Pricing Parisian options using Laplace transforms

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PRICING PARISIAN OPTIONS USING LAPLACE TRANSFORMS

CÉLINE LABART AND JÉRÔME LELONG

Abstract. In this work, we propose to price Parisian options using Laplace transforms. Not only do we compute the Laplace transforms of all the different Parisian options, but we also explain how to invert them numerically. We prove the accuracy of the numerical inversion.

1. Introduction

The analysis of structured financial products often leads to the pricing of exotic options. For instance, consider a re-callable convertible bond. The holder typically wants to recall the bond if ever the underlying stock has been traded above or below a given level for too long. Such a contract can be modelled with the help of Parisian options. Parisian options are barrier options that are activated or cancelled depending on the type of option if the underlying asset stays above or below the barrier long enough in a row. Parisian options are far less sensitive to influential agent on the market than standard barrier options. It is quite easy for an agent to push the price of a stock momentarily but not on a longer period so that it would affect the Parisian contract.
In this work, we study the pricing of European style Parisian options using Laplace transforms. Some other methods have already been proposed. On path dependent options, crude Monte Carlo techniques do usually not perform well. An improvement of this strategy using sharp large deviation estimates was proposed by Baldi et al. (2000). Techniques using a two dimensional partial differential equation have also drawn much attention, see for instance the works of Avellaneda and Wu (1999), Haber et al. (1999), or Wilmott (1998). The PDE approach is quite flexible and could even be used for American style Parisian option but the convergence is rather slow, which is badly suited for real time evaluation. In a quite similar state of mind, tree methods based on the framework of Cox et al. (1979) were investigated by Costabile (2002). An original concept of implied barrier was developed by Anderluh and van der Weide (2004), the idea is to replace the Parisian option by a standard barrier option with a suitably shifted barrier. The idea of using Laplace transforms to price Parisian options was introduced by Chesney et al. (1997). Their work is based on Brownian excursion theory in general and in particular on the study of the Azéma martingale (see Azéma and Yor (1989)) and the Brownian meander. The prices are then computed by numerically inverting the Laplace transforms. An original way of doing so was proposed by Quittard-Pinon et al. (2004). They approximate the Laplace transforms by negative power functions whose analytical inverse is well-known. But, there is no upper bound for the error due to the inversion.
In this work, we give the formulae of the Laplace transforms of the prices of the different Parisian options ready to be implemented. We also derive the formulae for the prices at any time after the emission time. We prove an accuracy result for the numerical inversion of the Laplace transforms to find the prices back.
First, we define the Parisian contract and introduce some material related to the excursion theory. Then, we present a few parity relationships which enable to reduce the pricing of the eight different types of Parisian options to the pricing of the down and in call — when

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the barrier is smaller than the initial value — and the up and in call — when the barrier is greater than the initial value. The Laplace transforms of the prices of the two latter options are computed in Sections 4 and 5. Section 6 is devoted to the pricing at any time after the emission time of the option. At this stage, we are able to compute the Laplace transforms of the prices of all the different Parisian options, we only need a method to accurately invert them. In Section 7, we study in details the numerical inversion of Laplace transforms as introduced by Abate and Whitt (Winter 1995) and prove an upper bound for the error. Finally, the last section is devoted to the comparison of our method with the enhanced Monte Carlo method of Baldi et al. (2000) whose implementation in PREMIA[1] has been used for the comparison. We have also implemented our method in PREMIA.

2. Definitions

2.1. **Some notations.** We consider a Brownian motion $W = \{W_t, t \geq 0\}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, which models a financial market. We assume that $\mathbb{Q}$ is the risk neutral measure and that $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of W. We denote by T the maturity time. In this context, we assume that the dynamics of an asset price is given by the process S

$$\forall t \in [0,T], \quad S_t = x\,\mathrm{e}^{(r-\delta-\sigma^2/2)t+\sigma W_t},$$

where $r > 0$ is the interest rate, $\delta > 0$ the dividend rate, $\sigma > 0$ the volatility and $x > 0$ the initial value of the stock. The Cameron-Martin-Girsanov Theorem (see Karatzas and Shreve (1991)) enables to state the following proposition for a finite time horizon $[0,T]$ with $T > 0$.

Proposition 1. *Let $m = \frac{1}{\sigma}\left(r - \delta - \frac{\sigma^2}{2}\right)$ and $\mathbb{P}$ be a new probability, which makes $Z = \{Z_t = W_t + mt, 0 \leq t \leq T\}$ a $\mathbb{P}$-Brownian motion. The change of probability is given by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}_{|\mathcal{F}_T} = \mathrm{e}^{mZ_T - \frac{m^2}{2}T},$$

and under $\mathbb{P}$, the dynamics of S is given by

$$\forall t \in [0,T], \quad S_t = x\ \mathrm{e}^{\sigma Z_t}.$$

Remark 1. *Since the drift term linking W and Z is deterministic, $\mathcal{F}_t$ is also the natural filtration of Z.*

Before explaining what a Parisian option is, we introduce the notion of excursion.

Definition 1 (Excursion)**.** *For any $L > 0$ and $t > 0$, we define*

$$g^S_{L,t} = \sup\{u \leq t\,:\, S_u = L\} \qquad d^S_{L,t} = \inf\{u \geq t\,:\, S_u = L\}.$$

with the conventions $\sup\emptyset = 0$ and $\inf\emptyset = +\infty$. The trajectory of S between $g^S_{L,\,t}$ and $d^S_{L,\,t}$ is the excursion at level L, straddling time t.

Obviously, such an excursion can also be described in terms of the Brownian motion Z. For a given barrier L for the process S, we introduce the corresponding barrier b for Z defined by

$$b = \frac{1}{\sigma}\log\left(\frac{L}{x}\right).$$

[1]PREMIA is a pricing software developed by the MathFi team of INRIA Rocquencourt, see `http://www.premia.fr`.

Definition 2 (Stopping times T_b, T_b^- and T_b^+). *Let $b \in \mathbb{R}$ and $t > 0$, we define the hitting time of level b by*

$$T_b(Z) = \inf\{u > 0 \, : \, Z_u = b\}.$$

In order to define $T_b^-(Z)$ and $T_b^+(Z)$, we introduce g_t^b and d_t^b

$$g_t^b = \sup\{u \leq t \, : \, Z_u = b\}, \quad d_t^b = \inf\{u \geq t \, : \, Z_u = b\}.$$

Let $T_b^-(Z)$ denote the first time the Brownian motion Z makes an excursion longer than some time D below the level b

$$T_b^-(Z) = \inf\{t > 0 \, : \, (t - g_t^b) \, \mathbf{1}_{\{Z_t < b\}} \geq D\}.$$

For the excursion above b, we define

$$T_b^+(Z) = \inf\{t > 0 \, : \, (t - g_t^b) \, \mathbf{1}_{\{Z_t > b\}} \geq D\}. \tag{1}$$

When no confusion is possible, we write T_b, T_b^- and T_b^+ instead of $T_b(Z)$, $T_b^-(Z)$ and $T_b^+(Z)$.

Remark 2. *Note that $g_t^b = g_{L,t}^S$ and $d_t^b = d_{L,t}^S$. Moreover, we can also write*

$$T_b^-(Z) = \inf\{t > D \, : \, \forall s \in [t - D, t] \; Z_t < b\}.$$

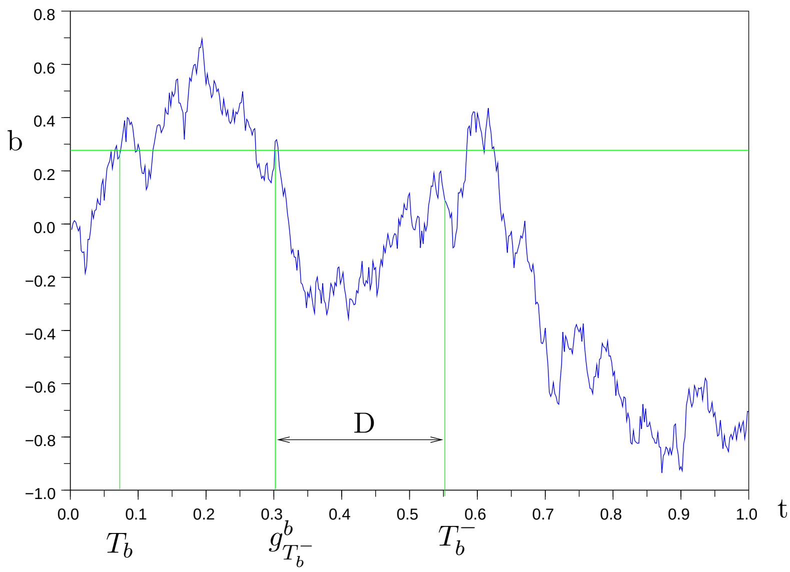

FIGURE 1. Excursion of Brownian Motion

Definition 3 (Laplace transform). *The Laplace Transform of a function f defined for all $t \geq 0$ is the function $\hat{f}$ defined by*

$$\hat{f}(\lambda) = \int_0^{+\infty} e^{-\lambda t} f(t) dt.$$

We also recall an elementary property of the Laplace transform of the convolution of two functions.

Proposition 2. *Let f and g be two functions defined on $\mathbb{R}^+$ whose Laplace transforms exist on (σ_f, ∞) and (σ_g, ∞) respectively, then the Laplace transform of the convolution $f \star g$ defined by $(f \star g)(t) = \int_0^t g(u) f(t-u) du$ exists on $(\max(\sigma_f, \sigma_g), \infty)$ and is given by*

$$\widehat{f \star g}(\lambda) = \widehat{f}(\lambda)\widehat{g}(\lambda).$$

Parisian options can be seen as barrier options where the condition involves the time spent in a row above or below a certain level and not only a hitting time. As for barrier options, which can be activated or cancelled (depending whether they are In or Out) when the asset S hits the barrier, Parisian options can be activated (In options) or cancelled (Out options) after S has spent more than a certain time in an excursion. Parisian options are defined in the following way

Definition 4 (Definition of θ, k and d)**.** *In the following, we define*

$$\theta = \sqrt{2\lambda}, \quad k = \frac{1}{\sigma} \log\left(\frac{K}{x}\right), \quad d = \frac{b-k}{\sqrt{D}}.$$

Definition 5 (Parisian Options)**.** *A Parisian option is defined by three characteristics:*

- *Up or Down,*
- *In or Out,*
- *Call or Put.*

Combining the above characteristics together enables to distinguish eight types of Parisian options. For example, PDIC denotes a Parisian Down and In call, whereas PUOP denotes a Parisian Up and Out put.

In the following section, we present Parisian Down options.

2.2. **Parisian Down options.**

2.2.1. *Parisian Down and In options.* The owner of a Down and In option receives the payoff if and only if S makes an excursion below level L older than D before maturity time T. The price of a Down and In option at time 0 with payoff $\phi(S_T)$ is given by

$$\mathrm{e}^{-rT} \mathbb{E}_{\mathbb{Q}} \left(\phi(S_T) \mathbf{1}_{\{T_b^- < T\}} \right) = \mathrm{e}^{-(r + \frac{m^2}{2})T} \mathbb{E}_{\mathbb{P}} \left(\mathbf{1}_{\{T_b^- < T\}} \phi(x\, \mathrm{e}^{\sigma Z_T}) \, \mathrm{e}^{m Z_T} \right). \tag{2}$$

For the sake of clearness, we introduce the following notation

Definition 6 (the star notation)**.** *For any function f, we define*

$$f^\star(t) = e^{(r + \frac{1}{2}m^2)t} f(t). \tag{3}$$

From (2), we define the price of a Parisian Down and In call.

Definition 7 (Parisian Down and In call)**.** *Let $PDIC(x, T; K, L; r, \delta)$ denote the value of a Parisian Down and In call. Then,*

$$PDIC(x, T; K, L; r, \delta) = \mathrm{e}^{-(r + \frac{1}{2}m^2)T} \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{T_b^- < T\}} (x\, \mathrm{e}^{\sigma Z_T} - K)^+ \, \mathrm{e}^{m Z_T}).$$

Using notation (3), we obtain

$$PDIC^\star(x, T; K, L; r, \delta) = \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{T_b^- < T\}} (x\, \mathrm{e}^{\sigma Z_T} - K)^+ \, \mathrm{e}^{m Z_T}).$$

2.2.2. *Parisian Down and Out options.* A Down and Out Parisian option becomes worthless if S reaches L and remains constantly below level L for a time interval longer than D before maturity time T. The price of a Down and Out option at time 0 with payoff $\phi(S_T)$ is given by

$$\mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{Q}}\left(\phi(S_T)\mathbf{1}_{\{T_b^->T\}}\right) = \mathrm{e}^{-(r+\frac{m^2}{2})T}\,\mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{T_b^->T\}}\phi(x\,\mathrm{e}^{\sigma Z_T})\,\mathrm{e}^{mZ_T}\right). \tag{4}$$

From (4), we define the price of a Parisian Down and Out call.

Definition 8 (Parisian Down and Out call)**.** *Let $PDOC(x,T;K,L;r,\delta)$ denote the value of a Parisian Down and Out call. Then,*

$$PDOC(x,T;K,L;r,\delta) = \mathrm{e}^{-(r+\frac{1}{2}m^2)T}\,\mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{T_b^->T\}}(x\,\mathrm{e}^{\sigma Z_T}-K)^+\,\mathrm{e}^{mZ_T}).$$

Using notation (3), we obtain

$$PDOC^\star(x,T;K,L;r,\delta) = \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{T_b^->T\}}(x\,\mathrm{e}^{\sigma Z_T}-K)^+\,\mathrm{e}^{mZ_T}).$$

3. Relationship between prices

Parisian option prices cannot be computed directly. We are only able to give closed formulae for their Laplace transforms w.r.t. the maturity time T. As we have seen it in the above definitions, Parisian option prices depend on many parameters. The computation of the Laplace transform of one option price (say PDOC) w.r.t T requires to distinguish several cases, depending on the relative positions of x, L and K. The sign of b $(=\frac{1}{\sigma}\log(\frac{L}{x}))$ plays an important role. In Section 3.2, we explain why computing the value of $\widehat{PDOC}^\star$ when $b>0$ can be reduced to computing the value of $\widehat{PDOC}^\star$ with $b=0$. As we will see it in Section 3.1, there also exists an In and Out parity relationship between the prices. This means that we can deduce the value of $PDOC^\star$ from the value of $PDIC^\star$. The following scheme explains how to deduce the Laplace transforms of the different Parisian call prices one from the others. Moreover, in Section 3.3, we state a call put parity relationship, which enables to deduce the Parisian put prices from the corresponding call prices through the Black Scholes formula.

3.1. **In and Out parity.** This part is devoted to make precise the way we compute the value of $\widehat{PDOC}^\star$ from the value of $\widehat{PDIC}^\star$. The technique developed below remains valid for Parisian Up calls. We recall Definitions 7 and 8,

$$PDIC^\star(x,T;K,L;r,\delta) = \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{T_b^-<T\}}(x\,\mathrm{e}^{\sigma Z_T}-K)^+\,\mathrm{e}^{mZ_T}).$$
$$PDOC^\star(x,T;K,L;r,\delta) = \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{T_b^->T\}}(x\,\mathrm{e}^{\sigma Z_T}-K)^+\,\mathrm{e}^{mZ_T}).$$

By summing the two previous equalities, we get

$$PDIC^\star(x,T;K,L;r,\delta) + PDOC^\star(x,T;K,L;r,\delta) = \mathbb{E}_{\mathbb{P}}((x\,\mathrm{e}^{\sigma Z_T}-K)^+\,\mathrm{e}^{mZ_T}). \tag{5}$$

Definition 9. *Let us define*

$$BSC^\star(x,T;K;r,\delta) = \mathbb{E}_{\mathbb{P}}((x\,\mathrm{e}^{\sigma Z_T}-K)^+\,\mathrm{e}^{mZ_T}).$$

BSC is the price of a Black Scholes call option.

From (5), we get

$$\widehat{PDOC}^\star(x,\lambda;K,L;r,\delta) = \widehat{BSC}^\star(x,\lambda;K;r,\delta) - \widehat{PDIC}^\star(x,\lambda;K,L;r,\delta).$$

Then, if we manage to get closed formulae for both $\widehat{PDIC}^\star$ and $\widehat{BSC}^\star$, we can easily deduce a closed formula for $\widehat{PDOC}^\star$. Since the pricing of a Parisian option can only be achieved through the numerical inversion of its Laplace transform, it makes sense to

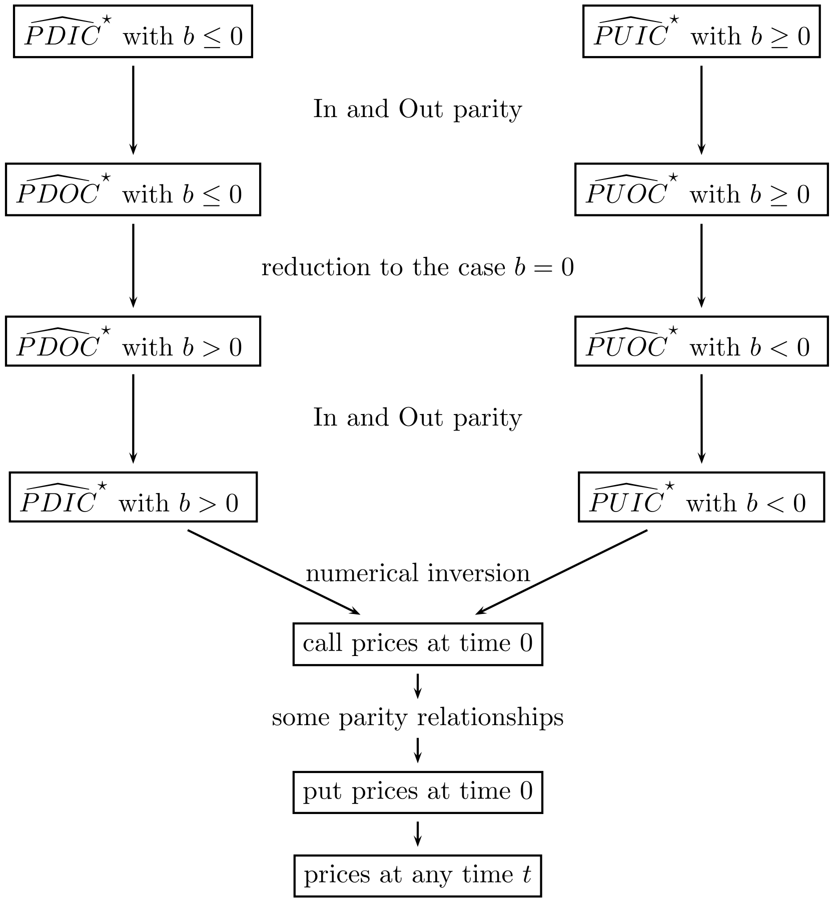

FIGURE 2. Computation scheme of Parisian option prices

compute the Laplace transform of BSC — even though it can also be accessed through the Black Scholes formula (see Black and Scholes (1973)) — to be able to implement the different parity relationships straightaway.

The following proposition gives the value of $\widehat{BSC}^{\star}(x, \lambda; K; r, \delta)$

Proposition 3. *For $K \geq x$,*

$$\widehat{BSC}^{\star}(x, \lambda; K; r, \delta) = \frac{K}{\theta} \, \mathrm{e}^{(m-\theta)k} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right).$$

For $K \leq x$,

$$\widehat{BSC}^{\star}(x, \lambda; K; r, \delta) = \frac{2K}{m^2 - \theta^2} - \frac{2x}{(m+\sigma)^2 - \theta^2} + \frac{K \, \mathrm{e}^{(m+\theta)k}}{\theta} \left(\frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta} \right).$$

k is defined in Definition 4.

Proof. From Definition 9

$$BSC^{\star}(x, T; K; r, \delta) = \int_{-\infty}^{+\infty} \mathrm{e}^{mz} (x \, \mathrm{e}^{\sigma z} - K)^{+} \frac{1}{\sqrt{2\pi T}} \, \mathrm{e}^{-\frac{z^2}{2T}} \, dz.$$

Then,

$$\widehat{BSC}^{\star}(x,\lambda;K;r,\delta) = \int_{-\infty}^{+\infty} \mathrm{e}^{mz}(x\,\mathrm{e}^{\sigma z} - K)^+ \int_0^{+\infty} \frac{\mathrm{e}^{-\lambda t}}{\sqrt{2\pi t}}\,\mathrm{e}^{-\frac{z^2}{2t}}\,dt\,dz. \tag{6}$$

The computation of the second integral on the right hand side is given in Appendix B. Combining (20) and (6), we find

$$\widehat{BSC}^{\star}(x,\lambda;K;r,\delta) \;=\; \int_k^{+\infty} \mathrm{e}^{mz}(x\,\mathrm{e}^{\sigma z} - K)\frac{\mathrm{e}^{-|z|\theta}}{\theta}dz. \tag{7}$$

- In the case $K \geq x$, $k \geq 0$ and the result easily follows.
- In the case $K \leq x$, we split the integral in (7) into two parts

$$\widehat{BSC}^{\star}(x,\lambda;K;r,\delta) \;=\; \int_k^0 \mathrm{e}^{mz}(x\,\mathrm{e}^{\sigma z} - K)\frac{\mathrm{e}^{z\theta}}{\theta}dz + \int_0^{\infty} \mathrm{e}^{mz}(x\,\mathrm{e}^{\sigma z} - K)\frac{\mathrm{e}^{-z\theta}}{\theta}dz,$$

  and an easy computation yields the result.

■

3.2. **Reduction to the case** $b = 0$**.** Assume that we know the value of $\widehat{PDOC}^{\star}$ with $b \leq 0$. This section aims at proving that computing $\widehat{PDOC}^{\star}$ with $b > 0$ boils down to computing the value of $\widehat{PDOC}^{\star}$ with $b = 0$, as suggested in Figure 2. First, we state a Proposition which links $PDOC^{\star}$ with $b > 0$ to $PDOC^{\star}$ with $b = 0$.

Proposition 4. *The price of a Parisian Down and Out call in the case $b > 0$ is given by*

$$PDOC^{\star}(x,T;K,L;r,\delta) = L\,\mathrm{e}^{mb}\int_0^D PDOC^{\star,0}(T-u;K/L;r,\delta)\mu_b(du)$$

where $\mu_b(du)$ is the law of T_b and

$$PDOC^{\star,0}(T;K;r,\delta) = \mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{T_0^- \geq T\}}(\mathrm{e}^{\sigma Z_T} - K)^+\,\mathrm{e}^{mZ_T}\right).$$

Remark 3. *Note that $PDOC^{\star,0}(T;K;r,\delta) = PDOC(1,T;K,1;r,\delta)$.*

Proof. First, we recall the value of $PDOC^{\star}$

$$PDOC^{\star}(x,T;K,L;r,\delta) = \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{T_b^- > T\}}(x\,\mathrm{e}^{\sigma Z_T} - K)^+\,\mathrm{e}^{mZ_T}).$$

Since Z starts from 0 and b is positive, $T_b < D$ on the set $\{T_b^- \geq T\}$. In fact, if T_b were strictly greater than D, it would mean that Z would not have crossed b before D and then T_b^- would be equal to D, which is impossible since we are on the set $\{T_b^- \geq T\}$, and $T > D$. Therefore, we can write

$$PDOC^{\star}(x,T;K,L;r,\delta) = \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{T_b^- \geq T\}}\mathbf{1}_{\{T_b \leq D\}}(x\,\mathrm{e}^{\sigma Z_T} - K)^+\,\mathrm{e}^{mZ_T}).$$

Introducing Z_{T_b}, we can also write

$$PDOC^{\star}(x,T;K,L;r,\delta) =$$
$$\mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{T_b \leq D\}}\mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{T_b^- - T_b \geq T - T_b\}}(x\,\mathrm{e}^{\sigma Z_T - Z_{T_b} + b} - K)^+\,\mathrm{e}^{m(Z_T - Z_{T_b} + b)} \mid \mathcal{F}_{T_b}]\right).$$

To compute the inner expectation in the previous formula, we rely on the strong Markov property. Let $B = \{B_t = Z_{T_b+t} - Z_{T_b}, t \geq 0\}$. B is independent of $\mathcal{F}_{T_b}$ and one can easily prove that $T_b^-(Z) - T_b(Z) = T_0^-(B)$ a.s. on the set $\{T_b^- \geq T\}$.
Hence, we find

$$PDOC^{\star}(x,T;K,L;r,\delta) = \mathbb{E}[\mathbf{1}_{\{T_b \leq D\}}\mathbb{E}[\mathbf{1}_{\{T_0^- \geq T-t\}}(x\,\mathrm{e}^{\sigma(B_{T-t}+b)} - K)^+\,\mathrm{e}^{m(B_{T-t}+b)}]_{|t=T_b}].$$

We get

$$PDOC^\star(x,T;K,L;r,\delta) = \int_0^D \mathbb{E}_\mathbb{P}\left(\mathbf{1}_{\{T_0^- \geq T-u\}}(x\,\mathrm{e}^{\sigma(B_{T-u}+b)} - K)^+\,\mathrm{e}^{m(B_{T-u}+b)}\right)\mu_b(du),$$

where $\mu_b(du)$ is the law of T_b. As $b = \frac{1}{\sigma}\ln\left(\frac{L}{x}\right)$, we get

$$PDOC^\star(x,T;K,L;r,\delta) = L\,\mathrm{e}^{mb}\int_0^D \mathbb{E}_\mathbb{P}\left(\mathbf{1}_{\{T_0^- \geq T-u\}}(\mathrm{e}^{\sigma B_{T-u}} - K/L)^+\,\mathrm{e}^{mB_{T-u}}\right)\mu_b(du),$$

and the result follows. ∎

By using Proposition 4, we can state the following formula for the Laplace transform of $PDOC^\star(x,T;K,L;r,\delta)$.

Proposition 5. *The Laplace transform of $PDOC^\star$ when $b > 0$ is given by*

$$\widehat{PDOC}^\star(x,\lambda;K,L;r,\delta) = L\,\mathrm{e}^{mb}\,\widehat{PDOC}^{\star,0}(\lambda;K/L;r,\delta)\int_0^D \mathrm{e}^{-\lambda u}\,\mu_b(du),$$

where

$$\int_0^D \mathrm{e}^{-\lambda u}\,\mu_b(du) = \mathrm{e}^{-\theta b}\,\mathcal{N}\left(\theta\sqrt{D} - \frac{b}{\sqrt{D}}\right) + \mathrm{e}^{\theta b}\,\mathcal{N}\left(-\theta\sqrt{D} - \frac{b}{\sqrt{D}}\right).$$

Proof. From Proposition 4, we have

$$PDOC^\star(x,T;K,L;r,\delta) \quad = \quad \mathrm{e}^{-\lambda T}\;L\,\mathrm{e}^{mb}\int_0^D PDOC^{\star,0}(T-u;K/L;r,\delta)\mu_b(du)\mathbf{1}_{\{T>D\}}.$$

Using Proposition 2, it is quite easy to show that

$$\widehat{PDOC}^\star(x,\lambda;K,L;r,\delta) \quad = \quad L\,\mathrm{e}^{mb}\int_0^D \mu_b(du)\,\mathrm{e}^{-\lambda u}\;\widehat{PDOC}^{\star,0}(\lambda;K/L;r,\delta).$$

We refer the reader to Appendix A for the computation of $\int_0^D \mu_b(du)\,\mathrm{e}^{-\lambda u}$. ∎

3.3. **Call put parity.** In this part, we explain how to deduce the put prices from the call prices using a parity relationship.

Proposition 6. *The following relationships hold*

$$PDOP(x,T;K,L;r,\delta) = xK\;PUOC\left(\frac{1}{x},T;\frac{1}{K},\frac{1}{L};\delta,r\right),$$

$$PUOP(x,T;K,L;r,\delta) = xK\;PDOC\left(\frac{1}{x},T;\frac{1}{K},\frac{1}{L};\delta,r\right),$$

$$PUIP(x,T;K,L;r,\delta) = xK\;PDIC\left(\frac{1}{x},T;\frac{1}{K},\frac{1}{L};\delta,r\right),$$

$$PDIP(x,T;K,L;r,\delta) = xK\;PUIC\left(\frac{1}{x},T;\frac{1}{K},\frac{1}{L};\delta,r\right).$$

Proof. Let us consider a Parisian Down and Out put

$$PDOP(x,T;K,L;r,\delta) = \mathbb{E}\left(\mathrm{e}^{mZ_T}(K - x\,\mathrm{e}^{\sigma Z_T})^+\,\mathbf{1}_{\{T_b^- > T\}}\right)\mathrm{e}^{-\left(r+\frac{m^2}{2}\right)T}.$$

One notices that the first time the Brownian motion Z makes an excursion below b longer than D is equal to the first time the Brownian motion $-Z$ makes above $-b$ an excursion longer than D. Therefore, by introducing the new Brownian motion $W = -Z$, we can rewrite

$$\begin{aligned}
PDOP(x,T;K,L;r,\delta) &= \mathbb{E}\left(\mathrm{e}^{-mW_T}(K - x\,\mathrm{e}^{-\sigma W_T})^+\,\mathbf{1}_{\{T^+_{-b}>T\}}\right)\mathrm{e}^{-\left(r+\frac{m^2}{2}\right)T},\\
&= xK\,\mathbb{E}\left(\mathrm{e}^{-(m+\sigma)W_T}\left(\frac{1}{x}\,\mathrm{e}^{\sigma W_T}-\frac{1}{K}\right)^+\,\mathbf{1}_{\{T^+_{-b}>T\}}\right)\mathrm{e}^{-\left(r+\frac{m^2}{2}\right)T}.
\end{aligned}$$

Let us introduce $m' = -(m+\sigma)$, $\delta' = r$, $r' = \delta$ and $b' = -b$. With these relations, we can easily check that $m' = \frac{1}{\sigma}\left(r' - \delta' - \frac{\sigma^2}{2}\right)$ and that $r' + \frac{m'^2}{2} = r + \frac{m^2}{2}$. Moreover, we notice that the barrier L' corresponding to $b' = -b$ is $\frac{1}{L}$.

Therefore, $\mathbb{E}\left(\mathrm{e}^{-(m+\sigma)W_T}\left(\frac{\mathrm{e}^{\sigma W_T}}{x}-\frac{1}{K}\right)^+\,\mathbf{1}_{\{T^+_{-b}>T\}}\right)\mathrm{e}^{-\left(r+\frac{m^2}{2}\right)T}$ is in fact the price of an Up and Out call $PUOC\left(\frac{1}{x},T;\frac{1}{K},\frac{1}{L};\delta,r\right)$. Finally, we come up with the following relation

$$PDOP(x,T;K,L;r,\delta) = xK\ PUOC\left(\frac{1}{x},T;\frac{1}{K},\frac{1}{L};\delta,r\right).$$

The three other assertions in Proposition 6 can be proved in the same way. ■

4. Valuation of Parisian call options

Looking at Figure 2, we notice that we only need to compute $\widehat{PDIC}^{\star}$ with $b \le 0$ and $\widehat{PUIC}^{\star}$ with $b \ge 0$. With these values we can deduce the prices of all the other Parisian call options.

4.1. **The valuation of a Parisian Down and In call option with $b \le 0$.** Before computing the Laplace transform of $PDIC^\star$ in Section 4.1.2, we state some preliminary results in Section 4.1.1. We give a new expression for $PDIC^\star$ in Proposition 7 and we state in Lemma 1 a key result for the computation of $\widehat{PDIC}^{\star}$.

4.1.1. *Preliminary results.*

Proposition 7.

$$PDIC^\star(x,T;K,L;r,\delta) = \int_k^\infty e^{my}(xe^{\sigma y}-K)h_b^-(T,y)dy,$$

where

$$h_b^-(t,y) = \int_{-\infty}^b \frac{b-z}{D}\,\mathrm{e}^{-\frac{(z-b)^2}{2D}}\,\gamma^-(t,z-y)dz,$$

and

$$\gamma^-(t,x) = \mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{T_b^-<t\}}\frac{\mathrm{e}^{-\frac{x^2}{2(t-T_b^-)}}}{\sqrt{2\pi(t-T_b^-)}}\right).$$

Proof. Remember that the value of $PDIC^\star$ is given by

$$PDIC^\star(x,T;K,L;r,\delta) \;=\; \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{T_b^-<T\}}(x\,\mathrm{e}^{\sigma Z_T}-K)^+\,\mathrm{e}^{mZ_T}).$$

By conditioning with respect to $\mathcal{F}_{T_b^-}$, we can write

$$PDIC^\star(x,T;K,L;r,\delta) = \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{T_b^-<T\}}\mathbb{E}_{\mathbb{P}}[x\,\mathrm{e}^{\sigma(Z_T-Z_{T_b^-}+Z_{T_b^-})}-K)^+\,\mathrm{e}^{m(Z_T-Z_{T_b^-}+Z_{T_b^-})}|\mathcal{F}_{T_b^-}]).$$

First, we deal with the conditional expectation. Let $B_t = Z_{t+T_b^-} - Z_{T_b^-}$ for $t \geq 0$. B is independent of $\mathcal{F}_{T_b^-}$. So, we obtain

$$\mathbb{E}_{\mathbb{P}}\left[(x\,\mathrm{e}^{\sigma(Z_T - Z_{T_b^-} + Z_{T_b^-})} - K)^+\,\mathrm{e}^{m(Z_T - Z_{T_b^-} + Z_{T_b^-})}|\mathcal{F}_{T_b^-}\right] =$$
$$\mathbb{E}_{\mathbb{P}}\left[(x\,\mathrm{e}^{\sigma(B_{T-\tau}+z)} - K)^+\,\mathrm{e}^{m(B_{T-\tau}+z)}\right]_{|z=Z_{T_b^-},\,\tau=T_b^-},$$

and

$$\mathbb{E}_{\mathbb{P}}\left[(x\,\mathrm{e}^{\sigma(B_{T-\tau}+z)} - K)^+\,\mathrm{e}^{m(B_{T-\tau}+z)}\right] = \frac{1}{\sqrt{2\pi(T-\tau)}}\left(\int_{-\infty}^{\infty}\mathrm{e}^{mu}(x\,\mathrm{e}^{\sigma u} - K)^+\,\mathrm{e}^{-\frac{(u-z)^2}{2(T-\tau)}}\,du\right).$$

Hence, we get

$$PDIC^\star(x,T;K,L;r,\delta) = \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{T_b^- < T\}}\mathcal{P}_{T-T_b^-}(f_x)(Z_{T_b^-})),$$

with

$$f_x(z) = \mathrm{e}^{mz}(x\,\mathrm{e}^{\sigma z} - K)^+,$$

and

$$\mathcal{P}_t(f_x)(z) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} f_x(u)\exp\left(-\frac{(u-z)^2}{2t}\right)du.$$

As recalled by Chesney et al. (1997), the random variables $Z_{T_b^-}$ and T_b^- are independent. Denoting the law of $Z_{T_b^-}$ by $\nu^-(dz)$ leads to

$$\begin{aligned} PDIC^\star(x,T;K,L;r,\delta) &= \int_{-\infty}^{\infty}\mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{T_b^- < T\}}\mathcal{P}_{T-T_b^-}(f_x)(z))\nu^-(dz), \\ &= \int_{-\infty}^{\infty} f_x(y)h_b^-(T,y)dy, \end{aligned}$$

where

$$h_b^-(t,y) = \int_{-\infty}^{\infty}\mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{T_b^- < t\}}\frac{\exp\left(-\frac{(z-y)^2}{2(t-T_b^-)}\right)}{\sqrt{2\pi(t-T_b^-)}}\right)\nu^-(dz).$$

Using the expression of $\nu^-(dz)$ given in Appendix C, we know that

$$h_b^-(t,y) = \int_{-\infty}^{b}\frac{b-z}{D}\,\mathrm{e}^{-\frac{(z-b)^2}{2D}}\,\mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{T_b^- < t\}}\frac{\mathrm{e}^{-\frac{(z-y)^2}{2(t-T_b^-)}}}{\sqrt{2\pi(t-T_b^-)}}\right)dz,$$

and the result follows. ■

Definition 10. *Let $\psi : \mathbb{R} \to \mathbb{R}$ denote*

$$\psi(z) \triangleq \int_0^{+\infty} xe^{-\frac{x^2}{2}+zx}dx = 1 + z\sqrt{2\pi}e^{\frac{z^2}{2}}\mathcal{N}(z).$$

Remark 4. *For the numerical inversion of Laplace transforms, it is important to notice that ψ is analytic on the complex plane.*

We can easily prove the following Lemma.

Lemma 1. *Let $K_{\lambda,D} : \mathbb{R} \to \mathbb{R}$ be defined as*

$$K_{\lambda,D}(a) = \int_0^{+\infty} v\,\mathrm{e}^{-\frac{v^2}{2D} - |a-v|\theta}\,dv.$$

Then,

$$K_{\lambda,D}(a) = \begin{cases} e^{\theta a} D\psi(-\theta\sqrt{D}) & \text{if } a \le 0, \\ e^{-\theta a} D\psi(\theta\sqrt{D}) - D\theta\sqrt{2\pi D} e^{\lambda D} \Big\{ \mathcal{N}(\theta\sqrt{D} - \frac{a}{\sqrt{D}}) e^{-\theta a} + & \\ \qquad \mathcal{N}(-\theta\sqrt{D} - \frac{a}{\sqrt{D}}) e^{\theta a} \Big\} & \text{otherwise.} \end{cases}$$

4.1.2. *The Laplace transform of $PDIC^\star(x, T; K, L; r, \delta)$.*

Theorem 1. *The value of $\widehat{PDIC}^\star$ is given by the following formula*

$$\widehat{PDIC}^\star(x, \lambda; K, L; r, \delta) = \frac{\mathrm{e}^{\theta b}}{D\theta\psi(\theta\sqrt{D})} \int_k^\infty e^{my}(xe^{\sigma y} - K) K_{\lambda,D}(b - y) dy. \tag{8}$$

For any $\lambda > \frac{(m+\sigma)^2}{2}$ and for $K > L$, we get

$$\widehat{PDIC}^\star(x, \lambda; K, L; r, \delta) = \frac{\psi(-\theta\sqrt{D})\, \mathrm{e}^{2b\theta}}{\theta\psi(\theta\sqrt{D})} K\, \mathrm{e}^{(m-\theta)k} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right), \tag{9}$$

and for $K \le L$

$$\begin{aligned} \widehat{PDIC}^\star(x, \lambda; K, L; r, \delta) = \frac{\mathrm{e}^{(m+\theta)b}}{\psi(\theta\sqrt{D})} \Bigg(\frac{2K}{m^2 - \theta^2} \left[\psi(m\sqrt{D}) - m\sqrt{2\pi D}\, \mathrm{e}^{\frac{Dm^2}{2}} \mathcal{N}(m\sqrt{D} + d) \right] \\ - \frac{2L}{(m+\sigma)^2 - \theta^2} \left[\psi((m+\sigma)\sqrt{D}) - (m+\sigma)\sqrt{2\pi D}\, \mathrm{e}^{\frac{D}{2}(m+\sigma)^2} \mathcal{N}\left((m+\sigma)\sqrt{D} + d \right) \right] \Bigg) \\ + \frac{K\, \mathrm{e}^{(m+\theta)k}}{\theta\psi(\theta\sqrt{D})} \left(\frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta} \right) \left[\psi(\theta\sqrt{D}) - \theta\sqrt{2\pi D}\, \mathrm{e}^{\lambda D} \mathcal{N}(\theta\sqrt{D} - d) \right] \\ + \frac{\mathrm{e}^{\lambda D} \sqrt{2\pi D}}{\psi(\theta\sqrt{D})} K\, \mathrm{e}^{2b\theta}\, \mathrm{e}^{(m-\theta)k} \mathcal{N}(-d - \theta\sqrt{D}) \left(\frac{1}{m+\sigma-\theta} - \frac{1}{m-\theta} \right). \end{aligned} \tag{10}$$

Proof. (9) and (10) easily follow from (8):

- if $K > L$, $b - y < 0$ $\forall y \in [k, \infty]$. Then, using Lemma 1 and (8) yields

$$\widehat{PDIC}^\star(x, \lambda; K, L; r, \delta) = \frac{\psi(-\theta\sqrt{D})\, \mathrm{e}^{2b\theta}}{\theta\psi(\theta\sqrt{D})} \int_k^\infty \mathrm{e}^{(m-\theta)y} (x\, \mathrm{e}^{\sigma y} - K) dy,$$

  and the result easily follows.
- if $K < L$, $b - y$ is positive on $[k, b]$ and negative on $[b, \infty]$. We have to split the integral

$$I \triangleq \int_k^{+\infty} e^{my}(xe^{\sigma y} - K) K_{\lambda,D}(b - y) dy.$$

  appearing in (8).

$$I = \int_k^b e^{my}(xe^{\sigma y} - K) K_{\lambda,D}(b - y) dy + \int_b^{+\infty} e^{my}(xe^{\sigma y} - K) K_{\lambda,D}(b - y) dy \triangleq I_1 + I_2.$$

$$I_1 = D\psi(-\theta\sqrt{D}) e^{\theta b} \int_b^{+\infty} e^{my}(xe^{\sigma y} - K) e^{-\theta y} = D\psi(-\theta\sqrt{D}) e^{mb} \left(\frac{K}{m-\theta} - \frac{L}{m+\sigma-\theta} \right).$$

The integral I_2 can be split into three terms

$$\begin{aligned}I_{21} &= D\psi(\theta\sqrt{D})\int_k^b e^{my}(xe^{\sigma y}-K)e^{\theta(y-b)}dy,\\ I_{22} &= -D\theta\sqrt{2\pi D}e^{\lambda D}\int_k^b e^{my}(xe^{\sigma y}-K)e^{\theta(y-b)}\mathcal{N}(\theta\sqrt{D}-\frac{b-y}{\sqrt{D}})dy,\\ I_{23} &= -D\theta\sqrt{2\pi D}e^{\lambda D}\int_k^b e^{my}(xe^{\sigma y}-K)e^{\theta(b-y)}\mathcal{N}(-\theta\sqrt{D}-\frac{b-y}{\sqrt{D}})dy.\end{aligned}$$

An easy computation leads to

$$I_{21}=D\psi(\theta\sqrt{D})e^{-\theta b}\left\{e^{(m+\theta)b}\left[\frac{L}{m+\sigma+\theta}-\frac{K}{m+\theta}\right]+Ke^{(m+\theta)k}\left[\frac{1}{m+\theta}-\frac{1}{m+\sigma+\theta}\right]\right\}.$$

I_{22} and I_{23} are computed in the following way: we change variables (we introduce $v=\theta\sqrt{D}-\frac{b-y}{\sqrt{D}}$ (for the valuation of I_{22})) and we use the following equality $\int_{a_1}^{a_2}\mathcal{N}(v)e^{bv}dv=\frac{1}{b}[\mathcal{N}(a_2)e^{a_2 b}-\mathcal{N}(a_1)e^{a_1 b}-e^{\frac{b^2}{2}}(\mathcal{N}(a_2-b)-\mathcal{N}(a_1-b))]$, for $a_1,a_2,b\in\mathbb{R}$, $b\neq 0$ and $a_1\leq a_2$.

We refer to Proposition 11, to prove that λ must be greater than $\frac{(m+\sigma)^2}{2}$. Let us prove (8). Using Proposition 7, we get

$$\widehat{PDIC}^{\star}(x,\lambda;K,L;r,\delta)=\int_k^\infty e^{my}(xe^{\sigma y}-K)\int_0^\infty \mathrm{e}^{-\lambda t}\,h_b^-(t,y)dtdy.$$

We would like to compute $\widehat{h_b^-}(\lambda,y)=\int_0^\infty \mathrm{e}^{-\lambda t}\,h_b^-(t,y)dt$. Using the definition of $h_b^-(t,y)$ in Proposition 7 yields

$$\widehat{h_b^-}(\lambda,y)=\int_{-\infty}^b \frac{b-z}{D}\,\mathrm{e}^{-\frac{(z-b)^2}{2D}}\int_0^\infty \mathrm{e}^{-\lambda t}\,\gamma^-(t,z-y)dt\,dz. \tag{11}$$

So, we need to compute the Laplace transform of $\gamma^-(t,x)$.

$$\int_0^\infty \mathrm{e}^{-\lambda t}\,\gamma^-(t,x)dt=\mathbb{E}_{\mathbb{P}}\left(\int_{T_b^-}^\infty \mathrm{e}^{-\lambda t}\,\frac{\mathrm{e}^{-\frac{x^2}{2(t-T_b^-)}}}{\sqrt{2\pi(t-T_b^-)}}dt\right).$$

The change of variables $u=t-T_b^-$ gives

$$\int_0^\infty \mathrm{e}^{-\lambda t}\,\gamma^-(t,x)dt=\mathbb{E}_{\mathbb{P}}(\mathrm{e}^{-\lambda T_b^-})\int_0^\infty \mathrm{e}^{-\lambda u}\,\frac{\mathrm{e}^{-\frac{x^2}{2u}}}{\sqrt{2\pi u}}du.$$

Using results from Appendices A and B, we get

$$\int_0^\infty \mathrm{e}^{-\lambda t}\,\gamma^-(t,x)dt=\frac{\mathrm{e}^{-(|x|-b)\theta}}{\theta\psi(\theta\sqrt{D})}.$$

Thanks to (11), we can rewrite

$$\widehat{h_b^-}(\lambda,y)=\frac{\mathrm{e}^{b\theta}}{D\theta\psi(\theta\sqrt{D})}\int_{-\infty}^b (b-z)\,\mathrm{e}^{-\frac{(z-b)^2}{2D}-|z-y|\theta}\,dz.$$

By changing variables $v=b-z$, we obtain

$$\widehat{h_b^-}(\lambda,y)=\frac{\mathrm{e}^{b\theta}}{D\theta\psi(\theta\sqrt{D})}\int_0^\infty v\,\mathrm{e}^{-\frac{v^2}{2D}-|b-v-y|\theta}\,dv,$$

and (8) follows. ∎

5. The Parisian Up calls

This section is devoted to the computation of the Laplace transforms of the Parisian Up call prices. We will go exactly through the same points as in the previous section but dealing with an Up and In call with $b \geq 0$ instead of a Down and In call with $b \leq 0$.

5.1. The valuation of a Parisian Up and In call with $b \geq 0$.

The owner of an Up and In option receives the payoff if S makes an excursion above the level L longer than D before the maturity time T, which is exactly the same as saying that the Brownian motion Z makes an excursion above b longer than D. Using the previous notations, the price of a Parisian Up and In call is given by

$$PUIC^\star(x,T;K,L;r,\delta) = \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\{T_b^+<T\}}(x\,\mathrm{e}^{\sigma Z_T} - K)^+\,\mathrm{e}^{mZ_T}),$$

where T_b^+ is defined by (1). The valuation of $\widehat{PUIC}^\star$ in the case $b \geq 0$ is similar to the valuation of $\widehat{PDIC}^\star$ in the case $b \leq 0$ (see previous Section). Before computing the Laplace transform of $PUIC^\star$ in Theorem 2, we give a new expression for $PUIC^\star$.

Proposition 8.

$$PUIC^\star(x,T;K,L;r,\delta) = \int_k^\infty e^{my}(xe^{\sigma y} - K)h_b^+(T,y)dy,$$

where

$$h_b^+(t,y) = \int_b^\infty \frac{z-b}{D}\,\mathrm{e}^{-\frac{(z-b)^2}{2D}}\,\gamma^+(t,z-y)dz,$$

and

$$\gamma^+(t,x) = \mathbb{E}_{\mathbb{P}}\left(\mathbf{1}_{\{T_b^+<t\}}\frac{\mathrm{e}^{-\frac{x^2}{2(t-T_b^+)}}}{\sqrt{2\pi(t-T_b^+)}}\right).$$

The proof of Proposition 8 is the same as the proof of Proposition 7. We only need to replace T_b^- by T_b^+.

Theorem 2. *The value of $\widehat{PUIC}^\star$ is given by the following formula*

$$\widehat{PUIC}^\star(x,\lambda;K,L;r,\delta) = \frac{\mathrm{e}^{-\theta b}}{D\theta\psi(\theta\sqrt{D})}\int_k^\infty e^{my}(xe^{\sigma y} - K)K_{\lambda,D}(y-b)dy.$$

For any $\lambda > \frac{(m+\sigma)^2}{2}$, we get for $K > L$

$$\begin{aligned}\widehat{PUIC}^\star(x,\lambda;K,L;r,\delta) = {} & 2\,\mathrm{e}^{(m-\theta)b}\,\frac{\sqrt{2\pi D}}{\psi(\theta\sqrt{D})}\left[\frac{K}{m^2-\theta^2}\,\mathrm{e}^{\frac{Dm^2}{2}}\,m\mathcal{N}(m\sqrt{D}+d)\right. \\ & \left. - \frac{L}{(m+\sigma)^2-\theta^2}\,\mathrm{e}^{\frac{D(m+\sigma)^2}{2}}(m+\sigma)\mathcal{N}((m+\sigma)\sqrt{D}+d)\right] \\ & + \frac{\mathrm{e}^{-2b\theta}}{\psi(\theta\sqrt{D})}K\,\mathrm{e}^{(m+\theta)k}\,\mathrm{e}^{\lambda D}\sqrt{2\pi D}\mathcal{N}(d-\theta\sqrt{D})\left(\frac{1}{m+\sigma+\theta} - \frac{1}{m+\theta}\right) \\ & + \frac{\mathrm{e}^{(m-\theta)k}}{\theta\psi(\theta\sqrt{D})}K\left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta}\right)\left(\psi(\theta\sqrt{D}) - \theta\sqrt{2\pi D}\,\mathrm{e}^{\lambda D}\mathcal{N}(d+\theta\sqrt{D})\right).\end{aligned}$$

and for $K \leq L$

$$\widehat{PUIC}^{\star}(x,\lambda;K,L;r,\delta) = \frac{2\,\mathrm{e}^{(m-\theta)b}}{\psi(\theta\sqrt{D})}\left[\frac{K}{m^2-\theta^2}\psi(m\sqrt{D}) - \frac{L}{(m+\sigma)^2-\theta^2}\psi((m+\sigma)\sqrt{D})\right] + \frac{\mathrm{e}^{-2b\theta}\,\psi(-\theta\sqrt{D})}{\theta\psi(\theta\sqrt{D})}Ke^{(m+\theta)k}\left(\frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma}\right).$$

Even if the computations involved in the proof of Theorem 2 are different from the one of Theorem 1, we dare omit the proof here as the scheme of the proof of Theorem 1 applies to the case of Up and In call.

6. Prices at any time t

So far, we have explained how to compute the prices at time 0 of the Laplace transforms of the different Parisian option prices w.r.t. maturity time. From a practical point of view, the real stake is to be able to hedge these options. This requires to compute the option prices at any given time t between 0 and the maturity time T. In this part, we explain how to deduce the prices at any time $t > 0$ from the prices at time 0.
In the following, we have chosen to focus on the Down and In call but the formula we obtain can easily be extended to the other options by means of parity relationships. We assume in the following computations that the relevant excursion has not occurred yet, otherwise the Parisian option has turned into the corresponding vanilla option and its price at time t is of common knowledge.

6.1. **Down and In call.** We introduce the r.v. D_t to count the time already spent in the excursion below b straddling time t

$$D_t = \begin{cases} t - g_t^b & \text{if } S_t \leq b, \\ 0 & \text{if } S_t > b. \end{cases}$$

Note that D_t is $\mathcal{F}_t$-measurable.
Let $PDIC(t, D_t, S_t, T; K, L; r, \delta)$ be the price of a Down and In call at time t. We know that

$$PDIC(t, D_t, S_t, T; K, L; r, \delta) = \mathrm{e}^{-r(T-t)}\,\mathbb{E}_{\mathbb{Q}}\left((x\,\mathrm{e}^{\sigma(W_T+mT)} - K)^+\,\mathbf{1}_{\{T_b^- \leq T\}}|\mathcal{F}_t\right). \qquad (12)$$

Proposition 9. *On the set $\{T_b^- > t\}$,*

$$\begin{aligned} &PDIC(t, D_t, S_t, T; K, L; r, \delta) \\ &= \mathrm{e}^{-\left(r+\frac{m^2}{2}\right)T'}\Bigg\{\mathbf{1}_{\{S_t>L\}}\mathbb{E}\left(\mathrm{e}^{mZ'_{T'}}(x\,\mathrm{e}^{\sigma Z'_{T'}} - K)^+\,\mathbf{1}_{\{T'^-_{b'} \leq T'\}}\right)_{|x=S_t} \\ &\quad + \mathbf{1}_{\{S_t\leq L\}}\mathbf{1}_{\{D-D_t\leq T'\}}\mathbb{E}\left(\mathrm{e}^{mZ'_{T'}}(x\,\mathrm{e}^{\sigma Z'_{T'}} - K)^+\,\mathbf{1}_{\{T'_{b'}\geq D-d\}}\right)_{|x=S_t,d=D_t} \\ &\quad + \mathbf{1}_{\{S_t\leq L\}}\mathbb{E}\left(\mathrm{e}^{mZ'_{T'}}(x\,\mathrm{e}^{\sigma Z'_{T'}} - K)^+\,\mathbf{1}_{\{T'_{b'}\leq D-d\}}\mathbf{1}_{\{T'^-_{b'}\leq T'\}}\right)_{|x=S_t,d=D_t}\Bigg\}. \qquad (13) \end{aligned}$$

where Z' is a $\mathbb{P}-$Brownian motion independent of $\mathcal{F}_t$ and

$$T' = T - t, \quad b' = \frac{1}{\sigma}\ln\left(\frac{L}{S_t}\right), \quad T'^-_{b'} = T^-_{b'}(Z'), \quad T'_{b'} = T_{b'}(Z').$$

Proof. We can change the probability measure as we did at the beginning to make $Z = \{W_t + mt; t \geq 0\}$ a $\mathbb{P}-$Brownian motion ($\mathbb{P}$ is defined in Proposition 1). $\mathbb{E}$ denotes the expectation under $\mathbb{P}$. Then, by changing the probability in Equation (12) we can write

$$\begin{aligned}
&PDIC(t, D_t, S_t, T; K, L; r, \delta) \\
&\quad= e^{-r(T-t)} \frac{\mathbb{E}\left(e^{mZ_T - \frac{1}{2}m^2 T}(x\, e^{\sigma Z_T} - K)^+ \mathbf{1}_{\{T_b^- \leq T\}} | \mathcal{F}_t\right)}{e^{mZ_t - \frac{1}{2}m^2 t}}, \\
&\quad= e^{-r(T-t)} \frac{\mathbb{E}\left(e^{mZ_t}\, e^{m(Z_T - Z_t) - \frac{1}{2}m^2 T}(x\, e^{\sigma Z_T} - K)^+ \mathbf{1}_{\{T_b^- \leq T\}} | \mathcal{F}_t\right)}{e^{mZ_t - \frac{1}{2}m^2 t}}, \\
&\quad= e^{-(r + \frac{m^2}{2})(T-t)}\, \mathbb{E}\left(e^{m(Z_T - Z_t)}(S_t\, e^{\sigma(Z_T - Z_t)} - K)^+ \mathbf{1}_{\{T_b^- \leq T\}} | \mathcal{F}_t\right).
\end{aligned}$$

We introduce $Z'_s = Z_{t+s} - Z_t$ for all $s \geq 0$. Z' is a $\mathbb{P}-$Brownian motion independent of $\mathcal{F}_t$.

$$PDIC(t, D_t, S_t, T; K, L; r, \delta) = e^{-(r + \frac{m^2}{2})T'}\, \mathbb{E}\left(e^{mZ'_{T'}}(S_t\, e^{\sigma Z'_{T'}} - K)^+ \mathbf{1}_{\{T_b^- \leq T\}} | \mathcal{F}_t\right).$$

The indicator $\mathbf{1}_{\{T_b^- \leq T\}}$ can be split up in several parts describing the different possible evolutions of Z' (see Figure 3). Either Z_t is not smaller than b and a whole excursion must be completed before T', or Z is already in an excursion below b. In the latter case, there are two possibilities corresponding to the two curves in Figure 3: either the current excursion will last longer than D (green curve), or Z will cross b before $D - D_t$ (blue curve) and a new excursion has to completed before T'. Then, on the set $\{T_b^- > t\}$, the indicator $\mathbf{1}_{\{T_b^- \leq T\}}$ can be rewritten as follows

$$\mathbf{1}_{\{T_b^- \leq T\}} = \mathbf{1}_{\{Z_t > b\}} \mathbf{1}_{\{T_{b'}'^- \leq T'\}} + \mathbf{1}_{\{Z_t \leq b\}} \left(\mathbf{1}_{\{T'_{b'} \geq D - D_t\}} \mathbf{1}_{\{D - D_t \leq T'\}} + \mathbf{1}_{\{T'_{b'} < D - D_t\}} \mathbf{1}_{\{T_{b'}'^- \leq T'\}} \right).$$

To find Equation (13), it is sufficient to notice that both T'_b and $T_b'^-$ are independent of Z', whereas S_t and D_t are $\mathcal{F}_t-$measurable. ■

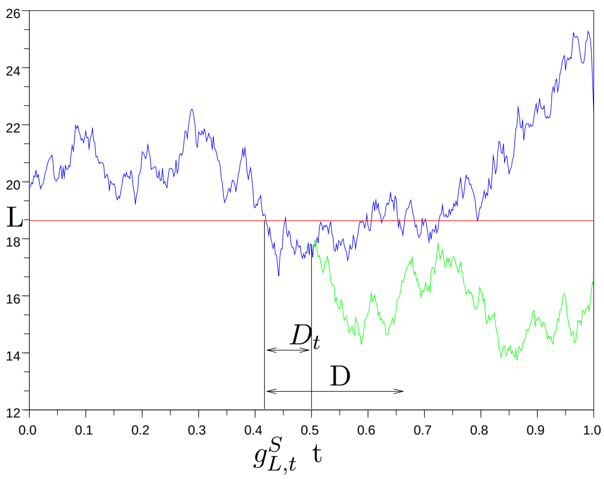

FIGURE 3. Possible evolutions of an asset price

In the sequel, we use the following decomposition based on Proposition 9

$$PDIC(t, D_t, S_t, T; K, L; r, \delta) \triangleq e^{-\left(r+\frac{m^2}{2}\right)T'} \{\mathbf{1}_{\{S_t>L\}} E_1(S_t, T') + \mathbf{1}_{\{S_t\leq L\}} E_2(S_t, D_t, T') + \mathbf{1}_{\{S_t\leq L\}} E_3(S_t, D_t, T')\}. \quad (14)$$

From Equation (13), we notice that E_1 is the star price of a Parisian Down and In call,

$$E_1(x, T') = PDIC^\star(x, T'; K, L; r, \delta). \quad (15)$$

Proposition 10. *On the set $\{T_b^- > t\}$, the price of a Down and In call at time t is given by*

$$PDIC^\star(t, D_t, S_t, T; K, L; r, \delta) = \mathbf{1}_{\{Z_t>b\}} PDIC^\star(S_t, T-t, K, L; r, \delta) + \mathbf{1}_{\{Z_t\leq b\}}(\mathbf{1}_{\{D-D_t\leq T-t\}} BSC^\star(S_t, T-t; K; r, \delta) + g(S_t, D_t, T-t)) \quad (16)$$

where the function g is characterised by its Laplace transform

$$\widehat{g}(S_t, D_t, \lambda) = e^{mb'} \int_0^{D-D_t} \mu_{b'} e^{-\lambda u}\, du \left(L\, \widehat{PDIC}^{\star,0}(\lambda; \frac{K}{L}; r, \delta) - \widehat{BSC}^\star(L, \lambda; K; r, \delta) \right).$$

Proof. Let us go back to Equation (14). E_1 is already known (see (15)) and gives the first term on the r.h.s of (16). First, we deal with E_2 and after with E_3.

Step 1 : Laplace transform of E_2.

$$\begin{aligned} E_2(x, d, t) &= \mathbb{E}\left(e^{mZ_t}(x\, e^{\sigma Z_t} - K)^+ \mathbf{1}_{\{T'_{b'}\geq D-d\}} \mathbf{1}_{\{D-d\leq t\}}\right) \\ &= \mathbf{1}_{\{D-d\leq t\}} BSC^\star(x, t; K; r, \delta) - \mathbf{1}_{\{D-d\leq t\}} \mathbb{E}\left(e^{mZ_t}(x\, e^{\sigma Z_t} - K)^+ \mathbf{1}_{\{T'_{b'}\leq D-d\}}\right) \\ &\triangleq E_{21}(x, d, t) - E_{22}(x, d, t). \end{aligned}$$

The term E_{21} corresponds to the first half of the second term on the r.h.s of (16). By conditioning w.r.t $\mathcal{F}_{T'_{b'}}$ and introducing $X_u = Z_{u+T'_{b'}} - b'$, which is a Brownian motion independent of $\mathcal{F}_{T'_{b'}}$, we get

$$\begin{aligned} E_{22}(x, d, t) &= \mathbf{1}_{\{D-d\leq t\}} \mathbb{E}\left(\mathbf{1}_{\{T'_{b'}\leq D-d\}} \mathbb{E}\left(e^{mX_{t-\tau}}\, e^{mb'}(x\, e^{+\sigma b'}\, e^{\sigma X_{t-\tau}} - K)^+\right) |_{\tau=T'_{b'}}\right), \\ &= \mathbf{1}_{\{D-d\leq t\}} \int_0^{D-d} e^{mb'}\, \mathbb{E}(e^{mX_{t-u}}(x\, e^{\sigma b'}\, e^{\sigma X_{t-u}} - K)^+) \mu_{b'}(u) du, \end{aligned}$$

where $\mu_{b'}$ is the density function of the hitting time $T'_{b'}$.
Using Proposition 2, it is quite easy to show that

$$\widehat{E_{22}}(S_t, D_t, \lambda) = e^{mb'}\, \widehat{BSC^\star}(L, \lambda; K; r, \delta) \int_0^{D-D_t} e^{-\lambda u}\, \mu_{b'}(u) du.$$

Step 2: Laplace transform of E_3. From Equation (14),

$$E_3(x, d, t) = \mathbb{E}\left(e^{mZ'_t}(x\, e^{\sigma Z'_t} - K)^+ \mathbf{1}_{\{T'_{b'}\leq D-d\}} \mathbf{1}_{\{T'^-_{b'}\leq t\}}\right)$$

To compute E_3, we condition w.r.t $\mathcal{F}_{T'_{b'}}$ and introduce $X_u = Z'_{u+T'_{b'}} - b'$. X is a Brownian motion independent of $\mathcal{F}_{T'_{b'}}$. Hence, we get

$$\begin{aligned} E_3(x, d, t) &= \mathbb{E}\left(\mathbb{E}\left(e^{mZ'_t}(x\, e^{\sigma Z'_t} - K)^+ \mathbf{1}_{\{T'_{b'}\leq D-d\}} \mathbf{1}_{\{T'^-_{b'}\leq t\}} | \mathcal{F}_{T'_{b'}}\right)\right), \\ &= e^{mb'}\, \mathbb{E}\left(\mathbf{1}_{\{T'_{b'}\leq D-d\}} \mathbb{E}\left(e^{mX_{t-T'_{b'}}}(x\, e^{\sigma b'}\, e^{\sigma X_{t-T'_{b'}}} - K)^+ \mathbf{1}_{\{T'^-_{b'}\leq t\}} | \mathcal{F}_{T'_{b'}}\right)\right). \end{aligned}$$

Moreover on the set $\{T'_{b'} \leq D-d\}$, $T'^{-}_{b'}(Z') = T'_{b'}(Z') + T_0^-(X)$ a.s.. Hence, we find

$$\begin{aligned} E_3(x,d,t) &= \mathrm{e}^{mb'}\,\mathbb{E}\left(\mathbf{1}_{\{T'_{b'}\leq D-d\}}\mathbb{E}\left(\mathrm{e}^{mX_{t-T'_{b'}}}(x\,\mathrm{e}^{\sigma b'}\,\mathrm{e}^{\sigma X_{t-T'_{b'}}} - K)^+\,\mathbf{1}_{\{T_0^- \leq t-T'_{b'}\}}|\mathcal{F}_{T'_{b'}}\right)\right) \\ &= \mathrm{e}^{mb'}\,\mathbb{E}\left(\mathbf{1}_{\{\tau\leq D-d\}}\mathbb{E}\left(\mathrm{e}^{mX_{t-\tau}}(x\,\mathrm{e}^{\sigma b'}\,\mathrm{e}^{\sigma X_{t-\tau}} - K)^+\,\mathbf{1}_{\{T_0^- \leq t-\tau\}}\right)_{|\tau=T'_{b'}}\right) \\ &= \mathrm{e}^{mb'}\int_0^{D-d}\mathbb{E}\left(\mathrm{e}^{mX_{t-\tau}}(x\,\mathrm{e}^{\sigma b'}\,\mathrm{e}^{\sigma X_{t-\tau}} - K)^+\,\mathbf{1}_{\{T_0^- \leq t-\tau\}}\right)\mu_{b'}(\tau)d\tau. \end{aligned}$$

Using Proposition 2, one can show that

$$\widehat{E_3}(x,d,\lambda) = x\,\mathrm{e}^{(m+\sigma)b'}\,\widehat{PDIC}^{\star,0}(\lambda; K\,\mathrm{e}^{-\sigma b'}/x; r,\delta)\int_0^{D-d}\mathrm{e}^{-\lambda\tau}\,\mu_{b'}(\tau)d\tau.$$

Finally, we get

$$\widehat{E_3}(S_t,D_t,\lambda) = L\,\mathrm{e}^{mb'}\,\widehat{PDIC}^{\star,0}(\lambda; K/L; r,\delta)\int_0^{D-D_t}\mathrm{e}^{-\lambda\tau}\,\mu_{b'}(\tau)d\tau.$$

Noticing that $E_3(S_t,D_t,\lambda) - E_{22}(S_t,D_t,\lambda) = \widehat{g}(S_t,D_t,\lambda)$ ends the proof. ■

6.2. **Other Parisian options.** The price at time t of an Up and In call can be computed by closely following what has been done for the Down and In call and it is sufficient to replace $PDIC$ by $PUIC$ in the above formula. All the other Parisian option prices can be deduced using either an In and Out parity or a call put parity relationship.

7. The inversion of Laplace transforms

This section is devoted to the numerical inversion of the Laplace transforms computed previously. We recall that the Laplace transforms are computed with respect to the maturity time. We explain how to recover a function from its Laplace transform using a contour integral. The real problem is how to numerically evaluate this complex integral. This is done in two separate steps involving two different approximations. First, as explained in Section 7.2 we replace the integral by a series. The first step creates a discretisation error, which is handled by Proposition 12. Secondly, one has to compute a non-finite series. This can be achieved by simply truncating the series but it leads to a tremendously slow convergence. Here, we prefer to use the Euler acceleration as presented in Section 7.3. Proposition 13 states an upper-bound for the error due to the accelerated computation of the non finite series. Theorem 1 gives a bound for the global error.

7.1. **Analytical prolongations.** Because the Laplace inversion is performed in the complex plane, we have to extend to the complex plane the expressions obtained for the Laplace transforms computed above. To do so, we introduce the analytic prolongation of the normal cumulative distribution function on the complex plane. From Proposition 11, it is quite easy to show that the expressions obtained for a real value of the Laplace parameter are still valid for a complex one with the function $\mathcal{N}$ defined by Lemma 2.

Proposition 11 (abscissa of convergence)**.** *The abscissa of convergence of the Laplace transforms of the star prices of Parisian options is smaller than* $\frac{(m+\sigma)^2}{2}$*. All these Laplace transforms are analytic on the complex half plane* $\{z \in \mathbb{C} : \mathcal{R}\mathrm{e}\,(z) > \frac{(m+\sigma)^2}{2}\}$.

Proof. It is sufficient to notice that the star price of a Parisian option is bounded by $\mathbb{E}(\mathrm{e}^{mZ_T}(x\,\mathrm{e}^{\sigma W_T} + K))$.

$$\mathbb{E}(\mathrm{e}^{mZ_T}(x\,\mathrm{e}^{\sigma W_T} + K)) \leq K\,\mathrm{e}^{\frac{m^2}{2}T} + x\,\mathrm{e}^{\frac{(m+\sigma)^2}{2}T} = \mathcal{O}(\mathrm{e}^{\frac{(m+\sigma)^2}{2}T}).$$

Hence, Widder (1941, Theorem 2.1) yields that the abscissa of convergence of the Laplace transforms of the star prices is smaller that $\frac{(m+\sigma)^2}{2}$. The second part of the proposition ensues from Widder (1941, Theorem 5.a). ■

Lemma 2 (Analytical prolongation of $\mathcal{N}$)**.** *The unique analytic prolongation of the normal cumulative distribution function on the complex plane is defined by*

$$\mathcal{N}(x+iy) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-\frac{(v+iy)^2}{2}} dv. \tag{17}$$

Proof. It is sufficient to notice that the function defined above is holomorphic on the complex plane (and hence analytic) and that it coincides with the normal cumulative distribution function on the real axis. ■

With the definition of $\mathcal{N}$ given by Equation (17), it is clear that all the expressions obtained so far for the Laplace transforms are also valid for complex values of λ satisfying $\mathcal{R}\mathrm{e}\,(\lambda) > \frac{(m+\sigma)^2}{2}$ since their are analytic on the complex half plane $\{z \in \mathbb{C} : \mathcal{R}\mathrm{e}\,(z) > \frac{(m+\sigma)^2}{2}\}$.

7.2. **The Fourier series representation.** Thanks to Widder (1941, Theorem 9.2), we know how to recover a function from its Laplace transform.

Theorem 3. *Let f be a continuous function defined on $\mathbb{R}^+$ and α a positive number. Assume that the function $f(t)\,\mathrm{e}^{-\alpha t}$ is integrable. Then, given the Laplace transform $\hat{f}$, f can be recovered from the contour integral*

$$f(t) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \mathrm{e}^{st}\, \hat{f}(s) ds, \quad t > 0. \tag{18}$$

The variable α has to be chosen greater than the abscissa of convergence of $\hat{f}$. In our case, α must be chosen strictly greater than $(m+\sigma)^2/2$.

For any real valued function satisfying the hypotheses of Theorem 3, we introduce a trapezoidal discretisation of Equation (18) of step π/t.

$$f_{\pi/t}(t) = \frac{e^{\alpha t}}{2t} \widehat{f}(\alpha) + \frac{e^{\alpha t}}{t} \sum_{k=1}^{\infty} (-1)^k \,\mathcal{R}\mathrm{e}\left(\widehat{f}\left(\alpha + i\frac{k\pi}{t} \right) \right). \tag{19}$$

Proposition 12. *If f is a continuous bounded function satisfying $f(t) = 0$ for $t < 0$, we have*

$$\left| e_{\pi/t}(t) \right| \triangleq \left| f(t) - f_{\pi/t}(t) \right| \leq \|f\|_\infty \, \frac{e^{-2\alpha t}}{1 - e^{-2\alpha t}}.$$

A proof of Proposition 12 can be found in Labart and Lelong (2008).

Remark 5. *For the upper bound in Proposition 12 to be smaller than $10^{-8}\,\|f\|_\infty$, one has to choose $2\alpha t = 18.4$. In fact, this bound holds for any choice of the discretisation step h satisfying $h < 2\pi/t$.*

Simply truncating the series in the definition of $f_{\pi/t}$ to compute the trapezoidal integral is far too rough to provide a fast and accurate numerical inversion. One way to improve the convergence of the series is to use the Euler summation.

7.3. **The Euler summation.** To improve the convergence of a series S, we use the Euler summation technique as described by Abate et al. (1999), which consists in computing the binomial average of q terms from the p-th term of the series S. The binomial average obviously converges to S as p goes to infinity. The following proposition describes the convergence rate of the binomial average to the infinite series $f_{\pi/t}(t)$ when p goes to ∞.

Proposition 13. *Let f be a function of class $\mathcal{C}^{q+4}$ such that there exists $\epsilon > 0$ s.t. $\forall k \leq q+4,\ f^{(k)}(s) = \mathcal{O}(e^{(\alpha-\epsilon)s})$, where α is the abscissa of convergence of $\widehat{f}$. We define $s_p(t)$ as the approximation of $f_{\pi/t}(t)$ when truncating the non-finite series in (19) to p terms*

$$s_p(t) = \frac{e^{\alpha t}}{2t}\widehat{f}(\alpha) + \frac{e^{\alpha t}}{t}\sum_{k=1}^{p}(-1)^k \mathcal{R}\mathrm{e}\left(\widehat{f}\left(\alpha + i\frac{\pi k}{t}\right)\right),$$

and $E(q,p,t) = \sum_{k=0}^{q} C_q^k 2^{-q} s_{p+k}(t)$. Then,

$$\left|f_{\pi/t}(t) - E(q,p,t)\right| \leq \frac{te^{\alpha t}\,|f'(0) - \alpha f(0)|}{\pi^2}\frac{p!\,(q+1)!}{2^q\,(p+q+2)!} + \mathcal{O}\left(\frac{1}{p^{q+3}}\right),$$

when p goes to infinity.

Using Propositions 12 and 13, we get the following result concerning the global error on the numerical computation of the price of a Parisian call option.

Corollary 1. *Let f be the price of a Parisian call option. Using the notations of Proposition 13, we have*

$$|f(t) - E(q,p,t)| \leq S_0\,\frac{e^{-2\alpha t}}{1-e^{-2\alpha t}} + \frac{e^{\alpha t}t\,|f'(0) - \alpha f(0)|\,p!\,(q+1)!}{\pi^2 2^q\,(p+q+2)!} + \mathcal{O}\left(\frac{1}{p^{q+3}}\right),$$

for any $\alpha > \frac{(m+\sigma)^2}{2}$.

We refer the reader to Labart and Lelong (2008) for a proof of Corollary 1 and Proposition 13.
For $2\alpha t = 18.4$ and $q = p = 15$, the global error is bounded by $S_0 10^{-8} + t\,|f'(0) - \alpha f(0)|\,10^{-11}$. As one can see, the method we use to invert Laplace transforms provides a very good accuracy with few computations.

Remark 6. *Considering the case of call options in Theorem 1 is sufficient since put prices are computed using parity relationships and their accuracy is hung up to the one of call prices.*

8. A few graphs

In this section, we perform a few numerical experiments with the method we have studied so far and compare it with the enhanced Monte Carlo method of Baldi et al. (2000).
First, we consider a dynamic delta hedging simulation of a Parisian Up and Out call. We simulate an asset path and try to hedge along this trajectory. For this purpose, we use the formulae to derive the price of Parisian options at any time strictly positive. The delta simply ensues from a finite difference scheme. The discrete delta hedging proves quite efficient even though as one can see it on Figure 4, there are huge variations in the hedging portfolio when the option is about to be activated or cancelled. This phenomena introduces some hedging error because the hedging is performed in discrete time. In this example, the hedging portfolio could be rebalanced three times a day.
Now, we would like to compare the prices obtained with our method with the prices given by the Monte Carlo method of Baldi et al. (2000). The Monte Carlo computation uses 10000 samples and 250 discretisation steps between 0 and T. Figure 5 shows the evolution w.r.t the delay of the price of a Down and Out put computed either with the invert Laplace transform method or the enhanced Monte Carlo method. The evolution of the prices provided by our method is much smoother than the one given by Monte Carlo. As one can see, the accuracy of the Monte Carlo method has nothing to do with the accuracy of our method. Let us recall that our prices are accurate up to 10^{-6} (when $S_0 = 100$) as stated in Theorem 1. Concerning the computational costs of the two methods, the invert Laplace transform method runs a thousand times faster than the corrected Monte Carlo.

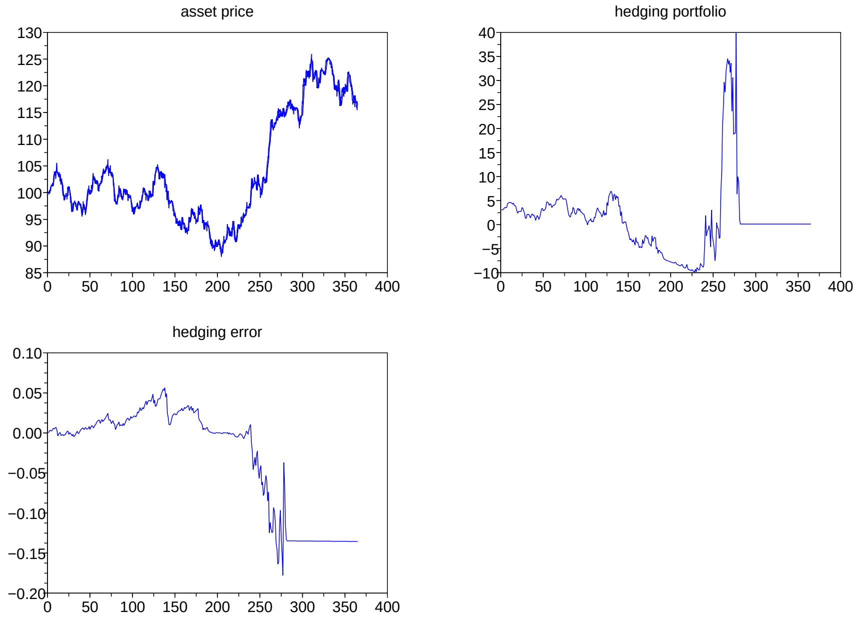

FIGURE 4. Example of delta hedging of a PUOC
$S_0 = 100 \quad K = 100 \quad T = 1 \quad L = 110$
$D = 20$ day $\quad \sigma = 0.2 \quad r = 0.025 \quad \delta = 0$

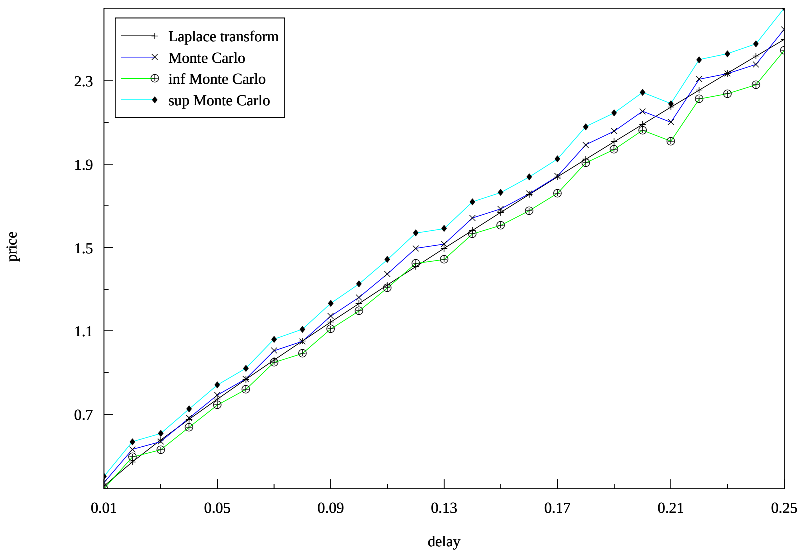

FIGURE 5. Comparison with improved Monte Carlo method in the case of a PDOP
$S_0 = 100 \quad K = 100 \quad T = 1 \quad L = 90$
$\sigma = 0.2 \quad r = 0.025 \quad \delta = 0$

9. Conclusion

In this work, we provide all the Laplace transforms of the different Parisian option prices, be it through explicit formulae or parity relationships. We also explain how to invert these formulae to compute the prices. The detailed study of the inversion algorithm enables to prove the accuracy and then the efficiency of the method. The efficiency is confirmed by the comparison with the enhanced Monte Carlo, which in fact is already very efficient when one thinks of how difficult it is to price Parisian options.

Appendix A. The Laplace transform of μ_b in the case $b > 0$

We already know that $\mu_b(u) = \frac{|b|}{\sqrt{2\pi u^3}} e^{\left(\frac{-b^2}{2u}\right)}$. We use the notation $\theta = \sqrt{2\lambda}$.

$$\int_0^D e^{-\lambda u}\mu_b(du) \quad = \quad \int_0^D e^{-\frac{\theta^2}{2}u}\frac{b}{\sqrt{2\pi u^3}}e^{\frac{-b^2}{2u}}du$$

The change of variable $t = \sqrt{\frac{b}{\theta}}\frac{1}{\sqrt{u}}$ leads to

$$\begin{aligned}
\int_0^D e^{-\lambda u}\mu_b(u)du &= \int_{\frac{\sqrt{b}}{\sqrt{\theta D}}}^{+\infty}\sqrt{\frac{2b\theta}{\pi}}\ \exp\left(\frac{-\theta b}{2}\left(\frac{1}{t^2}+t^2\right)\right)dt,\\
&= \int_{\frac{\sqrt{b}}{\sqrt{\theta D}}}^{+\infty}\sqrt{\frac{2b\theta}{\pi}}\ \exp\left(\frac{-\theta b}{2}\left(\frac{1}{t}-t\right)^2\right)e^{-\theta b}dt,\\
&\quad \text{a new change of variable } v = \tfrac{1}{t}-t \text{ gives}\\
&= \sqrt{\frac{b\theta}{2\pi}}e^{-\theta b}\int_{-\infty}^{\sqrt{\frac{\theta D}{b}}-\frac{\sqrt{b}}{\sqrt{\theta D}}} e^{\frac{-\theta b}{2}v^2}\left(1-\frac{v}{\sqrt{v^2+4}}\right)dv,\\
&\quad \text{we set } u = \sqrt{\theta b}v\\
&= \frac{1}{\sqrt{2\pi}}e^{-\theta b}\int_{-\infty}^{\theta\sqrt{D}-\frac{b}{\sqrt{D}}} e^{-u^2/2}\left(1-\frac{u}{\sqrt{u^2+4\theta b}}\right)du.
\end{aligned}$$

A last change of variable $v = \sqrt{u^2+4\theta b}$ ends the computation

$$\hat{\mu_b}(\lambda) = e^{-\theta b}\mathcal{N}\left(\theta\sqrt{D}-\frac{b}{\sqrt{D}}\right) + e^{\theta b}\mathcal{N}\left(-\theta\sqrt{D}-\frac{b}{\sqrt{D}}\right).$$

If we let D go to infinity, we can deduce the Laplace transform of T_b, for any real b

$$\mathbb{E}[e^{-\lambda T_b}] = e^{-\sqrt{2\lambda}|b|}.$$

Appendix B. The valuation of $\int_0^{+\infty} \mathrm{e}^{-\lambda u}\frac{\mathrm{e}^{-\frac{x^2}{2u}}}{\sqrt{2\pi u}}du$

Once again we introduce $\theta = \sqrt{2\lambda}$.
The change of variable $u = \frac{|x|t^2}{\theta}$ straightly gives the new expression

$$\begin{aligned}
\int_0^{+\infty} \mathrm{e}^{-\lambda u}\ \frac{\mathrm{e}^{-\frac{x^2}{2u}}}{\sqrt{2\pi u}}du &= \int_0^{+\infty}\sqrt{\frac{2\mid x\mid}{\pi\theta}}\exp\left(-\frac{\theta\mid x\mid}{2}\left(\frac{1}{t^2}+t^2\right)\right)dt,\\
&= \sqrt{\frac{2\mid x\mid}{\pi\theta}}e^{-\theta\,|x|}\int_0^{+\infty}\exp\left(-\frac{\theta\mid x\mid}{2}\left(\frac{1}{t}-t\right)^2\right)dt.
\end{aligned}$$

Once again, we can use the change of variable $s = u - \frac{1}{u}$, which maps $[0,+\infty[$ into $]-\infty,+\infty[$ and we have $du = \frac{ds}{2}\left(1+\frac{s}{\sqrt{s^2+4}}\right)$. The second term is odd, so its integral over $\mathbb{R}$ cancels and we get

$$\sqrt{\frac{|x|}{2\pi\theta}}\,\mathrm{e}^{-\theta\,|x|}\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{\theta\,|x|}{2}s^2}\,ds.$$

Finally, we obtain

$$\int_0^{+\infty} e^{-\lambda u} \frac{\mathrm{e}^{-\frac{x^2}{2u}}}{\sqrt{2\pi u}} du = \frac{1}{\theta} e^{-\theta\,|x|}. \tag{20}$$

Appendix C. Some results around Brownian meanders

In this part, we recall some useful results about Brownian motion and it excursion theory. We are interested in the law of $(T_b^-, Z_{T_b^-})$ and $(T_b^+, Z_{T_b^+})$. Such results can be found in Azéma and Yor (1989); Revuz and Yor (1999); Chung (1976).
In the following, we consider a standard Brownian motion Z.

C.1. **Case** $b = 0$**.**

$$\mathbb{P}(Z_{T^-} \in dx) = -\frac{x}{D} \mathrm{e}^{-\frac{x^2}{2D}} \mathbf{1}_{\{x<0\}} dx \quad \text{and} \quad \mathbb{P}(Z_{T^+} \in dx) = \frac{x}{D} \mathrm{e}^{-\frac{x^2}{2D}} \mathbf{1}_{\{x>0\}} dx.$$

$$\mathbb{E}\left(\mathrm{e}^{-\frac{1}{2}\lambda^2 T^-}\right) = \frac{1}{\psi(\lambda\sqrt{D})} \quad \text{and} \quad \mathbb{E}\left(\mathrm{e}^{-\frac{1}{2}\lambda^2 T^+}\right) = \frac{1}{\psi(\lambda\sqrt{D})}.$$

This kind of formula goes back to the work of Wendel (1964).

C.2. **Case** $b < 0$**.** This case can be reduced to the previous one with the help of the stopping time T_b. By introducing a new Brownian motion $W = \{W_t = Z_{T_b+t} - b; t \geq 0\}$ independent of $\mathcal{F}_{T_b}$, we can write $T_b^- = T_b + T^-(W)$ a.s.. T_b and $T_0^-(W)$ are independent, hence we find

$$\mathbb{E}(\mathrm{e}^{-\frac{1}{2}\lambda^2 T_b^-}) = \mathbb{E}(\mathrm{e}^{-\frac{1}{2}\lambda^2 T_b})\mathbb{E}(\mathrm{e}^{-\frac{1}{2}\lambda^2 T_0^-(W)}).$$

As $\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b)) = \exp(-|b|\lambda)$, we get

$$\mathbb{E}\left(\mathrm{e}^{-\frac{1}{2}\lambda^2 T_b^-}\right) = \frac{\mathrm{e}^{b\lambda}}{\psi(\lambda\sqrt{D})}.$$

Concerning the law of $Z_{T_b^-}$, we have

$$\mathbb{P}(Z_{T_b^-} \in dx) = \mathbb{P}(W_{T_b^-(Z)-T_b(Z)} \in dx - b) = \mathbb{P}(W_{T_0^-} \in dx - b).$$

Finally, we obtain

$$\nu^-(dx) = \mathbb{P}(Z_{T_b^-} \in dx) = \frac{b-x}{D} \mathrm{e}^{-\frac{(x-b)^2}{2D}} \mathbf{1}_{\{x<b\}} dx.$$

C.3. **Case** $b > 0$**.** Following closely the above reasoning, we find

$$\mathbb{E}\left(\mathrm{e}^{-\frac{1}{2}\lambda^2 T_b^+}\right) = \frac{\mathrm{e}^{-b\lambda}}{\psi(\lambda\sqrt{D})}.$$

$$\nu^+(dx) = \mathbb{P}(Z_{T_b^+} \in dx) = \frac{x-b}{D} \mathrm{e}^{-\frac{(x-b)^2}{2D}} \mathbf{1}_{\{x>b\}} dx.$$

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