An example in Combinatorial Cohomology

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Abstract

Steenrod cohomology operations are algebraic tools for distinguishing non–homeomorphic topological spaces. In this paper, starting off from the general method developed in [4] for Steenrod squares and Steenrod reduced powers, we present an explicit combinatorial formulation for the Steenrod reduced power $\mathcal{P}_1^p : H^q(X;\mathbf{F}_p) \to H^{qp-1}(X;\mathbf{F}_p)$, at cocycle level, where p is an odd prime, q a non–negative integer, X a simplicial set and $\mathbf{F}_p$ the finite field with p elements. We design an algorithm for computing $\mathcal{P}_1^p$ on the cohomology of the classifying space of $\mathbf{Z}_p$ and we generalize this process to any simplicial set at cohomology level.

1 Introduction

The relation between cohomology operations and simplicial cochain operations was initialized by L. Kristensen [6]. He worked with acyclic models and inductive computations making use of the explicit structure of the cohomology of Eilenberg–MacLane spaces known by the results of Serre and Cartan [11]. An alternative of this method is to obtain this relation using algebraic fibrations with a cartesian product of n copies of a given simplicial set X as the base space and a subgroup of the symmetric group S_n as the fiber space. This last point of view has been established in [10] and [4] and it closely follows the original work of Steenrod about cohomology operations [11].

In this paper, we analyze in detail the underlying combinatorial structure of the Steenrod reduced power $\mathcal{P}_1^p : H^q(X;\mathbf{F}_p) \to H^{qp-1}(X;\mathbf{F}_p)$, at coycle level, p being an odd prime, q a positive integer, X a simplicial set and $\mathbf{F}_p$ the finite field with p elements. This study is based on the results obtained in [4] for Steenrod reduced powers. Our motivation in the very near future is to give an explicit combinatorial picture of all Steenrod cohomology operations. Some results in this way have been given in [3].

2 A simplicial description of $\mathcal{P}_1^p$ at cochain level

First of all, we give some simplicial and algebraic preliminaries in order to facilitate the understanding of the paper. This material can be found, for example, in [8], [12] and [7]. A fixed ground ring will be denoted by Λ. Let X be a *simplicial set* (a graded set, $X_0, X_1, \dots$, endowed with face operators $\partial_j : X_n \to X_{n-1}$ and degeneracy operators $s_i : X_n \to X_{n+1}$, $1 \le j \le n$, verifying several "commutativity" relations). Let $C_*(X)$ be the chain complex $\{C_n(X),\, d_n\}$ in which $C_n(X)$ is the graded Λ-module generated by X_n and $d_n = \sum_{i=0}^{n}(-1)^i\partial_i$. Let G be a group and let $C_*^N(X;G)$ be the *normalized chain complex* $\{C_n^N(X),\, d_n\}$ where $C_n^N(X) = C_n(X)/s(C_{n-1}(X))$ and $s(C_{n-1}(X))$ is the graded Λ–module generated by all the degenerate simplices of $C_n(X)$. The *cochain complex* associated to $C_*^N(X)$ is denoted by $C^*(X;G) = \{C^n(X;G), \delta_n\}$, where $C^n(X;G)$ is the free Λ–module generated by all the Λ-module maps from $C_n^N(X)$ to G, and δ_n is defined by $\delta_n(c)(x) = c(d_{n+1}(x))$. In this way, the *cohomology* of X is $H^*(X;G) = \mathrm{Ker}\ \delta_n/\mathrm{Im}\ \delta_{n-1}$.

From now on, we consider that the field $\mathbf{F}_p$, with p a fixed odd prime, is the ground ring.

We are able to state the main result of this paper.

Theorem 2.1. *Let X be a simplicial set. An expression of $\mathcal{P}_1^p : H^q(X;\mathbf{F}_p) \to H^{pq-1}(X;\mathbf{F}_p)$, at coycle level, in terms of face operators is:*

$$
\begin{aligned}
\mathcal{P}_1^p(c)(x) = \sum_{1\le j\le p-1}\ \sum_{jq\le i\le (j+1)q-1} \quad & (-1)^{(i+1)(q+1)+1} \\
& c(\partial_{q+1}\cdots\partial_{pq-1}x) \\
& \bullet c(\partial_0\cdots\partial_{q-1}\partial_{2q+1}\cdots\partial_{pq-1}x) \\
& \vdots \\
& \bullet c(\partial_0\cdots\partial_{(j-2)q-1}\partial_{(j-1)q+1}\cdots\partial_{pq-1}x) \\
& \bullet c(\partial_0\cdots\partial_{(j-1)q-1}\partial_{i-q+1}\cdots\partial_{i-1}\partial_{(j+1)q}\cdots\partial_{pq-1}x) \\
& \bullet c(\partial_0\cdots\partial_{(j+1)q-2}\partial_{(j+2)q}\cdots\partial_{pq-1}x) \\
& \vdots \\
& \bullet c(\partial_0\cdots\partial_{(p-2)q-2}\partial_{(p-1)q}\cdots\partial_{pq-1}x) \\
& \bullet c(\partial_0\cdots\partial_{(p-1)q-2}x) \\
& \bullet c(\partial_0\cdots\partial_{i-q-1}\partial_{i+1}\cdots\partial_{pq-1}x)
\end{aligned}
$$

where c is a q–cocycle, $x \in C_{pq-1}^N(X)$ and $\bullet$ is the natural product in $\mathbf{F}_p$.

The key of our combinatorial approach is the description given in [4] for Steenrod reduced powers in terms of the component morphisms of a given Eilenberg–Zilber contraction [2]. These morphisms can be expressed in terms of face and degeneracy operators of X. The problem of this description is that the number of summands involved in it has an exponential nature.

The proof of the previous theorem is, on one hand, a simplification of that combinatorial formulation based on two facts. The first one is that any composition of face and degeneracy operators can be expressed in a "canonical" form:

$$s_{j_t}\cdots s_{j_1}\partial_{i_1}\cdots\partial_{i_s}$$

where $j_t > \cdots > j_1 \ge 0$ and $i_s > \cdots > i_1 \ge 0$; and the second one is that taking into account that $\mathcal{P}_1^p(c)(x)$ lies in the tensor product of p copies of X, we only have to consider those summands written in the canonical form without degeneracy operators.

On the other hand, bearing in mind that c is a q–cochain, the non–trivial summands of the formula are those summands with exactly $pq - q - 1$ face operators in each factor.

Assuming that face operators are evaluated in constant time, the following result gives us a first measure of the computational complexity of this formula.

Proposition 2.2. *Let X be a simplicial set. If c is a q–cocycle, then the number of face operators taking part in the formula for $\mathcal{P}_1^p(c)$ of the theorem above is $p(p-1)q[(p-1)q-1]$.*

It is clear that at least in the case in which X has a finite number of non–degenerate simplices in each degree, our method can be seen as a real algorithm for calculating $\mathcal{P}_1^p$.

3 Algorithm for computing $\mathcal{P}_1^p$ at cohomology level

First of all, since the ground ring is $\mathbf{F}_p$, the cohomology of any simplicial set is free. Then, using the results from the section above, we can describe a general algorithm for calculating $\mathcal{P}_1^p$ on the cohomology of a simplicial set X.

Algorithm 3.1. Algorithm for computing $\mathcal{P}_1^p$ on $H^*(X, \mathbf{F}_p)$

Input:

A simplicial set X.

A contraction (f^*, g^*, ϕ^*) from $C^*(X; \mathbf{F}_p)$ onto $H^*(X; \mathbf{F}_p)$.

An element α in $H^q(X; \mathbf{F}_p)$.

Output:

The element $f^*(\mathcal{P}_1^p(g^*(\alpha)))$ in $H^{pq-1}(X, \mathbf{F}_p)$.

The assumption of having a contraction from $C^*(X; \mathbf{F}_p)$ to $H^*(X; \mathbf{F}_p)$ is not a restriction in the case in which X is a simplicial set of finite type, because we can apply "Veblen" algorithm [9] (a classical matrix algorithm for obtaining the cohomology group of X and a contraction connecting $C^*(X; \mathbf{F}_p)$ and $H^*(X; \mathbf{F}_p)$).

Now, as an example, we design an algorithm to compute elements of the cohomology of the classifying space of $\mathbf{Z}_p$, $\bar{W}_*(\mathbf{Z}_p)$. This space is a particular simplicial set defined as follows:

$$\bar{W}_0(\mathbf{Z}_p) = \star, \qquad \bar{W}_n(\mathbf{Z}_p) = \mathbf{Z}_p^{\times n},$$

$$\partial_i(g) = \star, \quad i = 0, 1$$

$$\partial_0(g_{n-1}, \dots, g_0) = (g_{n-2}, \dots, g_0) \qquad \partial_n(g_{n-1}, \dots, g_0) = (g_{n-1}, \dots, g_1)$$

$$\partial_i(g_{n-1}, \dots, g_0) = (g_{n-1}, \dots, g_{n-i-1} + g_{n-i}, g_{n-i-2}, \dots, g_0)$$

$$s_i(g_{n-1}, \dots, g_0) = (g_{n-1}, \dots, g_{n-i}, 0, g_{n-i-1}, \dots, g_0)$$

where 0 denotes the identity in $\mathbf{Z}_p$.

There exists an explicit homotopy equivalence, a contraction $c = (f, g, \phi)$, from $C_*(\bar{W}(\mathbf{Z}_p))$ onto $E(u,1) \otimes \Gamma(v,2)$ [1] (where $E(u,1)$ and $\Gamma(v,2)$ are Cartan's elementary complexes: $E(u,1)$ is an exterior algebra with one generator u in dimension 1, and $\Gamma(v,1)$ is a divided power algebra with one generator v in dimension 2).

The expressions of f and g are:

$$f(x_1, y_1, \ldots, x_m, y_m) = \left\{\Pi_{i=1}^m s^2(x_i, y_i)\right\} \gamma_m(v)$$

$$f(x_1, y_1, \ldots, x_m, y_m, z) = \left\{s^1(z)\Pi_{i=1}^m s^2(x_i, y_i)\right\} u\gamma_m(v),$$

where $\quad s^2(i,j) = \begin{cases} 0 & i+j < h \\ 1 & i+j \geq h \end{cases} \quad$ and $\quad s^1(i) = i,\ 0 \leq i < h,$

$$g(u) = (1),$$

$$g(\gamma_k(v)) = \sum_{x_1,\ldots,x_k \in \mathbf{Z}_p} (1, x_1, \ldots, 1, x_k), \qquad g(u\gamma_k(v)) = \sum_{x_1,\ldots,x_k \in \mathbf{Z}_p} (1, x_1, \ldots, 1, x_k, 1)\,,$$

Notice that the image of the morphism f over an element of $C_*(\bar{W}(\mathbf{Z}_p))$ has one summand but the image of g over an element of $E \otimes \Gamma$ of dimension n has $p^{\lfloor n/2 \rfloor}$ summands.

Applying Hom functor, we can obtain a contraction $c^* = (f^*, g^*, \phi^*)$ from

$$\text{Hom }(C_*(\bar{W}(\mathbf{Z}_p), \mathbf{F}_p) = C^*(\bar{W}(\mathbf{Z}_p)); \mathbf{F}_p) \text{ onto Hom }(E \otimes \Gamma, \mathbf{F}_p) = E(\bar{u}, 1) \otimes P(\bar{v}, 2)$$

where $P(\bar{v}, 2)$ is a polynomial algebra with one generator $\bar{v}$ in dimension 2,

$$f^* : C^*(\bar{W}(\mathbf{Z}_p)) \to E \otimes P\,, \quad f^*(c) = c\, g\,, \qquad g^* : E \otimes P \to C^*(\bar{W}(\mathbf{Z}_p))\,, \quad g^*(c') = c' f$$

and $\phi^* : C^*(\bar{W}(\mathbf{Z}_p)) \to C^*(\bar{W}(\mathbf{Z}_p))\,,\ \phi^*(c) = c\,\phi$.

Since the ground ring is $\mathbf{F}_p$, then $E \otimes P$ is the cohomology of $\bar{W}(\mathbf{Z}_p)$. In this way, our description from Theorem 2.1 of $\mathcal{P}_1^p$, at cocycle level, can be seen as a real cohomology operation via this contraction.

Algorithm 3.2. Algorithm for computing $\mathcal{P}_1^p$ on $H^*(\bar{W}_*(\mathbf{Z}_p), \mathbf{F}_p)$.

Input:

An element α of degree q in $E \otimes P =$Hom $(E \otimes \Gamma; \mathbf{F}_p) = H^*(\bar{W}(\mathbf{Z}_p); \mathbf{F}_p)$.

Output:

The element $f^*(\mathcal{P}_1^p(g^*(\alpha)))$ in $H^{pq-1}(\bar{W}(\mathbf{Z}_p), \mathbf{F}_p)$.

Looking through the formula of f, g and $\mathcal{P}_1^p$, it is not difficult to see that the output of this last algorithm for every α in $H^*(\bar{W}(\mathbf{Z}_p); \mathbf{F}_p)$ is the unique element of $E \otimes P$ in dimension $* \cdot q - 1$.

This work was partially presented in a communication in [5].

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