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Input a rational parametrization $\mathcal{P}(t) = (\wp_1(t) : \wp_2(t) : \wp_3(t))$, of a plane algebraic curve $\mathcal{C}$.
Output a rational proper parametrization $\mathcal{Q}(t)$ of $\mathcal{C}$, and a rational function $R(t)$ such that $\mathcal{P}(t) = \mathcal{Q}(R(t))$.
Steps

1. Compute a μ-basis of $\mathcal{P}$. Let $\mathbf{p}(t)$, $\mathbf{q}(t)$ be this μ-basis.
2. Compute $p^{\mathcal{P}}(t,s) = \mathbf{p}(t) \cdot \mathcal{P}(s)$, $\quad q^{\mathcal{P}}(t,s) = \mathbf{q}(t) \cdot \mathcal{P}(s)$.
3. Compute $G^{\mathcal{P}}(t,s) = \gcd(p^{\mathcal{P}}(t,s), q^{\mathcal{P}}(t,s)) = C_m(t)s^m + \cdots + C_0(t)$. Let $m := \deg_t(G^{\mathcal{P}}(t,s))$.
4. If $m = 1$, return $\mathcal{Q}(t) = \mathcal{P}(t)$, and $R(t) = t$. Otherwise go to Step 5.
5. Consider $R(t) = \frac{C_i(t)}{C_j(t)} \in \mathbb{K}(t)$, such that $C_j(t)$, $C_i(t)$ are not associated polynomials (i.e., $C_j(t) \neq kC_i(t)$, $k \in \mathbb{K}$).
6. For $i = 1, 2$, compute

$$L_i(s, x_i) = \mathrm{Res}_t(x_i\wp_3(t) - \wp_i(t), sC_j(t) - C_i(t)) = (q_{i2}(s)x_i - q_{i1}(s))^{\deg(R)}.$$

7. Return $\mathcal{Q}_a(t) = (q_{11}(t)/q_{12}(t), q_{21}(t)/q_{22}(t))$ or the equivalent projective parametrization $\mathcal{Q}(t)$, and $R(t) = C_i(t)/C_j(t)$.

The main idea of the result that we develop in this paper consists in computing a reparametrization of two auxiliary parametrizations (defining two space curves), $\mathcal{P}_1$ and $\mathcal{P}_2$, directly defined from a given rational parametrization of the surface $\mathcal{P}$ (see Definition 2). Moreover, using that the degree of a rational map is multiplicative under composition, we get some results that relate the degree of the rational map that is induced by $\mathcal{P}$ with the degree of $\mathcal{Q}$, and the degree of the rational maps induced by $\mathcal{P}_1$ and P_2. In addition, we also prove the relation with the degree w.r.t. the variables t_1, t_2 of $R(\bar{t}) \in \mathbb{K}(\bar{t})^2$.

To start with, we first provide the following lemma that analyzes the behavior of the μ-basis under reparametrizations.

Lemma 1. *Let $\tilde{\mathbf{p}}(\bar{t})$, $\tilde{\mathbf{q}}(\bar{t})$ and $\tilde{\mathbf{r}}(\bar{t})$ be a μ-basis for a parametrization $\mathcal{Q}(\bar{t})$ of a surface $\mathcal{V}$. Let $R(\bar{t}) \in (\mathbb{K}(\bar{t}) \setminus \mathbb{K})^2$. Subsequently, $\mathbf{p}(\bar{t}) = \tilde{\mathbf{p}}(R(\bar{t}))$, $\mathbf{q}(\bar{t}) = \tilde{\mathbf{q}}(R(\bar{t}))$, $\mathbf{r}(\bar{t}) = \tilde{\mathbf{r}}(R(\bar{t}))$ is a μ-basis for the reparametrization $\mathcal{P}(\bar{t}) = \mathcal{Q}(R(\bar{t}))$.*

Proof. Taking into account that $\tilde{\mathbf{p}}(\bar{t})$, $\tilde{\mathbf{q}}(\bar{t})$ and $\tilde{\mathbf{r}}(\bar{t})$ is a μ-basis for $\mathcal{Q}(\bar{t})$, from Theorem 1, it follows that $[\tilde{\mathbf{p}}, \tilde{\mathbf{q}}, \tilde{\mathbf{r}}] = k\,\mathcal{Q}(\bar{t})$ for some non-zero constant k. Therefore, we easily get that $[\mathbf{p}, \mathbf{q}, \mathbf{r}] = k\,\mathcal{P}(\bar{t})$ for some non-zero constant k. Hence, from Theorem 1, we conclude that $\mathbf{p}(\bar{t})$, $\mathbf{q}(\bar{t})$, $\mathbf{r}(\bar{t})$ is a μ-basis for $\mathcal{P}(\bar{t})$. $\square$

In the next proposition, we assume that we know $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ and $R(\bar{t}) = (r_1(\bar{t}), r_2(\bar{t})) \in (\mathbb{K}(\bar{t}) \setminus \mathbb{K})^2$, and we present a method for computing $\tilde{\mathbf{p}}$, $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{r}}$ from $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, respectively. We state Proposition 1 for $\mathbf{p} = (p_1, p_2, p_3, p_4)$ and $\tilde{\mathbf{p}} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4)$. One reasons, similarly, to obtain $\tilde{\mathbf{q}}$ from $\mathbf{q}$ and $\tilde{\mathbf{r}}$ from $\mathbf{r}$.

We assume w.l.o.g that $p_4 \neq 0$m which implies that $\tilde{p}_4 \neq 0$ (otherwise, we consider another non-zero component of $\mathbf{p}$). Thus, we may write

$$(p_1/p_4, p_2/p_4, p_3/p_4) = (\tilde{p}_1/\tilde{p}_4, \tilde{p}_2/\tilde{p}_4, \tilde{p}_3/\tilde{p}_4)(R(\bar{t})).$$

Let us assume that $\gcd(p_1, p_4) = \gcd(\tilde{p}_1, \tilde{p}_4) = 1$ (otherwise, we simplify these rational functions). In addition, we note that, if $p_1 = 0$, then we easily get that $\tilde{p}_1 = 0$. For the case of $p_1 \neq 0$, we consider $(r_1, r_2, p_1/p_4)$ that can be seen as an affine parametrization of the surface defined by the irreducible polynomial $\tilde{p}_1(x_1, x_2) - x_3\tilde{p}_4(x_1, x_2) \in \mathbb{K}[x_1, x_2, x_3]$ (note that $\gcd(\tilde{p}_1, \tilde{p}_4) = 1$ and $\tilde{p}_4\,\tilde{p}_1 \neq 0$). Hence, we only have to compute the implicit equation of that surface by applying, for instance, the method that is presented in [44].

Reasoning, similarly, $(r_1, r_2, p_i/p_4)$ can be seen as a parametrization of the surface defined by the irreducible polynomial $\tilde{p}_i(x_1, x_2) - x_3\tilde{p}_4(x_1, x_2) \in \mathbb{K}[x_1, x_2, x_3]$, for $i = 2, 3$. Summarizing, we have the following proposition.

Proposition 1. *Under the conditions that are stated above, it holds that the implicit equation of the parametrization $(r_1, r_2, p_i/p_4)$ is given as $\tilde{p}_i(x_1, x_2) - x_3\tilde{p}_4(x_1, x_2)$, for $i = 1, 2, 3$.*

In Remark 3, we apply the same idea that is stated in Proposition 1, but for the particular case of curves.

Given a rational projective parametrization $\mathcal{N}(\bar{t})$ of a surface over $\mathbb{K}$, in Definition 2 we introduce some auxiliary parametrizations over $\mathbb{K}(t_i)$ that are defined from $\mathcal{N}$.

Definition 2. *Let $\mathcal{N}(\bar{t})$ be a parametrization with coefficients in $\mathbb{K}$. We define the partial parametrizations associated to $\mathcal{N}$ as the parametrizations $\mathcal{N}_i(t_j) := \mathcal{N}(\bar{t})$ with coefficients in $\mathbb{K}[t_i]$ (i.e., $\mathcal{N}_i$ is defined over $\mathbb{K}[t_i]$), for $i, j \in \{1, 2\}$ and $i \neq j$.*

We note that the partial parametrization $\mathcal{N}_i(t_j)$ $(i \neq j)$ determines a space curve in $\overline{\mathbb{K}(t_i)}^3$, where $\overline{\mathbb{K}(t_i)}$ is the algebraic closure of $\mathbb{K}[t_i]$. In addition, we note that Definition 2 can also be stated for any $N(\bar{t}) \in \mathbb{K}(\bar{t})^2$. That is, given $N(\bar{t}) \in \mathbb{K}(\bar{t})^2$, one may consider $N_i(t_j) := N(\bar{t}) \in (\mathbb{K}[t_i])(t_j)^2$ (i.e., N is seen defined over $\mathbb{K}[t_i]$ and in the variable t_j), for $i, j \in \{1, 2\}$ and $i \neq j$. Similarly, one also may adapt Definition 2 for any polynomial $n(\bar{t}) \in \mathbb{K}(\bar{t})$.

The properness of the input parametrization $\mathcal{P}$ of a surface $\mathcal{V}$ can be characterized by means of the properness of its partial parametrizations. In particular, it is proved that $\mathcal{P}$ is birational if and only if its associated partial parametrizations, $\mathcal{P}_i$, are proper and $\mathcal{P}_i^{-1} \in \mathbb{K}(\bar{x}) \setminus \mathbb{K}(t_i)$, where $\bar{x} = (x_1, x_2, x_3, x_4)$ (see [32]).

In the following, given a rational affine parametrization $\mathcal{P}(\bar{t})$ of a surface $\mathcal{V}$, we develop an algorithm that computes a rational parametrization $\mathcal{Q}(\bar{t})$ of $\mathcal{V}$, and $R(\bar{t}) \in \mathbb{K}(\bar{t})^2$, such that $\mathcal{P}(\bar{t}) = \mathcal{Q}(R(\bar{t}))$. The algorithm is based on the computation of polynomial gcds and univariate resultants whose computing time performance is very satisfactory.

We prove that the partial parametrizations that correspond to the output parametrization, $\mathcal{Q}(\bar{t})$, are proper (see Theorem 5), and we get properties relating the degree of $\phi_{\mathcal{P}}$ with the degree of the rational map $\phi_{\mathcal{Q}}$, and the degree of $R(\bar{t})$ (see Theorem 6). More precisely, we prove that

$$\deg(\phi_{\mathcal{P}}) = \deg(\phi_{\mathcal{Q}})\deg_{t_1}(S)\deg_{t_2}(T)$$

where

$$R(\bar{t}) = (S(\bar{t}), T(S(\bar{t}), t_2)), \;\; S, T \in \mathbb{K}(\bar{t}).$$

In Corollaries 2 and 3, we analyze in which conditions $\deg(\phi_{\mathcal{Q}}) = 1$ or, otherwise, the degree of the rational map induced by $\mathcal{Q}(\bar{t})$ is lower than the degree that is induced by the input parametrization $\mathcal{P}(\bar{t})$.

In Theorem 5, we have that the partial parametrizations associated to the output parametrization, $\mathcal{Q}(\bar{t})$, are proper (see [32]) but we can not ensure that $\mathcal{Q}$ is proper.

Theorem 5. *The partial parametrizations $\mathcal{Q}_1(t_2)$ and $\mathcal{Q}_2(t_1)$ associated to the parametrization $\mathcal{Q}$ computed by Algorithm 2 are proper.*

Algorithm 2 Proper Reparametrization for Surfaces using μ-Basis

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Input a rational parametrization $\mathcal{P}(\bar{t}) = \left(\wp_1(\bar{t}) : \wp_2(\bar{t}) : \wp_3(\bar{t}) : \wp_4(\bar{t})\right)$ of an algebraic surface $\mathcal{V}$.
Output a rational parametrization $\mathcal{Q}(\bar{t})$ of $\mathcal{V}$, and $R(\bar{t}) \in (\mathbb{K}(\bar{t}) \setminus \mathbb{K})^2$ such that $\mathcal{P}(\bar{t}) = \mathcal{Q}(R(\bar{t}))$.
Steps

1. Compute a μ-basis of $\mathcal{P}$. Let $\mathbf{p}(\bar{t})$, $\mathbf{q}(\bar{t})$, $\mathbf{r}(\bar{t})$ be this μ-basis.
2. Apply Algorithm 1 to $\mathcal{P}_2(t_1)$. If $\mathcal{P}_2$ is not proper, then the algorithm returns the proper parametrization $\mathcal{M}_2(t_1)$, and $S_2(t_1) \in (\mathbb{K}[t_2])(t_1)$ ($S_2(t_1) = S(t_1, t_2)$ seen with coefficients in $\mathbb{K}[t_2]$), such that $\mathcal{P}_2(t_1) = \mathcal{M}_2(S_2(t_1))$. Otherwise, the algorithm returns $\mathcal{M}(\bar{t}) = \mathcal{P}(\bar{t})$ (i.e $\mathcal{M}_2(t_1) = \mathcal{P}_2(t_1)$, and $S_2(t_1) = t_1$).
3. Apply Algorithm 1 to $\mathcal{M}_1(t_2)$. If $\mathcal{M}_1$ is not proper, the algorithm returns the proper parametrization $\mathcal{Q}_1(t_2)$, and $T_1(t_2) \in (\mathbb{K}[t_1])(t_2)$ ($T_1(t_2) = T(t_1, t_2)$ seen with coefficients in $\mathbb{K}[t_2]$) such that $\mathcal{M}_1(t_2) = \mathcal{Q}_1(T_1(t_2))$. Otherwise, the algorithm returns $\mathcal{Q}(\bar{t}) = \mathcal{M}(\bar{t})$ (i.e $\mathcal{Q}_1(t_2) = \mathcal{M}_1(t_2)$, and $T_1(t_2) = t_2$). Then,
$$\mathcal{P}(\bar{t}) = \mathcal{M}(S(\bar{t}), t_2) = \mathcal{Q}(t_1, T(\bar{t}))(S(\bar{t}), t_2) = \mathcal{Q}(S(\bar{t}), T(S(\bar{t}), t_2)).$$
4. Return the rational parametrization $\mathcal{Q}(\bar{t})$ of the surface $\mathcal{V}$, and
$$R(\bar{t}) = (S(\bar{t}), T(S(\bar{t}), t_2)) \in \mathbb{K}(\bar{t})^2.$$

From Algorithm 2, and while using that the degree of a rational map is multiplicative under composition, we deduce some properties that relate the degree the rational map $\phi_{\mathcal{P}}$ with the degree of the rational maps $\phi_{\mathcal{Q}}$, $\phi_{\mathcal{M}}$, $\phi_{\mathcal{P}_i}$, $i = 1, 2$, and with $\deg(R)$, where $R(\bar{t}) = (S(\bar{t}), T(S(\bar{t}), t_2))$. One reasons, similarly as in [32].

Theorem 6. *It holds that*

$$\deg(\phi_{\mathcal{P}}) = \deg(\phi_{\mathcal{Q}})\deg_{t_1}(S(\bar{t}))\deg_{t_2}(T(\bar{t})), \quad \textit{and}$$

$$\deg(\phi_{\mathcal{P}_2}) = \deg_{t_1}(S(\bar{t})), \quad \deg(\phi_{\mathcal{M}_1}) = \deg_{t_2}(T(\bar{t})).$$

In addition,

$$\deg(\phi_{\mathcal{P}}) = \deg(\phi_{\mathcal{M}})\deg_{t_1}(S(\bar{t})), \quad \deg(\phi_{\mathcal{M}}) = \deg(\phi_{\mathcal{Q}})\deg_{t_2}(T(\bar{t})).$$

Corollary 2. *The following statements are equivalent:*

1. $\mathcal{Q}$ *is proper.*
2. $\deg(\phi_{\mathcal{M}}) = \deg_{t_2}(T)$.
3. $\deg(\phi_{\mathcal{P}}) = \deg_{t_1}(S)\deg_{t_2}(T)$.

Finally, in Corollary 3, we show in which conditions Algorithm 2 does not return a better reparametrization than the input one (in the sense of the degree of the rational map that is induced by the rational parametrization).

Corollary 3. *It holds that* $\deg(\phi_{\mathcal{Q}}) = \deg(\phi_{\mathcal{P}})$ *if and only if* $\deg(\phi_{\mathcal{P}_2}) = \deg(\phi_{\mathcal{M}_1}) = 1$. *In particular, if* $\deg(\phi_{\mathcal{P}_i}) = 1$ *for* $i = 1, 2$, *then* $\deg(\phi_{\mathcal{Q}}) = \deg(\phi_{\mathcal{P}})$.

We observe that, while using previous results, one may easily analyze whether some families of surfaces can be properly reparametrized using the approach presented in this paper. For instance, if $\deg(\phi_{\mathcal{P}_i}) \neq 1$ for some $i = 1, 2$, and $\deg(\phi_{\mathcal{P}}) = n$, where n is a prime number, then $\deg(\phi_{\mathcal{Q}}) = 1$.

To finish this section, we illustrate Algorithm 2 with one example. The times of our implementation performance is similar to the times that were obtained in [32].

Example 4. *Let* $\mathcal{V}$ *be the rational surface that is defined by the parametrization*

$$\mathcal{P}(\bar{t}) = (\wp_4(\bar{t})) : \wp_2(\bar{t}) : \wp_3(\bar{t}) : \wp_4(\bar{t}) =$$

$$\left(t_2^4 t_1^2 - t_1^4 : -t_2^2 + t_2^6 + t_2^2 t_1^4 : -t_1^2 + t_2^2 t_1^2 + t_2^4 t_1^2 - t_1^4 : -t_1^2 + t_2^4 t_1^2 + t_1^4\right).$$

For this purpose, in Step 1 of Algorithm 2, we compute a μ*-basis of* $\mathcal{P}$ *and we get that is given by*

$\mathbf{p}(\bar{t}) = (t_2^2 t_1^2 + 2t_1^2 + 4t_2^8 + 6t_2^6 - 4t_2^2 - 4, -2t_1^2, -2t_2^2 t_1^2 - t_1^2 - 4t_2^8 - 4t_2^6 + 4t_2^2 + 2, t_2^2 t_1^2 + t_1^2 + 2t_2^6 - 2)$

$\mathbf{q}(\bar{t}) = (-2t_2^2 t_1^2 - 3t_1^2 + t_2^2 + 1, 0, 2t_2^2 t_1^2 + 2t_1^2 - t_2^2 - 1, -t_1^2 + 1)$

$\mathbf{r}(\bar{t}) = (-2t_2^4 - t_2^2 + 2, 0, 2t_2^4 - 1, -t_2^2 + 1)$.

Using Theorem 4, one gets that $\deg(\phi_{\mathcal{P}}) = 4$. *Now, we apply Algorithm 1 to* $\mathcal{P}_2(t_1)$. *We obtain that*

$$G^{\mathcal{P}_2}(t_1, s_1) = s_1^2 - t_1^2 \in (\mathbb{K}[t_2])[t_1, s_1],$$

and $S_2(t_1) = -t_1^2 \in (\mathbb{K}[t_2])[t_1]$ *(remind that* $S_2(t_1) = S(t_1, t_2)$ *is seen as a polynomial in the variable* t_1 *and with coefficients in* $\mathbb{K}[t_2]$*). Subsequently, we determine the polynomials*

$$L_i(s_1, x_i) = \mathrm{Res}_{t_1}(x_i \wp_4(\bar{t}) - \wp_i(\bar{t}), s_1 - S_2(t_1)) = (m_{i,2}(s_1)x_i - m_{i,1}(s_1))^{\deg_{t_1}(S)}$$

for $i = 1, 2, 3$. *We obtain that*

$$\mathcal{M}(\bar{t}) = ((-t_2^4 - t_1)t_1 : t_2^2(t_2^4 - 1 + t_1^2) : (1 - t_2^2 - t_2^4 - t_1)t_1 : t_1(1 - t_2^4 + t_1)).$$

Now, in Step 3 of the algorithm, we apply Algorithm 1 to $\mathcal{M}_1(t_2)$. *For this purpose, we first compute a* μ*-basis of* $\mathcal{M}$ *and we get that it is given by (see Remark 3)*

$\mathbf{p}_{\mathcal{M}}(\bar{t}) = (4t_2^2 + 4 - 4t_2^8 - 6t_2^6 + t_2^2 t_1 + 2t_1, -2t_1, -4t_2^2 - 2 + 4t_2^8 + 4t_2^6 - 2t_2^2 t_1 - t_1, -2t_2^6 + 2 + t_2^2 t_1 + t_1)$

$\mathbf{q}_{\mathcal{M}}(\bar{t}) = (t_2^2 + 1 + 2t_2^2 t_1 + 3t_1, 0, -t_2^2 - 1 - 2t_2^2 t_1 - 2t_1, 1 + t_1)$
$\mathbf{r}_{\mathcal{M}}(\bar{t}) = (-2 + 2t_2^4 + t_2^2, 0, -2t_2^4 + 1, -1 + t_2^2).$

We obtain that

$$G^{\mathcal{M}_1}(t_2, s_2) = s_2^2 - t_2^2 \in (\mathbb{K}[t_1])[t_2, s_2]$$

that is, $\mathcal{M}_1$ is not proper. Afterwards, we compute $T_1(t_2) = -t_2^2 \in (\mathbb{K}[t_1])[t_2]$, and the polynomials

$$L_i(s_2, x_i) = \mathrm{Res}_{t_2}(x_i m_4(\bar{t}) - m_i(\bar{t}), s_2 - T_1(t_2)) = (q_{i,2}(s_2)x_i - q_{i,1}(s_2))^{\deg_{t_2}(T)}$$

for $i = 1, 2, 3$ (remind that $T_1(t_2) = T(t_1, t_2)$ is seen as a polynomial in the variable t_2 and with coefficients in $\mathbb{K}[t_1]$). We obtain that

$$\mathcal{Q}(\bar{t}) = \Big((-t_1 - t_2^2)t_1 : -t_2(t_2^2 - 1 + t_1^2) : (-t_1 + 1 + t_2 - t_2^2)t_1 : t_1(1 + t_1 - t_2^2)\Big).$$

In Step 4, the algorithm returns the parametrization $\mathcal{Q}(\bar{t})$, and $R(\bar{t}) = (S(\bar{t}), T(S(\bar{t}), t_2)) = (-t_1^2, -t_2^2)$. We observe that

$$\deg_{t_2}(T) = 2, \quad \textit{and} \quad \deg_{t_1}(S) = 2.$$

Thus, since $\deg(\phi_{\mathcal{P}}) = 4$, by Theorem 6, we conclude that $\deg(\phi_{\mathcal{Q}}) = 1$ and, hence, $\mathcal{Q}$ is proper.

Remark 3. *Using Lemma 1, we may compute a μ-basis, $\mathbf{p}_{\mathcal{M}}, \mathbf{q}_{\mathcal{M}}, \mathbf{r}_{\mathcal{M}}$, of $\mathcal{M}$ from the μ-basis, $\mathbf{p}, \mathbf{q}, \mathbf{r}$, of $\mathcal{P}$. Remind that $\mathcal{P}(\bar{t}) = \mathcal{M}(S_2(t_1), t_2)$, and $S_2(t_1) = S(t_1, t_2)$ is seen as a polynomial in the variable t_1 with coefficients in $\mathbb{K}[t_2]$. Thus, we have a particular case (case of curves) of the reasoning that is presented in Proposition 1 (which is stated for surfaces). More precisely, we write $\mathbf{p} = (p_1, p_2, p_3, p_4)$ and $\mathbf{p}_{\mathcal{M}} = (p_{m_1}, p_{m_2}, p_{m_3}, p_{m_4})$. Observe that the implicit equation of the parametrization $(S_2(t_1), p_i/p_4)$ (seen with coefficients in $\mathbb{K}(t_2)$ and in the variable t_1) is given by the polynomial $p_{m_i}(x_1, t_2) - x_2 p_{m_4}(x_1, t_2) \in (\mathbb{K}(t_2))[x_1, x_2]$ for $i = 1, 2, 3$ (i.e., the coefficients of the the implicit equation are in $\mathbb{K}(t_2)$). In order to compute this implicit equation, we may use that*

$$\mathrm{Res}_{t_1}(x_i p_4(\bar{t}) - p_i(\bar{t}), x_1 - S_2(t_1)) = (p_{m_4}(x_1, t_2)x_i - p_{m_i}(x_1, t_2))^{\deg_{t_1}(S)}, \quad i = 1, 2, 3$$

(see, e.g., [44]). Similarly one reasons to get $\mathbf{q}_{\mathcal{M}}$ from $\mathbf{q}$ and $\mathbf{r}_{\mathcal{M}}$ from $\mathbf{r}$. Observe that this is a particular case of the result presented in Proposition 1 (we apply the same idea stated in Proposition 1, but for the particular case of curves).

5. Implicitization Using μ-Basis

In the following, we assume that we are in the affine space (i.e., $x_4 = 1$; this simplifies the time on the computations), and we consider the polynomials

$$G_1^{\mathcal{P}}(\bar{t}, \bar{x}) := p^{\mathcal{P}}(\bar{t}, \bar{x})r_3(\bar{t}) - r^{\mathcal{P}}(\bar{t}, \bar{x})p_3(\bar{t}) \in \mathbb{K}[\bar{t}, x_1, x_2]$$
$$G_2^{\mathcal{P}}(\bar{t}, \bar{x}) := q^{\mathcal{P}}(\bar{t}, \bar{x})r_3(\bar{t}) - r^{\mathcal{P}}(\bar{t}, \bar{x})q_3(\bar{t}) \in \mathbb{K}[\bar{t}, x_1, x_2]$$
$$G_3^{\mathcal{P}}(\bar{t}, \bar{x}) := r^{\mathcal{P}}(\bar{t}, \bar{x}) \in \mathbb{K}[\bar{t}, \bar{x}], \qquad \bar{x} = (x_1, x_2, x_3).$$

Note that, then, we may write

$$G_1^{\mathcal{P}}(\bar{t}, \bar{x}) := x_1(p_1 r_3 - p_3 r_1) + x_2(p_2 r_3 - p_3 r_2) + (p_4 r_3 - p_3 r_4)$$
$$G_2^{\mathcal{P}}(\bar{t}, \bar{x}) := x_1(q_1 r_3 - q_3 r_1) + x_2(q_2 r_3 - q_3 r_2) + (q_4 r_3 - q_3 r_4)$$
$$G_3^{\mathcal{P}}(\bar{t}, \bar{x}) := x_1 r_1 + x_2 r_2 + x_3 r_3 + r_4.$$

In addition, let

$$S_{12}^{\mathcal{P}}(t_1, \bar{x}) = \mathrm{pp}_{\bar{x}}(\mathrm{Res}_{t_2}(G_1^{\mathcal{P}}, G_2^{\mathcal{P}})) \in \mathbb{K}[t_1, x_1, x_2],$$

$$T_{12}^{\mathcal{P}}(t_2, \overline{x}) = \mathrm{pp}_{\overline{x}}(\mathrm{Res}_{t_1}(G_1^{\mathcal{P}}, G_2^{\mathcal{P}})) \in \mathbb{K}[t_2, x_1, x_2].$$

Finally, $\mathcal{F}_{\mathcal{P}_{12}}(x_1, x_2)$ denotes the fiber of a point $Q_{12} := \pi_{12}(Q) = (x_1, x_2)$, where $Q = (x_1, x_2, x_3) \in \mathcal{V}_a$ and $\pi_{12}(\mathcal{V}_a)$ is the $(1,2)$-projection of $\mathcal{V}_a$. That is

$$\mathcal{F}_{\mathcal{P}_{12}}(Q_{12}) = \mathcal{P}_{12}^{-1}(Q_{12}) = \{\,\overline{t} \in \mathbb{K}^2 \mid \mathcal{P}_{12}(\overline{t}\,) = Q_{12}\},$$

where $\mathcal{P}_{12} := (\wp_1/\wp_4, \wp_2/\wp_4) := \pi_{12}(\mathcal{P}_a)$.

Lemma 2. *It holds that*

$$\deg_{t_1}(S_{12}^{\mathcal{P}}) = \deg_{t_2}(T_{12}^{\mathcal{P}}) = \mathrm{Card}(\mathcal{F}_{\mathcal{P}_{12}}(x_1, x_2)).$$

Proof. Because $\mathbf{p}, \mathbf{q}, \mathbf{r}$ is a μ-basis of $\mathcal{P}(\,\overline{t}\,)$, we have

$$\begin{array}{c} p_1\wp_1 + p_2\wp_2 + p_3\wp_3 + p_4\wp_4 = 0, \\ q_1\wp_1 + q_2\wp_2 + q_3\wp_3 + q_4\wp_4 = 0, \\ r_1\wp_1 + r_2\wp_2 + r_3\wp_3 + r_4\wp_4 = 0. \end{array} \quad (1)$$

Consider a generic point $Q = (x_1, x_2, x_3)$ on the variety generated by $(\wp_1 : \wp_2 : \wp_4)$ and the associated polynomials

$$p^{\mathcal{P}}(\,\overline{t}, \overline{x}\,) = \mathbf{p}(\,\overline{t}\,) \cdot Q, \text{ and } q^{\mathcal{P}}(\,\overline{t}, \overline{x}\,) = \mathbf{q}(\,\overline{t}\,) \cdot Q,$$

where $\mathbf{p}(\,\overline{t}\,) = (p_1r_3 - p_3r_1, p_2r_3 - p_3r_2, p_4r_3 - p_3r_4)$, and $\mathbf{q}(\,\overline{t}\,) = (q_1r_3 - q_3r_1, q_2r_3 - q_3r_2, q_4r_3 - q_3r_4)$. It holds that $p^{\mathcal{P}}(\,\overline{t}, \mathcal{P}(\,\overline{t}\,)) = q^{\mathcal{P}}(\,\overline{t}, \mathcal{P}(\,\overline{t}\,)) = 0$. In fact, $p^{\mathcal{P}}(\,\overline{t}, \mathcal{P}(\,\overline{t}\,)) = (p_1r_3 - p_3r_1)\wp_1 + (p_2r_3 - p_3r_2)\wp_2 + (p_4r_3 - p_3r_4)\wp_4 = 0$ is derived by eliminating $\wp_3$ from the first and third equations in (1). Similarly, to find $q^{\mathcal{P}}(\,\overline{t}, \mathcal{P}(\,\overline{t}\,)) = 0$ from the last two equations in (1).

Thus, one may reason as in Theorem 4 and Corollary 1 (also see Remark 2) to get that

$$\deg_{t_1}(S_{12}^{\mathcal{P}}(t_1, \overline{x}\,)) = \deg_{t_2}(T_{12}^{\mathcal{P}}(t_2, \overline{x}\,)) = \mathrm{Card}(\mathcal{F}_{\mathcal{P}_{12}}(x_1, x_2))$$

(remind that $\mathcal{P}_{12} := (\wp_1/\wp_4, \wp_2/\wp_4) = \pi_{12}(\mathcal{P}_a)$). □

Theorem 7. *Let* $\mathbf{p}(\,\overline{t}\,)$, $\mathbf{q}(\,\overline{t}\,)$ *and* $\mathbf{r}(\,\overline{t}\,)$ *a* μ*-basis for* $\mathcal{P}(\,\overline{t}\,)$*. It holds that*

$$\mathrm{pp}_{x_3}(h(\,\overline{x}\,)) = f(\,\overline{x}\,)^{\deg(\phi_{\mathcal{P}})}$$

where

$$h(\,\overline{x}\,) = \mathrm{Cont}_{\{Z,W\}}(\mathrm{Res}_{t_2}(T_{12}^{\mathcal{P}}(t_2, \overline{x}\,), K(t_2, Z, W, \overline{x}\,))) \in \mathbb{K}[\,\overline{x}\,],$$

$$K(t_2, Z, W, \overline{x}\,) = \mathrm{Res}_{t_1}(S_{12}^{\mathcal{P}}(t_1, \overline{x}\,), H^{\mathcal{P}}(\,\overline{t}, Z, W, \overline{x}\,)) \in \mathbb{K}[t_2, Z, W, \overline{x}\,],$$

and

$$H^{\mathcal{P}}(\,\overline{t}, Z, W, \overline{x}\,) = G_3^{\mathcal{P}}(\,\overline{t}, \overline{x}\,) + ZG_1^{\mathcal{P}}(\,\overline{t}, \overline{x}\,) + WG_2^{\mathcal{P}}(\,\overline{t}, \overline{x}\,) \in \mathbb{K}[\,\overline{t}, Z, W, \overline{x}\,].$$

Proof. First, we recall that

$$\deg_{t_1}(S_{12}^{\mathcal{P}}) = \deg_{t_2}(T_{12}^{\mathcal{P}}) = \mathrm{Card}(\mathcal{F}_{\mathcal{P}_{12}}(\,\overline{x}\,)).$$

Let d_{12} be this quantity. Clearly, $d_{12} \geq 1$. In addition, let $m = \deg_{t_1}(H^{\mathcal{P}})$ and $k = \deg_{t_2}(H^{\mathcal{P}})$. Regarding $S_{12}^{\mathcal{P}}$ and $H^{\mathcal{P}}$ as polynomials in $\mathbb{K}(t_2, Z, W, \overline{x}\,)[t_1]$, and using that the resultant of two univariate polynomials is the product of the evaluations of one of them in the roots of the other, we get

$$K(t_2, Z, W, \overline{x}) = \mathrm{Res}_{t_1}(S_{12}^{\mathcal{P}}, H^{\mathcal{P}}) = A(\overline{x})^m \prod_{i=1}^{d_{12}} H^{\mathcal{P}}(\alpha_i, t_2, Z, W, \overline{x}),$$

where A is the leading coefficient of $S_{12}^{\mathcal{P}}$ w.r.t. t_1, and where $\{\alpha_1, \ldots, \alpha_{d_{12}}\}$ are the roots of $S_{12}^{\mathcal{P}}$ in the algebraic closure $\overline{\mathbb{K}(x_1, x_2)}$ of $\mathbb{K}(x_1, x_2)$ (we regard $S_{12}^{\mathcal{P}}$ as an univariate polynomial in t_1). Similarly,

$$\mathrm{Res}_{t_2}(T_{12}^{\mathcal{P}}, K) = B(\overline{x})^k \prod_{j=1}^{d_{12}} K(\beta_j, Z, W, \overline{x}),$$

where B is the leading coefficient of $T_{12}^{\mathcal{P}}$ w.r.t. t_2, and $\{\beta_1, \ldots, \beta_{d_{12}}\}$ are the roots of $T_{12}^{\mathcal{P}}$ in $\overline{\mathbb{K}(x_1, x_2)}$ (we regard $T_{12}^{\mathcal{P}}$ as a univariate polynomial in t_2). Therefore,

$$\mathrm{Res}_{t_2}(T_{12}^{\mathcal{P}}, K) = B^k A^{m \cdot d_{12}} \prod_{i=1}^{d_{12}} \prod_{j=1}^{d_{12}} H^{\mathcal{P}}(\alpha_i, \beta_j, Z, W, \overline{x}).$$

By Lemma 2, there exist d_{12} pairs of points (α_i, β_j) being in $\mathcal{F}_{\mathcal{P}_{12}}(x_1, x_2)$, and for each $U(x_1, x_2) \in \mathcal{F}_{\mathcal{P}_{12}}(x_1, x_2)$ it holds that $G_1^{\mathcal{P}}(U, \overline{x}) = G_2^{\mathcal{P}}(U, \overline{x}) = 0$. Thus,

$$\mathrm{Res}_{t_2}(T_{12}^{\mathcal{P}}, K) = B^k A^{m \cdot d_{12}} Q(\overline{x}, Z, W) \prod_{U \in \mathcal{F}_{\mathcal{P}_{12}}(x_1, x_2)} G_3^{\mathcal{P}}(U, \overline{x}),$$

where

$$Q(\overline{x}, Z, W) = \prod_{\substack{1 \le i, j \le d_{12} \\ (\alpha_i, \beta_j) \notin \mathcal{F}_{\mathcal{P}_{12}}(x_1, x_2)}} H(\alpha_i, \beta_j, Z, W, \overline{x}).$$

Note that for each root α_i there exists a unique b_j satisfying that the pair (α_i, β_j) is in the fiber. Furthermore, for $(\alpha_i, \beta_j) \notin \mathcal{F}_{\mathcal{P}_{12}}(x_1, x_2)$, either $G_1^{\mathcal{P}}(\alpha_i, \beta_j, \overline{x}) \neq 0$ or $G_2^{\mathcal{P}}(\alpha_i, \beta_j, \overline{x}) \neq 0$ (see Lemma 2). Hence, $Q(\overline{x}, Z, W)$ depends on Z or W. In addition, each $H^{\mathcal{P}}(\alpha_i, \beta_j, Z, W, \overline{x})$ does depend on Z or W.

Next, we show that $Q(\overline{x}, Z, W)$, regarded as polynomial in $\overline{\mathbb{K}[\overline{x}]}[\overline{x}, Z, W]$, is primitive w.r.t. the variables $\{Z, W\}$. For this purpose, we denote, by $N(x_3) \in \overline{\mathbb{K}[x_1, x_2]}[x_3]$, the content of Q w.r.t. $\{Z, W\}$. Thus, there exists $(\alpha_i, \beta_j) \notin \mathcal{F}_{\mathcal{P}_{12}}(x_1, x_2)$ satisfying that the polynomial N divides $H(\alpha_i, \beta_j, Z, W, \overline{x})$; that is, $N(x_3)$ divides $G_3^{\mathcal{P}}(\alpha_i, \beta_i, \overline{x}) + Z G_1^{\mathcal{P}}(\alpha_i, \beta_j, \overline{x}) + W G_2^{\mathcal{P}}(\alpha_i, \beta_j, \overline{x})$ and, then, $N(x_3)$ divides $G_1^{\mathcal{P}}(\alpha_i, \beta_j, x_1, x_2)$ and $G_2^{\mathcal{P}}(\alpha_i, \beta_j, x_1, x_2)$. Taking into account that at least one of them is not zero, we get that $N \in \overline{\mathbb{K}[x_1, x_2]}$ and, thus, Q is primitive w.r.t. $\{Z, W\}$. Now, using that

$$h(\overline{x}) = \mathrm{Cont}_{\{Z, W\}}(\mathrm{Res}_{t_2}(T_{12}^{\mathcal{P}}, K)),$$

we obtain that

$$h(\overline{x}) = B^k A^{m \cdot d_{12}} \cdot N(x_1, x_2) \cdot \prod_{U \in \mathcal{F}_{\mathcal{P}_{12}}(x_1, x_2)} G_3^{\mathcal{P}}(U, \overline{x}),$$

where $N \in \overline{\mathbb{K}[x_1, x_2]}$. Thus,

$$\mathrm{pp}_{x_3}(h(\overline{x})) = \mathrm{pp}_{x_3}\left(\prod_{U \in \mathcal{F}_{\mathcal{P}_{12}}(x_1, x_2)} G_3^{\mathcal{P}}(U(\overline{x}), \overline{x}) \right).$$

Under these conditions, it holds that $\deg_{x_3}(\mathrm{pp}_{x_3}(h(\overline{x}))) = d_{12}$. Indeed, clearly one has $\deg_{x_3}(\mathrm{pp}_{x_3}(h(\overline{x}))) \leq d_{12}$. If $\deg_{x_3}(\mathrm{pp}_{x_3}(h(\overline{x}))) < d_{12}$, thus, there exists $U \in \mathcal{F}_{\mathcal{P}_{12}}(x_1, x_2)$, such that $r_3(U) = 0$. Moreover, $U \in \mathbb{K}^2$, which is impossible since $U \in \mathcal{F}_{\mathcal{P}_{12}}(x_1, x_2)$.

Now, we prove that

$$\mathrm{pp}_{x_3}\left(\prod_{U \in \mathcal{F}_{\mathcal{P}_{12}}(x_1,x_2)} G_3^{\mathcal{P}}(U(\overline{x}), \overline{x})\right) = f(\overline{x})^r.$$

Indeed, clearly one has that

$$\mathrm{pp}_{x_3}\left(\prod_{U \in \mathcal{F}_{\mathcal{P}_{12}}(x_1,x_2)} G_3^{\mathcal{P}}(U(\overline{x}), \overline{x})\right) = f(\overline{x})^r g(\overline{x}).$$

Furthermore, $r \geq \deg(\phi_{\mathcal{P}})$, since $G_3^{\mathcal{P}}(U(\overline{x}), \overline{x}) = G_3^{\mathcal{P}}(V(\overline{x}), \overline{x})$ for $U, V \in \mathcal{F}_{\mathcal{P}}(\overline{x})$ (observe that $\mathcal{F}_{\mathcal{P}}(\overline{x}) \subseteq \mathcal{F}_{\mathcal{P}_{12}}(x_1, x_2))$). Thus, since $\deg_{x_3}(f) = d_{12}/\deg(\phi_{\mathcal{P}})$ (see [44]) and $\deg_{x_3}(\mathrm{pp}_{x_3}(h(\overline{x}))) = d_{12}$, we get that

$$d_{12}/\deg(\phi_{\mathcal{P}}) \cdot r + \deg(g) = d_{12}$$

, which implies that $d_{12}(1 - r/\deg(\phi_{\mathcal{P}})) = \deg(g)$ and, hence, $r \leq \deg(\phi_{\mathcal{P}})$. Because $r \geq \deg(\phi_{\mathcal{P}})$, we conclude that $\deg(g) = 0$ and $r = \deg(\phi_{\mathcal{P}})$. $\square$

In the following examples, we illustrate the above theorem. These examples are taken from [5].

Example 5. *Let $\mathcal{V}$ be the rational surface that is defined by the parametrization*

$$\mathcal{P}(\bar{t}) = (t_2^2 t_1 - t_1^2 : -t_2 + t_2^3 + t_2 t_1^2 : -t_1 + t_2 t_1 + t_2^2 t_1 - t_1^2 : -t_1 + t_2^2 t_1 + t_1^2).$$

We determine the polynomials $p^{\mathcal{P}}(\bar{t}, \overline{x}) = \mathbf{p}(\bar{t}) \cdot \overline{x}$, $q^{\mathcal{P}}(\bar{t}, \overline{x}) = \mathbf{q}(\bar{t}) \cdot \overline{x}$, $r^{\mathcal{P}}(\bar{t}, \overline{x}) = \mathbf{r}(\bar{t}) \cdot \overline{x}$, where the μ-basis is given by

$\mathbf{p}(\bar{t}) = (t_2 t_1 + 2t_1 + 4t_2^4 + 6t_2^3 - 4t_2 - 4, -2t_1, -2t_2 t_1 - t_1 - 4t_2^4 - 4t_2^3 + 4t_2 + 2, t_2 t_1 + t_1 + 2t_2^3 - 2)$

$\mathbf{q}(\bar{t}) = (-2t_2 t_1 - 3t_1 + t_2 + 1, 0, 2t_2 t_1 + 2t_1 - t_2 - 1, -t_1 + 1)$

$\mathbf{r}(\bar{t}) = (-2t_2^2 - t_2 + 2, 0, 2t_2^2 - 1, -t_2 + 1).$

Now, we determine

$$G_1^{\mathcal{P}}(\bar{t}, \overline{x}) := p^{\mathcal{P}}(\bar{t}, \overline{x}) r_3(\bar{t}) - r^{\mathcal{P}}(\bar{t}, \overline{x}) p_3(\bar{t}) \in \mathbb{K}[\bar{t}, x_1, x_2]$$
$$G_2^{\mathcal{P}}(\bar{t}, \overline{x}) := q^{\mathcal{P}}(\bar{t}, \overline{x}) r_3(\bar{t}) - r^{\mathcal{P}}(\bar{t}, \overline{x}) q_3(\bar{t}) \in \mathbb{K}[\bar{t}, x_1, x_2]$$
$$G_3^{\mathcal{P}}(\bar{t}, \overline{x}) := r^{\mathcal{P}}(\bar{t}, \overline{x}) \in \mathbb{K}[\bar{t}, \overline{x}], \quad \overline{x} = (x_1, x_2, x_3)$$

and we compute

$S_{12}^{\mathcal{P}}(t_1, \overline{x}) = \mathrm{pp}_{\overline{x}}(\mathrm{Res}(G_1^{\mathcal{P}}, G_2^{\mathcal{P}}, t_2)) = -t_1 + 2t_1^2 + x_1 + 2x_1^2 - x_2^2 t_1^2 x_1^2 + x_2^2 t_1^2 x_1 - 4x_2^2 t_1^3 x_1 + 4x_2^2 t_1^4 x_1 + 4x_2^2 t_1^3 x_1^2 - x_2^2 t_1^2 x_1^3 - 4x_2^2 t_1^4 x_1^3 + 4x_2^2 t_1^3 x_1^3 - 4x_2^2 t_1^4 x_1^2 + 4x_2^2 t_1^4 - 4x_2^2 t_1^3 + x_2^2 t_1^2 - 5x_1 t_1^4 - t_1^5 x_1 - 2x_1^4 t_1 + 2x_1^3 t_1^4 + 2x_1^3 t_1^5 + 2x_1^2 t_1^5 + x_1^5 t_1^2 + 3x_1^5 t_1^4 - x_1^4 t_1^5 - 3x_1^5 t_1^3 - 7x_1^3 t_1 + x_1 t_1^3 - x_1^5 t_1^5 + 16x_1^3 t_1^2 - 4x_1^2 t_1^4 - 6x_1^2 t_1^3 + 8x_1^4 t_1^2 - 5x_1 t_1 + 14x_1^2 t_1^2 - 9x_1^2 t_1 + 7x_1 t_1^2 + 6x_1^4 t_1^4 - 11x_1^4 t_1^3 - 14x_1^3 t_1^3 - t_1^5 + x_1^3 - 2t_1^4 + t_1^3,$

$T_{12}^{\mathcal{P}}(t_2, \overline{x}) = \mathrm{pp}_{\overline{x}}(\mathrm{Res}(G_1^{\mathcal{P}}, G_2^{\mathcal{P}}, t_1)) = -2t_2^5x_1 + 4x_1t_2^3 - 2t_2^4x_2 + t_2^2x_2 + t_2^5 + x_1^2t_2^5 - x_1^2t_2^3 + 2x_1t_2^4x_2 - 3x_1t_2^2x_2 + x_1x_2 - t_2 - 2x_1t_2 + t_2^3.$

Also, let

$$H^{\mathcal{P}}(\overline{t}, Z, W, \overline{x}) = G_3^{\mathcal{P}}(\overline{t}, \overline{x}) + ZG_1^{\mathcal{P}}(\overline{t}, \overline{x}) + WG_2^{\mathcal{P}}(\overline{t}, \overline{x}) \in \mathbb{K}[\overline{t}, Z, W, \overline{x}]$$

and

$$K(t_2, Z, W, \overline{x}) = \mathrm{Res}_{t_1}(S_{12}^{\mathcal{P}}(t_1, \overline{x}), H^{\mathcal{P}}(\overline{t}, Z, W, \overline{x})) \in \mathbb{K}[t_2, Z, W, \overline{x}].$$

Finally, we compute

$$h(\overline{x}) = \mathrm{Cont}_{\{Z,W\}}(\mathrm{Res}_{t_2}(T_{12}^{\mathcal{P}}(t_2, \overline{x}), K(t_2, Z, W, \overline{x}))) \in \mathbb{K}[\overline{x}],$$

and

$\mathrm{pp}_{x_3}(h(\overline{x})) = f(\overline{x})^{\deg(\phi_{\mathcal{P}})} = -1 - 46x_1^2x_2x_3 + 38x_1x_2x_3^2 - 8x_1x_2x_3 + 4x_2^2x_1^2x_3 - 12x_2^2x_1x_3 + 10x_1^3x_2x_3 + x_1^2x_2x_3^2 - 10x_1x_2x_3^3 + 4x_1x_2^2x_3^2 - 10x_2x_3^3 + 8x_2^2x_3^2 - 4x_2^2x_3 + 5x_2x_3^2 - 5x_1^4x_2 + 19x_1^3x_2 - 4x_2^2x_3^3 + 4x_2x_3^4 + 2x_1^2x_2 - 2x_3^2x_1^3 + 47x_3^2x_1^2 - 12x_1^4x_3 + 14x_1x_3^2 + 11x_1^2x_3^3 - 6x_1x_3^4 - 32x_1x_3^3 - 22x_3x_1^3 - 12x_3x_1^2 + 4x_3x_1 - 4x_1^3x_2^2 + 4x_1^2x_2^2 + 4x_1x_2^2 - x_1x_2 + 7x_3^4 - 5x_3^3 - 5x_3^2 + 8x_1^5 + 2x_1^2 + 2x_1^3 + x_3^5 + 3x_3 - 2x_1 + x_2.$

Observe that we may conclude that $\deg(\phi_{\mathcal{P}}) = 1$ *and, thus,* $\mathcal{P}(\overline{t})$ *is a proper rational parametrization.*

We have implemented this method while using Maple 2016 on a Lenovo ThinkPad Intel(R) Core(TM) i7-7500U CPU @ 2.70 *GHz* 2.90 *GHz and* 16 *GB of RAM, OS-Windows 10 Pro. The time, in CPU seconds, for this example is* 10.907 *and using Gröbner basis, we get* 0.187.

Example 6. *Let* $\mathcal{V}$ *be the rational surface defined by the parametrization*

$\mathcal{P}(\overline{t}) = (t_1^2 + t_2t_1 + 2t_2^2 - 2t_2^2t_1 : t_1^2 + 2t_2t_1 + t_2t_1^2 + 2t_2^2 - t_2^2t_1 + 2t_2^2t_1^2 : -t_1^2 + t_2t_1 + 2t_2t_1^2 + 2t_2^2 - t_2^2t_1 - 2t_2^2t_1^2 : 2t_2t_1 - 2t_2t_1^2 - 2t_2^2t_1 - t_2^2t_1^2).$

We determine the polynomials $p^{\mathcal{P}}(\overline{t}, \overline{x}) = \mathbf{p}(\overline{t}) \cdot \overline{x}$, $q^{\mathcal{P}}(\overline{t}, \overline{x}) = \mathbf{q}(\overline{t}) \cdot \overline{x}$, $r^{\mathcal{P}}(\overline{t}, \overline{x}) = \mathbf{r}(\overline{t}) \cdot \overline{x}$, *where the* μ*-basis is given by*

$\mathbf{p}(\overline{t}) = (-1344390t_2^2t_1 + 34075368t_2t_1 - 22657890t_1 - 5710808t_2^3 - 181563t_2^2 - 23392736t_2 - 4984080, 1344390t_2^2t_1 - 25195836t_2t_1 + 10711400t_1 + 3569255t_2^3 + 1074194t_2^2 + 18408656t_2 + 4984080, 1344390t_2^2t_1 - 17483628t_2t_1 - 11946490t_1 + 2141553t_2^3 - 892631t_2^2 + 4984080t_2, 9704572t_2 - 11391246t_2t_1 + 6590790t_1 + 2855404t_2^3 + 6075203t_2^2 - 2492040)$

$\mathbf{q}(\overline{t}) = (-229530t_2t_1 - 50278t_1 + 139288t_2^2 - 174717t_2 + 194136, 131160t_2t_1 + 155206t_1 - 87055t_2^2 + 85766t_2 - 194136, 65580t_2t_1 + 104928t_1 - 52233t_2^2 + 88951t_2, 131160t_2t_1 + 100556t_1 - 69644t_2^2 - 58603t_2 + 97068)$

$\mathbf{r}(\overline{t}) = (-8t_2^3 + 11t_2^2 - 4t_2 + 4, 5t_2^3 - 6t_2^2 + 8t_2 - 4, 3t_2^3 - 5t_2^2 - 4t_2, 4t_2^3 + t_2^2 + 2).$

Now, we determine

$$G_1^{\mathcal{P}}(\overline{t}, \overline{x}) := p^{\mathcal{P}}(\overline{t}, \overline{x})r_3(\overline{t}) - r^{\mathcal{P}}(\overline{t}, \overline{x})p_3(\overline{t}) \in \mathbb{K}[\overline{t}, x_1, x_2]$$

$$G_2^{\mathcal{P}}(\overline{t}, \overline{x}) := q^{\mathcal{P}}(\overline{t}, \overline{x})r_3(\overline{t}) - r^{\mathcal{P}}(\overline{t}, \overline{x})q_3(\overline{t}) \in \mathbb{K}[\overline{t}, x_1, x_2]$$

$$G_3^{\mathcal{P}}(\overline{t}, \overline{x}) := r^{\mathcal{P}}(\overline{t}, \overline{x}) \in \mathbb{K}[\overline{t}, \overline{x}], \quad \overline{x} = (x_1, x_2, x_3)$$

and we compute $S_{12}^{\mathcal{P}}(t_1, \overline{x})$ *and* $T_{12}^{\mathcal{P}}(t_2, \overline{x})$*. Additionally, let*

$$H^{\mathcal{P}}(\bar{t}, Z, W, \overline{x}) = G_3^{\mathcal{P}}(\bar{t}, \overline{x}) + ZG_1^{\mathcal{P}}(\bar{t}, \overline{x}) + WG_2^{\mathcal{P}}(\bar{t}, \overline{x}) \in \mathbb{K}[\bar{t}, Z, W, \overline{x}]$$

and

$$K(t_2, Z, W, \overline{x}) = \mathrm{Res}_{t_1}(S_{12}^{\mathcal{P}}(t_1, \overline{x}), H^{\mathcal{P}}(\bar{t}, Z, W, \overline{x})) \in \mathbb{K}[t_2, Z, W, \overline{x}].$$

Finally, we compute

$$h(\overline{x}) = \mathrm{Cont}_{\{Z,W\}}(\mathrm{Res}_{t_2}(T_{12}^{\mathcal{P}}(t_2, \overline{x}), K(t_2, Z, W, \overline{x}))) \in \mathbb{K}[\overline{x}],$$

and from $\mathrm{pp}_{x_3}(h(\overline{x}))$*, we get that*

$$f(\overline{x}) = -449792 + 51270879x_3x_1x_2^2 + 13092929x_3^2x_1x_2 - 3482416x_3x_1^2x_2 + 22904376x_3x_1x_2 + 675054x_1x_3^6 - 862596x_2x_3^6 - 29225028x_1^2x_2x_3^3 + 11830146x_1x_2x_3^3 + 32231373x_1x_2^2x_3^3 + 110760512x_1^5x_2x_3 - 90099948x_1^4x_2^2x_3 - 129717124x_1^4x_2x_3^2 + 40844546x_1^3x_2^3x_3 + 124810810x_1^3x_2^2x_3^2 + 74702726x_1^3x_2x_3^3 - 19905824x_1^2x_2^4x_3 - 64077866x_1^2x_2^3x_3^2 - 75736662x_1^2x_2^2x_3^3 + 18645124x_1x_2^4x_3^2 + 9481980x_1x_2^5x_3 + 33323142x_1x_2^3x_3^3 - 40980736x_1^6x_2 - 50645760x_1^6x_3 + 34182816x_1^5x_2^2 + 54875936x_1^5x_3^2 - 24633612x_1^4x_2^3 - 26906244x_1^4x_3^3 + 30171008x_1^3x_2^4 - 25238190x_1^2x_2^5 + 9961900x_1x_2^6 - 1918400x_2^6x_3 + 30499314x_1x_2^4x_3 - 28601154x_3^2x_1^2x_2^2 + 22849453x_3^2x_1x_2^3 + 27085291x_3^2x_1^3x_2 - 2534290x_2^5x_3^2 - 5374808x_2^4x_3^3 + 3235863x_1^3x_3^3 - 9477510x_2^3x_3^3 + 2815992x_1^2x_3^3 - 8545002x_2^2x_3^3 + 3831717x_1x_3^3 - 5284450x_2x_3^3 - 15996836x_3^2x_1^3 + 119044501x_1x_2^4 - 25566784x_1^5x_2 + 9070776x_1^4x_2^2 + 80443033x_1^3x_2^3 + 9622080x_1^5x_3 - 109224707x_1^2x_2^4 + 53021640x_1x_2^5 - 9557440x_2^5x_3 - 19184968x_3^2x_1^4 - 6794107x_3^2x_2^4 + 16282880x_1^4x_2 + 119771644x_1^3x_2^2 - 205506824x_1^2x_2^3 + 49430528x_3x_1^4 + 41274000x_1^4x_3x_2 - 63080603x_1^3x_3x_2^2 - 3378276x_1^2x_3x_2^3 - 96074840x_3x_1^3x_2 + 9101736x_3x_1^2x_2^2 + 53646214x_3x_1x_2^3 + 10909032x_3^2x_1^2x_2 - 21171376x_3x_2^4 - 10777540x_3^2x_2^3 + 7054782x_3^4x_1^3 - 6762492x_2^2x_3^4 + 2808045x_1^2x_3^4 + 16635132x_3^2x_1x_2^2 - 1041561x_1x_3^4 - 2701863x_2x_3^4 - 5920710x_3^4x_2^3 - 3201822x_3^5x_2^2 - 367011x_3^5x_1 - 1776762x_3^5x_2 - 2007234x_3^5x_1^2 + 10217583x_2x_1x_3^4 - 26748144x_3^4x_2x_1^2 + 23463306x_3^4x_2^2x_1 + 6779862x_3^5x_2x_1 + 4466880x_1 - 3813952x_2 - 520768x_3 - 16392064x_1^2 - 13928144x_2^2 - 857264x_3^2 + 23714880x_1^3 - 28078300x_2^3 - 573568x_1^4 - 33492106x_2^4 - 24141568x_1^5 - 23550085x_2^5 - 196096x_1^6 - 9046200x_2^6 + 39690x_3^5 - 638436x_3^4 - 1440780x_3^3 + 3591536x_3x_1 - 2849856x_3x_1^2 + 4138196x_3^2x_1 + 31582352x_1x_2 - 4613120x_3x_2 - 90566848x_1^2x_2 + 92152884x_1x_2^2 - 15211892x_3x_2^2 + 88688752x_1^3x_2 - 195019560x_1^2x_2^2 + 140970299x_1x_2^3 - 25290352x_3x_1^3 - 24626550x_3x_2^3 - 5042468x_3^2x_2 + 1087832x_3^2x_1^2 - 10534970x_3^2x_2^2 + 18348032x_1^7 - 1475500x_2^7 - 119313x_3^6 - 96066x_3^7,$$

and $\deg(\phi_{\mathcal{P}}) = 1$*. That is,* $\mathcal{P}(\bar{t})$ *is a proper rational parametrization.*

The time, in CPU seconds, for this example is 71.703*, and using Gröbner basis, we get a time that is* > 5000.

Remark 4. *In order to improve the time of computations, one may compute the polynomial*

$$h(\overline{x}) = \mathrm{Cont}_{\{Z,W\}}(\mathrm{Res}_{t_2}(T_{12}^{\mathcal{P}}(t_2, \overline{x}), K(t_2, Z, W, \overline{x})))$$

as $\gcd(R_1, R_2)$*, where*

$$R_i = \mathrm{Res}_{t_2}(T_{12}^{\mathcal{P}}(t_2, \overline{x}), K(t_2, a_i, b_i, \overline{x})), \ i = 1, 2$$

and $a_i, b_i \in \mathbb{K}$ *are random constants. The answer is correct with a probability of almost one, since, taking into account the behavior of the gcd under specializations, this property holds in an open Zariski subset (see e.g., Lemmas 7 and 8 in [45]).*

6. Conclusions

The μ-basis has shown as a bridge tool between the parametric form and the implicit form of curves and surfaces. Moreover, the μ-basis has also been introduced into applications in singularities analysis and collision detections. The μ-basis theory of curves are more complete than that of surfaces, but surfaces would certainly attract more attention, although the discussion is more difficult. We study the μ-basis further for improper rational surfaces. The results are essential to the theoretical completeness of the μ-basis of surface.

We show how the μ-basis allows for computing the inversion of a given proper parametrization $\mathcal{P}(\bar{t})$ of an algebraic surface. If $\mathcal{P}(\bar{t})$ is not proper, we show how the degree of the rational map that is induced by $\mathcal{P}(\bar{t})$ can be computed as well as the elements of the fiber. Furthermore and directly from $\mathcal{P}(\bar{t})$, we propose a method to find a μ-basis for a proper reparametrization $\mathcal{Q}(\bar{t})$ with some assumptions. If $\mathcal{P}(\bar{t})$ is improper, we give some partial results in finding a proper reparametrization of $\mathcal{V}$. Finally, we show how the μ-basis of a given not being necessarily proper parametrization also allows for computing the implicit equation of a given surface bysubsequence univariate resultants.

As the further work, the numerical consideration could be an interesting extension of the μ-basis theory. One possible way would consist in generalizing the symbolic computation to numerical situation using the ideas and techniques that have already been implemented in some other problems, such as the numerical proper reparametrization of surfaces [46].

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