On the computability of the p-local homology of twisted cartesian products of Eilenberg-Mac Lane spaces

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Abstract

Working in the framework of the Simplicial Topology, a method for calculating the p-local homology of a twisted cartesian product $X(\pi, m, \tau, \pi', n) = K(\pi, m) \times_\tau K(\pi', n)$ of Eilenberg-Mac Lane spaces is given. The chief technique is the construction of an explicit homotopy equivalence between the normalized chain complex of X and a free DGA-module of finite type M, via homological perturbation. If X is a commutative simplicial group (being its inner product the natural one of the cartesian product of $K(\pi, m)$ and $K(\pi', n)$), then M is a DGA-algebra. Finally, in the special case $K(\pi, 1) \hookrightarrow X \xrightarrow{p} K(\pi', n)$, we prove that M can be a small twisted tensor product.

1 Introduction

In Simplicial Topology [14], fibre bundles simplify its own structure since they can be considered as twisted cartesian products (TCPs) of two simplicial sets (fibre) $\times_\tau$ (base), where the function τ produces a "torsion" on the 0-face of the cartesian product. An example of principal TCP is $X(\pi, m, \tau, \pi', n) = K(\pi, m) \times_\tau K(\pi', n)$, where $K(\pi, m)$ and $K(\pi', n)$ are Eilenberg-Mac Lane spaces.

We are interested here in a PTCP of the form $X(\pi, m, \tau, \pi', n)$ where π and π' are finitely generated abelian groups and $m, n \in \mathbf{N}$. More precisely, we want to calculate the p-local homology of X. Our approach is based essentially in the use of techniques

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First author is supported by a grant from the Junta de Andalucía

from Homological Perturbation Theory (see [8], [9], [11]). Working over $\mathbf{Z}$ localised over a prime p and combining results from the papers [18], [16] and [17], it is easy to establish an explicit contraction (special homotopy equivalence) from the normalized chain complex of a $K(\pi, m)$ to a tensor product $G(\pi, m)$ of elementary complexes defined by Cartan ([4], exposé 11) corresponding to admissible sequences. With these data at hand, a "constructive" version of the Serre spectral sequence applied to this case, provides us that the p-local homology of X is reduced to that of a small "modified" (from a differential point of view) tensor product $M = (G(\pi, m) \otimes G(\pi', n),\ d_{G\pi} \otimes 1 + 1 \otimes d_{G\pi'} + d_\tau)$, where the formula of d_τ is given by the Basic Perturbation Lemma.

Let us note that the cartesian product $K(\pi, m) \times K(\pi', n)$ shows a structure of commutative simplicial group in a natural way. If $X(\pi, m, \tau, \pi', n)$, endowed with this same product, is a simplicial group, we see here that the general method can be substantially improved. More concretely, we obtain that M is a DGA-algebra and that the morphism d_τ is completely determined by knowing its images on the generators of this DGA-algebra M.

Finally, taking as point of departure $X(\pi, 1, \tau, \pi', n)$, we prove that its p-local homology can be derived from that of a twisted tensor product $G(\pi, 1) \otimes_{t'} \bar{B}(G(\pi', n-1))$, where $\bar{B}(\ )$ is the reduced bar construction (see [13]) and t' is a Brown's cochain.

This paper is organized as follows: Section 2 describes the main algebraic and simplicial objects we intend to deal with and gives same important results which play an essential role in the sequel. Possibly, the main result in this section is: *Working over $\mathbf{Z}_{(p)}$, there is a semi-full algebra contraction from $C(K(\pi, n)$ to a tensor product $G(\pi, n)$ of Cartan elementary complexes corresponding to admissible sequences.* Section 3 is devoted to describe a constructive version of the Serre spectral sequence for computing the p-local homology of $X(\pi, m, \tau, \pi', n)$. If X is a simplicial group (being its inner product the natural one of the cartesian product of the Eilenberg-Mac Lane simplicial groups, then we see that the previous method can be enormously improved. Finally, we analyze in detail the case $X(\pi, 1, \tau, \pi', n)$, concluding that the p-local homology of X is determined by a twisted tensor product $G(\pi, 1) \otimes_{t'} \bar{B}(G(\pi', n-1))$.

2 Preliminaries

Let Λ be a commutative ring with $1 \neq 0$. A *simplicial set* X is a collection of simplices, each one having a dimension (positive integer), endowed with a collection of face operators and a collection of degeneracy operators satisfying several conditions of compatibility [14]. If X has only one 0-simplex, then X is called reduced.

If G is a simplicial set provided with a group structure, compatible in a natural sense with the simplicial structure, then G is called a simplicial group. To define a twisted cartesian product, several ingredients are needed: a base space B, a fibre space F, both of which are simplicial sets, a structural group G which is a simplicial group, acting on the fibre ($g * f \in F$; $g \in G$, $f \in F$) and finally a twisting operator (or geometric torsion) τ which looks like a function $B \to G$, satisfying some coherence conditions. The total space is denoted $F \times_\tau B$ and is essentially the product of B by F twisted according to the τ-operator. The twisted cartesian product (TCP), $F \times_\tau B$, is a simplicial set where the n-simplices, face and degeneracy operators are the following ones:

$$\begin{gathered}(F \times_\tau B)_n = F_n \times B_n;\\ \partial_0(f, b) = (\tau b * \partial_0 f, \partial_0);\\ \partial_i(f, b) = (\partial_i a, \partial_i) \quad \text{for } i > 0;\\ s_j(f, b) = (s_j f s_j b), \;\; \text{for } i \geq 0;\end{gathered}$$

In the case that $F = G$, $F \times_\tau B$ is called a principal twisted cartesian product (PTCP).

If π is an ordinary (discrete) abelian group, one can consider $G\pi$ as a simplicial group, where $(G\pi)_n = \pi \quad n \geq 0$ and the face and degeneracy operators are all the identity maps. Then the simplicial group $K(\pi, n)$ is defined as follows: $K(\pi, 0) = G\pi$ and $K(\pi, n+1) = \bar{W}K(\pi, n)$, where $\bar{W}$ denotes the classifying construction [14]. An important property of the Eilenberg-MacLane spaces is that they have only one non null homotopy group , $\pi_n(K(\pi, n)) = \pi$.

Given a simplicial set X, there is a DG-module canonically associated to X, that we denote by $C_*(X)$ (called chain complex of X) where the n-component is the free Λ-module $C_n(X) = \Lambda[X_n]$ generated by the n-simplices of X, and the differential operator $d_n : C_n(X) \to C_{n-1}(X)$ is the alternated sum $d_n = \sum_{0 \leq i \leq n} (-1)^i \partial_i$. The homology groups $H_*(X)$ are the quotient groups $H_n(X) = ker\, d_n / Im\, d_{n+1}$. The subset consisting of all the degenerated simplices of $C_*(X)$, $s(C_*(X))$ is a submodule verifying $d_n(s(C_*(X))_n) \subset s(C_*(X))_{n-1}$, so that, $C(X) = \{C_*(X)/s(C_*(X))\}$ is a DG-module, called the normalized DG-module canonically associated to X. It is verified that $H_*(X) = H_*(C_*(X)) = H_*(C(X))$.

From a algebraic point view, our main objets are commutative (connected) DGA-(co)algebras. The differential, product (resp. coproduct), augmentation and coaugmentation of a DGA-algebra A will be denoted respectively by d_A, μ_A (resp. Δ_A , ϵ_A and η_A). In what follows, the Koszul sign conventions will be used. A morphism $\rho : A_* \to A_{*-1}$ is called *derivation* if it is compatible with the algebra structures on A. The degree of an element $a \in A$ is denoted by $|a|$.

We need here the *reduced bar construction* or simply *bar construction* $\bar{B}(A)$ of a DGA-algebra A and the *cobar construction* $\bar{\Omega}(C)$ of a DGA-coalgebra C (see [13]). Recall that, the bar construction is a coalgebra, $\bar{B}(A) = T^c(S(\text{Ker } \epsilon_A)))$, where $T^c(\ )$ is the tensor coalgebra and $S(\ )$ is the suspension functor. If A is commutative then a product $*$ (called shuffle product) can be defined on $\bar{B}(A)$, such that the reduced bar construction has a Hopf algebra structure. Recall too, that the cobar contruction is an algebra, $\bar{\Omega}(A) = T^a(S^{-1}(\text{Ker } \epsilon_A)))$, where $T^a(\ )$ is the tensor algebra and $S^{-1}(\ )$ is the desuspension functor.

We deal with a special type of homotopy equivalence: a *contraction* (see [6], [12]) is a data set $c : \{N, M, f, g, \phi\}$, (we denote by $N \Rightarrow M$) where $f : N \to M$, $g : M \to N$ are morphisms of DGA-modules (called, respectively, *projection* and *inclusion*) and

$\phi : N \to N$ is a morphism of graded modules of degree +1 (called *homotopy operator*), and these data are required to satisfy the rules: **(r1)** $fg = 1_M$, **(r2)** $f\phi = 0$; **(r3)** $\phi g = 0$, **(r4)** $\phi d_N + d_N \phi + gf = 1_N$ and **(r5)** $\phi\phi = 0$.

For instance, let us consider the Eilenberg-Zilber contraction:

Theorem 1 (Eilenberg-Zilber Theorem [7]) *Let X and Y be simplicial sets. We will consider the morphisms:*

Alexander-Whitney,

$$AW : C(X \times Y) \longrightarrow C(X) \otimes C(Y)$$

Eilenberg-MacLane,

$$EML : C(X) \otimes C(Y) \longrightarrow C(X \times Y)$$

And Shih,

$$SHI : C(X \times Y) \longrightarrow C(X \times Y)$$

The morphisms are defined by the formulas:

$$\begin{aligned}
AW(a_n \times b_n) &= \sum_{i=0}^{n} \partial_{i+1} \cdots \partial_n a_n \otimes \partial_0 \cdots \partial_{i-1} b_n, \\
EML(a_p \otimes b_q) &= \sum_{(\alpha,\beta)\in\{(p,q)-shuffles\}} (-1)^{sg(\alpha,\beta)} (s_{\beta_q} \cdots s_{\beta_1} a_p \times s_{\alpha_p} \cdots s_{\alpha_1} b_q), \\
SHI(a_n \times b_n) &= \\
&= -\sum (-1)^{m+sg(\alpha,\beta)} (s_{\beta_q+m} \cdots s_{\beta_1+m} s_{m-1} \partial_{n-q+1} \cdots \partial_n a_n \times \\
&\qquad \times s_{\alpha_{p+1}+m} \cdots s_{\alpha_1+m} \partial_m \cdots \partial_{m+p-1} b_n);
\end{aligned}$$

where $m = n - p - q$, $sg(\alpha,\beta) = \sum_{i=1}^{p} (\alpha_i - (i-1))$, and the last sum is taken over the indices $0 \le q \le n-1$, $0 \le p \le n-q-1$ and $(\alpha,\beta) \in \{(p+1,q)$-shuffles$\}$.

Then, the data set

$$EZ_{X,Y} : \{C(X \times Y),\, C(X) \otimes C(Y),\, AW_{X,Y},\, EML_{X,Y},\, SHI_{X,Y}\} \qquad (1)$$

is a contraction.

From now on, a contraction of the type (1) will be called Eilenberg-Zilber contraction.

The first definition of the SHI operator is given in an inductive way in [6]. The above explicit formula for the SHI operator is given by Rubio in [19]. An appendix in [17] is devoted to prove that this explicit formula satisfies the inductive definition given in [6].

We recall the concept of a perturbation datum. Let N be a graded module and let $f : N \to N$ be a morphism of graded modules. The morphism f is *pointwise* *nilpotent* if for all x, $x \in N$, $x \neq 0$, a positive integer n exists (in general, the number n depends on the element x) such that $f^n(x) = 0$. A *perturbation of a DGA-module* N is a morphism of graded modules $\delta : N \to N$ of degree -1, such that $(d_N + \delta)^2 = 0$ and $\xi_N \delta = 0$. A *perturbation datum of the contraction* $c : \{N, M, f, g, \phi\}$ is a perturbation δ of the DGA-module N verifying the composition $\phi\delta$ is pointwise nilpotent.

We now introduce the main tool in Homological Perturbation Theory: the Basic Perturbation Lemma ([8], [11],[9], [2]).

Theorem 2 *(***BPL***)*

Let $c : \{N, M, f, g, \phi\}$ *be a contraction and* $\delta : N \to N$ *a perturbation datum of* r*. Then, a new contraction*

$$r_\delta : \{(N, d_N + \delta, \epsilon_N, \eta_N),\ (M, d_M + d_\delta, \epsilon_M, \eta_M),\ \ f_\delta,\ g_\delta,\ \phi_\delta\}$$

is defined by the formulas: $d_\delta = f\delta\Sigma_r^\delta g$; $f_\delta = f(1 - \delta\Sigma_r^\delta \phi)$; $g_\delta = \Sigma_r^\delta g$; $\phi_\delta = \Sigma_r^\delta \phi$; *where*

$$\Sigma_r^\delta \ = \ \sum_{i \geq 0} (-1)^i \ (\phi\delta)^i \ = \ 1 - \phi\delta + \phi\delta\phi\delta - \cdots + (-1)^i (\phi\delta)^i + \cdots .$$

Let us note that $\Sigma_r^\delta(x)$ is a finite sum for each $x \in N$, because of the pointwise nilpotency of the composition $\phi\delta$. Moreover, it is obvious that the morphism d_δ is a perturbation of the DGA-module $(M, d_M, \epsilon_M, \eta_M)$.

Theorem 3 (Brown's theorem [3]) *Let* $F \times_\tau B$ *be a twisted cartesian product with structural group* G*. It is possible to establish the following contraction:*

$$C(F \times_\tau B) \Rightarrow C(F) \otimes_t C(B) \tag{2}$$

In [20], Shih proved that the contraction (2) is obtained from $C(F \times B) \Rightarrow C(F) \otimes C(B)$ via perturbation, with $\delta(f, b) = (\tau b * \partial_0 f,\, \partial_0 b) - (\partial_0 f,\, \partial_0 b)$ as perturbation datum.

Note 1 $C(F) \otimes_t C(B)$ *is a twisted tensor product in the Brown's sense, that is, its differential* d *can be obtained by the action of a Brown cochain* $t : C(B)_n \to C(G)_{n-1}$*; therefore,* $d = d_F \otimes 1 + 1 \otimes d_B + (\mu \otimes 1)(1 \otimes t \otimes 1)(1 \otimes \Delta)$*;* μ *is the morphism induced by the operation* $*$ *from* G *to* F*, and* Δ *is the coproduct in* $C_*^N(B)$ *[3], [20].*

Theorem 4 *[14] Let* $F \times_\tau B$ *be a PTCP. If* $\tau(b) = e_0 \quad \forall b \in B_1$*, then* $t(b) = 0 \quad \forall b \in B_1$ *non degenerated; the element* e_0 *denote the unit of* G_0*.*

Theorem 5 *[15] Let* $t : C \to A$ *be a Brown's cochain (*C *is a DGA-coalgebra and* A *is a DGA-algebra). Then* t *admits a lifting (a DGA-algebra morphism)* $T : \bar{\Omega}(C) \to A$*. If* $T : \bar{\Omega}(C) \to A$ *is a DGA-algebra morphism, then* $T_{|C}$ *is a Brown's cochain.*

The perturbation problem for more complex structures has been largely considered. The most significant results for DGA-algebras (or DGA-coalgebras) can be found in [11], [12] and [17]. First, we review several notions.

Definition 1 *[17]* Let A and A' be two DGA-algebras and $c : \{A, A', f, g, \phi\}$ a contraction. The projection f is *a quasi algebra projection* if the following conditions hold:

$$f\mu_A(\phi \otimes \phi) \;=\; 0, \qquad f\mu_A(\phi \otimes g) \;=\; 0,$$
$$f\mu_A(g \otimes \phi) \;=\; 0,$$

The homotopy operator ϕ is a *a quasi algebra homotopy* if the following conditions hold:

$$\phi\mu_A(\phi \otimes \phi) \;=\; 0, \qquad \phi\mu_A(\phi \otimes g) \;=\; 0,$$
$$\phi\mu_A(g \otimes \phi) \;=\; 0,$$

Definition 2 *[11]* Let A and A' be two DGA-algebras and $c : \{A, A', f, g, \phi\}$ a contraction. The homotopy operator ϕ is said to be *an algebra homotopy* if

$$\phi\mu_A \;=\; \mu_A(1_A \otimes \phi + \phi \otimes gf).$$

Definition 3 *[17]* Let A and A' be two DGA-algebras and $c : \{A, A', f, g, \phi\}$ a contraction. We say that r is

- *a semi-full algebra contraction* if f is a quasi algebra projection, g is a morphism of DGA-algebras and ϕ is a quasi algebra homotopy.
- *an almost-full algebra contraction* if f and g are morphisms of DGA-algebras and ϕ is a quasi algebra homotopy.
- *a full algebra contraction* if f and g are morphisms of DGA-algebras and ϕ is an algebra homotopy.

Obviously, full and almost-full algebra contractions are, in particular, semi-full algebra contractions. It is not difficult to prove that both sets of semi-full and almost-full algebra contractions are closed by composition and tensor product of contractions.

Now, we introduce an important example of a semi-full algebra contraction. Recall that if X and Y are simplicial groups then $C(X)$ and $C(Y)$ are DGA-algebras. An interesting problem is the multiplicative behaviour of (1) when X and Y are simplicial groups. It is possible to enunciate the following theorem.

Theorem 6 *[1] If G and G' are simplicial groups, then the Eilenberg-Zilber contraction:*

$$EZ_{G,G'} : \{C(G \times G'),\, C(G) \otimes C(G'),\, AW_{G,G'},\, EML_{G,G'},\, SHI_{G,G'}\}$$

is an almost-full algebra contraction.

Eilenberg and MacLane in [6] showed that the projection ($AW_{G,G'}$) and the inclusion ($EM_{G,G'}$) are DGA-algebra morphisms if G and G' are abelian. We show in [1] that $AW_{G,G'}$ and $EM_{G,G'}$ are DGA-algebra morphisms in the non-abelian case and the homotopy operator ($SHI_{G,G'}$) is a quasi-homotopy operator.

Definition 4 *[9]* Let A and A' be two DGA-algebras and $c : \{A, A', f, g, \phi\}$ a contraction. An *algebra perturbation datum δ of c* is a perturbation datum of this contraction which is also a derivation.

The following result tells us that the set of semi-full algebra contractions is closed by homological perturbation. This theorem plays a key role in the proof of the main theorem of this paper.

Theorem 7 (SF-APL) *([17])*

Taking as data a semi-full algebra contraction r and an algebra perturbation datum δ of r, the perturbed contraction r_δ is an algebra contraction of the same type, where the product on ${A'}_\delta$ is the original product $\mu_{A'}$.

Using the theorem above and Theorem 3, it is possible to state the following result.

Theorem 8 *[1] Let F and B be simplicial groups and let us suppose that the PTCP $F \times_\tau B$, endowed with the natural inner product of the cartesian product $F \times B$, is a simplicial group. Then the contraction (2) is a semi-full algebra contraction.*

Now, we enunciate several results which allow us to determine the p-local homology of a PTCP of Eilenberg-Mac Lane spaces.

First of all, we give various theorems from [17] and [18].

Theorem 9 *[18] Let G be a commutative simplicial group. There exists a semi-full algebra contraction c_{wb} from $C(W(G))$ to the bar construction $B(C(G))$.*

The inclusion of this contraction is a morphism of DGA-coalgebras (see [14]).

Theorem 10 *[17] Let c be an algebra contraction from a commutative DGA-algebra A to other commutative DGA-algebra A', in which the inclusion is a DGA-algebra morphism. Then there exists a semi-full algebra contraction $\bar{B}(c)$ from $\bar{B}(A)$ to $\bar{B}(A')$.*

Furthermore, in [10] is proved that $\bar{B}(c)$ is a full coalgebra contraction, (where A and A' can be non commutative DGA-algebras).

Analogous results can be established on the cobar construction. We are interested in the following result:

Theorem 11 *[12] Let c be an algebra contraction from a DGA-coalgebra C to other DGA-coalgebra C',where the inclusion g is a DGA-coalgebra morphism. Then there exists a full algebra contraction $\bar{\Omega}(c)$ from $\bar{\Omega}(C)$ to $\bar{\Omega}(C')$.*

If we note $\bar{\Omega}(g)$ the inclusion of the contraction $\bar{\Omega}(c)$, the following identity holds:

$$\bar{\Omega}(g)|_{C'} = g. \tag{3}$$

The following "endogamic" process will allow us to get a complete determination of the p-local homology of an Eilenberg-Mac Lane space.

Theorem 12 *[17] Let us consider $\Lambda = \mathbf{Z}_{(p)}$. Let A be a Cartan's elementary complex (see [4]). Then there exists a semifull algebra contraction from $\bar{B}(A)$ to a tensor product of Cartan's elementary complexes.*

The starting points in order to construct an explicit contraction from $C(K(\pi, n))$, being π an finitely generated abelian group, are:

Theorem 13 *[6]*

- *There is an almost full algebra contraction from $B(\Lambda[\mathbf{Z}])$ to the exterior algebra $E(u, 1)$.*
- *There is an almost full algebra contraction from $\bar{B}(\Lambda[\mathbf{Z}_{p^r}])$ to the twisted tensor product $E(u, 1) \otimes_\rho \Gamma(v, 2)$, where ρ is a derivation defined by $\rho(v) = p^r \cdot u$ and $\rho(u) = 0$.*

Now, combining these results, it is possible to set up the following chain of DGA-algebra contractions:

$$C(K(\pi,1)) \Rightarrow C(\bar{W}(K(\pi,0))) \Rightarrow \bar{W}_N(C(K(\pi,0))) \Rightarrow \bar{B}(\Lambda[\pi]) \Rightarrow G(\pi,1)$$

and hence,

$$\begin{aligned}&C(K(\pi,n+1)) \Rightarrow C(\bar{W}(\bar{W}^n(K(\pi,0)))) \Rightarrow \bar{W}_N(\bar{W}_N^n(C(K(\pi,0))))\\ &\Rightarrow \bar{B}(\bar{W}_N^n(C(K(\pi,0))) \Rightarrow \bar{B}(\bar{B}^n(\Lambda[\pi])) \Rightarrow \bar{B}(G(\pi,n)) \Rightarrow G(\pi,n+1)\end{aligned}$$

Thanks to them, the following theorem extracted from [16] can be proved without difficulty:

Theorem 14 *Let* $\Lambda = \mathbf{Z}_{(p)}$ *be the ground ring. Let* π *and* n *be a finitely generated abelian group and a non-negative integer, respectively. There is a semi-full algebra contraction* $c_{\pi,n}$ *from* $C(K(\pi,n))$ *to a tensor product* $G(\pi,n)$*. The latter is composed by Cartan's elementary complexes corresponding to admissible sequences.*

Obviously, the p-local homology of $K(\pi,n)$ is the the p-local homology of $G(\pi,n)$ and this last homology can be computed easily.

Note 2 $G(\pi,n)$ *is a free DGA-algebra of finite type.*

Let us take remark that we use in the proof of the theorem above that there is a semifull algebra contraction from $\bar{B}(G(\pi,n))$ to $G(\pi,n+1)$.

It is not hard to deduce that in the case $n=1$, we have

Theorem 15 *[17] Let* Λ *be a commutative ring with* $1 \neq 0$ *and* π *a finitely generated abelian group. There is an almost-full algebra contraction* $c_{\pi,1}$ *from* $C(K(\pi,1))$ *to the tensor product* $G(\pi,1)$*.*

The projection of the last contraction is a DGA-algebra morphism because of the facts that there is an algebra isomorphism between $C(W(K(\pi',0)))$ and the bar construction $B(C(K(\pi',0)))$ and Th. 13.

Using the previous results, we prove the following Theorem which plays an essential role in the proof of Theorem 19

Theorem 16 *Under the hypothesis of Theorem 14, there exists a full algebra contraction* $c_{n,n-1}$ *from* $\bar{\Omega}(K(\pi,n))$ *to* $\bar{\Omega}\bar{B}(G(\pi,n-1))$*.*

Proof

First, we have the following isomorphism of DGA-coalgebras:

$$C(K(\pi,n) \Rightarrow C(\bar{W}(K(\pi,n-1)) \tag{4}$$

By Theorem 9, we establish the following contraction, where the inclusion is a morphism of DGA-coalgebras:

$$C(\bar{W}(K(\pi,n-1))) \Rightarrow \bar{B}C(K(\pi,n-1)). \tag{5}$$

By theorem 14, there is a semi-full algebra contraction:

$$C(K(\pi,n-1)) \Rightarrow G(\pi,n-1)$$

.

Using Theorem 10, we obtain the following full coalgebra contraction:

$$\bar{B}C(K(\pi, n-1)) \Rightarrow \bar{B}G(\pi, n-1). \tag{6}$$

Now, composing contractions (4), (5) and (6), we obtain the following one, where the inclusion is a DGA-coalgebra morphism:

$$C(K(\pi, n) \Rightarrow \bar{B}G(\pi, n-1)$$

The last contraction satisfies the hypothesis of Theorem 11, and, therefore, there exists the following full algebra contraction:

$$\bar{\Omega}C(K(\pi, n) \Rightarrow \bar{\Omega}\bar{B}G(\pi, n-1) \tag{7}$$

$\square$

3 Computation of $H_*(X, \mathbf{Z}_{(p)})$

In the sequel, the ground ring will be $\mathbf{Z}_{(p)}$. In this section, we are interested in the determination of the p-local homology of the PTCP $X(\pi, m, \tau, \pi', n)$. First, Brown's theorem provides us an explicit contraction between the normalized chain complex of $X(\pi, 1, \tau, \pi', n)$ and a twisted tensor product $C(K(\pi, 1)) \otimes_t C(K(\pi', n))$.

Theorem 17 *There is a contraction c_τ from $C(X(\pi, m, \tau, \pi', n))$ to a free DG-module of finite type M. Furthermore, M has the form $(G(\pi, m) \otimes G(\pi', n), d_{G(\pi,m)} \otimes 1 + 1 \otimes d_{G(\pi',n)} + d_\tau)$.*

Proof.

Applying Theorem 3 for $F = K(\pi, m)$ and $B = K(\pi', n)$, we get the following explicit contraction:

$$C(K(\pi, m) \times_\tau K(\pi', n)) \Rightarrow C(K(\pi, m)) \otimes_t C(K(\pi', n)) \tag{8}$$

Let us recall that $t : C(K(\pi, n)_p \to C(K(\pi, m))_{p-1}$ is a Brown's cochain and if $m > 0$ then $C(K(\pi, m))$ is a reduced simplicial group. It is clear that, in this situation, the hypothesis of Theorem 4 are satisfied and, therefore, the following identity holds:

$$t(b_1) = 0, \quad \forall b_1 \in K(\pi', n)_1 \tag{9}$$

By Theorem 14, we know that there exist the following contractions:

$$c_{\pi,m} : C(K(\pi, m)) \Rightarrow G(\pi, m) \quad \text{and} \quad c_{\pi',n} : C(K(\pi', n)) \Rightarrow G(\pi', n).$$

Now, we can do the tensor product of these last two contractions and establish the contraction:

$$c_\otimes : C(K(\pi, m)) \otimes C(K(\pi', n)) \Rightarrow G(\pi, m) \otimes G(\pi', n)$$

Let us note that the complex $C(K(\pi,m)) \otimes_t C(K(\pi',n))$ is equal to $(C(K(\pi,m)) \otimes C(K(\pi',n),\, d_{K\pi} \otimes 1 + 1 \otimes d_{K\pi'} + d_t)$ (see Note 1). Thanks to (9), it is easy to see that the morphism d_t is a perturbation datum for the contraction $c_{\otimes}$. Now applying the BPL (Theorem 2) to the contraction $c_{\otimes}$, taking as perturbation datum d_t, the following contraction is obtained:

$$(c_{\otimes})_{d_t} : C(K(\pi,m)) \otimes_t C(K(\pi',n)) \Rightarrow (G(\pi,m) \otimes G(\pi',n),\, d_{G\pi} \otimes 1 + 1 \otimes d_{G\pi'} + d_{\tau}) \quad (10)$$

Finally, composing the contractions (8) and (10), we have:

$$c_{\tau} : C(K(\pi,m) \times_{\tau} K(\pi',n)) \Rightarrow (G(\pi,m) \otimes G(\pi',n),\, d_{G\pi} \otimes 1 + 1 \otimes d_{G\pi'} + d_{\tau})$$

□

There is a classical result (see [14]) telling us that if X is a commutative simplicial group then X is homotopically equivalent to a banal cartesian product of Eilenberg-Mac Lane spaces. Let us suppose that our $X(\pi,m,\tau,\pi',n)$ is a simplicial group, considering as inner product the natural one of the cartesian product $K(\pi,m) \times K(\pi',n)$. In the light of the previous result, it seems to be possible to transforms a simplicial group $X(\pi,m,\tau,\pi',n)$ into a simple cartesian product by means of simplicial techniques. We use here a different approach in order to compute the p-local homology of X in this case.

Theorem 18 *Let us suppose that $X(\pi,m,\tau,\pi',n)$, endowed with the natural inner product of the cartesian product $X(\pi,m,0,\pi',n)$, is a commutative simplicial group. Then, c_{τ} is a semi-full algebra contraction.*

Proof.

By Theorem 8, the contraction (8) is a semi-full algebra contraction. Theorem 14 tells us that the contractions $c_{\pi,m}$ and $c_{\pi',n}$ are semi-full algebra contractions. It is clear that the morphism d_t is a algebra perturbation datum.

Now, taking into account that the set of semi-full algebra contractions is closed by perturbation(see theorem 7), composition, and tensor product of contractions, it is easy to conclude the desired result. □

Note 3 *Let us observe that in Theorem 18, the differential operator d_{τ}, obtained by perturbation, is completely determined if we know its images on the generators of the algebra $G(\pi,m) \otimes G(\pi',n)$. In this way, the improvement in the computation of the differential is enormous in comparison with the general case of Theorem 17.*

Up to now, our approach in this note is based in essence merely on the application of the basic perturbation techniques. Now, we are concerned with the behaviour of Brown's cochains in this context. The fact that the projection of the algebra contraction $c_{\pi',1}$ is a DGA-algebra morphism plays an essential role in the proof of the following theorem:

Theorem 19 *There exists an explicit contraction from $C(X(\pi,1,\tau,\pi',n))$ to a twisted tensor product $G(\pi,1)\otimes_{t'}\bar{B}(G(\pi',n-1))$*

Proof.

On the one hand, Theorem 14 and Theorem 16 gives us the following contractions:

$c_{\pi,1}\ :\ \{C(K(\pi,1)),\ G(\pi,1),\ f_{\pi,1},\ g_{\pi,1},\ \phi_{\pi,1}\}$ an almost-full algebra contraction

$c_{n,n-1}\ :\ \{C(K(\pi',n)),\ \bar{B}(G(\pi',n{-}1)),\ f_{n,n-1},\ g_{n,n-1},\phi_{n,n-1}\}$ $g_{n,n-1}$ is a coalgebra morphism

$\Omega(c_{n,n-1})\ :\ \{\bar{\Omega}(C(K(\pi',n))),\ \bar{\Omega}(\bar{B}(G(\pi',n-1))),\ \bar{f},\ \bar{g},\ \bar{\phi}\}$ a full algebra contraction

As in Theorem 17, it is possible to build the contraction

$$C(K(\pi,1)\times_\tau K(\pi',n))\Rightarrow(G(\pi,1)\otimes\bar{B}(G(\pi',n-1)),\ d_{G\pi}\otimes 1+1\otimes d_{BG\pi'}+d_{\tau'}).$$

Let us recall that the first step in the construction process ofthe contraction above is based on Brown's theorem, i.e

$$C(K(\pi,1)\times_\tau K(\pi',n)\Rightarrow C(K(\pi,1)\otimes_t C(K(\pi',n))$$

Let us recall that the Brown cochain $t: C(K(\pi,n)\rightarrow C(K(\pi',1))$ admits a lifting (a DGA-algebra morphism) $T:\Omega(C(K(\pi',n)))\rightarrow C(K(\pi,1))$ (see theorem 5).

Now, we define the morphism

$$T':\bar{\Omega}(\bar{B}(G(\pi',n-1)))\rightarrow G(\pi,1)$$

$$T'=f_{\pi',1}T\bar{g}$$

It is manifest that T' is a DGA-algebra morphism, since $f_{\pi',1},\ T,\ \bar{g}$ are morphism of DGA-algebras. Obviously, T' is the lifting of the Brown cochain $t'=T'_{|_{\bar{B}(G(\pi,n-1))}}$.

Now, we show that the DG-module $(G(\pi,1)\otimes_{t'}\bar{B}(G(\pi',n-1))$ coincides with $(G(\pi,1)\otimes\bar{B}(G(\pi',n-1)),\ d_{G\pi}\otimes 1+1\otimes d_{\bar{B}G\pi'}+d_{\tau'})$, determined in Th. 17.

First, for clarity, we need to fix some notations . In the sequel, f_1 , g_1 , ϕ_1 , f_n , g_n and ϕ_n denote $f_{\pi,1},\ g_{\pi,1},\ \phi_{\pi,1},\ f_{n,n-1}$, $g_{n,n-1}$ and $\phi_{n,n-1}$, respectively. We denote the coproducts on $C(K(\pi',n))$ and on $\bar{B}(G(\pi',n-1))$ by Δ and Δ_B, respectively. Finally, μ and μ_G denote the products on $C(K(\pi,1))$ and on $G(\pi,1)$, respectively.

Now, all is reduced to proving that $d_{t'}=(\mu_G\otimes 1)(1\otimes t'\otimes 1)(1\otimes\Delta_B)$ is equal to $d_{\tau'}$.

Now, we describe how $d_{\tau'}$ is obtained. We proceed a step at a time.

Step 1. First of all, we start with the contraction

$$C(K(\pi,1))\otimes C(K(\pi',n)\Rightarrow G(\pi,1)\otimes\bar{B}(G(\pi',n-1)) \tag{11}$$

Step 2, Perturbing the contraction 11, with $d_t=(\mu\otimes 1)(1\otimes t\otimes 1)(1\otimes\Delta)$ as perturbation datum, then we get:

$$C(K(\pi,1))\otimes_t C(K(\pi',n)\Rightarrow(G(\pi,1)\otimes\bar{B}(G(\pi',n-1)),d_{G\pi}\otimes 1+1\otimes d_{BG\pi'}+d_{\tau'})$$

Hence, (see theorem 2)

$$d_{\tau'} = (f_1 \otimes f_n) d_t \left[\sum_{i \geq 0} (-1)^i (\phi_1 \otimes g_n f_n + 1 \otimes \phi_n) d_t)^i \right] (g_1 \otimes g_n)$$

By a simple inspection, we have

$$d_{\tau'} = (\mu_G \otimes 1)(1 \otimes f_1 t g_n \otimes 1)(1 \otimes \Delta_B) \tag{12}$$

In order to prove the identity above, we have to take into account that f_1 is a morphism of DGA-algebras and g_n is a morphism of DGA-coalgebras.

Alternatively, we have that

$$d_{t'} = \mu_G (1 \otimes t' \otimes 1)(1 \otimes \Delta) \tag{13}$$

The identity (12)=(13) can be deduced from the following facts:

1. $T' = f_1 T \bar{g}$
2. $t' = T'_{|_{\bar{B}(G(\pi',n-1)))}}$
3. $\bar{g}_{|_{\bar{B}(G(\pi',n-1)))}} = g_n$ (see (3)).

$\square$

Finally, in the light of the Proute work [15] on the homotopical behaviour of Brown's cochains, we are able to conjecture that there is a contraction from the twisted tensor product $C(K(\pi,1)) \otimes_t C(K(\pi',n))$ to the twisted tensor product $HM = G(\pi,1) \otimes_{t'} G(\pi',n)$, where $t' : G(\pi',n) \to G(\pi,1)$ is a twisting γ-cochain (see [15]).

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