A 4D counter-example showing that DWCness does not imply CWCness in n-D

Nicolas Boutry[1], Rocio Gonzalez-Diaz[2], Laurent Najman[3], and Thierry Géraud[1]

[1] EPITA Research and Development Laboratory (LRDE), EPITA, France
[2] Universidad de Sevilla, Sevilla, Spain
[3] Université Paris-Est, LIGM, Équipe A3SI, ESIEE, France

Abstract. In this paper, we prove that the two flavours of well-composedness called Continuous Well-Composedness (shortly CWCness), stating that the boundary of the continuous analog of a discrete set is a manifold, and Digital Well-Composedness (shortly DWCness), stating that a discrete set does not contain any critical configuration, are not equivalent in dimension 4. To prove this, we exhibit the example of a configuration of 8 tesseracts (4D cubes) sharing a common corner (vertex), which is DWC but not CWC. This result is surprising since we know that CWCness and DWCness are equivalent in 2D and 3D. To reach our goal, we use local homology.

Keywords: well-composed · topological manifolds · critical configurations · digital topology · local homology

1 Introduction

Digital well-composedness (shortly DWCness) is a nice property in digital topology, because it implies the equivalence of $2n$ and $(3^n - 1)$ connectivities [3] in a subset of $\mathbb{Z}^n$ and in its complement at the local and global points of view. A well-known application of this flavour of well-composedness is the tree of shapes [7,9]: usual natural or synthetic images generally contain many critical configurations and this way we cannot ensure that the hierarchy induced by the inclusion relationship between these shapes does not draw a graph without cycles. On the contrary, when an image is DWC, no cycle is possible and then we obtain a tree, called the *tree of shapes* [7,9,16].

On the other side, continuously well-composed (shortly CWC) images are known as "counterparts" of n-dimensional manifolds in the sense that the boundary of their continuous analog does not have singularities (called "pinches"), which is a very strong topological property. The consequence is that some geometric differential operators can be computed directly on the discrete sets, which simplifies or makes specific algorithms faster [12,13].

These two flavours of well-composednesses, known to be equivalent in 2D and in 3D, are not equivalent in 4D, and this is what we are going to prove in this paper. Section 2 recalls the material relative to discrete topology and local homology necessary to the proof detailed in Section 3. Section 4 concludes the paper.

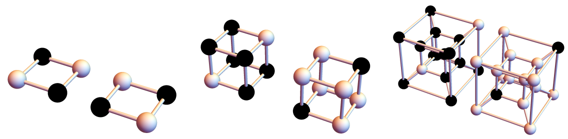

Fig. 1: Examples of primary and secondary critical configurations in 2D/3D/4D blocks S. Black bullets correspond to the points of the digital set $\mathcal{X}$ and the white bullets correspond to the points of $S \setminus \mathcal{X}$.

2 Discrete topology

As usual in discrete topology, we will only work with *digital sets*, that is, finite subsets of $\mathbb{Z}^n$ or subsets $\mathcal{X}$ of $\mathbb{Z}^n$ whose complementary set $\mathcal{X}^c = \mathbb{Z}^n \setminus \mathcal{X}$ is finite.

2.1 Digital well-composedness

Let $n \geq 2$ be a (finite) integer called the *dimension*. Now, let $\mathbb{B} = \{e^1, \dots, e^n\}$ be the (orthonormal) canonical basis of $\mathbb{Z}^n$. We use the notation v_i, where i belongs to $[\![1,n]\!] := \{i \in \mathbb{Z} \ ; \ 1 \leq i \leq n\}$, to determine the i^{th} coordinate of a vector $v \in \mathbb{Z}^n$. We recall that the L^1-*norm* of the vector $v \in \mathbb{Z}^n$ is denoted by $\| \cdot \|_1$ and is equal to $\sum_{i \in [\![1,n]\!]} |v_i|$ where $| \cdot |$ is the *absolute value*. Also, the L^∞-*norm* is denoted by $\| \cdot \|_\infty$ and is equal to $\max_{i \in [\![1,n]\!]} |v_i|$.

For a given point $p \in \mathbb{Z}^n$, the $2n$-*neighborhood* in $\mathbb{Z}^n$, denoted by $\mathcal{N}_{2n}(p)$, is equal to $\{q \in \mathbb{Z}^n \ ; \ \|p - q\|_1 \leq 1\}$. Also, the $(3^n - 1)$-*neighborhood* in $\mathbb{Z}^n$, denoted by $\mathcal{N}_{3^n-1}(p)$, is equal to $\{q \in \mathbb{Z}^n \ ; \ \|p - q\|_\infty \leq 1\}$. Let ξ be a value in $\{2n, 3^n - 1\}$. The *starred ξ-neighborhood* of $p \in \mathbb{Z}^n$ is noted $\mathcal{N}^*_\xi(p)$ and is equal to $\mathcal{N}_\xi(p) \setminus \{p\}$. An element of the starred ξ-neighborhood of $p \in \mathbb{Z}^n$ is called a ξ-*neighbor* of p in $\mathbb{Z}^n$. Two points $p, q \in \mathbb{Z}^n$ such that $p \in \mathcal{N}^*_\xi(q)$ or equivalently $q \in \mathcal{N}^*_\xi(p)$ are said to be ξ-*adjacent*. A finite sequence $(p^0, \dots, p^k)$ of points in $\mathbb{Z}^n$ is a ξ-*path* if and only if p^0 is ξ-adjacent only to p^1, p^k is ξ-adjacent only to p^{k-1}, and if for $i \in [\![1, k-1]\!]$, p^i is ξ-adjacent only to p^{i-1} and to p^{i+1}. A digital set $\mathcal{X} \subset \mathbb{Z}^n$ is said ξ-*connected* if for any pair of points $p, q \in \mathcal{X}$, there exists a ξ-path joining them into $\mathcal{X}$. A ξ-connected subset C of $\mathcal{X}$ which is *maximal in the inclusion sense*, that is, there is no ξ-connected subset of $\mathcal{X}$ which is greater than C, is said to be a ξ-*component* of $\mathcal{X}$.

For any $p \in \mathbb{Z}^n$ and any $\mathcal{F} = (f^1, \dots, f^k) \subseteq \mathbb{B}$, we denote by $S(p, \mathcal{F})$ the set:

$$\Big\{ p + \sum_{i \in [\![1,k]\!]} \lambda_i f^i \ ; \ \lambda_i \in \{0,1\}, \forall i \in [\![1,k]\!] \Big\}.$$

We call this set the *block* associated with the pair $(p, \mathcal{F})$; its *dimension*, denoted by $\dim(S)$, is equal to k. More generally, a set $S \subset \mathbb{Z}^n$ is said to be a *block* if there exists a pair $(p, \mathcal{F}) \in \mathbb{Z}^n \times \mathcal{P}(\mathbb{B})$ such that $S = S(p, \mathcal{F})$. We say that two points $q, q' \in \mathbb{Z}^n$ belonging to a block S are *antagonists* in S if the distance between them equals the maximal distance using the L^1 norm between two points in S; in this case we write $q = \text{antag}_S(q')$. Note that the antagonist of a point q in a block S containing q exists and is unique. Two points that are antagonists in a block of dimension $k \geq 0$ are said to be *k-antagonists*; k is then called the *order of antagonism* between these two points. We say that a digital subset $\mathcal{X}$ of $\mathbb{Z}^n$ contains a *critical configuration* in a block S of dimension $k \in [\![2, n]\!]$ if there exists two points $\{q, q'\} \in \mathbb{Z}^n$ that are antagonists in S such that $\mathcal{X} \cap S = \{q, q'\}$ (*primary case*) or such that $S \setminus \mathcal{X} = \{q, q'\}$ (*secondary case*). Figure 1 depicts examples of critical configurations.

Definition 1 (digital well-composedness [3]). *A digital set $\mathcal{X} \subset \mathbb{Z}^n$ is said to be* digitally well-composed (DWC) *if it does not contain any critical configuration.*

This property is *self-dual*: for any digital set $\mathcal{X} \subset \mathbb{Z}^n$, $\mathcal{X}$ is digitally well-composed iff $\mathcal{X}^c$ is digitally well-composed.

2.2 Basics in topology and continuous well-composedness

Let $(X,\mathcal{U})$ be a *topological space* [11,1]. The elements of the set X are called the *points* and the elements of the topology $\mathcal{U}$ are called the *open sets*. In practice, we will abusively say that X is a topological space, assuming it is supplied with $\mathcal{U}$. An open set which contains a point of X is said to be a *neighborhood* of this point. Let X be a topological space, and let T be a subset of X. A set $T \subseteq X$ is said *closed* if it is the complement of an open set in X. A function $f : X \to Y$ between two topological spaces X and Y is *continuous* if for every open set $V \subset Y$, the inverse image $f^{-1}(V) = \{x \in X \ ; \ f(x) \in V\}$ is an open subset of X. The function f is a *homeomorphism* if it is bicontinuous and bijective. The *continuous analog* $\text{CA}(p)$ of a point $p \in \mathbb{Z}^n$ is the closed unit cube centered at this point with faces parallel to the coordinate planes $\text{CA}(p) = \{q \in \mathbb{R}^n \ ; \ \|p - q\|_\infty \leq 1/2\}$. The *continuous analog* $\text{CA}(\mathcal{X})$ *of a digital set* $\mathcal{X} \subset \mathbb{Z}^n$ is the union of the continuous analogs of the points belonging to the set $\mathcal{X}$, that is, $\text{CA}(\mathcal{X}) = \bigcup_{p \in \mathcal{X}} \text{CA}(p)$. Then, we will denote $\text{bdCA}(\mathcal{X})$ the topological boundary of $\text{CA}(\mathcal{X})$, that is, $\text{bdCA}(\mathcal{X}) = \text{CA}(\mathcal{X}) \setminus \text{Int}(\text{CA}(\mathcal{X}))$, where $\text{Int}(\text{CA}(\mathcal{X}))$ is the union of all open subsets of $\text{CA}(\mathcal{X})$.

Definition 2 (continuous well-composedness [14,15]). *Let $\mathcal{X} \subset \mathbb{Z}^n$ be a digital set. We say that $\mathcal{X}$ is* continuously well-composed (CWC) *if the boundary of its continuous analog* $\text{bdCA}(\mathcal{X})$ *is a topological $(n-1)$-manifold, that is, if for any point $p \in \mathcal{X}$, the (open) neighborhood of p in* $\text{bdCA}(\mathcal{X})$ *is homeomorphic to $\mathbb{R}^{n-1}$.*

This property is *self-dual*: for any digital set $\mathcal{X} \subset \mathbb{Z}^n$, $\text{bdCA}(\mathcal{X}) = \text{bdCA}(\mathcal{X}^c)$ and then $\mathcal{X}$ is continuously well-composed iff $\mathcal{X}^c$ is continuously well-composed.

2.3 Homomorphisms

Recalls about *Abelian groups* and *homomorphisms* can be found in [10]. A homomorphism f is called an *isomorphism* if it is bijective. Two free Abelian groups are said *isomorphic* if there exists an isomorphism between them; for A and B two free Abelian groups, we write $A \simeq B$ when A and B are isomorphic. Let A be a free Abelian group and B a subgroup of A. For each $a \in A$, defined the *equivalence class* $[a] := \{a + b\ ;\ b \in B\}$. The *quotient group* A/B is defined as $A/B := \{[a]\ ;\ a \in A\}$.

Theorem 1 (First Isomorphism Theorem [10])**.** *Let A and B be two free Abelian groups and $f : A \to B$ a homomorphism. Then $A/\ker f \simeq \operatorname{im} f$.*

2.4 Cubical sets

An *elementary interval* is a closed subinterval of $\mathbb{R}$ of the form $[l, l+1]$ or $\{l\}$ for some $l \in \mathbb{Z}$. Elementary intervals that consist of a single point are *degenerate*, while those of length 1 are *non-degenerate*. An *elementary cube* h is a finite product of elementary intervals, that is, $h = h_1 \times \cdots \times h_d = \times_{i \in [\![1,d]\!]} h_i \subset \mathbb{R}^d$ where each h_i is an elementary interval. The set of elementary cubes in $\mathbb{R}^d$ is denoted by $\mathcal{K}^d$. The set of all elementary cubes is $\mathcal{K} := \bigcup_{d=1}^{\infty} \mathcal{K}^d$. Let $h = \times_{i \in [\![1,d]\!]} h_i \subset \mathbb{R}^d$ be an elementary cube. The elementary interval h_i is referred to as the ith *component* of h. The *dimension* of h is defined to be the number of non-degenerate components in h and is denoted by $\dim(h)$. Also, we define $\mathcal{K}_k := \{h \in \mathcal{K}\ ;\ \dim(h) = k\}$ and $\mathcal{K}_k^d := \mathcal{K}_k \cap \mathcal{K}^d$. A set $X \subset \mathbb{R}^d$ is *cubical* if X can be written as a finite union of elementary cubes. If X is a cubical set, we adopt the following notation $\mathcal{K}(X) := \{h \in \mathcal{K}\ ;\ h \subseteq X\}$ and $\mathcal{K}_k(X) := \{h \in \mathcal{K}(X)\ ;\ \dim(h) = k\}$.

2.5 Homology

Let $X \subseteq \mathbb{R}^d$ be a cubical set. The k-chains of X, denoted by $C_k(X)$, is the free Abelian group generated by $\mathcal{K}_k(X)$. The *boundary homomorphism* $\partial_k^X : C_k(X) \to C_{k-1}(X)$ is defined on the elementary cubes of $\mathcal{K}_k(X)$ and extended to $C_k(X)$ by linearity (see [10]). The chain complex $\mathcal{C}(X)$ is the graded set $\{C_k(X), \partial_k^X\}_{k \in \mathbb{Z}}$. A k-chain $z \in C_k(X)$ is called a *cycle* in X if $\partial_k^X z = 0$. The set of all k-cycles in X, which is denoted by $Z_k(X)$, is $\ker \partial_k^X$ and forms a subgroup of $C_k(X)$. A k-chain $z \in C_k(X)$ is called a *boundary* in X if there exists $c \in C_{k+1}(X)$ such that $\partial_{k+1}^X c = z$. Thus the set of boundary elements in $C_k(X)$, which is denoted by $B_k(X)$, consists of the image of ∂_{k+1}^X. Since ∂_{k+1}^X is a homomorphism, $B_k(X)$ is a subgroup of $C_k(X)$. Since $\partial_k^X \partial_{k+1}^X = 0$, every boundary is a cycle and thus $B_k(X)$ is a subgroup of $Z_k(X)$. We say that two cycles $z_1, z_2 \in Z_k(X)$ are *homologous* and write $z_1 \sim z_2$ if $z_1 - z_2$ is a boundary in X, that is, $z_1 - z_2 \in B_k(X)$. The *equivalence classes* are then the elements of the quotient group $\mathbb{H}_k(X) = Z_k(X)/B_k(X)$ called the *k-th homology group* of X. The homology of X is the collection of all homology groups of X. The shorthand notation for this is $\mathbb{H}(X) := \{H_k(X)\}_{k \in \mathbb{Z}}$. Given $z \in Z_k(X)$, $[z]$ is the homology class of z in X. A sequence of vertices $V_0, \ldots, V_n \in \mathcal{K}_0(X)$ is an *edge path* in X if there exists edges $E_1, \ldots, E_n \in \mathcal{K}_1(X)$ such that V_{i-1}, V_i are the two faces of E_i for $i = 1, \ldots, n$. For $V, V' \in \mathcal{K}_0(X)$, we write $V \sim_X V'$

if there exists an edge path $V_0, \ldots, V_n \in \mathcal{K}_0(X)$ in X such that $V = V_0$ and $V' = V_n$. We say that X is *edge-connected* if $V \sim_X V'$ for any $V, V' \in \mathcal{K}_0(X)$. For $V \in \mathcal{K}_0(X)$ we define the *edge-connected component of V in X* as the union of all edge-connected cubical subsets of X that contain V. We denote it $\operatorname{ecc}_X(V)$. The following result states that in the context of cubical sets, edge-connectedness is equivalent to (topological) connectedness[4].

Theorem 2 (Theorem 2.55 of [10]). *Let X be a cubical set. Then $\mathbb{H}_0(X)$ is a free Abelian group. Furthermore, if $\{V_i\ ;\ i \in [\![1,n]\!]\}$ is a collection of vertices in X consisting of one vertex from each component of X, then*

$$\left\{[\widehat{V_i}] \in \mathbb{H}_0(X)\ ;\ i \in [\![1,n]\!]\right\}$$

forms a basis for $\mathbb{H}_0(X)$ (where $\widehat{V_i}$ is the algebraic element associated to V_i).

This way, edge-connected components of X are (topologically) connected components of X and conversely.

2.6 Relative homology

Now, we recall some background in matter of *relative homology*. A pair of cubical sets X and A with the property that $A \subseteq X$ is called *cubical pair* and is denoted by (X, A). Relative homology is used to compute how the two spaces A, X differ from each other. Intuitively, we want to compute the homology of X *modulo* A: we want to ignore the set A and everything connected to it. In other words, we want to work with chains belonging to $C(X)/C(A)$, which leads to the following definition.

Definition 3 (Definition 9.3 of [10]). *Let (X, A) be a cubical pair. The* relative chains of X modula A *are the elements of the quotient groups $C_k(X, A) := C_k(X)/C_k(A)$. The equivalence class of a chain $c \in \mathcal{C}(X)$ relative to $\mathcal{C}(A)$ is denoted by $[c]_A$. Note that for each k, $C_k(X, A)$ is a free Abelian group. The* relative chain complex of X modulo A *is given by $\left\{C_k(X, A), \partial_k^{(X,A)}\right\}_{k\in\mathbb{Z}}$ where $\partial_k^{(X,A)} : C_k(X, A) \to C_{k-1}(X, A)$ is defined by $\partial_k^{(X,A)}[c]_A := [\partial_k^X c]_A$. Obviously, this map satisfies that $\partial_k^{(X,A)}\partial_{k+1}^{(X,A)} = 0$. The relative chain complex gives rise to the* relative k-cycles*: $Z_k(X, A) := \ker \partial_k^{(X,A)}$, the* relative k-boundaries $B_k(X, A) := \operatorname{im} \partial_{k+1}^{(X,A)}$*, and finally the* relative homology groups*: $\mathbb{H}_k(X, A) := Z_k(X, A)/B_k(X, A)$.*

Proposition 4 (Proposition 9.4 of [10]). *Let X be an (edge-)connected cubical set and let A be a nonempty cubical set of X. Then, $\mathbb{H}_0(X, A) = 0$.*

[4] A set X is said *connected* if it is not the union of two disjoint open non-empty sets.

2.7 Exact sequences

A sequence of groups and homomorphisms $\cdots \to G_3 \xrightarrow{\psi_3} G_2 \xrightarrow{\psi_2} G_1 \to \ldots$ is said *exact* at G_2 if $\operatorname{im}\psi_3 = \ker\psi_2$. It is an *exact sequence* if it is exact at every group. If the sequence has a first or a last element, then it is automatically exact at that group. A *short exact sequence* is an exact sequence of the form $0 \to G_3 \xrightarrow{\psi_3} G_2 \xrightarrow{\psi_2} G_1 \to 0$. A *long exact sequence* is an exact sequence with more than three nonzero terms.

Example 5 (Example 9.21 of [10])**.** *The* short exact sequence of the pair (X, A) *is:*

$$0 \longrightarrow C_k(A) \xrightarrow{\iota_k} C_k(X) \xrightarrow{\pi_k} C_k(X, A) \longrightarrow 0$$

where ι_k is the inclusion map and π_k is the quotient map.

Lemma 6 (Exact homology sequence of a pair [10]))**.** *Let (X, A) be a cubical pair. Then there is a long exact sequence*

$$\cdots \to \mathbb{H}_{k+1}(A) \xrightarrow{\iota_*} \mathbb{H}_{k+1}(X) \xrightarrow{\pi_*} \mathbb{H}_{k+1}(X, A) \xrightarrow{\partial_*} \mathbb{H}_k(A) \to \ldots$$

where $\iota : \mathcal{C}(A) \hookrightarrow \mathcal{C}(X)$ is the inclusion map and $\pi : \mathcal{C}(X) \to \mathcal{C}(X, A)$ is the quotient map.

2.8 Manifolds and local homology

A subset X of $\mathbb{R}^n$ is said to be a *(n-dimensional) homology manifold* if for any $x \in X$ the homology groups $\{\mathbb{H}_i(X, X \setminus \{x\})\}_{i\in\mathbb{Z}}$ satisfy:

$$\mathbb{H}_i(X, X \setminus \{x\}) = \begin{cases} \mathbb{Z} & \text{when } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 7 ([17])**.** *A topological manifold is a homology manifold.*

2.9 Homotopical equivalence

Let X, Y be two topological spaces, and f, g be two continuous functions from X to Y. We say that f and g are *homotopic* if there exists a continuous function $H : X \times [0, 1] \to Y$ such that for any $x \in X$, $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Furthermore, we say that X and Y are *homotopically equivalent* if there exist $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to Id_X and $f \circ g$ is homotopic to Id_Y.

3 DWCness does not imply CWCness

It is well-known that DWCness and CWCness are equivalent in 2D and 3D (see, for example, [4]). In this section, we prove that there exists at least one set $\mathcal{X} \subset \mathbb{Z}^4$ which is DWC but not CWC.

To this aim, we will start with the definition of the set $\mathcal{X}$ and we will check that $\mathcal{X}$ is DWC. Then, to prove that $\mathcal{X}$ is not CWC, we will prove that $X = \text{bdCA}(\mathcal{X})$ (up to a translation) is not a homology manifold and conclude that it is not topological manifold by Theorem 7. To compute the homology groups $\{\mathbb{H}_i(X, X \setminus \{x^0\}\}_{i\in\mathbb{Z}}$, where x^0 is a particular point in X (detailed hereafter), we need to compute $\{\mathbb{H}_i(X \setminus \{x^0\})\}_{i\in\mathbb{Z}}$ and $\{\mathbb{H}_i(X)\}_{i\in\mathbb{Z}}$. However, $X \setminus \{x^0\}$ is not a cubical set, then we need to find a cubical set $\widetilde{X}(x^0)$ which is homotopy equivalent to $X\setminus\{x^0\}$ to compute its homology groups using the CHomP software package [8]. After having defined $\widetilde{X}(x^0)$ and having proven that it is a cubical set, we will show that $X \setminus \{x^0\}$ and $\widetilde{X}(x^0)$ are homotopically equivalent. Then, we will compute the homology groups of $\widetilde{X}(x^0)$ and of X; this way we will deduce $\{\mathbb{H}_i(X, X \setminus \{x^0\})\}_{i\in\mathbb{Z}}$ using the long exact sequence of the pair $(X, X \setminus \{x^0\})$. At this moment, we will see that X is not a homology 3-manifold, which will make us able to conclude that $\mathcal{X}$ is not CWC since the boundary of its continuous analog is not a topological 3-manifold. This way, we will conclude that DWCness does not imply CWCness in 4D.

3.1 Choosing a particular DWC set $\mathcal{X} \subset \mathbb{Z}^4$

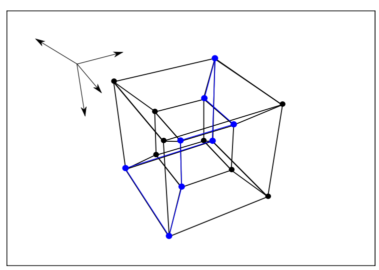

Fig. 2: A set $\mathcal{X} \subset \mathbb{Z}^4$ depicted by blue points which is DWC and not CWC. Blue lines show that the blue points are $2n$-connected ($n = 4$).

We recall that it is well-known in the community of discrete topology that CWCness and DWCness are equivalent in 2D and 3D as developed in [2,4]. For this reason, we chose a digital set $\mathcal{X}$ in $\mathbb{Z}^4$ to study the relation between these two flavours of well-composedness in higher dimensions.

As depicted in Figure 2, we can define the digital subset of $\mathbb{Z}^4$:

$$\begin{aligned}\mathcal{X} := \{&\{0,0,0,0\},\{0,0,0,1\},\{0,0,1,1\},\{0,1,1,1\},\\ &\{1,1,1,1\},\{1,1,1,0\},\{1,1,0,0\},\{1,0,0,0\}\}.\end{aligned}$$

Let us check that $\mathcal{X}$ is DWC (see Figure 2). It is easy to observe that it does not contain any 2D critical configuration. Now, to observe that there is no primary or secondary 3D critical configuration, we can simply look at the eight 3-faces (including the interior and the exterior cubes): since each one contains exactly four points of $\mathcal{X}$, they contain neither a primary critical configuration (made of two points) nor a secondary critical configuration (made of six points in the 3D case). Finally, we observe that the only 4D block that we have to consider is $\{0,1\}^4$ which contains eight points of $\mathcal{X}$, and eight points of $\mathcal{X}^c$, concluding that $\mathcal{X}$ contains neither a primary nor a secondary 4D critical configuration.

Property 8. *The digital set $\mathcal{X}$ is DWC.*

3.2 Finding a cubical set $\widetilde{X}(x^0)$ homotopy equivalent to $X \setminus \{x^0\}$

Let us start with the following proposition.

Proposition 9. *Let X be a cubical set in $\mathbb{R}^n$ and x^0 be a point of $X \cap \mathbb{Z}^n$. Then, the set:*

$$\widetilde{X}(x^0) := \{x \in X \setminus \{x^0\} \; ; \; \|x - x^0\|_\infty \geq 1\}$$

is cubical.

Proof. Our aim is to prove that $\widetilde{X}(x^0)$ is equal to $\bigcup\{h \in \mathcal{K}(X) \; ; \; x^0 \notin h\}$. This way, we will be able to conclude that $\widetilde{X}(x^0)$ is equal to $\bigcup\{h \in \mathcal{K}(X \setminus \{x^0\})\}$ and then it is a cubical set (since it is made of cubes and closed under inclusion).
First, let us prove that:

$$\widetilde{X}(x^0) \subseteq \cup\{h \in \mathcal{K}(X) \; ; \; x^0 \notin h\}.$$

Let $x \in X \setminus \{x^0\}$ be a point such that $\|x - x^0\|_\infty \geq 1$. Then, there exists $i^* \in [\![1,n]\!]$ such that $\|x - x^0\|_\infty = |x_{i^*} - x^0_{i^*}| \geq 1$. Then two cases are possible:

(1) $x_{i^*} > x^0_{i^*}$, then $x_{i^*} \geq x^0_{i^*} + 1$,
(2) $x_{i^*} < x^0_{i^*}$, then $x_{i^*} \leq x^0_{i^*} - 1$.

Since $x \in X \setminus \{x^0\} \subset X$ where X is a cubical set, there exists a smaller face $h^* \in \mathcal{K}(X)$ (in the inclusion sense) such that $x \in h^* := \times_{i \in [\![1,n]\!]}[\lfloor x_i \rfloor, \lceil x_i \rceil]$. Then, $x^0 \in h^*$ iff for each $i \in [\![1,n]\!]$, $x^0_i \in [\lfloor x_i \rfloor, \lceil x_i \rceil]$. However, since in case (1), $x^0_{i^*} \leq x_{i^*} - 1 < \lfloor x_{i^*} \rfloor$, and in case (2), $x^0_{i^*} \geq x_{i^*} + 1 > \lceil x_{i^*} \rceil$, then $x^0 \notin h^*$. Obviously, $h^* \in \mathcal{K}(X)$: otherwise, all the cubes containing h^* do not belong to $\mathcal{K}(X)$, and then $x \notin X \setminus \{x^0\}$. This way, there exists a cube $h \in \mathcal{K}(X)$ such that $x^0 \notin h$ and $x \in h$.
Second, let us prove that: $\widetilde{X}(x^0) \supseteq \bigcup\{h \in \mathcal{K}(X) \; ; \; x^0 \notin h\}$. Let p be an element of $\bigcup\{h \in \mathcal{K}(X) \; ; \; x^0 \notin h\}$. In other words, $p \in h \in \mathcal{K}(X)$ and $x^0 \notin h$. Since $p \in h \in \mathcal{K}(X)$ and $x^0 \notin h$ then $p \in X \setminus \{x^0\}$. Now, let us write $h = \times_{i \in [\![1,n]\!]}[h^{\min}_i, h^{\max}_i]$, where $h^{\min}, h^{\max} \in \mathbb{Z}^n$. Since $x^0 \notin h$, there exists $i^* \in [\![1,n]\!]$ such that $x^0_{i^*} \notin [h^{\min}_{i^*}, h^{\max}_{i^*}]$. Furthermore, we have:

(a) either $x^0_{i^*} \leq h^{\min}_{i^*} - 1$,

(b) or $x^0_{i^*} \geq h^{\max}_{i^*} + 1$.

Since $p \in h$, for each $i \in [\![1,n]\!]$, we have $p_i \in [h^{\min}_i, h^{\max}_i]$, and then $p_{i^*} \in [h^{\min}_{i^*}, h^{\max}_{i^*}]$. Then, in case (a), $x^0_{i^*} \leq h^{\min}_{i^*} - 1 \leq p_{i^*} - 1$, which leads to $p_{i^*} - x^0_{i^*} \geq 1$, and in case (b), $x^0_{i^*} \geq h^{\max}_{i^*} + 1 \geq p_{i^*} + 1$, which leads to $x^0_{i^*} - p_{i^*} \geq 1$. In both cases, we obtain that $\|p - x^0\|_\infty \geq 1$. □

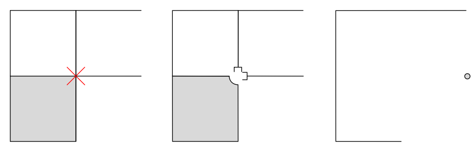

Fig. 3: $X \setminus \{x^0\}$ is homotopy equivalent to $\widetilde{X}(x^0)$: From left to right, a cubical set X (see the location of the central point x^0 in red), X minus its central point x^0 and the new cubical set $\widetilde{X}(x^0)$ homotopy equivalent to $X \setminus \{x^0\}$.

Now, let us prove that $X \setminus \{x^0\}$ and $\widetilde{X}(x^0)$ are homotopy equivalent (as depicted on Figure 3).

Proposition 10. *Let X be a cubical set in $\mathbb{R}^n$ and x^0 be a point of $X \cap \mathbb{Z}^n$. Then, $X \setminus \{x^0\}$ is homotopy equivalent to $\widetilde{X}(x^0)$.*

Proof. Let $f : X \setminus \{x^0\} \to \mathbb{R}^n$ be the function defined such as:

$$f(x) := \begin{cases} x & \text{when } \|x - x^0\|_\infty \geq 1, \\ x^0 + \frac{x - x^0}{\|x - x^0\|_\infty} & \text{otherwise,} \end{cases}$$

and let $g : \widetilde{X}(x^0) \to X \setminus \{x^0\}$ be the map from $\widetilde{X}(x^0)$ to $X \setminus \{x^0\}$ such that:

$$\forall x \in \widetilde{X}(x^0),\ g(x) = x,$$

which is possible since $\widetilde{X}(x^0) \subseteq X \setminus \{x^0\}$. Now, let us proceed step by step.

***Step 1**: $X \setminus \{x^0\}$ and $\widetilde{X}(x^0)$ are topological spaces.* The sets $X \setminus \{x^0\}$ and $\widetilde{X}(x^0)$ are topological spaces since they are subsets of $\mathbb{R}^n$ supplied with the usual Euclidian distance.

***Step 2**: f is a map from $X \setminus \{x^0\}$ to $\widetilde{X}(x^0)$.* Let x be an element of $X \setminus \{x^0\}$. When $\|x - x^0\|_\infty \geq 1$, $f(x) = x$. This way, $f(x) \in X \setminus \{x^0\}$ and $\|f(x) - x^0\|_\infty \geq 1$, then $f(x) \in \widetilde{X}(x^0)$. When $\|x - x^0\|_\infty < 1$, $f(x) = x^0 + \frac{x - x^0}{\|x - x^0\|_\infty}$. This way, $\|f(x) - x^0\|_\infty = 1$. Since $x \in X \setminus \{x^0\} \subset X$ with X a cubical set, there exists a cube $h \in \mathcal{K}(X)$ such that $x \in h$. Furthermore, this cube h contains x^0 since all the cubes containing a point of $\times_{i \in [\![1,n]\!]}]x^0_i - 1, x^0_i + 1[$ contain also x^0 (the cubes are defined relatively to integral coordinates). Since $h = \times_{i \in [\![1,n]\!]} h_i$, then for each $i \in [\![1,n]\!]$, $x_i \in h_i$, and

$$(f(x))_i = x_i^0 + \frac{x_i - x_i^0}{\|x - x^0\|_\infty}.$$

Let us prove that this last equality shows that $f(x) \in h$. Since n is finite, there exists some $i^* \in [\![1,n]\!]$ such that: $\|x - x^0\|_\infty = |x_{i^*} - x_{i^*}^0|$, then:

$$(f(x))_i = x_i^0 + \frac{x_i - x_i^0}{|x_{i^*} - x_{i^*}^0|} \quad (P1)$$

Let us assume without constraint that $x_{i^*} > x_{i^*}^0$, then $(f(x))_{i^*} = x_{i^*}^0 + 1$. However, $x_{i^*} > x_{i^*}^0$ implies that $h_{i^*} = [x_{i^*}^0, x_{i^*}^0 + 1]$ since h contains x^0. When $i \neq i^*$, since $x_i \in h_i$, and since $h \ni x^0 \in \mathbb{Z}^n$, $h_i = [x_i^0, x_i^0 + 1]$ or $h_i = [x_i^0 - 1, x_i^0]$. Let us assume without constraint that $x_i > x_i^0$, then $h_i = [x_i^0, x_i^0 + 1]$. Because of $(P1)$, it follows easily that $(f(x))_i \in h_i$ since $\frac{x_i - x_i^0}{|x_{i^*} - x_{i^*}^0|} \in [0,1]$.
Then, we have proven that when $\|x - x^0\|_\infty < 1$, there exists $h \in \mathcal{K}(X)$ such that for any $i \in [\![1,n]\!]$, $(f(x))_i \in h_i$, that is to say,

$$f(x) \in h. \quad (P2)$$

Also, $x \neq x^0$, which is equivalent to $f(x) \neq x^0$. Then, $f(x) \in \widetilde{X}(x^0)$.

Step 3: *$g \circ f$ is homotopic to* $\mathrm{Id}_{X \setminus \{x^0\}}$.

- We can observe that $g \circ f : X \setminus \{x^0\} \to X \setminus \{x^0\}$ is the continuous function defined as:

$$g \circ f = \begin{cases} x & \text{when } \|x - x^0\|_\infty \geq 1, \\ x^0 + \frac{x - x^0}{\|x - x^0\|_\infty} & \text{otherwise.} \end{cases}$$

- Let $H : (X \setminus \{x^0\}) \times [0,1] \to \mathbb{R}^n$ defined such that for any $x \in X \setminus \{x^0\}$ and any $\lambda \in [0,1]$,
$$H(x, \lambda) := \lambda x + (1 - \lambda) g \circ f(x),$$
then:
  - H is continuous as a composition of continuous functions,
  - H is a function from $(X \setminus \{x^0\}) \times [0,1]$ to $X \setminus \{x^0\}$:
    - when $\|x - x^0\|_\infty \geq 1$, $H(x, \lambda) = x \in X \setminus \{x^0\}$,
    - when $\|x - x^0\|_\infty < 1$, $H(x, \lambda) = \lambda x + (1 - \lambda) f(x)$. However, we have seen that in this case, cf. $(P2)$, there exists a cube $h \in \mathcal{K}(X)$ such that $f(x) \in h$. Since h is a cube, it is convex, and then $H(x, \lambda) \in h$. Then, $H(x, \lambda) \in X$. Also, we can prove that $H(x, \lambda) \neq x^0$: the cases $\lambda = 0$ and $\lambda = 1$ are obvious; in the case $\lambda \in]0,1[$, we can see that

$$H(x, \lambda) = \lambda x + (1 - \lambda) \left(x^0 + \frac{x - x^0}{\|x - x^0\|_\infty} \right)$$

and then, by assuming without constraints that $x^0 = 0$ and that for any $i \in [\![1,n]\!]$, $x_i \geq 0$, we obtain that for any $i \in [\![1,n]\!]$:

$$(H(x,\lambda))_i = \lambda x_i + (1-\lambda)\frac{x_i}{\|x\|_\infty} = (\lambda(\|x\|_\infty - 1) + 1)\frac{x_i}{\|x\|_\infty},$$

then, because $(\|x\|_\infty - 1) < 1$ and $\frac{x_i}{\|x\|_\infty} \geq 0$, $(H(x,\lambda))_i$ is decreasing relatively to λ, and then

$$x_i \leq (H(x,\lambda))_i \leq \frac{x_i}{\|x\|_\infty}.$$

Since $x \neq 0$, there exists i^* such that $x_{i^*} \neq 0$, and then such that $(H(x,\lambda))_{i^*} > 0$ since $x_{i^*} > 0$. Therefore, $H(x,\lambda) \neq 0$, that is, $H(x,\lambda) \neq x^0$. Then, $H(x,\lambda) \in X \setminus \{x^0\}$.

- We can see that $H(x,0) = g \circ f(x), \forall x \in X \setminus \{x^0\}$,
- We can also observe that $H(x,1) = x, \forall x \in X \setminus \{x^0\}$.

Then $g \circ f$ is homotopic to $\mathrm{Id}_{X \setminus \{x^0\}}$.

Step 4*: $f \circ g$ is homotopic to* $\mathrm{Id}_{\widetilde{X}(x^0)}$. Since $f \circ g$ is equal to $\mathrm{Id}_{\widetilde{X}(x^0)}$, they are homotopic.

Step 5*: Conclusion.* $X \setminus \{x^0\}$ and $\widetilde{X}(x^0)$ are homotopically equivalent. □

Corollary 11. *Assuming the notations of Proposition 9, we can compute the homology groups of $X \setminus \{x^0\}$ based on the ones of the cubical set $\widetilde{X}(x^0)$. Indeed, for each $i \in \mathbb{Z}$, we have the following equality: $\mathbb{H}_i(X \setminus \{x^0\}) = \mathbb{H}_i\big(\widetilde{X}(x^0)\big)$.*

Proof. This follows from Propositions 9 and 10. □

3.3 Defining the cubical set $\varphi_d(\mathcal{X})$

Let us define the mapping $\varphi : \mathbb{Z}/2 \to \mathcal{K}^1$:

$$\forall x \in \mathbb{Z}/2, \quad \varphi(x) := \begin{cases} [x, x+1] & \text{when } x \in \mathbb{Z}, \\ \left\{x + \frac{1}{2}\right\} & \text{otherwise.} \end{cases}$$

Then we can define: $\forall x \in (\mathbb{Z}/2)^d$, $\varphi_d(x) := \times_{i=1}^{d} \varphi(x_i)$ where, as usual, x_i denotes the i^{th} coordinate of x; φ_d is then a bijection between $(\mathbb{Z}/2)^d$ and $\mathcal{K}^d$. Note that the underlying polyhedron of $\varphi_d(\mathcal{X})$ is equal to $\mathrm{CA}(\mathcal{X})$ up to a translation, and in this way they are topologically equivalent.

3.4 Choosing a particular point x^0 in the boundary X of $\varphi_4(\mathcal{X})$

Let us begin with a simple property.

Property 12. *The point $x^0 := (1,1,1,1)$ belongs to the boundary X of $\varphi_4(\mathcal{X})$.*

Proof. Let us recall that $\varphi_4(\mathcal{X})$ is a translation of the set $\mathrm{CA}(\mathcal{X})$ by a vector $v := (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Also, the point $p := (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in \mathbb{R}^4$ belongs to $\mathrm{CA}(\mathcal{X})$ and does not belong to $\mathrm{Int}(\mathrm{CA}(\mathcal{X}))$ since there does not exist any topological open ball $B(p, \varepsilon)$, $\varepsilon > 0$, in $\mathbb{R}^4$ centered at p and contained in $\mathrm{Int}(\mathrm{CA}(\mathcal{X}))$ since $B(p, \varepsilon)$ intersects $\mathrm{Int}(\mathrm{CA}(\mathcal{X}^c))$. This way, p belongs to $\mathrm{bdCA}(\mathcal{X})$. Finally, we obtain that the translation $x^0 = p + v$ of p by v belongs to the translation X of $\mathrm{bdCA}(\mathcal{X})$ by v. □

3.5 Computation of $\mathbb{H}(X, X \setminus \{x^0\})$

Let us compute the relative homology groups $\mathbb{H}_i(X, A)$ for each $i \in \mathbb{Z}$.

Obviously, since $\mathcal{C}_k(X, X \setminus \{x^0\}) = 0$ for $k \in \mathbb{Z} \setminus [\![0, 4]\!]$, then $\mathbb{H}_k(X, X \setminus \{x^0\}) = 0$. Also, thanks to Proposition 4, we know that $\mathbb{H}_0(X, X \setminus \{x^0\}) = 0$. To compute the other relative homology groups, we will use the exact homology sequence of $(X, X \setminus \{x^0\})$ discussed in Lemma 6: the sequence

$$\dots \to \mathbb{H}_{k+1}(X \setminus \{x^0\}) \overset{\iota_*}{\to} \mathbb{H}_{k+1}(X) \overset{\pi_*}{\to} \mathbb{H}_{k+1}(X, X \setminus \{x^0\}) \overset{\partial_*}{\to} \mathbb{H}_k(X \setminus \{x^0\}) \to \dots$$

is exact, and then by computing the homology groups $\mathbb{H}_k(X)$ and $\mathbb{H}_k(X \setminus \{x^0\})$ and using the First Isomorphism Theorem, we will be able to compute the local homology groups $\mathbb{H}_k(X, X \setminus \{x^0\})$ and to deduce if X is a homology 3-manifold or not.

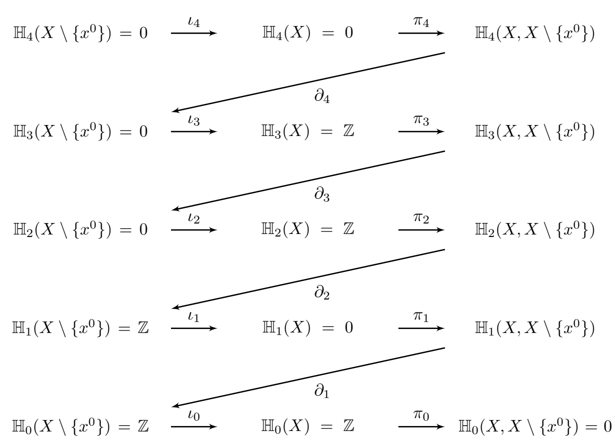

Fig. 4: Long exact sequence of the cubical pair $(X, X \setminus \{x^0\})$.

Using CHomP [8], we compute the local homology groups $\mathbb{H}(\widetilde{X}(x^0))$ and $\mathbb{H}_i(X)$ for $i \in [\![0, 4]\!]$. Using Corollary 11 we obtain $\mathbb{H}_i(X \setminus \{x^0\})$, and replacing this information in the long exact sequence discussed in Lemma 6, we obtain Figure 4.

Let us compute $\mathbb{H}_4(X, X \setminus \{x^0\})$. By exactness, $\text{im}\,\pi_4 = 0 = \ker \partial_4$ and $\mathbb{H}_4(X, X \setminus \{x^0\}) / \ker \partial_4 \simeq \text{im}\,\partial_4 = 0$, then $\mathbb{H}_4(X, X \setminus \{x^0\}) = 0$.

Now, let us compute $\mathbb{H}_3(X, X \setminus \{x^0\})$. By exactness, $\text{im}\,\iota_3 = 0 = \ker \pi_3$, and $\mathbb{Z} / \ker \pi_3 \simeq \text{im}\,\pi_3 \simeq \mathbb{Z}$, then $\text{im}\,\pi_3 = \ker \partial_3 = \mathbb{H}_3(X, X \setminus \{x^0\}) \simeq \mathbb{Z}$.

Concerning $\mathbb{H}_2(X, X \setminus \{x^0\})$, by exactness, $\text{im}\,\iota_2 = 0 = \ker \pi_2$, and $\mathbb{Z} / \ker \pi_2 \simeq \text{im}\,\pi_2 \simeq \mathbb{Z} \simeq \ker \partial_2$. Also, $\ker(\iota_1) = \mathbb{Z} = \text{im}\,\partial_2$ and $\mathbb{H}_2(X, X \setminus \{x^0\}) / \ker \partial_2 \simeq \text{im}\,\partial_2$ imply that: $\mathbb{H}_2(X, X \setminus \{x^0\}) \simeq \mathbb{Z}^2$.

Finally, let us compute $\mathbb{H}_1(X, X \setminus \{x^0\})$. By exactness, $\text{im}\,\pi_1 = 0 = \ker \partial_1$. Also, $\ker \pi_0 = \mathbb{Z} \simeq \text{im}\,\iota_0$, and $\mathbb{Z} / \ker \iota_0 \simeq \text{im}\,\iota_0$ imply that $\ker \iota_0 = 0 = \text{im}\,\partial_1$. Then, $\mathbb{H}_1(X, X \setminus \{x^0\}) = 0$.

3.6 Our final observation

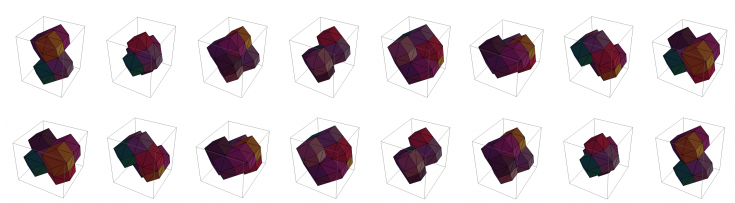

Fig. 5: Projection in the 3D space of the continuous analog of the 4D counter-example. Each color corresponds to a same (projected) hypercube. Note that the pinch is not observable in 3D.

Since we have $\mathbb{H}_2(X, X \setminus \{x^0\}) \simeq \mathbb{Z}^2 \neq 0$, X is not a homology 3-manifold, and then it is not a topological 3-manifold, which implies that DWCness does not imply CWCness in 4D, which contradicts the conjecture arguing that DWCness and CWCness are equivalent in nD on cubical grids [2]. See Figure 5 for some 3D projections of the continuous analog of our 4D counter-example. Furthermore, this counter-example shows that a digital set which is *well-composed in the sense of Alexandrov (AWC)* [16,6] is not always CWC, since it has been proven in [5] that AWCness and DWCness are equivalent in nD.

4 Conclusion

The counter-example presented in this paper shows that in 4D, DWCness does not imply CWCness. It shows how much it is important to explicit which flavour of well-composedness we consider when we work with nD discrete images.

Furthermore, two questions arise in a natural way. First, is it possible to find a generic counter-example showing that DWCness does not imply CWCness in any dimension greater than 3 (like the product of the set $\mathcal{X}$ with $\{0,1\}^{n-4}$). Second, does CWCness imply DWCness in nD? This last question seems intuitive but we will show in a future report that it is far from being so simple.

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