THE SEMI-CLASSICAL APPROXIMATION IN A HIERARCHY OF MODELS WITH INSTANTONS

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THESIS

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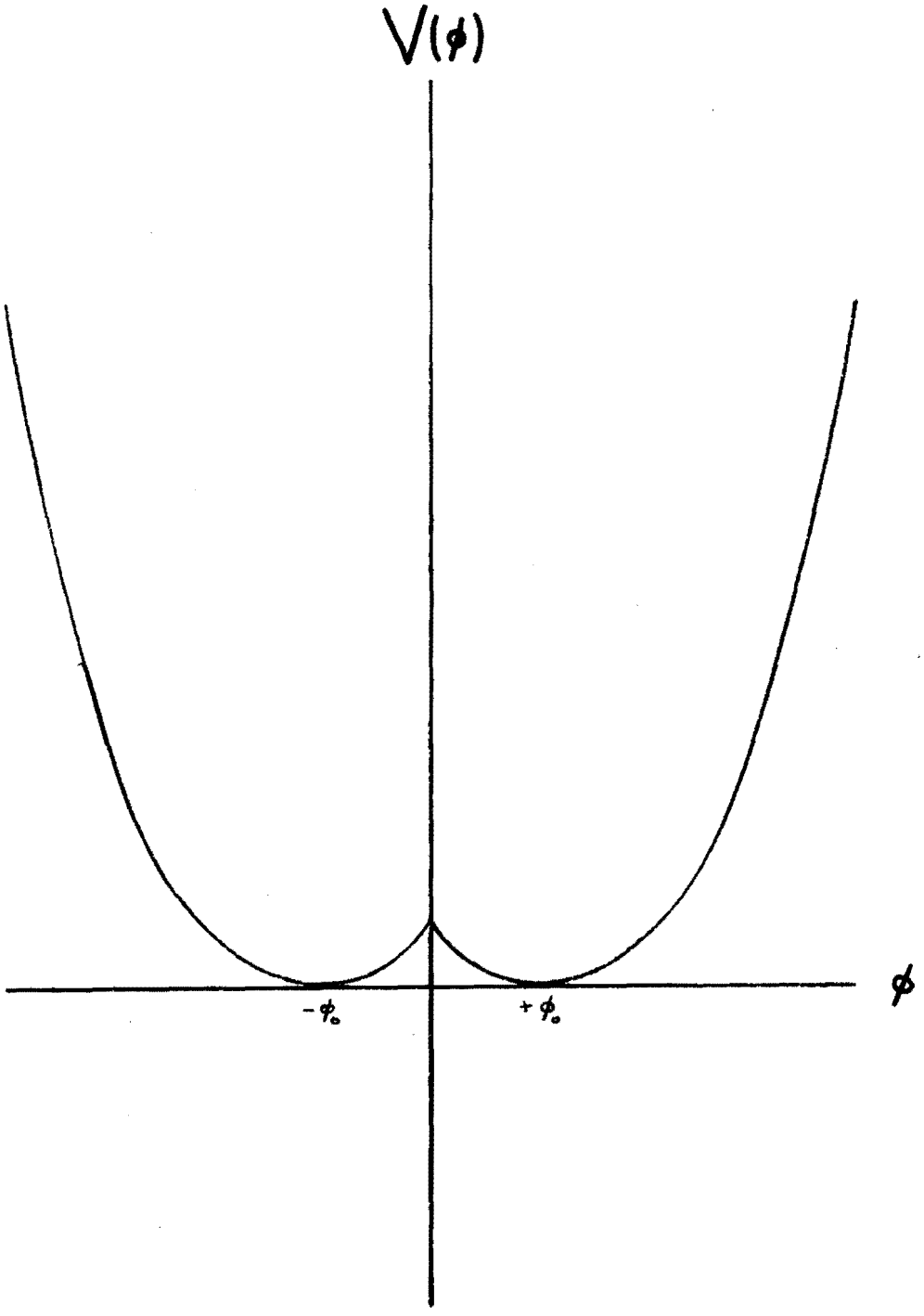
I. INTRODUCTION

Quantum Chromodynamics (QCD) is by far the most attractive candidate for a theory of the strong interactions. The approximate scaling noticed at high energies in deep inelastic electron scattering is suggestive of a theory with asymptotic freedom, and non-Abelian gauge theories are the only theories in four-dimensions that are asymptotically free. Unfortunately, at this time, a deep understanding of the particle spectrum of the theory is lacking. This is largely due to the non-Abelian, and hence, non-linear nature of the theory. The ability of the gluon field to interact directly with itself necessarily complicates matters to the point where the usual perturbation method of calculating is totally inadequate to describe the strong coupling, large distance region in QCD.

Much recent work has been devoted to the task of finding non-perturbative methods of analyzing non-linear theories. Some of these include lattice gauge theories, the large N expansion, and the semi-classical approximation. From the last of these, it has lately been realized that the structure of the vacuum plays an important role in determining the features of a theory [1]. A model which illustrates this in a simple manner is the one-dimensional double-well potential, Fig. 1 [2,3]. Classically, it has

Figure 1. A simple double-well potential.


V(φ)
φ
-φ₀
+φ₀

two ground states (or vacua). A particle with lowest energy in this potential would be located at either $\phi = +\phi_0$ or $\phi = -\phi_0$. Quantum mechanically this is not so. A particle located in one well has a finite probability of tunneling into the other well. Over an infinite time, an infinite number of tunnelings can have occurred. Thus, the true ground state of the model must be a superposition of the two classical vacua. The localized, imaginary time solutions to the equations of motion of the model (called instantons) describe the most probable tunneling path from one classical vacuum to the other. The semi-classical method of approximation uses these instantons in a saddle point calculation to find the zeroeth order quantum effects in a theory.

Quantum Chromodynamics contains multiple classical vacua, and hence also contains instantons which describe the tunnelings between these vacua. A classical vacuum has $\vec{E} = \vec{B} = 0$. However, this does not uniquely determine A_μ, but in fact only requires A_μ to be a pure gauge, i.e., the vector potential A_μ satisfies $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0$. This implies for SU(2) gauge theory that $A = g^\dagger \partial_\mu g$, with g belonging to SU(2). To specify that pure gauge a further label is needed, the topological charge. The topological charge counts the number of times that the angles of SU(2), $\theta(x,t)$, cover the group as $\vec{x}$ moves over all three dimensional space. Now a

classical vacuum has not only $\vec{E} = \vec{B} = 0$ but a unique topological charge. To go from one vacuum to another requires crossing a potential barrier, and instantons describe the tunneling through the barriers. Once again, the true, quantum mechanical vacuum must be a superposition of all these classical vacua, the so-called θ-vacuum [4,5,6].

At present, it is not clear how to use the instantons in QCD to do general calculations of physical quantities, or to gain a deeper understanding of the theory. The structure of QCD is sufficiently complicated that it is difficult to manipulate its instantons and to carry out calculations utilizing them. We thus try to sharpen our skills upon simpler models, ones that we hope are sufficiently similar to QCD to retain its salient features, but yet are simple enough to allow for exact calculations that teach us the proper role of instantons in a QCD-like theory.

This thesis deals with a hierarchy of models that resemble QCD in varying degrees. Chapter II outlines the basic ideas of the semi-classical method, and illustrates it on a simple one dimensional model. The instantons in the model are used to calculate the energy difference between the ground and first excited states of the model, and demonstrate the restoration of symmetry. Chapter III describes the

non-linear σ model. This model is a two dimensional model with a U(1) gauge degree of freedom. It is remarkably like QCD for reasons which are not completely understood. Recently, the semi-classical approximation to the non-linear σ model has been shown to be equivalent to a Coulomb gas. We show that this Coulomb gas picture and the dilute gas picture, where the instantons are assumed not to interact when they are sufficiently far apart, are exactly equivalent, even when the instantons can no longer be considered to be dilute. This result could have important implications for further understanding of multi-instanton phenomena. Finally, in Chapter IV, we introduce a new class of four dimensional models which are generalizations of the non-linear σ models that are also non-Abelian gauge theories. These help explain the remarkable similarities between the non-linear σ model and non-Abelian gauge theories. It is hoped that these theories will be useful in understanding the instantons in QCD and in bridging the gap between the non-linear σ model and QCD.

II. THE SEMI-CLASSICAL APPROXIMATION

In this chapter, we will introduce the path integral approach to the semi-classical method as it applies to one dimensional systems only. The generalization to field theory is straightforward and we will use it in later chapters.

The path integral approach is in the same spirit as the one dimensional WKB approximation. In WKB, the wave-function in Schrodinger's equation is written as a magnitude and a phase, $\psi(x)=A(x)\exp[iS(x)/\hbar]$, with A determined by S. Writing S as an expansion in powers of $\hbar$, only the lowest order, S_0, is kept. Here, S_0 becomes the classical action, and ignoring higher order terms in $\hbar$ is equivalent to taking the $\hbar \longrightarrow 0$ limit. However, this treatment is definitely more than classical physics, for it can be carried out in regions (E<V) where classical particles cannot penetrate. In these regions (the tunneling regions) we find that S(x) is imaginary, leading to non-oscillatory behavior in the wave-function [7].

In the path integral approach, lowest order quantum effects are also picked up. Here, we use path integrals to calculate $I=\langle \phi'(t) | \phi(0) \rangle$, the amplitude for a particle at ϕ with t=0 to propagate to ϕ' at t=t. We use ϕ to denote

the position coordinate. Various expectation values (or correlation functions) can be calculated using these path integrals. Using the time evolution operator, $|\phi(t)\rangle = \exp(-iHt/\hbar)|\phi(0)\rangle$, we see $I=\langle\phi'|\exp(-iHt/\hbar)|\phi\rangle$. As a path integral,

$$I=\int\prod_x d\phi(t)\ \exp[(i/\hbar)\int L(\phi)dt]. \qquad (2.1)$$

Here L is the classical Lagrangian, so that the expression in the exponential in (2.1) is just $i/\hbar$ times the classical action, $S[\phi]$.

Since we will be examining tunneling effects, that is, looking at regions where the particles have imaginary momenta, it is convenient to Wick rotate to imaginary time, $t\rightarrow -i\tau$. The quantity of interest is now $I=\langle\phi'|\exp(-H\tau/\hbar)|\phi\rangle$. For τ large, this picks out the ground state to ground state amplitude, $I=\langle 0|\exp(-H\tau/\hbar)|0\rangle$. More practically, switching to imaginary time allows us to isolate some interesting quantities. For example, the correlation function $\langle 0|\phi(\tau)\phi(0)|0\rangle = \sum_n |\langle 0|\phi|n\rangle|^2 \exp[-(E_n-E_0)\tau]$. In the limit $\tau\longrightarrow\infty$, we see that this goes as $|\langle 0|\phi|0\rangle|^2 + |\langle 0|\phi|1\rangle|^2 \exp[-(E_1-E_0)\tau]$. In a double well potential, $\langle 0|\phi|0\rangle=0$ will indicate that left-right symmetry remains unbroken and that the true ground state retains this symmetry, even though the classical vacua

do not. The exponential will then pick off the energy splitting between the ground state and the first excited state. This is precisely the calculation we will demonstrate in this chapter.

The imaginary time equivalent of (2.1) is

$$I=\int \prod_{\tau} d\phi(\tau)\ \exp[(-1/\hbar)\ \int L_E(\phi)\,d\tau]. \tag{2.2}$$

Here L_E is the Euclidean Lagrangian,

$$L_E=(d\phi/dt)^2/2 + V(\phi), \tag{2.3}$$

for the particle moving in a potential, $V(\phi)$.

The path integral approach can now be interpreted as a saddle point calculation of the integral in (2.2). Stationary points (local maxima) of the integrand are found, and then the integral is approximated by a gaussian about that point. The stationary point is found by solving the classical equations of motion for L_E. For a double-well potential, this configuration (the instanton) describes the tunneling from one of the classical vacua to the other. For a potential with only a single classical vacuum, this procedure is the first step of perturbation theory. For example, with $V(\phi)= m^2\phi^2/2 + b\,\phi^4/4$, the path integral is

$$I=\int\prod_t d\phi(t)\ \exp[-\int[(d\phi/dt)^2/2 + V(\phi)]dt]. \qquad (2.4)$$

(We will be using t to denote imaginary time.)

The minimum of the action lies at $\phi=0$. A gaussian approximation around this point does not include the ϕ^4 term. Higher orders of b can be found by expanding the exponential in a Taylor series in b, and then doing the remaining gaussian integrals. This expansion is precisely the Feynman perturbation series expansion.

For a potential with multiple vacua, the situation is not so simple, for the minima in the action (maxima in the integrand) do not occur with ϕ=constant. Given an instanton configuration (which minimizes the action), we can calculate the amplitude for making a transition through that instanton and compare it to making no transition at all. This amplitude is

$$W= \frac{\int[\exp[-\int L(\phi_i+\eta)dt]]\prod d\eta(t)}{\int[\exp[-\int L(\phi_0+\eta)dt]]\prod d\eta(t)}\ . \qquad (2.5)$$

Here, ϕ_i is the instanton configuration and η, assumed to be small, is the deviation from that configuration. The integral in the denominator represents the amplitude for the system to remain in the classical vacuum represented by

$\phi = +\phi_0$. Keeping only $O(\eta^2)$ fluctuations in the numerator and denominator gives W as

$$W = \exp[-\int L(\phi_i)\,dt] \tag{2.6}$$

$$\frac{\int \exp[-\int \eta(t')M(t',t)\,\eta(t)\,dt'dt/2]\,\pi\, d\eta(t)}{\exp[-\int \eta(t')M_0(t',t)\,\eta(t)\,dt'dt/2]\,\pi\, d\eta(t)},$$

where, with S the classical action,

$$M(t',t) = \left[\frac{\delta^2 S}{\delta\phi(t')\,\delta\phi(t)}\right]_{\phi=\phi_i} \tag{2.7}$$

and

$$M_0(t',t) = \left[\frac{\delta^2 S}{\delta\phi(t')\,\delta\phi(t)}\right]_{\phi=\phi_0} . \tag{2.8}$$

The necessity of finding finite action solutions is clear, since the first term in W would vanish with an infinite action. Since L is local, the operators $M(t',t)$ and $M_0(t',t)$ can be written as

$$M(t',t) = \delta(t'-t)M \tag{2.9}$$

and

$$M_0(t',t) = \delta(t'-t)M_0 \tag{2.10}$$

where M and M_o are second order differential operators.

We now proceed to use the semi-classical method to calculate the correlation function $D(T,0)=\langle 0 | \phi(T)\,\phi(0) | 0\rangle$ for $T \longrightarrow \infty$ for a simple double-well potential.

The path integral expression for this is

$$\langle 0 | \phi(T)\,\phi(0) | 0\rangle = \frac{\int \phi(T)\,\phi(0) \exp[-S(\phi)]\,\pi\, d\phi(t)}{\int \exp[-S(\phi)]\,\pi\, d\phi(t)} . \qquad (2.11)$$

In the semi-classical method, $\phi = \phi_i + \eta$ as before, and therefore, the lowest order contribution in the numerator will come from replacing $\phi(T)\,\phi(0)$ by $\phi_i(T)\,\phi_i(0)$. These do not contribute in the gaussian integration over η. The denominator of (2.11) is calculated as in (2.6).

Since this is a saddle point integration, to get the final value of the integral, we must do a gaussian approximation at each maximum of the integrand and then sum over them all. A variety of minima of the action are possible. For instance, the particle can simply remain in one vacuum. Also possible is the one-instanton configuration, with the partice tunneling through the potential barrier once. Finally the particle can start in

one well, tunnel through to the other well, sit there a while, and then return. In other words, any number of tunnelings through the barrier is possible. These multiple tunnelings are not exact minima of the action, but approach a minimum as the exponential of the distance between them. They therefore make important contributions to the path integral.

The two path integrals composing $D(T,0)$ are thus the sums of the contributions of no instantons, one instanton, two instantons, etc., and we find

$$D(T,0) = \frac{I_0(T,0)+I_1(T,0)+I_2(T,0)+\ldots}{J_0+J_1+J_2+J_3+\ldots}\ , \qquad (2.12)$$

where here $I_n(T,0)$ and J_n are the n-instanton contributions to the path integral in the numerator and denominator of (2.11), respectively.

The zero instanton solution has the particle remaining in one well, $\phi = \phi_0$. Thus, J_0 is given by the denominator of W in (2.5),

$$J_0 = \int \exp[-\int L(\phi_0+\eta)\,dt]\,\mathcal{D}\eta\,(t)\,. \qquad (2.13)$$

$I_0(T,0)$ is similarly given by

$$I_0 = \int \phi_0(T)\, \phi_0(0) \exp[-\int L(\phi_0 + \eta)\, dt] \prod d\eta(t)$$

$$= \phi_0^2 J_0 \qquad (2.14)$$

since ϕ_0 is constant.

Now, in general the path integrals I_n or J_n have undetermined normalizations. However, in an expression like (2.5), these undetermined normalizations divide out. We therefore write

$$D(T,0) = \frac{\phi_0^2 + (I_1/J_0) + (I_2/J_0) + \ldots}{1 + (J_1/J_0) + (J_2/J_0) + \ldots} \; . \qquad (2.15)$$

The previously introduced W is just the term (J_1/J_0).

We will first calculate the contribution of one instanton (a single tunneling) to both the numerator, (I_1/J_0), and denominator, (J_1/J_0), in (2.15), and then go on to handle multiple tunnelings.

The double-well potential is shown in Fig. 1 [8],

$$V(\phi) = [m^2(\phi - \phi_0)^2\, \Theta(\phi) + m^2(\phi + \phi_0)^2\, \Theta(-\phi)]/2, \qquad (2.16)$$

where $\Theta(\phi)$ is the usual step function. We notice that the

two classical vacua are located at $\phi=+\phi_0$ and at $\phi=-\phi_0$. The instanton, or "kink," which leads from $\phi=-\phi_0$ at $t=-\infty$ to $\phi=+\phi_0$ at $t=+\infty$ while crossing $\phi=0$ at t=a, is given by

$$\phi_i = \phi_0 [1-\exp(-m|t-a|)] \epsilon(t-a). \tag{2.17}$$

We will refer to the place where the kink has $\phi=0$ as its location. The action of the kink is

$$S_i = m \phi_0^2 . \tag{2.18}$$

The operators M and M_0 are given by

$$M=-(d^2/dt^2)+m^2-2m\,\delta(t-a) \tag{2.19}$$

and

$$M_0=-(d^2/dt^2)+m^2, \tag{2.20}$$

respectively. The operator M_0 does not involve the discontinuous behavior at $\phi=0$, since the amplitude in the denominator of W is the amplitude to remain at $\phi=-\phi_0$. In the approximation of keeping only gaussian fluctuations around $\phi=-\phi_0$, the potential for this process is just $m^2(\phi+\phi_0)^2/2$, which leads to (2.20) for M_0.

With the model thus defined, we proceed to calculate $W=(J_1/J_0)$ for one instanton. W is given by

$$W=\exp(-S_i)(K/K_0), \tag{2.21}$$

where we have introduced the notation K and K_0 for the integrals in the numerator and denominator of (2.6). Thus,

$$K=\int[\exp[-\int\eta(t)M\,\eta(t)dt/2]]\prod d\eta\,(t), \tag{2.22}$$

and

$$K_0=\int\exp[-\int\eta(t)M_0\,\eta(t)dt/2]]\prod d\eta\,(t). \tag{2.23}$$

We will calculate W by relating the path integral to the Fredholm determinant for scattering in the potential $-2m\delta(t)$.

The selection of the kink located at t=0 is of course arbitrary. Owing to the translational invariance of the action, the tunneling will proceed with equal probability via the kink at t=a. The function $\eta_0=a\,\partial\phi_i/\partial t$ represents, to first order in a, the fluctuation which tranlates the kink at t=0 to t=a. Since the action is translation invariant, η_0 must be a zero mode of M, i.e.,

$$M\eta_0=[(-d^2/dt^2)+m^2-2m\,\delta(t)]\eta_0=0, \tag{2.24}$$

From (2.17) we have

$$\eta_0 = a\,\phi_0\, m\, \exp(-m|t|), \tag{2.25}$$

which is easily seen to satisfy (2.24).

We proceed to evaluate K and K_0 by expanding η in eigenfunctions of M and M_0, respectively. We put the system in a large (one dimensional) box of volume V, by restricting t to the range $-V/2<t<V/2$. For the calculation of K, we expand η as

$$\eta = \sum_n c_n \psi_n , \tag{2.26}$$

where

$$M\psi_n = E_n \psi_n , \tag{2.27}$$

and likewise for evaluating K_0, we expand η as

$$\eta = \sum_n c_n^{(0)} \psi_n^{(0)} , \tag{2.28}$$

where

$$M_0 \psi_n^{(0)} = E_n^{(0)} \psi_n^{(0)} . \tag{2.29}$$

Substituting into the expressions for K and K_o, and changing variables of integration, we have

$$\frac{K}{K_o} = \frac{\int \exp(-\sum E_n c_n^2/2) \pi\, dc_n}{\int \exp(-\sum E_n^{(o)} [c_n^{(o)}]^2/2) \pi\, dc_n} . \qquad (2.30)$$

Since the coefficients c_n and $c_n^{(o)}$ are related by a unitary transformation, the Jacobian involved in the change of variables of integration cancels out of numerator and denominator in (2.30).

The presence of the box shifts the lowest eigenvalue of M infinitesimally to a value $E_o = O[\exp(-mV)]$. However, the integration in K over the coefficient c_o belonging to this eigenvalue is still essentially non-gaussian, which reflects the translation invariance of the action. The functional integration is in effect summing up paths through kinks translated to locations a, each weighted in the overall amplitude W by the same action factor, $\exp(-S_i)$. Ignoring terms of $O[\exp(-mV)]$, the discrete eigenfunction of M with zero energy is

$$\psi_o = \sqrt{m}\, \exp(-m|t|), \qquad (2.31)$$

which has been normalized so that

$$\int_{-\infty}^{+\infty} |\psi_o|^2 dt = 1. \qquad (2.32)$$

Comparing (2.25) and (2.31), we see that $c_0 = a\,\phi_0\sqrt{m}$ and $dc_0 = \phi_0\sqrt{m}\,da$. We interpret the integration over a in the standard manner as giving an overall volume factor, V. It is perhaps worth pointing out that integrating over c from $-\infty$ to $+\infty$ and using the fact that E_0 is not literally zero to make the integral converge would be absurd, since such large values of c_0 clearly correspond to allowing the translation parameter a to be much larger than the volume V [9].

Carrying out the gaussian integrations in K, and doing likewise for K_0, we have

$$W = (\phi_0\sqrt{m}\int da)\,[(2\pi/E_0^{(0)})\prod_n (E_n/E_n^{(0)})]^{-1/2}\,\exp(-S_i)\,. \tag{2.33}$$

Since the operators M and M_0 differ only by the potential $-2m\delta(t)$, we can evaluate the quantity inside the square brackets in terms of the Fredholm scattering determinant scattering at energy $-m^2$. The energy is $-m^2$ because of the m^2 in M_0. We have in the large-volume limit

$$(1/E_0^{(0)})\prod_n (E_n/E_n^{(0)}) = \tag{2.34}$$

$$(1/m^2)\exp[(-2/\pi)\int k\;\delta(k)\,(k^2+m^2)^{-1}\,dk],$$

where the phase shift δ is given by

$$\delta(k) = \mathrm{Arctan}(m/k), \tag{2.35}$$

with $\delta(\infty)=0$. The fact that the final result involves an integral just reflects the continuous spectra of M and M_o. Substituting for $\delta(k)$, we can evaluate W as

$$W=[m\ \phi_o\ (m/\pi)^{1/2}\int da]\exp(-S_i). \tag{2.36}$$

For (I_1/J_o), the one instanton contribution is calculated as for (J_1/J_o). All gaussian integrations over η remain the same, since the lowest order contribution in the non-exponential part comes from the instantons themselves. The one difference is in the "a" integration. For (J_1/J_o), we simply got $\int da=V$. In (I_1/J_o), we must calculate $\int \phi_i(T;a)\ \phi_i(0;a)da$, where $\phi_i(t;a)$ is given in (2.17). For large T, this integral is

$$\int \phi_i(T;a)\ \phi_i(0;a)da=\phi_o^2[V-2T+O(T\ \exp(-mT))]. \tag{2.37}$$

Calling $P=m(m/\pi)^{1/2}\phi_o$, the expression for D(T,0) to this order is now

$$D(T,0)=\phi_o^2\ \frac{1+(V-2T)P\ \exp(-S_i)+\ldots}{1+VP\ \exp(-S_i)+\ldots}\ . \tag{2.38}$$

In anticipation of the final result, we notice that the numerator and denominator contain the first terms of an exponential expansion, $\exp[(V-2T)P\cdot\exp(-S_i)]/\exp[VP\cdot\exp(-S_i)]$. These will end up giving the desired behavior, $\exp[-(E_1-E_0)T]$, as discussed on page 8.

For multiple instantons, we consider the instantons to be well separated relative to their size. Then the total action for a configuration will be just nS_i, where we have n instantons. Furthermore, the contributions from the gaussian fluctuations will just be the product of the contribution from each kink. To see this, the potential for the applicable M operator simply contains n widely separated single-instanton potentials. In a formula like (2.34), the total phase shift is just the phase shift accumulated from each single-instanton potential, leading to the desired result.

As in the one instanton case, the locations of the n kinks are also arbitrary. There exist n eigenfunctions of M which have almost zero eigenvalues. These correspond to the fluctuations which translate each of the kinks separately. When we expand η in eigenfunctions of M, we will find n c's which have zero eigenvalues. These integrations are replaced by n integrations over kink locations, $dc_j = \phi_0\sqrt{m}\,da_j$, for each kink location, a_j.

Carrying out the gaussian integrations, we find

$$J_n/J_0 = (1/n!)P^n \exp(-nS_i) \int da_1\, da_2 \ldots da_n \qquad (2.39)$$

$$= (1/n!)[VP\cdot\exp(-S_i)]^n \ .$$

The (n!) appears to avoid double counting instanton configurations while varying the a's.

To compute (I_n/J_0), we again consider the kinks to be well separated, so that the n instanton configuration is

$$\phi_{ni}(t) \cong \phi_0 \prod_{j=1}^{n} [\,\phi_i(t-a_j)/\phi_0\,]. \qquad (2.40)$$

This has n kinks, each one located at one of the a_j. Then the integrations over the a's give

$$I_n/J_0 = (1/n!)P^n \exp(-nS_i) \quad x \qquad (2.41)$$

$$\int da_1 \ldots da_n \prod_{j=1}^{n} \phi_i(T-a_j)\,\phi_i(a_j)$$

$$= (1/n!)\left[\int \phi_i(T;a)\,\phi_i(0;a)\,da\ P\cdot\exp(-S_i)\right] \ .$$

Thus the expression for the correlation function is

$$D(T,0) = \phi_0^2 \frac{\sum_n (1/n!)[(V-2T)P\cdot\exp(-S_i)]^n}{\sum_n (1/n!)[VP\cdot\exp(-S_i)]^n} \qquad (2.42)$$

$$= \exp[-2TP\cdot\exp(-S_i)] \ .$$

In the large T limit, we have now demonstrated that the correlation function decays exponentially, and that the energy splitting is $2P\cdot\exp(-m\phi_0^2)$. The lack of a constant term indicates that the ground state is left-right symmetric.

III. THE NON-LINEAR SIGMA MODEL

The next model we consider is the non-linear σ model. This is a two dimensional (one space, one time) model which is both asymptotically free [10] and contains instantons [11], just as gauge theory does. By studying this model we hope to gain insights into QCD. Although the σ model does not have confinement, the analogous problem is to understand the absence of symmetry breaking and the existence of a purely massive spectrum in two dimensions. The action for the σ model is

$$S=(1/2f)\int(\partial_\mu n^a)\cdot(\partial_\mu n^a)\,d^2x, \quad (a=1,2,3) \tag{3.1}$$

subject to the constraint $(n^a)(n^a)=1$. Here $n^a(x)$ is a unit vector in three dimensional group space, f is the dimensionless coupling constant, and we have already changed from real time (t) to imaginary time (x_4). This model is equivalent to the Heisenberg ferromagnet, where each $n^a(x)$ represents the spin of a ferromagnet located at $\vec{x}$.

This model contains instantons. To find them, we look for finite action solutions of the equations of motion from (3.1). Now, the values of $n^a(x)$ can be considered to range over the surface of a three dimensional sphere, $\tilde{S}^2$. However, the two dimensional coordinate space with the point

at infinity identified is topologically equivalent to another three dimensional sphere, S^2. Therefore, any configuration, $n^a(x)$, can be thought of as a mapping of the coordinate sphere, S^2, onto the group sphere, $\tilde{S}^2$. It is well known that there exist an infinite number of classes of these mappings, where a mapping in one class cannot be continuously deformed into a mapping in a different class. The integer which characterizes the degree of a mapping is

$$Q=(1/8\pi)\int \epsilon_{abc}\,\epsilon_{\mu\nu}\, n^a\, \partial_\mu n^b\, \partial_\nu n^c\, d^2x. \qquad (3.2)$$

By switching to spherical coordinates on the group sphere, $n^a(x)=$ [sin $\theta(x)\cos\phi(x)$, sin $\theta(x)\sin\phi(x)$, cos $\theta(x)$], this becomes

$$Q=(1/4\pi)\int \sin\theta(x)\, d\,\theta(x)\, d\,\phi(x). \qquad (3.3)$$

Thus Q, called the topological charge, counts the number of times that the coordinate sphere, S^2, is covered by the mapping.

In each topological sector, the action has a specific minimum. Consider the inequality,

$$(\partial_\mu n^a + \epsilon_{abc}\,\epsilon_{\mu\nu}\, n^b \partial_\nu n^c)^2 \geq 0. \qquad (3.4)$$

Squaring and integrating over all space, we find

$$\int (\partial_\mu n^a)^2 \, d^2x \geq \int \epsilon_{abc} \epsilon_{\mu\nu} n^a \partial_\mu n^b \partial_\nu n^c \, d^2x, \tag{3.5}$$

or, for a mapping of degree Q,

$$S_Q \geq 4\pi Q/f. \tag{3.6}$$

Thus, the action in each sector is limited by (3.6). A configuration for which the bound is saturated will necessarily solve the equations of motion for $n^a(x)$, since that configuration will be at a local minimum. To saturate the bound, we let

$$\partial_\mu n^a + \epsilon_{abc} \epsilon_{\mu\nu} n^b \partial_\nu n^c = 0. \tag{3.7}$$

The solution to (3.7) is not immediately apparent. It is convenient to introduce new variables which are the stereographic projection of $n^a(x)$ onto a two dimensional plane,

$$n_1 = 2w_1 \, (w_1^2 + w_2^2 + 1)^{-1},$$

$$n_2 = 2w_2 \, (w_1^2 + w_2^2 + 1)^{-1},$$

and

$$n_3 = (w_1^2 + w_2^2 - 1)(w_1^2 + w_2^2 + 1)^{-1} . \tag{3.8}$$

These variables satisfy the constraint, $(n^a)^2 = 1$. In terms of the w_i, (3.7) becomes

$$\partial_\mu w_i = \epsilon_{ij} \epsilon_{\mu\nu} \partial_\nu w_j , \tag{3.9}$$

or more familiarly,

$$\frac{\partial w_1}{\partial x_1} = \frac{\partial w_2}{\partial x_2} \quad , \quad \frac{\partial w_1}{\partial x_2} = -\frac{\partial w_2}{\partial x_1} . \tag{3.10}$$

These are just the Cauchy-Riemann conditions. Their solution is $w = w_1 + iw_2 = f(z)$ with $z = x_1 + ix_2$. In general, $f(z)$ can be an arbitrary function of z, but requiring that S be finite only allows those $f(z)$'s which are rational functions of z. If $f(z)$ is written as $p(z)/q(z)$ with p and q polynomials, the topological charge is just the maximal degree of p and q.

The most general form for a configuration with topological charge Q is therefore

$$w(z) = c \prod_{i=1}^{Q} (z - a_i)(z - b_i)^{-1} . \tag{3.11}$$

Here a_i, b_i, and c are all complex. When $z=a_i$, (for a_i one of the zeroes of w), $\hat{n}$ points directly down ($\hat{n}=-\hat{e}_3$), and when $z=b_i$, $\hat{n}$ points directly up ($\hat{n}=+\hat{e}_3$).

There are two ways of writing the most general instanton with topological charge Q, depending on the boundary conditions one wishes to place on $\hat{n}$ at spatial infinity, $|z|\longrightarrow\infty$. We will first exhibit the two cases for a single instanton solution. For the first case, we let the $\hat{n}$ vector, which we will call the spin, at infinity be arbitrary, and

$$w=c(z-a_1)(z-b_1)^{-1}. \tag{3.12}$$

Here, a_1, b_1, and c are all complex. At spatial infinity, the spin is oriented such that w=c. From (3.8), it is clear that when z is near a_1, the spin points down, and likewise when z is near b_1, the spin points up. We will see later that this solution can be interpreted as a positive charge located at a_1 and a negative charge located at b_1.

The other case pins the spin up at infinity. This is made most general by performing a global O(3)/O(2) rotation, $R(\theta,\phi)$, on the spins to orient the spin at infinity in any desired direction, $\hat{n}\rightarrow R(\theta,\phi)\hat{n}$. The extra possible O(2) rotation is divided out since it represents the

rotation parallel to the 3-axis, which does nothing to a spin already pointing up. Denoting the solution with the spin up at infinity as w_u, we write

$$w_u = (z-z_1)(r_1)^{-1}. \tag{3.13}$$

Here, z_1 and r_1 are complex. Near $z=z_1$, the spin points down. The magnitude of r_1 is the size of the instanton. It is a measure of how fast the spin turns over from up at infinity to down at z_1. The phase of r_1 just orients the spins relative to the spatial axes.

We see that w_u is described by four real parameters, while w has six. Performing the rotation R on the $\hat{n}$ vector corresponding to w_u gives us two more parameters, θ and ϕ, equalizing the number of parameters for both cases.

The most general instanton with arbitrary spin at infinity is

$$w = c\prod_{i=1}^{Q}(z-a_i)(z-b_i)^{-1}. \tag{3.14}$$

This will be interpreted as a collection of positive charges at each a_i and negative charges at each b_i. For the spin pointing up at infinity, we have, for the most general

instanton,

$$w_u = [\sum_{i=1}^{Q} ((z-z_i)/r_i)^{-1}]^{-1}. \qquad (3.15)$$

When z is near any z_i, this looks like a single instanton located at z_i with size r_i, so this will be interpreted as a collection of single instantons located at each z_i. To make w_u most general, we must perform the rotation on the spin at infinity. The spin at infinity will point in the (θ, ϕ) direction when

$$w=[e^{i\phi}\cot(\theta/2)][w_u-\tan(\theta/2)][w_u+\cot(\theta/2)] . \qquad (3.16)$$

Let us now exhibit the general path integral for this model,

$$\int\prod_{x,a} dn^a\,\delta(n^2-1)\exp[-(1/2f)\int(\partial_\mu n^a)\cdot(\partial_\mu n^a)\,d^2x] \qquad (3.17)$$

$$=\int\prod_{x} dw^*\,dw(1+w^*w)^{-2}\exp[-(2/f)\int\partial_\mu w^*\,\partial_\mu w(1+w^*w)^{-2}].$$

The first path integral is expressed in terms of the unit spin vector, $\hat{n}$, with the δ-function imposing the proper constraint. The second path integral is expressed in terms of the previously introduced complex w, which automatically

satisfies the constraint. Ultimately, one wishes to compute correlation functions by inserting fields into the integrand of (3.17), but for now, we concern ourselves with the path integral as is, and consider its multi-instanton expansion. This will correspond to the quantity $(J_0 + J_1 + J_2 + \ldots)$ calculated in the last chapter.

We first consider the single instanton contribution to (3.17), which we will also call J_1. This calculation has been done in other places [12]; we will only sketch the method here.

As in the second chapter, w is expanded about its instanton solution, $w = w_i + \eta$, where w_i stands for an instanton configuration. We use the form (3.13) to express w_i. Again, only up to quadratic terms in η are kept. Letting (J_1/J_0) once again represent the amplitude for going through one instanton normalized by the amplitude with no instanton, we find

$$\frac{J_1}{J_0} = \exp(-S_i) \frac{\int \pi d\eta \exp[-(1/2f) \eta M \eta d^2x]}{\int \pi d\eta \exp[-(1/2f) \eta M_0 \eta d^2x]} . \qquad (3.18)$$

Here, $M(x)$ and $M_0(x)$ are second order differential operators. The single instanton action is

$$S_i = 4\pi/f, \qquad (3.19)$$

which can be seen from (3.6).

The small deviation, η , is expanded in eigenfunctions of M, and the product of the non-zero eigenvalues thereof is computed, with a like process occuring for M_o. Here, unlike the one dimensional case of Chapter II, the ratio of the eigenvalues of M to the corresponding free eigenvalues is infinite. This infinity is regulated using the Pauli-Villars technique. The ratio of the eigenvalues is just the ratio of the determinants of the M and M_o operators (det M/det M_o). This ratio is replaced by

$$(\det M/\det M_o)\prod_{i=1}^{R}[\det(M+m_i^2)/\det(M_o+m_i^2)]^{e_i}, \qquad (3.20)$$

where the m_i are large regulators of alternating metric, $e_i = \pm 1$. The integer R is chosen to render (3.20) finite, with $\sum_i e_i = -1$, $\sum_i e_i (m_i)^n = 0$, and $\sum_i e_i \ln m_i = -\ln \Lambda$, where Λ is the regulator mass. The infinity now appears as $\exp[\ln(|r_1|^2\Lambda^2)]$. The coupling constant f is then interpreted as the bare coupling at Λ and the infinity is removed by the one loop renormalization of the coupling constant, $4\pi/f(\Lambda) = 4\pi/f(\mu) + \ln(\Lambda^2/\mu^2)$. In the process, a renormalization point mass, μ , has appeared.

The operator M also has some zero eigenvalues. In Chapter II, a zero mode occured when the fluctuation η

corresponded to a simple translation of the instanton, i.e., the instanton was perturbed in a direction that did not change the total action of the configuration. In this model, there are six parameters which can be varied without changing the action. There are thus six zero modes of M; two correspond to translating the instanton, one to changing the size of the instanton, one to changing the orientation of the instanton relative to the spatial axes, and two which change the value of the spin at infinity. These parameters are all integrated, as "a" was integrated in (2.33).

Actually, the form of (J_1/J_0) can be easily guessed. It should be dimensionless, and must have both z_1 and r_1 integrated, so it must contain the factor $d^2z_1\, d^2r_1\, |r_1|^{-4}$. This is the only dimensionless quantity which preserves the translational invariance. From the action, we get $\exp[-4\pi/f(\mu)+\ln(|r_1|^2\mu^2)]$, with the coupling at the renormalization point, μ. This is the only renormalization group invariant expression possible, i.e., it satisfies $-4\pi/f(\mu)+\ln(|r_1|^2\mu^2)=-4\pi/f(\mu')+\ln(|r_1|^2\mu'^2)$. The final answer for (J_1/J_0) is

$$(J_1/J_0)=\int d^2z_1\, d^2r_1\ |r_1|^{-2}\exp[-S_1+2\ln(\mu)], \qquad (3.21)$$

where we have separated r_1 from μ in the logarithm. Notice that there is no integration over θ or ϕ in this equation.

Since θ and ϕ change the global orientation of the spins, the zero modes of M are also zero modes of M_o. Therefore, the same integration $\int \sin\theta \, d\theta \, d\phi$ occurs in both J_1 and J_o, and divides out.

The expression (3.21) has a new problem not encountered in Chapter II. For a one dimensional situation, the scale size of the instanton is fixed by the potential; here it is free to vary as it wishes, and the integration over r_1 diverges at both large and small r_1. Presumably, despite this, any physical quantities such as correlation functions will still be finite.

A program similar to the one in the previous chapter to calculate the effect of many instantons cannot be carried out. There, it was assumed that two instantons had very little overlap (the dilute gas approximation). This was a very good approximation since the instantons have exponential tails and the interaction between them falls exponentially at large distances. Here, the spins approach their classical vacuum values slowly, e.g., $n_3 = 1 + O(|r_1/z|^2)$ as $|z| \longrightarrow \infty$. So, there is no sharp distinction even between widely separated instantons. Furthermore, in integrating over the sizes of the instantons, one eventually will be considering instantons much larger than any separation. Dense, multi-instanton configurations thus must be calculated

exactly.

Fortunately, two remarkable papers, one by Berg and Luscher [13], the other by Fateev, Frolov, and Schwarz [14], have recently appeared which succeed in evaluating the quantum fluctuations around multi-instanton configurations. They write instantons in the form of (3.14) instead of the form of (3.15). Their formula for the contribution to the vacuum-vacuum amplitude for topological charge Q is

$$(1/Q!)^2\,[L\mu^4\exp(-4\pi/f(\mu))/f^2(\mu)]^Q \times \int\prod_{i=1}^{Q} d^2a_i\,d^2b_i \prod_{i<j}^{Q}[\,|a_i-a_j|^2\cdot|b_i-b_j|^2\,]\prod_{i,j}^{Q}|a_i-b_j|^{-2} \quad . \tag{3.22}$$

Here, L is a constant (the value of which is not germane to this discussion), μ is the renormalization point, and $f(\mu)$ is the renormalized coupling constant at the renormalization point. If the integrand of (3.22) is written as the exponential of a logarithm, (3.22) is easily recognized as a term in the grand canonical partition function of a neutral Coulomb gas, with positive charges located at each a_i and negative charges at each b_i. We will discuss the meaning of this result at the end of this chapter.

The purpose of this chapter is to transform the result of (3.22), written in terms of the a_i and b_i

parameters, into the more familiar language of locations and sizes of instantons, z_i and r_i. One expects a general form

$$(1/Q!)[L\exp(-4\pi/f(\mu))/f^2(\mu)]^Q \times \int\prod_{i=1}^{Q}[d^2z_i\, d^2r_i\, |r_i|^{-2}\exp(2\ln(\mu))]\exp(U_Q), \tag{3.23}$$

where $U_Q(z_1,z_2,\ldots,z_Q;r_1,r_2,\ldots,r_Q)$ represents the effective potential between instantons, arising from evaluating the quantum fluctuations about the Q instanton configuration. In the dilute gas region, $|r_i| << |a_i - a_j|$, U_Q would be expected to go to zero. Naturally, for Q=1, $U_1=0$.

The surprising result will be that $U_Q=0$ for all Q. In other words, the naive form one would write down with confidence only in the dilute gas region is an exact expression. Thus, even in regions where the instantons cannot be considered to be widely separated, the dilute gas formula nonetheless remains entirely accurate.

We will first demonstrate the transformation for the case of one instanton, showing the general relationship between the two sets of parameters, and then go on to prove the general result for any number of instantons.

The two forms for the instanton, w and w_u, must be

equivalent after performing the global rotation (3.16) upon w_u. Setting the two equal leads to

$$c\,\frac{z-a_i}{z-b_i} = e^{i\phi}\cot(\theta/2)\;\;\frac{(z-z_i)-r_i\tan(\theta/2)}{(z-z_i)+r_i\cot(\theta/2)}\,. \tag{3.24}$$

Equating coefficients gives

$$a_i = z_i + r_i\tan(\theta/2),$$

$$b_i = z_i - r_i\cot(\theta/2),$$

and

$$c = e^{i\phi}\cot(\theta/2). \tag{3.25}$$

Since c is a function of only θ and ϕ, it pertains only to the factor $\int\sin(\theta)\,d\theta\,d\phi$, which is divided out. The fact that the form (3.22) contains no term with c reflects the overall O(3) symmetry of the theory.

Inverting (3.25) gives

$$z_i = [a_i\cot(\theta/2)+b_i\tan(\theta/2)]\,[\tan(\theta/2)+\cot(\theta/2)]^{-1}$$

and

$$r_i = [a_i - b_i]\,[\tan(\theta/2)+\cot(\theta/2)]^{-1}, \tag{3.26}$$

so we see that z_i is effectively a center of mass coordinate for a_i and b_i, and r_i is an effective relative coordinate. So, considering an instanton to be composed of a pair of unlike charges, its location is given by the center of mass of those charges, and its size is proportional to the distance between them.

The single instanton Coulomb gas contribution to the partition function is

$$K \int d^2 a_i\, d^2 b_i \quad |a_i - b_i|^{-2}, \tag{3.27}$$

where we let K be the fugacity, $K = L(\mu^2/f^2)\exp(-4\pi/f)$. Then, changing variables gives

$$|a_i - b_i|^{-2} = G^{-1}\,|r_i|^{-2} \tag{3.28}$$

and

$$d^2 a_i\, d^2 b_i = |\partial(a_i, b_i)/\partial(z_i, r_i)|^2\; d^2 z_i\, d^2 r_i$$

$$= G^{+1}\; d^2 z_i\, d^2 r_i \quad . \tag{3.29}$$

Since it will be occuring quite often, we designate

$G=[\tan(\theta/2)+\cot(\theta/2)]$. In the above equations, we treat all variables as complex, and the complex Jacobian is computed in the standard way. Then, the volume elements are related by the square of the norm of the Jacobian.

Finally, we find

$$K\int d^2a_1\, d^2b_1\, |a_1-b_1|^{-2} = K\int d^2z_1\, d^2r_1\, |r_1|^{-2} \quad , \tag{3.30}$$

as promised. The dependence on c cancels, as it must, since on the right hand side of (3.30), everything is defined relative to the spin at infinity, and is therefore independent of c.

For two instantons, the solution with spin up at infinity is

$$w_u=[(z-z_1)(z-z_2)][r_1(z-z_2)+r_2(z-z_1)]^{-1} \quad . \tag{3.31}$$

Equating this with the other form for a two instanton solution gives

$$\frac{(z-a_1)(z-a_2)}{(z-b_1)(z-b_2)} = \tag{3.32}$$

$$\frac{(z-z_1)(z-z_2)-[r_1(z-z_2)+r_2(z-z_1)]\tan(\theta/2)}{(z-z_1)(z-z_2)+[r_1(z-z_2)+r_2(z-z_1)]\cot(\theta/2)} \quad .$$

From this equation, a_1, a_2, b_1, and b_2 can be solved for in terms of z_1, z_2, r_1, and r_2 simply by factoring the quadratic polynomials on the right hand side of the equation. This factoring could conceivably (but not practically) be done for three or four instanton solutions, but would fail (in general) for five or more. Therefore, with an eye on generalizing the procedure, a different method of making the variable change will be employed.

We first regroup the terms in (3.32), and use these regroupings to define new variables,

$$\frac{z^2-A_1 z+A_1 A_2}{z^2-B_1 z+B_1 B_2} = \frac{(z^2-Z_1 z+Z_1 Z_2)-(R_1 z-R_2)\tan(\theta/2)}{(z^2-Z_1 z+Z_1 Z_2)+(R_1 z-R_2)\cot(\theta/2)} \quad . \tag{3.33}$$

The new variables are chosen to be

$$A_1 = a_1 + a_2 \qquad B_1 = b_1 + b_2 \tag{3.34}$$

$$A_2 = a_1 a_2 \qquad B_2 = b_1 b_2$$

on the left hand side, and

$$Z_1 = z_1 + z_2 \qquad R_1 = r_1 + r_2 \tag{3.35}$$

$$Z_2 = z_1 z_2 \qquad R_2 = r_1 z_2 + r_2 z_1$$

on the right hand side. Simple inspection allows us to relate the two sets of variables,

$$A_1 = Z_1 + R_1 \tan(\theta/2), \qquad (3.36)$$

$$A_2 = Z_2 + R_2 \tan(\theta/2),$$

and

$$B_1 = Z_1 - R_1 \cot(\theta/2), \qquad (3.37)$$

$$B_2 = Z_2 - R_2 \cot(\theta/2).$$

For the general case of many instantons, (3.33-3.37) are easily generalized.

Let us now step back and survey the route by which we will travel from (a_i, b_i) to (z_i, r_i). The variable changes will be as follows:

$$(a_i, b_i) \longrightarrow (A_i, B_i) \longrightarrow (Z_i, R_i) \longrightarrow (z_i, r_i), \qquad (3.38)$$

and the quantity we wish to evaluate is the integrand of (3.22),

$$\frac{\prod_{i=1}^{2} d^2 a_i\, d^2 b_i \;\; |a_1 - a_2|^2 \cdot |b_1 - b_2|^2}{|a_1 - b_1|^2 \cdot |a_1 - b_2|^2 \cdot |a_2 - b_1|^2 \cdot |a_2 - b_2|^2}\,. \qquad (3.39)$$

The Jacobian of the volume factor occurs as

$$\prod_i d^2 a_i\, d^2 b_i = \left|\frac{\partial(a_i, b_i)}{\partial(z_i, r_i)}\right|^2 \prod_i d^2 z_i\, d^2 r_i \tag{3.40}$$

$$= \left|\frac{\partial(a_i, b_i)}{\partial(A_i, B_i)}\right|^2 \cdot \left|\frac{\partial(A_i, B_i)}{\partial(Z_i, R_i)}\right|^2 \cdot \left|\frac{\partial(Z_i, R_i)}{\partial(z_i, r_i)}\right|^2 \prod_i d^2 z_i\, d^2 r_i$$

$$= \left|\frac{\partial(A_i, B_i)}{\partial(a_i, b_i)}\right|^{-2} \cdot \left|\frac{\partial(A_i, B_i)}{\partial(Z_i, R_i)}\right|^2 \cdot \left|\frac{\partial(Z_i, R_i)}{\partial(z_i, r_i)}\right|^2 \prod_i d^2 z_i\, d^2 r_i .$$

These Jacobians all factor,

$$\frac{\partial(A_i, B_i)}{\partial(a_i, b_i)} = \frac{\partial(A_i)}{\partial(a_i)} \cdot \frac{\partial(B_i)}{\partial(b_i)} = (a_1 - a_2)(b_1 - b_2), \tag{3.41}$$

$$\frac{\partial(A_i, B_i)}{\partial(Z_i, R_i)} = \frac{\partial(A_1, B_1)}{\partial(Z_1, R_1)} \cdot \frac{\partial(A_2, B_2)}{\partial(Z_2, R_2)} = G^2,$$

and

$$\frac{\partial(Z_i, R_i)}{\partial(z_i, r_i)} = \frac{\partial(Z_i)}{\partial(z_i)} \cdot \frac{\partial(R_i)}{\partial(r_i)} = (z_1 - z_2)(z_1 - z_2),$$

giving us

$$\prod_i d^2 a_i\, d^2 b_i = \frac{|z_1 - z_2|^4 \cdot G^4 \cdot \prod d^2 z_i\, d^2 r_i}{|a_1 - a_2|^2 \cdot |b_1 - b_2|^2} . \tag{3.42}$$

For the corresponding steps with many instantons, we now use the fact, proven in the appendix, that the Jacobian in going from C_i to c_i is

$$|\partial(C_i)/\partial(c_i)|^2 = \prod_{i<j}^{n} |c_i - c_j|^2 \quad , \tag{3.43}$$

where the c_i are defined by

$$z^n - C_1 z^{n-1} + \ldots + (-1)^{n-1} C_{n-1} z + (-1)^n C_n \tag{3.44}$$

$$= (z-c_1)(z-c_2)\ldots(z-c_n) .$$

These describe precisely the relation between the pairs (A_i, a_i), (B_i, b_i), and (Z_i, z_i). Then, we find in general

$$|\partial(A_i)/\partial(a_i)|^2 = \prod_{i<j}^{k} |a_i - a_j|^2 \quad , \tag{3.45}$$

$$|\partial(B_i)/\partial(b_i)|^2 = \prod_{i<j}^{k} |b_i - b_j|^2 \quad ,$$

$$|\partial(A_i, B_i)/\partial(Z_i, R_i)|^2 = G^{2k} \quad ,$$

and

$$|\partial(Z_i)/\partial(z_i)|^2 = \prod_{i<j}^{k} |z_i - z_j|^2 \quad .$$

The $(R_i \longrightarrow r_i)$ transformation is linear in terms of these

variables, with the same coefficients as the Jacobian in $(Z_i \longrightarrow z_i)$, $R_i = \sum_j (\partial Z_i / \partial z_j) r_j$, so that

$$|\partial(R_i)/\partial(r_i)|^2 = \prod_{i<j}^{k} |z_i - z_j|^2 , \tag{3.46}$$

also.

Therefore, the general Jacobian of the volume factor is

$$\prod_{i=1}^{k} d^2 a_i \, d^2 b_i = \frac{\prod_{i<j}^{k} |z_i - z_j|^4 \cdot G^{2k} \prod d^2 z_i \, d^2 r_i}{\prod_{i<j}^{k} [\, |a_i - a_j|^2 \cdot |b_i - b_j|^2 \,]} . \tag{3.47}$$

To complete the transformation, we must find $\prod |a_i - b_j|^2$ in terms of the (z_i, r_i). For two instantons, we define a new set of variables, C_i, (i=1, 2, 3, 4), by

$$z^4 - C_1 z^3 + C_2 z^2 - C_3 z + C_4 = (z-a_1)(z-a_2)(z-b_1)(z-b_2)$$

$$= (z^2 - A_1 z + A_2)(z^2 - B_1 z + B_2) . \tag{3.48}$$

The second equality follows from the definitions of A_i and B_i. These C_i's are no more than an artificial set of variables which are used as a tool to calculate $\prod |a_i - b_j|^2$. By (3.43),

$$|\partial(C_i)/\partial(a_i,b_i)|^2 = |a_1-a_2|^2\cdot|b_1-b_2|\prod_{i,j=1}^{2}|a_i-b_j|^2 \ . \qquad (3.49)$$

However, from (3.43-3.44) we also have $|\partial(A_i)/\partial(a_i)|^2 = |a_1-a_2|^2$ and $|\partial(B_i)/\partial(b_i)|^2 = |b_1-b_2|^2$, so that, using the product rule for Jacobians, we finally get

$$\prod_{i,j=1}^{2}|a_i-b_j|^2 = |\partial(C_i)/\partial(A_i,B_i)|^2 \ , \qquad (3.50)$$

which is the desired quantity.

With a simple generalization of (3.48), it is simple to show the result for an arbitrary number of instantons,

$$\prod_{i,j=1}^{k}|a_i-b_j|^2 = |\partial(C_i)/\partial(A_i,B_i)|^2 \ . \qquad (3.51)$$

To evaluate (3.50-3.51), the relation between A, B, and C is found from the second equality of (3.48), and the Jacobian, written in terms of (A_i,B_i) is first converted to (Z_i,R_i) and finally to (z_i,r_i), yielding

$$\prod_{i,j=1}^{2}|a_i-b_j|^2 = |r_1|^2\cdot|r_2|^2\cdot|z_1-z_2|^4\cdot G^4 \qquad (3.52)$$

for two instantons, and

$$\prod_{i,j=1}^{k} |a_i - b_j|^2 = \left[\prod_{i=1}^{k} |r_i|^2\right]\left[\prod_{i<j}^{k} |z_i - z_j|^4\right] \cdot G^{2k} \qquad (3.53)$$

for the general case. Equation (3.52) will be derived here, illustrating the method used for the general case, but the general case, (3.53), will be proven in the appendix.

The Jacobian in (3.50) is

$$\partial(C_i)/\partial(A_i, B_i) = \begin{vmatrix} 1 & 1 & 0 & 0 \\ B_1 & A_1 & 1 & 1 \\ B_2 & A_2 & B_1 & A_1 \\ 0 & 0 & B_2 & A_2 \end{vmatrix} \qquad (3.54)$$

$$= \begin{vmatrix} 1 & 1 & 0 & 0 \\ Z_1 - R_1\cot(\theta/2) & Z_1 + R_1\tan(\theta/2) & 1 & 1 \\ Z_2 - R_2\cot(\theta/2) & Z_2 + R_2\tan(\theta/2) & Z_1 - R_1\cot(\theta/2) & Z_1 + R_1\tan(\theta/2) \\ 0 & 0 & Z_2 - R_2\cot(\theta/2) & Z_2 + R_2\tan(\theta/2) \end{vmatrix}.$$

Instead of evaluating this determinant as is, and then performing the variable changes, we will instead apply standard determinant identities on (3.54), and evaluate the determinant only after making the variable changes. One such identity says that any column of a determinant may have added

to it any multiple of any other column. Applying this to (3.54) leads to

$$\partial(C_i)/\partial(A_i,B_i) = G^2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ Z_1 & R_1 & 1 & 0 \\ Z_2 & R_2 & Z_1 & R_1 \\ 0 & 0 & Z_2 & R_2 \end{vmatrix} . \tag{3.55}$$

One factor of G came from each pair of columns. Calling the remaining determinant D, we have

$$D = \begin{vmatrix} r_1+r_2 & 1 & 0 \\ r_1 z_2+r_2 z_1 & z_1+z_2 & r_1+r_2 \\ 0 & z_1 z_2 & r_1 z_2+r_2 z_1 \end{vmatrix} . \tag{3.56}$$

Recombining the first two columns, we are left with

$$D = \begin{vmatrix} r_1 z_1+r_2 z_2 & r_1+r_2 \\ r_1 z_1 z_2+r_2 z_1 z_2 & r_1 z_2+r_2 z_1 \end{vmatrix} . \tag{3.57}$$

The interesting thing about the matrix corresponding to D is that it factorizes into three matrices. Thus, D is the product of the determinants of those three matrices,

$$D=\begin{vmatrix} 1 & 1 \\ z_2 & z_1 \end{vmatrix} \times \begin{vmatrix} r_1 & 0 \\ 0 & r_2 \end{vmatrix} \times \begin{vmatrix} z_1 & 1 \\ z_2 & 1 \end{vmatrix} \quad . \tag{3.58}$$

Each of these determinants is easily computed, giving (3.52) directly. This factorization occurs for the general multi-instanton case as well.

Thus, the integrand of (3.22) transforms into

$$\prod_{i=1}^{k} d^2 a_i \, d^2 b_i \prod_{i<j}^{k} [\, |a_i - a_j|^2 \cdot |b_i - b_j|^2 \,] \prod_{i,j}^{k} |a_i - b_j|^{-2} \tag{3.59}$$

$$= \prod_{i=1}^{k} [d^2 z_i \, d^2 r_i \, |r_i|^{-2} \,] .$$

Finishing the transformation requires one final factor. The Coulomb gas partition function contains a factor $(1/k!)^2$. Due to the exchange symmetry between all the a_i's and all the b_i's, each integration requires a factor of $(1/k!)$ to avoid double counting any configuration. However, with z_i and r_i, these two variables appear as a pair and must be interchanged as a pair. Thus, the partition function written in these variables requires only a single factor of $(1/k!)$.

We have now demonstrated the equivalence of the Coulomb and dilute instanton gas pictures, that is, that $U_Q=0$

for all Q. Such a result is completely unexpected, for while instantons do not interact classically, there is no reason that the quantum fluctuations should do likewise. Yet, by choosing the proper variables, z_i and r_i, we can cause the interaction to vanish. This is not surprising in and of itself, but that the proper variables are precisely those variables for a dilute instanton gas.

A fair amount is known about the classical neutral Coulomb gas [15]. The Coulomb gas at temperature T with k positive charges and k negative charges has a potential

$$V=-\sum_{i<j}^{k} \ln |a_i - a_j|^{q^2/k_B T} - \sum_{i<j}^{k} \ln |b_i - b_j|^{q^2/k_B T} + \sum_{i,j}^{k} \ln |a_i - b_j|^{q^2/k_B T}, \qquad (3.60)$$

so that we have the $T \longrightarrow (q^2/2k_B)$ limit of the Coulomb gas. The temperature $T=(q^2/2k_B)$ is the point where the gas condenses from a plasma phase to a molecular phase [16]. Furthermore, in the grand canonical ensemble at fixed chemical potential, the particle density diverges as $T \longrightarrow (q^2/2k_B)$. In the present context, the grand canonical ensemble corresponds to the sum over all values of Q, and the charged particles in the Coulomb gas correspond to points where the non-linear σ model $\hat{n}$ vector is $\pm\hat{e}_3$. Thus, we have a situation in which the instantons are sitting directly on top of each other, but as we have shown, without interacting in the least. The fact that we have infinite density might

be worrisome, except that physical quantities have finite limits as $T \longrightarrow (q^2/2k_\beta)$. By physical quantities we mean propagators, vertex functions, etc. The vacuum to vacuum path integral itself is just the shift above the free vacuum, and is not of direct physical interest. When a propagator is calculated, the divergences arising from the r_i integrations cancel in the ratio.

It has been shown that observables in the massive Thirring model have corresponding observables in the classical Coulomb gas [16,17]. The Thirring model has the Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi - (1/2)g(\bar{\psi}\gamma_\mu\psi)^2 , \qquad (3.61)$$

where $m = c(\mu/f(\mu))\exp[-4\pi/f(\mu)]$ (c is an uninteresting constant) and $g = (2k_\beta T/q^2 - 1)\pi$. Thus, the observables in the $T = (q^2/2k_\beta)$ Coulomb gas have corresponding observables in the <u>free</u> massive Dirac field theory. So, the instanton expansion has been sufficient to show dynamic mass generation in the non-linear σ model. In other words, by calculating the quantum fluctuations around solutions of the classical equations of motion, a model which on a purely classical level is scale invariant, and therefore massless, has been found to describe massive behavior.

However, the above instanton expansion cannot be telling us the whole story for the non-linear σ model. There also exist anti-instanton solutions to the field equations, found by reversing the sign in the inequality (3.4). A full description of the theory requires using a mixture of both instanton and anti-instanton solutions. We can show this by exhibiting a correlation function for which the expansion with only instantons fails miserably. First we consider the quantity

$$\langle \rho(x) \rangle = \sum_{Q=-\infty}^{+\infty} \langle \rho(x) \rangle_Q = 0. \tag{3.62}$$

Here, $\langle \rho(x) \rangle$ is the expectation value of the instanton density, and $\langle \rho(x) \rangle_Q$ is the expectation value of the instanton density in the Q instanton sector. If we let $\rho(x)$ contain configurations with instantons or anti-instantons only, with no mixtures of both, the plus and minus symmetry on Q forces the sum to zero.

Let us now consider

$$\langle \rho(x)\rho(0) \rangle = \sum_{Q=-\infty}^{+\infty} \langle \rho(x)\, \rho(0) \rangle_Q \ . \tag{3.63}$$

By cluster decomposition properties, as x gets large, $\langle \rho(x)\rho(0) \rangle$ should go to $\langle \rho(x) \rangle \langle \rho(0) \rangle$. However, with only instanton configurations, each term in the sum on the right

hand side of (3.63) is greater than zero, preventing $\langle\rho(x)\rho(0)\rangle$ from ever approaching zero. Therefore, including only instantons and only anti-instantons can lead to problems with some correlation functions. What is needed are configurations which contain <u>both</u> instantons and anti-instantons. This will accomplish the desired vanishing of $\langle\rho(x)\rangle$ on a local basis, and give the correlation function $\langle\rho(x)\rho(0)\rangle$ the correct cluster properties.

Where does this leave us? Classically, instantons do not interact, since $S_Q = QS_1$. However, instantons and anti-instantons do interact, that is, S(instanton,anti-instanton) $\neq$ S(instanton) + S(anti-instanton), so there must be an effective potential between instantons and anti-instantons. The effects of interactions between instantons and anti-instantons are important in any complete calculation. The fact that instantons do not interact either classically or quantum mechanically ensures that the instanton-anti-instanton classical effects will be dominant in any calculation. We will not discuss the difficult problem of computing instanton-anti-instanton interactions in this thesis.

One can hope that the situation in gauge theory will be similar. The non-interaction of instantons in the non-linear σ model raises hopes that they do not interact in

gauge theory, or if they do, that their effects will be small. These results give added credence to a program such as that of Callan, Dashen, and Gross [18,19,20] in calculating with an instanton-anti-instanton gas in gauge theories.

IV. THE SIGMA CROSS SIGMA MODEL

In this chapter, we introduce a new class of models which are four dimensional generalizations of the two dimensional non-linear σ model. The motive for doing such a thing is that we hope to lift two dimensional σ model results to these four dimensional theories and eventually relate those results to non-Abelian gauge theories. The reason that we will be able to relate the two is that while these models are still non-linear σ models, their actions are also in non-Abelian gauge theory form. These models thus provide a link between non-linear σ models on one hand and non-Abelian gauge theories on the other. They are topologically non-trivial, and possess the same SU(2) instantons as non-Abelian gauge theory. In addition, they have a 1/N expansion and are asymptotically free, though to a lesser degree than SU(2) gauge theory.

Actually, the non-linear σ model is just the first in a series of two dimensional models, the CP(N-1) models [21,22]. The new class of models we will introduce generalize the entire set. Before we generalize, we will make a short detour and introduce the CP(N-1) models, then go on to generalize the $O(3)\cong CP(1)$ σ model and finally extend it to generalize all CP(N-1) σ models.

The CP(N-1) models contain complex fields, $z_\alpha(x)$, $(\alpha=1,2,\ldots,N)$, with all fields related by a space-time dependent phase identified. This introduces a U(1) gauge field. The action is

$$S=(N/f)\int \overline{D_\mu z}\cdot D_\mu z \; d^2x, \qquad |z_\alpha|^2=1, \tag{4.1}$$

with the covariant derivative $D_\mu=\partial_\mu-\bar{z}\cdot\partial_\mu z$, so that the quantity $\bar{z}\cdot\partial_\mu z$ is acting like a U(1) gauge field.

Specializing to the case N=2, the O(3) non-linear σ model and the CP(1) model are equivalent. Here the fields, z_α, are complex spinors and members of the group $SU(2)\cong O(3)\cong CP(1)$ and are related to the σ model $\hat{n}$ vector. The U(1) gauge field appearing in the covariant derivative D_μ has its origen in the fact that the O(3) n vector may have an O(2)=U(1) rotation performed about its own axis without affecting it. The formula relating the $\hat{n}$ vector to z is

$$n^a=z_\alpha\sigma^a_{\alpha\beta}z_\beta. \tag{4.2}$$

Substituting this into the σ model action, (3.1), leads directly to the CP(1) action, (4.1). In terms of w, the stereographic projection of $\hat{n}$, z takes a simple form,

$$z=\begin{pmatrix} w \\ 1 \end{pmatrix}(1+w^*w)^{-1/2}. \tag{4.3}$$

Then, the gauge field is

$$A_\mu = \frac{1}{2} \frac{w^* \partial_\mu w - w \partial_\mu w^*}{(1+w^* w)} \quad . \tag{4.4}$$

The only role the gauge field plays in this theory is that the topological charge can be written in terms of the field strengths from A_μ,

$$Q = (-i/4\pi) \int \epsilon_{\mu\nu} F_{\mu\nu} \, d^2x. \tag{4.5}$$

The key to generalizing the O(3) σ model to four dimensions and obtaining similar instanton structure is to generalize the formula which bounds the action, (3.4). For four dimensional space it is natural to consider the group O(5), in complete analogy with using O(3) in a two dimensional model. We introduce a unit vector in O(5) group space, $n^a(x)$, (a=1, 2, 3, 4, 5), so that $n^a(x)$ lies on a five dimensional sphere, $\tilde{S}^4$. However, four dimensional coordinate space with the point at infinity identified is topologically equivalent to another five dimensional sphere, S^4. The mappings of coordinate space S^4 onto the O(4) group space $\tilde{S}^4$ fall into distinct classes that cannot be continuously deformed into one another. This is in complete analogy to the two dimensional case. The integer which characterizes

the degree of the mapping is now

$$Q=(1/64\pi^2)\int \epsilon_{abcde}\,\epsilon_{\mu\nu\lambda\sigma}\; n^a\,\partial_\mu n^b\,\partial_\nu n^c\,\partial_\lambda n^d\,\partial_\sigma n^e\,d^4x. \qquad (4.6)$$

Switching to spherical coordinates on the group sphere shows that Q, the topological charge, counts the number of times that the group sphere is covered by the mapping.

There is a natural route from the topological number, Q, to the action of the theory. In both the non-linear σ model in two dimensions and non-Abelian gauge theory in four dimensions, the action for instanton solutions is equal to the modulus of the topological charge. For the specific case of the O(3) σ model, this condition is true when the equality holds in (3.4). When the equality is squared, action terms come from the squares of the two components, while the topological charge comes from the cross term. We thus seek a similar inequality that will give the O(5) topological charge as a cross term and we will designate the squared terms as the action of our model. The requisite inequality is

$$(\partial_\mu n^a\,\partial_\nu n^b-\partial_\nu n^a\,\partial_\mu n^b+\epsilon_{abcde}\,\epsilon_{\mu\nu\lambda\sigma}\,n^c\,\partial_\lambda n^d\,\partial_\sigma n^e/2)^2\geq 0, \qquad (4.7)$$

which by the above procedure gives the action

$$S=(1/8g^2)\int(\partial_\mu n^a\,\partial_\nu n^b-\partial_\nu n^a\,\partial_\mu n^b)^2\,d^4x. \qquad (4.8)$$

As in the O(3) σ models, the inequality bounds the action,

$$S\geq 8\pi^2 Q/g^2. \qquad (4.9)$$

The bound is saturated by the instanton solutions. We will return to these later.

Thus, the action of this new model is given by the square of a quantity which is anti-symmetric in both its spatial and its group indices. This condition is forced on us by the ϵ-symbols in the topological charge, (4.6). The totally antisymmetric quantity $\partial_\mu n^a\,\partial_\nu n^b-\partial_\nu n^a\,\partial_\mu n^b = F^{ab}_{\mu\nu}$ represents the rate of sweeping out area on the ab plane tangent to the group sphere. For the non-linear σ model, the analogous quantity, $\partial_\mu n^a$, represented the rate of sweeping out length on the line in the a direction tangent to the group sphere. Since $F^{ab}_{\mu\nu}$ is essentially the 2-form which is the exterior product of two 1-forms, $\partial_\mu n^a$, these models are aptly called $\sigma\wedge\sigma$ models.

We note at this point that it is convenient to make the stereographic projection of $\hat{n}$ as we did for the O(3) model,

$$n_\alpha = 2w_\alpha \left(w_1^2 + w_2^2 + w_3^2 + w_4^2 + 1\right)^{-1}, \qquad (4.10)$$

and

$$n_5 = \left(w_1^2 + w_2^2 + w_3^2 + w_4^2 + 1\right)\left(w_1^2 + w_2^2 + w_3^2 + w_4^2 + 1\right)^{-1}.$$

This handles the constraint $(n^A)^2 = 1$, and will have later use when we hunt for the instantons of the theory.

The action, S, is terms of w is simply found,

$$S = (2/g^2) \int \left(\partial_\mu w_\alpha \partial_\nu w_\beta - \partial_\nu w_\alpha \partial_\mu w_\beta\right)^2 (1+w^2)^{-4} \, d^4x. \qquad (4.11)$$

As in the O(3) σ model, the $\sigma\Lambda\sigma$ model has a gauge degree of freedom. This is the freedom to rotate the O(5) $\hat{n}$ vector about its axis, which generates an $O(4)=SU(2)xSU(2)$ gauge invariance. We will only use one of the SU(2) subgroups and define a gauge field as in the CP(1) model. To do so, we need to generalize the complex $w=w_1+iw_2$ of (4.4) to accomodate the four w_α that occur in the stereographic projection of the O(5) $\hat{n}$ vector. The natural object for this is the quaternion. The quaternion $\tilde{w}$ is defined by $\tilde{w}=-i\tau_\alpha w_\alpha$, where $\tau_4=i$ and $\tau_i=\sigma_i$. The 2X2 matrices σ_i are just the Pauli spin matrices. Thus, $\tilde{w}$ is simply a 2X2 matrix.

In complete analogy to (4.4) we define the SU(2) gauge field

$$A_\mu = \frac{1}{2} \frac{\tilde{w}^\dagger \partial_\mu \tilde{w} - \partial_\mu \tilde{w}^\dagger \tilde{w}}{(1+\tilde{w}^\dagger \tilde{w})} \quad . \tag{4.12}$$

The set of gauge fields, A_μ, that can be found in this manner is not the complete set of all possible SU(2) Yang-Mills fields. We have the subset which is constrained by (4.12).

We can compute the Yang-Mills field strengths from the gauge field, (4.12), using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. It is straightforward to show

$$F_{\mu\nu} = \frac{\partial_\mu \tilde{w}^\dagger \partial_\nu \tilde{w} - \partial_\nu \tilde{w} \partial_\mu \tilde{w}}{(1+\tilde{w}^\dagger \tilde{w})^2} \tag{4.13}$$

$$= \frac{\partial_\mu w_\alpha \partial_\nu w_\beta - \partial_\nu w_\alpha \partial_\mu w_\beta}{(1+w^2)^2} \frac{\tau_\alpha^\dagger \tau_\beta - \tau_\beta^\dagger \tau_\alpha}{2} .$$

The second equation is in a form reminiscent of the action of the model, (4.11), and in fact, squaring $F_{\mu\nu}$ and taking its trace gives

$$S = (2/g^2) \int (\partial_\mu w_\alpha \partial_\nu w_\beta - \partial_\nu w_\alpha \partial_\mu w_\beta)^2 \, (1+w^2)^{-4} \, d^4x \tag{4.14}$$

$$= (1/2g^2) \int \mathrm{tr}[F_{\mu\nu} F_{\mu\nu}] d^4x .$$

Note that the last equality of (4.14) is the standard gauge theory action. As in the CP(1) model, the topological charge can also be related to the field strengths. We get

$$Q=-(1/32\pi^2)\int \epsilon_{\mu\nu\lambda\sigma}\, \mathrm{tr}\,[F_{\mu\nu}\, F_{\lambda\sigma}]\, d^4x, \qquad (4.15)$$

which is the standard gauge theory expression. It is interesting that the topological charge in gauge theory has an alternate definiton as the number of times $n^a(x)$ covers the O(5) sphere as $\vec{x}$ moves over all space.

We have now succeeded in writing our $\sigma\wedge\sigma$ model as a non-Abelian gauge theory with an SU(2) gauge field. It should again be stressed that the gauge theory form, (4.14), is not full SU(2) gauge theory, since the gauge fields are constrained by (4.12). It is also worth pointing out that the gauge field, A , exists also in a generic O(5) σ model with $S=\int(\partial_\mu n^a)^2\, d^4x$, but its action has no special connection with the gauge theory action, as also occurs in the two dimensional O(3) σ model.

We can also write A_μ in a form reminiscent of the CP(N-1) models by introducing a quaternionic spinor, ψ. Letting each entry in the spinor be a quaternionic 2X2 matrix, we find

$$\psi = \begin{pmatrix} \tilde{w} \\ 1 \end{pmatrix} (1+\tilde{w}^{\dagger}\tilde{w})^{-1/2} \tag{4.16}$$

in direct analogy with (4.3), giving

$$A_{\mu} = \psi^{\dagger} \partial_{\mu} \psi , \tag{4.17}$$

which exactly mimics the CP(1) case.

In fact, we can find a simple relation between the $\hat{n}$ vector and ψ, as was the case in (4.2),

$$n^{a} \delta_{ij} = \psi_{i\alpha}^{\dagger} \gamma^{a}_{\alpha\beta} \psi_{\beta j} . \tag{4.18}$$

The γ matrices are the standard Dirac matrices.

The O(5) model discussed up to now can be generalized as the O(3) σ model was generalized to the CP(N-1) models. All we have to do is extend the spinor ψ to having N quaternionic components. The gauge field A_{μ} is always an SU(2) gauge field, and is calculated by using the N-component ψ in (4.17). The field strengths and action are still given in the standard way, (4.14), so these theories are all SU(2) non-Abelian gauge theories, but are still constrained by (4.17).

Returning to topological properties, we try to find instanton solutions by saturating the bound of (4.7) and let

$$\partial_\mu n^a \partial_\nu n^b - \partial_\nu n^a \partial_\mu n^b = \mp \epsilon_{abcde} \epsilon_{\mu\nu\lambda\sigma} n^c \partial_\lambda n^d \partial_\sigma n^e / 2. \qquad (4.19)$$

The upper sign gives instanton solutions and the lower sign gives anti-instanton solutions. Expressed in terms of the projective coordinates, w_α, (4.19) is equivalent to

$$\partial_\mu w_\alpha \partial_\nu w_\beta - \partial_\nu w_\alpha \partial_\mu w_\beta = \pm \epsilon_{\mu\nu\lambda\sigma} \epsilon_{\alpha\beta\gamma\delta} \partial_\lambda w^\gamma \partial_\sigma w^\delta / 2. \qquad (4.20)$$

In two dimensions, minimum action configurations of the O(3) σ model were described by the Cauchy-Riemann equations. Equations (4.20) represent a generalization of the Cauchy-Riemann equations to four dimensions. They share with the Cauchy-Riemann equations the property of providing an angle preserving mapping from x_α to w_α, but the angle refers now to the angle between two infinitesimal elements of area, rather than infinitesimal line elements. (The angle between area elements is defined in the sense of an exterior or cross product).

Solutions of (4.20) are instanton or anti-instanton configurations in the O(5) model; they are also instanton or

anti-instanton configurations of the gauge field A_μ, since the topological charge can be written both in terms of $\hat{n}$, (4.3), and in terms of A_μ, (4.15). The instanton (anti-instanton) solutions of gauge theory are known to have self-dual (anti-self-dual) field strengths. Here the self dual nature follows directly from the expression for $F_{\mu\nu}(A(w))$, (4.13). Since $(\tau_\alpha^\dagger \tau_\beta - \tau_\beta^\dagger \tau_\alpha)/2$ is self-dual, a solution of (4.20) generates either a self-dual (instanton) or anti-self-dual (anti-instanton) $F_{\mu\nu}$.

Equations (4.20) have the elementary Q=1 solutions $w_\alpha = x_\alpha$. Substituting this into (4.12) produces the standard one instanton formula for the field strength in SU(2) gauge theory, $A_\mu = x_\alpha \partial_\mu x_\beta (1+x^2)^{-1} (\tau_\alpha^\dagger \tau_\beta - \tau_\beta^\dagger \tau_\alpha)/2 = \eta_{\mu\nu a} \sigma_a x_\nu (1+x^2)^{-1}$ [23], where $\eta_{\mu\nu a}$ is the familiar 't Hooft η-symbol [1]. The solution $w_\alpha = x_\alpha$ may be transformed into the most general Q=1 instanton form by performing translations and applying O(5) transformations, giving

$$\tilde{w} = \tilde{c}(\tilde{x}-\tilde{a})(\tilde{x}-\tilde{b})^{-1}, \qquad (4.21)$$

with $\tilde{w}$, $\tilde{x}$, $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ all quaternions, as the most general solution. This general solution just expresses the same A_μ in a different gauge than the $w_\alpha = x_\alpha$ solution.

Equations (4.20) do not have solutions for Q>1

[27]. The lack of Q>1 solutions to (4.20) does not necessarily mean that minima of the action with higher topological charge do not exist. The inequality (4.7) is not the only inequality which can be constructed which bounds the action, (4.8). The equations (4.20) have an obvious O(4) symmetry. However, O(4) contains within it two SU(2) groups O(4)=SU(2)xSU(2). It turns out that a restricted set of (4.20) corresponding to only one of these subgroups is sufficient to saturate the bound. However, it is not known at this time whether these restricted equations permit solutions of higher topological charge.

It should be pointed out that nonetheless it is possible to construct configurations with higher topological charge which get very close to saturating the bound on the action. For example, using quaternion notation, the trial form $\tilde{w}=(\tilde{x}+\tilde{a})^{-1}/2 + (\tilde{x}-\tilde{a})^{-1}/2$ has Q=2, and an action which converges rapidly to $16\pi^2/g^2$ as $|a|\to\infty$. The computed action is already within one part in 10^{-3} of its minimum value for $|a|=2$. This rapid attenuation could possibly allow a multi-instanton gas calculation despite lacking exact solutions.

Although in the O(5) $\sigma\Lambda\sigma$ model we do not know how to write an instanton solution for Q>1 (or even if they exist), we can write higher instanton solutions using the

generalized models with the N-component Ψ. The theory with an N-component Ψ can handle an instanton with $Q \leq N-1$. We can thus write a general SU(2) gauge theory instanton with arbitrary Q [24,25,26].

The O(5) $\sigma\Lambda\sigma$ model is asymptotically free. This can be seen by using the one instanton quantum fluctuations to find the one-loop coefficient of the Callan-Symanzic β-function, which describes coupling constant renormalization. The gaussian fluctuations about the instanton, (4.21), and the attendant zero modes must contribute a term

$$\mu^{8-n}\ e^{-8\pi^2/g^2(\mu)} \int d^4a\ d^4b\ |a-b|^{-n} \qquad (4.22)$$

to the vacuum-vacuum path integral. This is the only possible expression which is translationally invariant and dimensionless with a renormalization point, μ. It corresponds to the Q=1 formula in the non-linear σ model, (3.27).

Equation (4.21) allows the O(5) spin at infinity to be arbitrary. As in the last chapter, the instanton could also have been written with its spin pinned up at infinity and then globally rotated. The spin up solution is

$$\tilde{w}_\mu = (\tilde{x}-\tilde{z})\,(\tilde{r})^{-1}. \tag{4.23}$$

We use $\tilde{z}$ for the location of the instanton with size $\tilde{r}$. In terms of these parameters, the gaussian fluctuations contribute

$$e^{-8\pi^2/g^2(\mu)} \int d^4z\; d^4r\; |r|^{-8} \exp[\,\beta \ln|\mu r|\,], \tag{4.24}$$

where β will designate the coefficient of the β-function. This formula corresponds to (3.21) in the last chapter. What we are doing here is repeating for the O(5) model the demonstration in the O(3) model that the instanton contribution can be written in terms of either the parameters describing a pair of charges or the parameters describing an instanton location and size. The two pictures must be equivalent after we rotate the spins at infinity in $\tilde{w}_\mu$ to point in any desired direction. As in the last chapter, any final answer must be independent of those rotation parameters. So, using vector instead of quaternion notation, we make the formal identification that the relations between the variables from the last chapter,

$$z_\mu = [a_\mu \cot(\theta/2) + b_\mu \tan(\theta/2)]\cdot G^{-1}, \tag{4.25}$$

and

$$r_\mu = [a_\mu - b_\mu] \cdot G^{-1},$$

will still hold for the single instanton case. (We continue to use $G=[\tan(\theta/2)+\cot(\theta/2)]$). We note here that trying to imitate these steps for higher Q will not work. Since the quaternions of the $\sigma\Lambda\sigma$ model do not commute like the complex parameters in the O(3) σ model, we are unable to use analogies of (3.34-3.35).

Performing this variable change on (4.22) gives

$$\mu^{8-n} e^{-8\pi^2/g^2(\mu)} \int d^4a\, d^4b\, |a-b|^{-n} \tag{4.26}$$

$$= \mu^{8-n} e^{-8\pi^2/g^2(\mu)} \int d^4z\, d^4r\, |r|^{-n} \cdot G^{4-n}.$$

Demanding that (4.26) be independent of θ forces n=4, and then setting (4.24) and (4.26) equal gives us $\beta = 4$. Thus, the coupling constant renormalizes as

$$8\pi^2/g^2(\mu') = 8\pi^2/g^2(\mu) + 4\ln(\mu'/\mu) \tag{4.27}$$

and the theory is asymptotically free. Compare this with SU(2) gauge theory, for which

$$8\pi^2/g^2(\mu') = 8\pi^2/g^2(\mu) + (22/3)\ln(\mu'/\mu). \tag{4.28}$$

It makes sense that the coefficient in (4.26) is less than that in (4.27), since the O(5) $\sigma\Lambda\sigma$ model is a subset of the full SU(2) gauge theory. Another way of looking at it is that the constraint, (4.12), only allows a restricted set of quantum fluctuations, thus reducing the coefficient of the β-function.

These models form an important intermediate step between the non-linear σ model and non-Abelian gauge theory. Their very existence makes it likely that the results or methods used in the much simpler two dimensional models can be passed on to their four dimensional counterparts and eventually to gauge theories. Also, when it might not be possible to lift bodily a method in the simpler theory, that method can be a guide to the correct analysis of the more complicated models.

One can also speculate that these models are more than just models. We have introduced a whole new class of asymptotically free field theories. It could be that a constrained version of non-Abelian gauge theory will adequately describe nature.

V. CONCLUSIONS

This work has used the path integral approach to the semi-classical approximation to analyze a hierarchy of models which eventually leads to QCD. All the models have instantons, which are the primary foundation upon which the semi-classical approximation is built. By learning the effects of instantons in the simpler models of the hierarchy, we hope to apply the knowledge and insights we acquire there to the more complicated systems, and eventually arrive at a clearer understanding of non-Abelian gauge theories. The ultimate goal of this program is to learn how to handle a full instanton and anti-instanton calculation and show that it leads to confinement and the absence of symmetry breaking.

In a simple one dimensional double well potential, we saw how instantons provide a link between the various classical vacua of a system. Regular perturbation theory, which starts in one of the classical vacua, can never really accomodate other classical vacua, and falsely predicts the breaking of the left-right symmetry of the model. A full instanton treatment allow tunnelings between the classical vacua and correctly shows that the symmetry is not broken. It furthermore shows the separation of the first excited state above the quantum mechanical ground state.

The instantons in QCD are not just simple kinks as in the one dimensional case, but have a complicated topological structure. QCD has an ordered, countably infinite number of classical vacua, and the instantons describe tunnelings between them. The two dimensional non-linear σ model has a similar topological structure and multiple vacua, and provides a simpler environment in which to study this sort of instanton.

In the sector with only instantons, the fluctuations about instantons behave like a classical neutral Coulomb gas. We have shown that given the proper variables, namely the intuitively pleasing variables of locations and sizes (as in a dilute gas of instantons), the instantons do not interact at all. This very important result means that the instantons interact neither classically nor quantum mechanically, thereby making much easier the task of doing a full instanton-anti-instanton calculation. It has implications for QCD in that it suggests that the interaction between instantons in that theory could also be small or non-existent.

Finally, the new $\sigma\wedge\sigma$ models provide a new proving ground upon which we can test the applications of instantons. By being both σ models and gauge theories, they bridge the gap between the theories and strengthen the applicability of

results from the non-linear σ model to non-Abelian gauge theory. We have shown that the $\sigma\wedge\sigma$ models share the instanton structure and asymptotic freedom with both their cousins. While at this time not much else is yet known about these $\sigma\wedge\sigma$ models, it is probable that further research on these models will prove fruitful in understanding the proper role of instantons in such questions as confinement and the absence of symmetry breaking in Quantum Chromodynamics.

APPENDIX

This appendix will compute two quantities for general Q. First, it will show that, for the variable change described by

$$z^k - A_1 z^{k-1} + \ldots + (-1)^{k-1} A_{k-1} z + (-1)^k A_k = \prod_{i=1}^{k} (z - a_i), \qquad (A.1)$$

the Jacobian of the transformation is given by

$$|\partial(A)/\partial(a)|^2 = \prod_{i<j}^{k} |a_i - a_j|^2 . \qquad (A.2)$$

Second, with variables defined as generalizations of equations (3.33-3.37) it will show that

$$\prod_{i,j=1}^{k} |a_i - b_j|^2 = \prod_{i=1}^{k} |r_i|^2 \prod_{i<j}^{k} |z_i - z_j|^4 \cdot G^{2k} \qquad (A.3)$$

In this appendix we will use the notation that (A) will stand for the complete set of A_i, etc.

Tackling the first one, we define the variable $A_n^{(k)}$ ($n<k$) as the A_n defined when (A.1) is a polynomial in z of degree k. Then, the $A_n^{(k)}$ can be calculated in terms of the $A_n^{(k-1)}$ and a_k. We find

$$A_1^{(k)} = A_1^{(k-1)} + a_k \tag{A.4}$$

$$A_2^{(k)} = A_2^{(k-1)} + a_k A_1^{(k-1)}$$

$$\vdots$$

$$A_{k-1}^{(k)} = A_{k-1}^{(k-1)} + a_k A_{k-2}^{(k-1)}$$

$$A_k^{(k)} = a_k A_{k-1}^{(k-1)} \quad ,$$

or, in general, $A_n^{(k)} = A_n^{(k-1)} + a_k A_{n-1}^{(k-1)}$, where we define $A_0^{(k)} = 1$ and $A_l^{(k)} = 0 \quad (l > k)$. We notice at this point that $A_1^{(k)} = \sum_{j=1}^{k} a_j$ and $A_k^{(k)} = \prod_{j=1}^{k} a_j$, so that $(\partial A_1^{(k)} / \partial a_m) = 1 \quad (m \leq k)$.

Before getting to actually computing the Jacobian, we would like to produce one final formula which turns out to be highly useful,

$$(\partial A_i^{(k-1)} / \partial a_j) - A_{i-1}^{(k-1)} = -a_j (\partial A_{i-1}^{(k-1)} / \partial a_j) \tag{A.5}$$

$$(i = 2, 3, \ldots, k-2)$$

$$(j = 1, 2, \ldots, k-1)$$

$$-A_{k-1}^{(k-1)} = -a_j (\partial A_{k-1}^{(k-1)} / \partial a_j)$$

$$(j = 1, 2, \ldots, k-1).$$

The latter equation is trivially true, since $A_{k-1}^{(k-1)} = \prod_{j=1}^{k-1} a_j$, and the former equation can be proven by a simple inductive proof.

We now find

(A.6)

$$\frac{\partial(A^{(k)})}{\partial(a)} = \begin{vmatrix} \frac{\partial A_1^{(k-1)}}{\partial a_1} & \frac{\partial A_1^{(k-1)}}{\partial a_2} & \cdots & 1 \\ \frac{\partial A_2^{(k-1)}}{\partial a_1} + a_k \frac{\partial A_1^{(k-1)}}{\partial a_1} & \frac{\partial A_2^{(k-1)}}{\partial a_2} + a_k \frac{\partial A_1^{(k-1)}}{\partial a_2} & \cdots & A_1^{(k-1)} \\ \frac{\partial A_3^{(k-1)}}{\partial a_1} + a_k \frac{\partial A_2^{(k-1)}}{\partial a_1} & \frac{\partial A_3^{(k-1)}}{\partial a_2} + a_k \frac{\partial A_2^{(k-1)}}{\partial a_2} & \cdots & A_2^{(k-1)} \\ \vdots & \vdots & & \vdots \\ a_k \frac{A_{k-1}^{(k-1)}}{\partial a_1} & a_k \frac{\partial A_{k-1}^{(k-1)}}{\partial a_2} & \cdots & A_{k-1}^{(k-1)} \end{vmatrix} .$$

Recall that, for a determinant, a multiple of any column may be added to any other column without changing the value of that determinant. We thus subtract the k^{th} column from each of the other columns and use (A.5) on the results. This

leaves

$$\frac{\partial(A^{(k)})}{\partial(a)} = (-1)^{k-1} \begin{vmatrix} (a_k - a_1)\frac{\partial A_1^{(k-1)}}{\partial a_1} & \cdots & (a_k - a_{k-1})\frac{\partial A_1^{(k-1)}}{\partial a_{k-1}} \\ (a_k - a_1)\frac{\partial A_2^{(k-1)}}{\partial a_1} & \cdots & (a_k - a_{k-1})\frac{\partial A_2^{(k-1)}}{\partial a_{k-1}} \\ \vdots & & \vdots \\ (a_k - a_1)\frac{\partial A_{k-1}^{(k-1)}}{\partial a_1} & \cdots & (a_k - a_{k-1})\frac{\partial A_{k-1}^{(k-1)}}{\partial a_{k-1}} \end{vmatrix}$$

$$= (-1)^{k-1} \prod_{j=1}^{k-1} (a_k - a_j)\,[\partial(A^{(k-1)})/\partial(a)]. \qquad (A.7)$$

Equation (A.2) now follows inductively.

We now move on to showing (A.3). Our strategy is as follows: We will define C's as in (3.48) and compute $\partial(C)/\partial(A,B)$, which from (3.51) is $\prod |a_i - b_j|^2$. The Jacobian $\partial(C)/\partial(A,B)$ will be the determinant of a matrix written in terms of the variables A and B. Leaving the Jacobian in determinant form, we will successively change from (A,B) to (Z,R) to (z,r), simplifying as we go. Let us remind

ourselves of the relationships between these variables. The pairs (A, a), (B, b) and (Z, z) all are related as in (A.1) or (3.44). The variables R and r are simply related by $R_i = \sum_j (\partial Z_i / \partial z_j) \cdot r_j$. Additionally, all these variables are interconnected by $A_i = Z_i + R_i \tan(\theta/2)$ and $B_i = Z_i - R_i \cot(\theta/2)$. These are all exactly the relationships of chapter III. Thus the variables (A, B) in the Jacobian determinant will be changed to (Z, R) and finally to (z, r), while always keeping the Jacobian in determinant form, since this allows various determinant identities to be used in simplifying the expression. Finally, the matrix of the determinant will be found to be factorizable as in (3.58), with each term being a simple matrix. Hence the total determinant will be easily computable at this point.

With C's defined by

$$[z^{2k} - C_1 z^{2k-1} + \ldots + (-1)^{2k} C_{2k}] = \tag{A.8}$$

$$[z^k - A_1 z^{k-1} + \ldots + (-1)^k A_k][z^k - B_1 z^{k-1} + \ldots + (-1)^k B_k],$$

we find,

$$\frac{\partial(C)}{\partial(A,B)} = \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ B_1 & A_1 & 1 & 1 & \cdots & 0 & 0 \\ B_2 & A_2 & B_1 & A_1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & 1 & 1 \\ B_k & A_k & B_{k-1} & A_{k-1} & \cdots & B_1 & A_1 \\ 0 & 0 & B_k & A_k & \cdots & B_2 & A_2 \\ 0 & 0 & 0 & 0 & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & B_k & A_k \end{vmatrix}_{2k \times 2k} . \qquad (A.9)$$

The Jacobian is the determinant of a (2k)x(2k) matrix. After making the substitution like (3.36-3.37) the columns are manipulated pairwise and factors of G are extracted, yielding

$$\frac{\partial(C)}{\partial(A,B)} = G^k \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ Z_1 & R_1 & 1 & 0 & \cdots & 0 & 0 \\ Z_2 & R_2 & Z_1 & R_1 & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & 1 & 0 \\ Z_k & R_k & Z_{k-1} & R_{k-1} & \cdots & Z_1 & R_1 \\ 0 & 0 & Z_k & R_k & \cdots & Z_2 & R_2 \\ 0 & 0 & 0 & 0 & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & Z_k & R_k \end{vmatrix}_{2k \times 2k} . \qquad (A.10)$$

Notice that at this point we are really left with only a $(2k-1)x(2k-1)$ size matrix. After substituting for (R), everything is now written in terms of Z , z , r , and $(\partial Z_i / \partial z_j)$. We again manipulate columns pairwise, starting with the second and third columns. In the process the identity (A.5) is used repeatedly, and the size of the matrix is reduced to $(k)x(k)$. The determinant reduces to

$$D=\begin{vmatrix} \sum_j r_j z_j^{k-1}(\partial Z_1/\partial z_j) & \cdots & \sum_j r_j z_j^{0}(\partial Z_1/\partial z_j) \\ \sum_j r_j z_j^{k-1}(\partial Z_2/\partial z_j) & \cdots & \sum_j r_j z_j^{0}(\partial Z_2/\partial z_j) \\ \vdots & & \vdots \\ \sum_j r_j z_j^{k-1}(\partial Z_k/\partial z_j) & \cdots & \sum_j r_j z_j^{0}(\partial Z_k/\partial z_j) \end{vmatrix}_{k\times k} . \qquad (A.11)$$

The matrix corresponding to D is the product of $[\partial(Z)/\partial(z)]$, the diagonal matrix $(r_1, r_2, \ldots, r_k)$ and the matrix corresponding to D_3, where

$$D_3=\begin{vmatrix} z_1^{k-1} & z_1^{k-2} & \cdots & z_1 & 1 \\ z_2^{k-1} & z_2^{k-2} & \cdots & z_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ z_k^{k-1} & z_k^{k-1} & \cdots & z_k & 1 \end{vmatrix} . \qquad (A.12)$$

The matrix corresponding to D_3 is closely related to Vandermonde's matrix and hence has the determinant

$$D_3 = \prod_{i<j}^{k} (z_i - z_j) . \qquad (A.13)$$

The diagonal matrix obviously has determinant $D_2 = \prod_{i=1}^{k} r_i$, and the determinant of $[\partial(Z)/\partial(z)]$ has already been computed, $D_1 = D_3$. Assembling all these pieces leads directly to (A.3).

REFERENCES

1. G. 't Hooft, Phys. Rev. D14, 3432 (1976).

2. E. Gildener and A. Patrascioiu, Phys. Rev. D16, 423 (1977).

3. A. Polykov, Nuc. Phys. B120, 429 (1977).

4. R. Jackiw, Rev. Mod. Phys. 49, 681 (1977).

5. C. Callan, R. Dashen, D. Gross, Phys. Lett. 63B, 334 (1976).

6. R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976).

7. A. Messiah, "Quantum Mechanics," (North Holland Publishing Company, Amsterdam, 1965), p. 231.

8. R. Neinast and J. Stack, Phys. Rev. D, 19, 3785 (1979).

9. A. Jevicki, Nuc. Phys. B117, 365 (1976).

10. A. Polykov, Phys. Lett. 59B, 82 (1975).

11. A. Belavin and A. Polykov, JETP Lett. 22, 245 (1975).

12. A. Jevicki, Nuc. Phys. B127, 125 (1977).

13. B. Berg and M. Luscher, Commun. Math. Phys. 69, 57 (1979).

14. V. Fateev, I. Frolov, and A. Schwarz, Nuc. Phys. B154, 1 (1979).

15. C. Deutsch and M. Lavaud, Phys. Rev. A9, 2598 (1974).

16. J. Frolich, Commun. Math. Phys. 47, 233 (1976).

17. S. Coleman, Phys. Rev. D11, 2088 (1975).

18. C. Callan, R. Dashen, and D. Gross, Phys. Rev. D17, 2717 (1978).

19. C. Callan, R. Dashen, and D. Gross, Phys. Rev. D19, 1826 (1979).

20. C. Callan, R. Dashen, and D. Gross, Phys. Rev. D20, 3279 (1979).

21. M. Luscher, Phys. Lett. 78B, 465 (1978).

22. A. D'Adda, P. Di Vecchia, and M. Luscher, Nuc. Phys. B146, 63 (1978).

23. A. Belavin, A. Polykov, A. Schwartz, and Y. Tyupkin, Phys. Lett. 59B, 85 (1975).

24. M. Atiyah, H. Hitchen, V. Drinfeld, and Y. Manin, Phys. Lett. 65A, 185 (1978).

25. N. Christ, E. Weinberg, and N. Stanton, Phys. Rev. D18, 2013 (1978).

26. E. Corrigan, D. Fairlie, P. Goddard, and S. Templeton, Nuc. Phys. B140, 31 (1978).

27. S. Kobayashi and K. Nomizu, "Foundations of Differential Geometry," (Interscience, New York, 1963).

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