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Abstract

We present an algorithm to solve BSDEs based on Wiener Chaos Expansion and Picard's iterations. We get a forward scheme where the conditional expectations are easily computed thanks to chaos decomposition formulas. We use the Malliavin derivative to compute Z. Concerning the error, we derive explicit bounds with respect to the number of chaos and the discretization time step. We also present numerical experiments. We obtain very encouraging results in terms of speed and accuracy.

1 Introduction

In this paper, we are interested in the numerical approximation of solutions (Y, Z) to backward stochastic differential equations (BSDEs for short in the sequel). BSDEs have been introduced by J.-M. Bismut in [Bis73] in the linear case, whereas the nonlinear case has been considered later by É. Pardoux and S. Peng in [PP90]. A BSDE is an equation of the following form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s \cdot dB_s, \quad 0 \le t \le T, \tag{1.1}$$

where B is a d-dimensional standard Brownian motion, the terminal condition ξ is a real-valued $\mathcal{F}_T$–measurable random variable where $\{\mathcal{F}_t\}_{0\le t\le T}$ stands for the augmented filtration of the Brownian motion B and the generator f is a map from $[0, T] \times \mathbb{R} \times \mathbb{R}^d$ into $\mathbb{R}$. A solution to this equation is a pair of processes $\{(Y_t, Z_t)\}_{0\le t\le T}$ which is required to be adapted to the filtration $\{\mathcal{F}_t\}_{0\le t\le T}$. We will assume the conditions of Pardoux and Peng to ensure existence and uniqueness of solutions.

Our main objective in this study is the numerical approximation of the solution (Y, Z) to BSDE (1.1) (even though there exists a large literature on this subject). The first two contributions to this topic are due to D. Chevance [Che97], who considered generators independent of Z, and V. Bally [Bal97], who used a random time mesh. J. Ma and J. Yong [MY99] proposed numerical schemes based on the link between Markovian BSDEs and semilinear partial differential equations (PDEs). Another approach, based on Donsker's theorem and close to [Che97], was proposed by F. Coquet, V. Mackevicius and J. Mémin [CMM99] in the case of a generator f independent of Z; the general case was treated by Ph. Briand, B. Delyon and J. Mémin in [BDM01], who later extended it to a more general framework [BDM02], including the case of a "stepwise constant Brownian motion". This extension led to the formulas

$$Y_t = \mathbb{E}\left(Y_{t+h} \mid \mathcal{F}_t\right) + hf(t, Y_t, Z_t), \quad Z_t = h^{-1/2}\, \mathbb{E}\left(Y_{t+h}\left(B_{t+h} - B_t\right) \mid \mathcal{F}_t\right)$$

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known as the dynamic programming algorithm. Even though the convergence was proved in the case of path-dependent terminal condition ξ, the rate of convergence was left as an open question in [BDM02]. This problem was solved by J. Zhang [Zha04] and B. Bouchard and N. Touzi [BT04] in the case of Markovian BSDE, namely in the case of a terminal condition $\xi = g(X_T)$ where X is the solution to a stochastic differential equation (in [Zha04], the author considers the path-dependent case as well). Their result was generalized by E. Gobet and C. Labart [GL07] and also by E. Gobet and A. Makhlouf [GM10].

From a numerical point of view, the main difficulty in solving BSDEs is to efficiently compute conditional expectations. Several approaches have been proposed using various tools: the Malliavin calculus [BT04], regression methods [GLW05, GLW06] and quantization technics [BP03].

Finally, let us mention that there exists some works dealing with the discretization of solutions to BSDEs in a more general framework: forward-backward SDEs [DM06] and quadratic BSDEs [Ric11].

Let us now describe briefly the main points of our approach in the case of a real-valued Brownian motion. As already used in several quoted papers, see also [BD07, GL10, BSar], our starting point is the use of Picard's iterations: $(Y^0, Z^0) = (0, 0)$ and for $q \in \mathbb{N}$,

$$Y_t^{q+1} = \xi + \int_t^T f\left(s, Y_s^q, Z_s^q\right) ds - \int_t^T Z_s^{q+1} \cdot dB_s, \quad 0 \le t \le T.$$

It is well-known that the sequence (Y^q, Z^q) converges exponentially fast towards the solution (Y, Z) to BSDE (1.1). We write this Picard scheme in a forward way

$$Y_t^{q+1} = \mathbb{E}\left(\xi + \int_0^T f\left(s, Y_s^q, Z_s^q\right) ds \,\Big|\, \mathcal{F}_t\right) - \int_0^t f\left(s, Y_s^q, Z_s^q\right) ds,$$

$$Z_t^{q+1} = D_t Y_t^{q+1} = D_t \mathbb{E}\left(\xi + \int_0^T f\left(s, Y_s^q, Z_s^q\right) ds \,\Big|\, \mathcal{F}_t\right),$$

where $D_t X$ stands for the Malliavin derivative of the random variable X.

In order to compute the previous conditional expectation, we use a Wiener chaos expansion of the random variable

$$F^q = \xi + \int_0^T f\left(s, Y_s^q, Z_s^q\right) ds.$$

More precisely, we use the following orthogonal decomposition of the random variable F^q

$$F^q = \mathbb{E}\left[F^q\right] + \sum\nolimits_{k\ge1} \sum\nolimits_{|n|=k} d_k^n \prod\nolimits_{i\ge1} K_{n_i}\left(\int_0^T g_i(s) dB_s\right),$$

where K_l denotes the Hermite polynomial of degree l, $(g_i)_{i\ge1}$ is an orthonormal basis of $\mathrm{L}^2(0,T)$ and, if $n = (n_i)_{i\ge1}$ is a sequence of integers, $|n| = \sum_{i\ge1} n_i$. $(d_k^n)_{k\ge1,|n|=k}$ is the sequence of coefficients ensuing from the decomposition of F^q. Of course, from a practical point of view, we only keep a finite number of terms in this expansion:

- we work with a finite number of chaos, p;
- we choose a finite number of functions $g_1, \ldots, g_N$.

This leads to the following approximation with $n = (n_1, \ldots, n_N)$

$$F^q \simeq \mathbb{E}\left[F^q\right] + \sum\nolimits_{1\le k\le p} \sum\nolimits_{|n|=k} d_k^n \prod\nolimits_{1\le i\le N} K_{n_i}\left(\int_0^T g_i(s) dB_s\right).$$

One of the key points in using such a decomposition is that, for choices of simple functions g_1, $\ldots$, g_N, there exist explicit formulas for both

$$\mathbb{E}\left(F^q \,\big|\, \mathcal{F}_t\right) \quad \text{and} \quad Z_t^{q+1} = D_t \mathbb{E}\left(F^q \,\big|\, \mathcal{F}_t\right) ; \tag{1.2}$$

this plays a crucial role in our algorithm. Using these formulas and starting from M trajectories of the underlying Brownian motion we are able to construct M trajectories of the solution (Y, Z) to the BSDE.

In the following, the functions g_i are chosen as step functions:

$$g_i = \mathbf{1}_{]\bar{t}_{i-1},\bar{t}_i]}(t)/\sqrt{h}, \quad i = 1, \dots, N, \text{ where } \bar{t}_i := ih, \ h = \frac{T}{N}$$

and the previous formulas are really simple (see (2.8)-(2.9) and Proposition 2.7). Eventually, the main advantage of this method is that only one decomposition has to be computed per Picard iteration: the decomposition of F^q. Therein lies the main difference between our approach and the approach based on regression technics developed by C. Bender and R. Denk in [BD07]. In their paper, for a given Picard iteration q and for each time t_i of the mesh grid, two projections have to be computed, one for $Y^q_{t_i}$ and one for $Z^q_{t_i}$. The second difference comes from the way of computing Z^q. In our method, once the decomposition of F^q is computed, Z^q is given explicitly as the Malliavin derivative of Y^q. Let us also point out that our algorithm can handle fully path dependent terminal conditions.

The rest of the paper is organized as follows. Section 2 contains the notations and the preliminary results, Section 3 describes precisely the algorithm, Section 4 is devoted to the study of the convergence of the algorithm and finally Section 5 contains some numerical experiments. Some technical proofs are post-done to the appendix.

2 Preliminaries

2.1 Definitions and Notations

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an $\mathbb{R}^d$-valued Brownian motion B, we consider

- $\{(\mathcal{F}_t); t \in [0, T]\}$, the filtration generated by the Brownian motion B and augmented
- $\mathrm{L}^p(\mathcal{F}_T) := \mathrm{L}^p(\Omega, \mathcal{F}_T, \mathbb{P})$, $p \in \mathbb{N}^*$, the space of all $\mathcal{F}_T$-measurable random variables (r.v. in the following) $X : \Omega \longmapsto \mathbb{R}^d$ satisfying $\|X\|_p^p := \mathbb{E}(|X|^p) < \infty$.
- $\mathbb{E}_t(X)$ denotes $\mathbb{E}(X|\mathcal{F}_t)$ for any X in $\mathrm{L}^1(\mathcal{F}_T)$.
- $\mathrm{S}^p_T(\mathbb{R}^d)$, $p \in \mathbb{N}, p \geq 2$, the space of all càdlàg predictable processes $\phi : \Omega \times [0, T] \longmapsto \mathbb{R}^d$ such that $\|\phi\|^p_{\mathrm{S}^p} = \mathbb{E}(\sup_{t\in[0,T]} |\phi_t|^p) < \infty$.
- $\mathrm{H}^p_T(\mathbb{R}^d)$, $p \in \mathbb{N}, p \geq 2$, the space of all predictable processes $\phi : \Omega \times [0, T] \longmapsto \mathbb{R}^d$ such that $\|\phi\|^p_{\mathrm{H}^p_T} = \mathbb{E} \int_0^T |\phi_t|^p dt < \infty$.
- $\mathrm{L}^2(0, T)$, the space of all square integrable functions on $[0, T]$.
- $C^{k,l}$, the set of continuously differentiable functions $\phi : (t, x) \in [0, T] \times \mathbb{R}^d$ with continuous derivatives w.r.t. t (resp. w.r.t. x) up to order k (resp. up to order l).
- $C^{k,l}_b$, the set of continuously differentiable functions $\phi : (t, x) \in [0, T] \times \mathbb{R}^d$ with continuous and uniformly bounded derivatives w.r.t. t (resp. w.r.t. x) up to order k (resp. up to order l). The function ϕ is also bounded.
- $\|\partial^j_{sp} f\|^2_\infty$, the norm of the derivatives of $f([0, T] \times \mathbb{R}^d, \mathbb{R})$ w.r.t. all the space variables x which sum equals j : $\|\partial^j_{sp} f\|^2_\infty := \sum_{|k|=j} \|\partial^{k_1}_{x_1} \cdots \partial^{k_d}_{x_d} f\|^2_\infty$, where $|k| = k_1 + \cdots + k_d$.
- C^∞_p, the set of smooth functions $f : \mathbb{R}^n \longmapsto \mathbb{R}$ with partial derivatives of polynomial growth.

- $\|(\cdot,\cdot)\|^p_{\mathrm{L}^p}$, $p\in\mathbb{N}, p\geq 2$, the norm on the space $\mathrm{S}^p_T(\mathbb{R})\times\mathrm{H}^p_T(\mathbb{R}^d)$ defined by

$$\|(Y,Z)\|^p_{\mathrm{L}^p} := \mathbb{E}(\sup_{t\in[0,T]}|Y_t|^p)+\int_0^T \mathbb{E}(|Z_t|^p)dt. \tag{2.1}$$

We also recall some useful definitions related to Malliavin calculus. We use the notations of [Nua06].

- $\mathcal{S}$ denotes the class of random variables of the form $F = f(W(h_1),\cdots,W(h_n))$, where $f\in C^\infty_p(\mathbb{R}^{n\times d},\mathbb{R})$, for all $j\leq n$ $h_j=(h^1_j,\cdots,h^d_j)\in\mathrm{L}^2([0,T];\mathbb{R}^d)$ and for all $i\leq d$ $W^i(h^i_j)=\int_0^T h^i_j(t)dW^i_t$.
- $\mathbb{D}^{r,2}$ denotes the closure of $\mathcal{S}$ w.r.t. the following norm on $\mathcal{S}$

$$\|F\|^2_{\mathbb{D}^{r,2}} := \mathbb{E}|F|^2+\sum_{q=1}^r\sum_{|\alpha|_1=q}\mathbb{E}\left(\int_0^T\cdots\int_0^T\left|D^\alpha_{(t_1,\cdots,t_q)}F\right|^2dt_1\cdots dt_q\right)$$

  where α is a multi-index $(\alpha_1,\cdots,\alpha_q)\in\{1,\cdots,d\}^q$ $|\alpha|_1:=\sum_{i=1}^q\alpha_i=q$ and D^α represents the multi-index Malliavin derivative operator. We recall $\mathbb{D}^{\infty,2}=\cap_{r=1}^\infty\mathbb{D}^{r,2}$.

Remark 2.1. *When* $d=1$, $\|F\|^2_{\mathbb{D}^{r,2}} := \mathbb{E}|F|^2+\sum_{q=1}^r\mathbb{E}\left(\int_0^T\cdots\int_0^T\left|D^{(q)}_{(t_1,\cdots,t_q)}F\right|^2dt_1\cdots dt_q\right) = \mathbb{E}|F|^2+\sum_{q=1}^r\|D^{(q)}F\|^2_{\mathrm{L}^2(\Omega\times[0,T]^q)}$.

Let $m\in\mathbb{N}^*$ and $j\in\mathbb{N}, j\geq 2$. We also introduce the following notations

- $\mathcal{D}^{m,j}$ denotes the space of all $\mathcal{F}_T$-measurable r.v. such that

$$\|F\|^j_{m,j} := \sum_{1\leq l\leq m}\sum_{|\alpha|_1=l}\sup_{t_1\leq\cdots\leq t_l}\mathbb{E}[|D^\alpha_{t_1,\cdots,t_l}F|^j]<\infty$$

  where $\sup_{t_1\leq\cdots\leq t_l}$ means $\sup_{(t_1,\cdots,t_l):t_1\leq\cdots\leq t_l}$.

- $\mathcal{S}^{m,j}$ denotes the space of all couple of processes (Y,Z) belonging to $\mathrm{S}^j_T(\mathbb{R})\times\mathrm{H}^j_T(\mathbb{R}^d)$ and such that

$$\|(Y,Z)\|^j_{m,j} := \sum_{1\leq l\leq m}\sum_{|\alpha|_1=l}\sup_{t_1\leq\cdots\leq t_l}\|(D^\alpha_{t_1,\cdots,t_l}Y,D^\alpha_{t_1,\cdots,t_l}Z)\|^j_{\mathrm{L}^j}<\infty.$$

  We recall

$$\|(Y,Z)\|^j_{m,j} = \sum_{1\leq l\leq m}\sum_{|\alpha|_1=l}\sup_{t_1\leq\cdots\leq t_l}\left\{\mathbb{E}[\sup_{t_l\leq r\leq T}|D^\alpha_{t_1,\cdots,t_l}Y_r|^j]+\int_{t_l}^T\mathbb{E}[|D^\alpha_{t_1,\cdots,t_l}Z_r|^j]dr\right\}.$$

  We also denote $\mathcal{S}^{m,\infty}:=\cap_{j\geq 2}\mathcal{S}^{m,j}$.

2.2 Wiener Chaos Expansion

2.2.1 Notations and useful results

We refer to [Nua06] for more details on this section. Let us briefly recall the Wiener chaos expansion in the simple case of a real-valued Brownian motion. It is well known that every random variable $F\in\mathrm{L}^2(\mathcal{F}_T)$ has an expansion of the following form:

$$\begin{aligned}F =&\mathbb{E}[F]+\int_0^T u_1(s_1)dB_{s_1} \\ &+\int_0^T\int_0^{s_2}u_2(s_2,s_1)dB_{s_1}dB_{s_2}+\ldots+\int_0^T\int_0^{s_n}\cdots\int_0^{s_2}u_n(s_n,\ldots,s_1)dB_{s_1}\ldots dB_{s_n}+\ldots\end{aligned} \tag{2.2}$$

where the functions $(u_n, n \geq 1)$ are deterministic functions. There is an ambiguity for the definition of these functions u_n. We adopt in this paper the following point of view: the function u_n is defined on the simplex

$$\mathcal{S}_n(T) := \{(s_1, \cdots, s_n) \in [0,T]^n : 0 < s_1 < \ldots < s_n < T\}.$$

We define the iterated integral for a deterministic function $f \in \mathrm{L}^2(\mathcal{S}_n(T))$ as

$$J_n(f) := \int_0^T \int_0^{s_n} \cdots \int_0^{s_2} f(s_n, \cdots, s_1) dB_{s_1} \cdots dB_{s_n}.$$

Due to the Itô isometry, $\|J_n(f)\|^2 = \|f\|^2_{\mathrm{L}^2(\mathcal{S}_n(T))}$ and $\mathbb{E}[J_n(f)J_m(g)] = \delta_{nm} < f, g >_{\mathrm{L}^2(\mathcal{S}_n(T))}$. Then, $\|F\|^2 = \sum_{n\geq 0} \|u_n\|^2_{\mathrm{L}^2(\mathcal{S}_n(T))}$.

Definition. Let F be a random variable in $\mathrm{L}^2(\mathcal{F}_T)$ whose chaos expansion is given by (2.2). We introduce

- $P_n(F) := J_n(u_n)$ the Wiener chaos of order n of F.
- $\mathcal{C}_p(F) := \sum_{n \leq p} P_n(F)$ the chaos decomposition of F up to order p, i.e.

$$\mathcal{C}_p(F) = \mathbb{E}[F] + \int_0^T u_1(s_1) dB_{s_1} + \int_0^T \int_0^{s_2} u_2(s_2, s_1) dB_{s_1} dB_{s_2} + \ldots + \int_0^T \int_0^{s_p} \cdots \int_0^{s_2} u_p(s_p, \ldots, s_1) dB_{s_1} \ldots dB_{s_p}. \quad (2.3)$$

We state two Lemmas useful for the sequel.

Lemma 2.2 (Nualart)**.** *$F \in \mathbb{D}^{m,2}$ if and only if $\|D^m F\|^2_{\mathrm{L}^2(\Omega\times[0,T]^m)} = \sum_{n\geq 0}(n+m-1)\times\cdots\times n \times \mathbb{E}[|P_n(F)|^2] < \infty$. In this case, we have*

$$\sum_{n\geq 0}(n+m-1)\times\cdots\times n\times\mathbb{E}[|P_n(F)|^2] \leq \|F\|^2_{\mathbb{D}^{m,2}}.$$

From Lemma 2.2, we deduce

Lemma 2.3. *Let $F \in \mathbb{D}^{m,2}$. We have*

$$\mathbb{E}[|F - \mathcal{C}_p(F)|^2] \leq \frac{\|D^m F\|^2_{\mathrm{L}^2(\Omega\times[0,T]^m)}}{(p+m)\cdots(p+1)}.$$

Proof.

$$\mathbb{E}[|F - \mathcal{C}_p(F)|^2] = \sum_{k\geq p+1} \mathbb{E}[P_k(F)^2] = \sum_{k\geq p+1} (k+m-1)\cdots k \times \frac{1}{(k+m-1)\cdots k} \times \mathbb{E}[|P_k(F)|^2]$$
$$\leq \frac{1}{(p+m)\cdots(p+1)} \sum_{k\geq p+1} (k+m-1)\cdots k\mathbb{E}[|P_k(F)|^2].$$

□

The following Lemma gives some useful properties of the chaos decomposition.

Lemma 2.4.

- *Let F be a r.v. in $\mathrm{L}^2(\mathcal{F}_T)$. $\forall p \geq 1$, we have $\mathbb{E}(|\mathcal{C}_p(F)|^2) \leq \mathbb{E}(|F|^2)$. If F belongs to $\mathrm{L}^j(\mathcal{F}_T)$, $\forall j > 2$, $\mathbb{E}(|\mathcal{C}_p(F)|^j) \leq (1 + p(j-1)^{\frac{p}{2}})^j \mathbb{E}(|F|^j)$.*
- *Let H be in $\mathrm{H}^2_T(\mathbb{R})$. We have $\mathcal{C}_p\left(\int_0^T H_s ds\right) = \int_0^T \mathcal{C}_p(H_s) ds$.*
- *For all $F \in \mathbb{D}^{1,2}$ and for all $t \leq r$, $D_t \mathbb{E}_r[\mathcal{C}_p(F)] = \mathbb{E}_r[\mathcal{C}_{p-1}(D_t F)]$.*

The first result ensues from the fact that for $j > 2$ $\|P_n(F)\|_j \leq (j-1)^{\frac{n}{2}} \|F\|_j$ (see [Nua06, page 63]).

2.2.2 Wiener chaos expansion and Hermite polynomials

Another approach to Wiener chaos expansion uses Hermite polynomials. This approach can be easily generalized when considering d-dimensional Brownian motions, this is then the one we consider in the following. We present it for $d = 1$. Let $\{g_i\}_{i\geq 1}$ be an orthonormal basis of $\mathrm{L}^2(0,T)$. The Wiener chaos of order n, $P_n(F)$, is the L^2-closure of the vector field spanned by

$$\left\{ \prod_{i\geq 1} \sqrt{n_i!}\, K_{n_i} \left(\int_0^T g_i(s) dB_s \right) : |(n_i)_{i\geq 1}| := \sum n_i = n \right\}$$

where K_n is the Hermite polynomial of order n defined by the expansion:

$$e^{xt - t^2/2} = \sum_{n\geq 0} K_n(x)\, t^n.$$

with the convention $K_{-1} \equiv 0$. With this normalization, we have $K_n'(x) = K_{n-1}(x)$ for any integer n. It is well-known that $(K_n)_{n\geq 0}$ is a sequence of orthogonal polynomials in $\mathrm{L}^2(\mathbb{R}, \mu)$, where μ denotes the reduced centered Gaussian measure. Moreover, we have

$$\int_{\mathbb{R}} K_n^2(x) \mu(dx) = \frac{1}{n!}.$$

Every square integrable random variable F, measurable with respect to $\mathcal{F}_T$, admits the following orthogonal decomposition

$$F = d_0 + \sum\nolimits_{k\geq 1} \sum\nolimits_{|n|=k} d_k^n \prod\nolimits_{i\geq 1} K_{n_i} \left(\int_0^T g_i(s) dB_s \right), \tag{2.4}$$

where $n = (n_i)_{i\geq 1}$ is a sequence of positive integers and where $|n|$ stands for $\sum_{i\geq 1} n_i$. Taking into account the normalization of the Hermite polynomials we use, we get

$$d_0 = \mathbb{E}\left[F\right], \qquad d_k^n = n!\, \mathbb{E}\left[F \times \prod\nolimits_{i\geq 1} K_{n_i} \left(\int_0^T g_i(s) dB_s \right) \right],$$

where $n! = \prod_{i\geq 1} n_i!$. Before describing the chaos decomposition formulas we use in the algorithm, we give a Lemma useful in the sequel.

Lemma 2.5. *Let $g \in \mathrm{L}^2(0,T)$ and let $U_t = \int_0^t g^2(s) ds$. For $n \in \mathbb{N}$, let us define*

$$M_t^n = U_t^{n/2}\, K_n \left(B(g)_t / \sqrt{U_t} \right), \quad B(g)_t = \int_0^t g(s) dB_s.$$

Then $\{M_t^n\}_{0\leq t\leq T}$ is a martingale and

$$dM_t^n = g(t)\, M_t^{n-1} dB_t.$$

2.3 Chaos decomposition formulas

These formulas are based on the decomposition (2.4). To get tractable formulas, we consider a finite number of chaos and a finite number of functions $(g_1, \cdots, g_N)$. The $(g_i)_{1\leq i\leq N}$ functions are chosen such that we can quickly compute $\mathbb{E}(F|\mathcal{F}_t)$ and $D_t\mathbb{E}(F|\mathcal{F}_t)$ (as required in (1.2)). We develop in this Section the case $d = 1$, we refer to Section B.2 when $d > 1$.

The first step consists in considering a finite number of chaos. In order to approximate the random variable F, we consider its projection $\mathcal{C}_p(F)$ onto the first p chaos, namely

$$\mathcal{C}_p(F) = d_0 + \sum\nolimits_{1\leq k\leq p} \sum\nolimits_{|n|=k} d_k^n \prod\nolimits_{i\geq 1} K_{n_i} \left(\int_0^T g_i(s) dB_s \right). \tag{2.5}$$

Of course, we still have an infinite number of terms in the previous sum and the second step consists in working with only the first N functions $g_1, \ldots, g_N$ of an orthonormal basis of $\mathrm{L}^2(0,T)$.

Let us consider a regular mesh grid of N time steps $\mathcal{T} = \{\bar{t}_i = i\frac{T}{N}, i = 0, \cdots, N\}$ and the N step functions

$$g_i = \mathbf{1}_{]\bar{t}_{i-1},\bar{t}_i]}(t)/\sqrt{h}, \quad i = 1, \ldots, N, \text{ where } h := \frac{T}{N}. \tag{2.6}$$

We complete these N functions $g_1, \ldots, g_N$ into an orthonormal basis of $\mathrm{L}^2(0,T)$, $(g_i)_{i\geq 1}$. For instance, one can consider the Haar basis on each interval $(\bar{t}_{i-1}, \bar{t}_i)$, $i = 1, \ldots, N$. We implicitly assume that $N \geq p$. This leads to the following approximation

$$\mathcal{C}_p^N(F) = d_0 + \sum\nolimits_{1\leq k\leq p} \sum\nolimits_{|n|=k} d_k^n \prod\nolimits_{1\leq i\leq N} K_{n_i}\left(\int_0^T g_i(s)dB_s\right), \tag{2.7}$$

where $n = (n_1, \ldots, n_N)$ and $|n| = n_1 + \ldots + n_N$. Due to the simplicity of the functions g_i, $i = 1, \cdots, N$, we can compute explicitly

$$\int_0^T g_i(s)dB_s = G_i, \qquad \text{where } G_i = \frac{B_{\bar{t}_i} - B_{\bar{t}_{i-1}}}{\sqrt{h}}.$$

Roughly speaking this means that P_k, the k^{th} chaos, is generated by

$$\{K_{n_1}(G_1)\ldots K_{n_N}(G_N) : n_1 + \ldots + n_N = k\}.$$

Thus, the approximation we will use for the random variable F is

$$\mathcal{C}_p^N(F) = d_0 + \sum_{k=1}^{p} \sum_{|n|=k} d_k^n K_{n_1}(G_1)\ldots K_{n_N}(G_N) = d_0 + \sum_{k=1}^{p} \sum_{|n|=k} d_k^n \prod_{1\leq i\leq N} K_{n_i}(G_i), \tag{2.8}$$

where the coefficients d_0 and d_k^n are given by

$$d_0 = \mathbb{E}[F], \;\; d_k^n = n!\,\mathbb{E}\left[FK_{n_1}(G_1)\ldots K_{n_N}(G_N)\right]. \tag{2.9}$$

The following Lemma, similar to Lemma 2.4, gives some useful properties of the operator $\mathcal{C}_p^N$

Lemma 2.6. *Let F be a r.v. in $\mathrm{L}^2(\mathcal{F}_T)$ and H be in $\mathrm{H}^2_T(\mathbb{R})$. Then*

- $\forall (p,N) \in (\mathbb{N}^\star)^2$, $\mathbb{E}(|\mathcal{C}_p^N(F)|^2) \leq \mathbb{E}(|\mathcal{C}_p(F)|^2) \leq \mathbb{E}(|F|^2)$,
- $\mathcal{C}_p^N\left(\int_0^T H_s ds\right) = \int_0^T \mathcal{C}_p^N(H_s)ds.$
- *For all $t \leq r$, $D_t\mathbb{E}_r[\mathcal{C}_p^N(F)] = \mathbb{E}_r[\mathcal{C}_{p-1}^N(D_tF)]$.*

From (2.8), we deduce the expressions of $\mathbb{E}_t(\mathcal{C}_p^N F)$ and $D_t\mathbb{E}_t\left(\mathcal{C}_p^N(F)\right)$, useful for the approximation of (Y,Z) by the chaos decomposition (see (1.2)).

Proposition 2.7. *Let F be a real random variable in $\mathrm{L}^2(\mathcal{F}_T)$ and let r be an integer in $\{1, \cdots, N\}$. For all $\bar{t}_{r-1} < t \leq \bar{t}_r$, we have*

$$\mathbb{E}_t\left(\mathcal{C}_p^N F\right) = d_0 + \sum_{k=1}^{p} \sum_{|n(r)|=k} d_k^n \prod\nolimits_{i<r} K_{n_i}(G_i) \times \left(\frac{t-\bar{t}_{r-1}}{h}\right)^{\frac{n_r}{2}} K_{n_r}\left(\frac{B_t - B_{\bar{t}_{r-1}}}{\sqrt{t-\bar{t}_{r-1}}}\right),$$

$$D_t\mathbb{E}_t\left(\mathcal{C}_p^N(F)\right) = h^{-1/2} \sum_{k=1}^{p} \sum_{\substack{|n(r)|=k \\ n_r>0}} d_k^n \prod\nolimits_{i<r} K_{n_i}(G_i) \times \left(\frac{t-\bar{t}_{r-1}}{h}\right)^{\frac{n_r-1}{2}} K_{n_r-1}\left(\frac{B_t - B_{\bar{t}_{r-1}}}{\sqrt{t-\bar{t}_{r-1}}}\right),$$

where, if $r \leq N$ and $n = (n_1, \ldots, n_N)$, $n(r)$ stands for $(n_1, \ldots, n_r)$.

The proof of Proposition 2.7 is postponed to Section B.1.

Remark 2.8. *For $t = \bar{t}_r$ and $r \geq 1$, Proposition 2.7 leads to*

$$\mathbb{E}_{\bar{t}_r}\left(\mathcal{C}_p^N F\right) = d_0 + \sum_{k=1}^{p} \sum_{|n(r)|=k} d_k^n \prod\nolimits_{i\leq r} K_{n_i}\left(G_i\right)$$

$$D_{\bar{t}_r}\mathbb{E}_{\bar{t}_r}\left(\mathcal{C}_p^N F\right) = h^{-1/2}\sum_{k=1}^{p} \sum_{\substack{|n(r)|=k \\ n_r>0}} d_k^n \prod\nolimits_{i<r} K_{n_i}\left(G_i\right) \times K_{n_r-1}\left(G_r\right).$$

When $r = 0$, we get $\mathbb{E}_{\bar{t}_0}\left(\mathcal{C}_p^N F\right) = d_0$ and we define $D_{\bar{t}_0}\mathbb{E}_{\bar{t}_0}\left(\mathcal{C}_p^N F\right) = \frac{1}{\sqrt{h}} d_1^{e_1}$ (which is the limit of $D_t\mathbb{E}_t\left(\mathcal{C}_p^N F\right)$ when t tends to 0).

Let us end this subsection by some examples.

Example 2.9 (Case $p = 2$)**.** From (2.8)-(2.9), we have

$$\mathcal{C}_2^N(F) = d_0 + \sum_{j=1}^{N} d_1^{e_j} K_1(G_j) + \sum_{j=1}^{N}\sum_{i=1}^{j-1} d_2^{e_{ij}} K_1(G_i)K_1(G_j) + \sum_{j=1}^{N} d_2^{2e_j} K_2(G_j),$$

where e_j denotes the unit vector whose j^{th} component is one, and $e_{ij} = e_i + e_j$. For $j = 1, \cdots, N$ and $i = 1, \cdots, j-1$, it holds

$$d_1^{e_j} = \mathbb{E}(FK_1(G_j)),\ d_2^{e_{ij}} = \mathbb{E}(FK_1(G_i)K_1(G_j)),\ d_2^{2e_j} = 2\mathbb{E}(FK_2(G_j)).$$

Remark 2.8 leads to

$$\mathbb{E}_{\bar{t}_r}\left(\mathcal{C}_2^N F\right) = d_0 + \sum_{j=1}^{r} d_1^{e_j} K_1(G_j) + \sum_{j=1}^{r}\sum_{i=1}^{j-1} d_2^{e_{ij}} K_1(G_i)K_1(G_j) + \sum_{j=1}^{r} d_2^{2e_j} K_2(G_j),$$

$$D_{\bar{t}_r}\mathbb{E}_{\bar{t}_r}\left(\mathcal{C}_2^N F\right) = h^{-1/2}\left(d_1^{e_r} + d_2^{2e_r} K_1(G_r) + \sum_{i=1}^{r-1} d_2^{e_{ir}} K_1(G_i)\right).$$

3 Description of the algorithm

The algorithm is based on four types of approximations : Picard's iterations, a Wiener chaos expansion up to a finite order, the truncation of an $\mathrm{L}^2(0,T)$ basis in order to apply formulas of Proposition 2.7, and a Monte Carlo method to approximate the coefficients d_0 and d_k^n defined in (2.9). We present the first three steps of the approximation procedure in Section 3.1. The Monte Carlo method and the practical implementation are presented in Section 3.2.

3.1 Approximation procedure

3.1.1 Picard's iterations

The first step consists in approximating (Y, Z) — solution to (1.1) — by Picard's sequence $(Y^q, Z^q)_q$, built as follows : $(Y^0 = 0, Z^0 = 0)$ and for all $q \geq 1$

$$Y_t^{q+1} = \xi + \int_t^T f\left(s, Y_s^q, Z_s^q\right) ds - \int_t^T Z_s^{q+1} \cdot dB_s, \quad 0 \leq t \leq T. \tag{3.1}$$

From (3.1), under the assumptions that $\xi \in \mathbb{D}^{1,2}$ and $f \in C_b^{0,1,1}$, we express (Y^{q+1}, Z^{q+1}) as a function of the processes (Y^q, Z^q):

$$Y_t^{q+1} = \mathbb{E}_t\left(\xi + \int_t^T f\left(s, Y_s^q, Z_s^q\right) ds\right), \quad Z_t^{q+1} = D_t Y_t^{q+1}, \tag{3.2}$$

which can also be written

$$Y_t^{q+1} = \mathbb{E}_t\left(\xi + \int_0^T f\left(s, Y_s^q, Z_s^q\right) ds\right) - \int_0^t f\left(s, Y_s^q, Z_s^q\right) ds, \quad Z_t^{q+1} = D_t Y_t^{q+1}. \tag{3.3}$$

As recalled in the introduction, the computation of the conditional expectation is the cornerstone in the numerical resolution of BSDEs. Chaos decomposition formulas enable to circumvent this problem.

3.1.2 Wiener Chaos Expansion

Computing the chaos decomposition of the r.v. $F = \xi + \int_t^T f\left(s, Y_s^q, Z_s^q\right) ds$ (appearing in (3.2)) in order to compute Y_t^{q+1} is not judicious. F depends on t, and then the computation of Y^{q+1} on the grid $\mathcal{T} = \{\bar{t}_i = i\frac{T}{N}, i = 0, \cdots, N\}$ would require $N+1$ calls to the chaos decomposition function. To build an efficient algorithm, we need to call the chaos decomposition function as less as possible, since each call is computationally demanding and brings an approximation error due to the truncation and to the Monte-Carlo approximation (see next Sections). Then, we look for a r.v. F^q independent of t such that Y_t^{q+1} and Z_t^{q+1} can be expressed as functions of $\mathbb{E}_t(F^q)$, $D_t\mathbb{E}_t(F^q)$ and of Y^q and Z^q. Equation (3.3) gives a more tractable expression of Y^{q+1}. Let F^q be defined by $F^q := \xi + \int_0^T f(s, Y_s^q, Z_s^q) ds$. Then

$$Y_t^{q+1} = \mathbb{E}_t(F^q) - \int_0^t f\left(s, Y_s^q, Z_s^q\right) ds, \quad Z_t^{q+1} = D_t\mathbb{E}_t(F^q). \tag{3.4}$$

The second type of approximation consists in computing the chaos decomposition of F^q up to order p. Since F^q does not depend on t, the chaos decomposition function $\mathcal{C}_p$ is called only once per Picard's iteration.

Let $(Y^{q,p}, Z^{q,p})$ denote the approximation of (Y^q, Z^q) built at step q using a chaos decomposition with order p: $(Y^{0,p}, Z^{0,p}) = (0, 0)$ and

$$Y_t^{q+1,p} = \mathbb{E}_t\left[\mathcal{C}_p\left(F^{q,p}\right)\right] - \int_0^t f\left(s, Y_s^{q,p}, Z_s^{q,p}\right) ds, \quad Z_t^{q+1,p} = D_t\mathbb{E}_t\left[\mathcal{C}_p\left(F^{q,p}\right)\right], \tag{3.5}$$

where $F^{q,p} = \xi + \int_0^T f\left(s, Y_s^{q,p}, Z_s^{q,p}\right) ds$. In the sequel, we also use the following equality

$$Z_t^{q+1,p} = \mathbb{E}_t[D_t\mathcal{C}_p(F^{q,p})]. \tag{3.6}$$

3.1.3 Truncation of the basis

The third type of approximation comes from the truncation of the orthonormal $\mathrm{L}^2(0, T)$ basis used in the definition of $\mathcal{C}_p$ (2.5). Instead of considering a basis of $\mathrm{L}^2(0, T)$, we only keep the first N functions $(g_1, \cdots, g_N)$ defined by (2.6) to build the chaos decomposition function $\mathcal{C}_p^N$ (2.7). Proposition 2.7 gives us explicit formulas for $\mathbb{E}_t(\mathcal{C}_p^N F)$ and $D_t\mathbb{E}_t(\mathcal{C}_p^N F)$. From (3.5), we build $((Y^{q,p,N}, Z^{q,p,N})_q$ in the following way : $((Y^{0,p,N}, Z^{0,p,N}) = (0, 0)$ and

$$Y_t^{q+1,p,N} = \mathbb{E}_t(\mathcal{C}_p^N(F^{q,p,N})) - \int_0^t f\left(s, Y_s^{q,p,N}, Z_s^{q,p,N}\right) ds, \quad Z_t^{q+1,p,N} = D_t(\mathbb{E}_t(\mathcal{C}_p^N(F^{q,p,N}))), \tag{3.7}$$

where $F^{q,p,N} := \xi + \int_0^T f(s, Y_s^{q,p,N}, Z_s^{q,p,N}) ds$.

Equation (3.7) is tractable as soon as we know closed formulas for the coefficients d_k^n of the chaos decomposition of $\mathbb{E}_t(\mathcal{C}_p^N(F^{q,p,N}))$ and $D_t(\mathbb{E}_t(\mathcal{C}_p^N(F^{q,p,N})))$ (see Proposition 2.7). When it is not the case, we need to use a Monte-Carlo method to approximate these coefficients. The next Section is devoted to this method and to the practical implementation. In particular, we give the pseudo-code of the algorithm.

3.2 Implementation

In this Section, we first explain how to practically compute the chaos decomposition $\mathcal{C}_p^N(F)$ of a r.v. F. Then, we give the pseudo-code of the algorithm.

3.2.1 Monte-Carlo simulations of the chaos decomposition

Let F denote a r.v. of $\mathrm{L}^2(\mathcal{F}_T)$. Practically, when we are not able to compute exactly d_0 and/or the coefficients d_k^n of the chaos decomposition (2.8)-(2.9) of F, we use Monte-Carlo simulations to approximate them. Let $(F^m)_{1\leq m\leq M}$ be a M i.i.d. sample of F and $(G_1^m,\cdots,G_N^m)_{1\leq m\leq M}$ be a M i.i.d. sample of $(G_1,\cdots,G_N)$. We recall that d_0 and the coefficients $(d_k^n)_{1\leq k\leq p,|n|=k}$ are given by $d_0=\mathbb{E}[F]$ and $d_k^n=n!\,\mathbb{E}\left[FK_{n_1}(G_1)\dots K_{n_N}(G_N)\right]$ (see (2.9)). Then, they are solutions of

$$\underset{\mathbf{c}=(c_0,(c_k^n)_{1\leq k\leq p,|n|=k})}{\arg\min}\quad \mathbb{E}[|F-\psi(c,G)|^2], \tag{3.8}$$

where $\psi:(\mathbf{c},G)\longmapsto c_0+\sum_{k=1}^p\sum_{|n|=k}c_k^n\prod_{1\leq i\leq N}K_{n_i}(G_i)$. We propose two methods to approximate $\mathbf{d}:=(d_0,(d_k^n)_{1\leq k\leq p,|n|=k})$

- the first one consists in approximating the expectations of (2.9) by empirical means $\widehat{\mathbf{d_M}}:=(\hat{d}_0,\hat{d}^n_{k\,1\leq k\leq p,|n|=k})$ where

$$\widehat{d_0}:=\frac{1}{M}\sum_{m=1}^M F^m,\;\; \widehat{d_k^n}:=\frac{n!}{M}\sum_{m=1}^M F^mK_{n_1}(G_1^m)\cdots K_{n_N}(G_N^m), \tag{3.9}$$

- the second one is based on a sample average approximation

$$\overline{\mathbf{d_M}}:=(\overline{d_0},\overline{d_k^n}_{1\leq k\leq p,|n|=k})=\underset{c_0,(c_k^n)_{1\leq k\leq p,|n|=k}}{\arg\min}\frac{1}{M}\sum_{m=1}^M|F^m-\psi(\mathbf{c},G^m)|^2$$

Remark 3.1. *In terms of computation time, the first method is much faster than the second one.*

- *The first method requires $O(M\times p)$ computations per coefficient. Since we are looking for $O(N^p)$ coefficients, its computational cost is $O(M\times p\times N^p)$.*
- *The second method requires $O(M\times p\times N^p)$ computations to evaluate $\frac{1}{M}\sum_{m=1}^M|F^m-\psi(c,G^m)|^2$ (in fact, it requires the same number of computations as the first method, since the function ψ contains as much as additions as coefficients, and each addition contains as much as products as the associated coefficient). We still have to compute the argmin, which computational cost depends on the method we use.*

From a theoretical point of view, the second method gives better convergence results than the first one. For the first method, we only know that $\widehat{\mathbf{d_M}}$ converges to $\mathbf{d}$ a.s.. Concerning the second method, we know that $\overline{\mathbf{d_M}}$ converges to $\mathbf{d}$ a.s. and under regularity assumptions on ψ, the uniform strong law of large numbers gives the a.s. convergence of $\frac{1}{M}\sum_{m=1}^M|F^m-\psi(\overline{\mathbf{d_M}},G^m)|^2$ to $\mathbb{E}[|F-\psi(\mathbf{d},G)|^2]$.

In the following, $\mathcal{C}_p^{N,M}(F)$ denotes the approximation of the chaos decomposition of order p of F when using the first method to approximate the coefficients d_k^n:

$$\mathcal{C}_p^{N,M}(F)=\widehat{d_0}+\sum_{k=1}^p\sum_{|n|=k}\widehat{d_k^n}\prod_{1\leq i\leq N}K_{n_i}(G_i). \tag{3.10}$$

$\mathbb{E}_t(\mathcal{C}_p^{N,M}(F))$ and $D_t(\mathbb{E}_t(\mathcal{C}_p^{N,M}(F)))$ denote the conditional expectations obtained in Proposition 2.7 when $(d_0, d_k^n)_{1\le k\le p, |n|=k})$ are replaced by $(\widehat{d_0}, \widehat{d_k^n})_{1\le k\le p,|n|=k})$:

$$\mathbb{E}_t\left(\mathcal{C}_p^{N,M}F\right) := \widehat{d_0} + \sum_{k=1}^{p}\sum_{|n(r)|=k}\widehat{d_k^n}\prod\nolimits_{i<r}K_{n_i}(G_i)\times\left(\frac{t-\overline{t}_{r-1}}{h}\right)^{\frac{n_r}{2}}K_{n_r}\left(\frac{B_t-B_{\overline{t}_{r-1}}}{\sqrt{t-\overline{t}_{r-1}}}\right),$$

$$D_t\mathbb{E}_t\left(\mathcal{C}_p^{N,M}(F)\right) := h^{-1/2}\sum_{k=1}^{p}\sum_{\substack{|n(r)|=k\\ n_r>0}}\widehat{d_k^n}\prod\nolimits_{i<r}K_{n_i}(G_i)\times\left(\frac{t-\overline{t}_{r-1}}{h}\right)^{\frac{n_r-1}{2}}K_{n_r-1}\left(\frac{B_t-B_{\overline{t}_{r-1}}}{\sqrt{t-\overline{t}_{r-1}}}\right),$$

Remark 3.2. *When M samples of $\mathcal{C}_p^{N,M}(F)$ are needed, we can either use the same samples as the ones used to compute $\widehat{d_0}$ and $\widehat{d_k^n}$: $(\widehat{\mathcal{C}_p^N}(F))^m = \widehat{d_0} + \sum_{k=1}^p\sum_{|n|=k}\widehat{d_k^n}\prod_{1\le i\le N}K_{n_i}(G_i^m)$, or use new ones. In the first case, we only require M samples of F and $(G_1,\cdots,G_N)$. The coefficients $\widehat{d_k^n}$ and $\widehat{d_0}$ are not independent of $\prod_{1\le i\le N}K_{n_i}(G_i^m)$. The notation $\mathbb{E}_t(\mathcal{C}_p^{N,M}(F))$ introduced above cannot be linked to $\mathbb{E}\left(\mathcal{C}_p^{N,M}F|\mathcal{F}_t\right)$. In the second case, the coefficients $\widehat{d_k^n}$ and $\widehat{d_0}$ are independent of $\prod_{1\le i\le N}K_{n_i}(G_i^m)$ and we have $\mathbb{E}_t\left(\mathcal{C}_p^{N,M}F\right) = \mathbb{E}\left(\mathcal{C}_p^{N,M}F|\mathcal{F}_t\right)$. This second approach requires $2M$ samples of F and $(G_1,\cdots,G_N)$ and its variance increases with N. Practically, we use the first technique.*

We introduce the processes $(Y^{q+1,p,N,M}, Z^{q+1,p,N,M})$, useful in the following. It corresponds to the approximation of $(Y^{q+1,p,N}, Z^{q+1,p,N})$ when we use $\mathcal{C}_p^{N,M}$ instead of $\mathcal{C}_p^N$, i.e. when we use a Monte Carlo procedure to compute the coefficients d_k^n.

$$Y_t^{q+1,p,N,M} = \mathbb{E}_t(\mathcal{C}_p^{N,M}(F^{q,p,N,M})) - \int_0^t f\left(\theta_s^{q,p,N,M}\right)ds,\ \ Z_t^{q+1,p,N,M} = D_t(\mathbb{E}_t(\mathcal{C}_p^{N,M}(F^{q,p,N,M}))), \tag{3.11}$$

where $F^{q,p,N,M} := \xi + \int_0^T f(\theta_s^{q,p,N,M})ds$ and $\theta_s^{q,p,N,M} = \left(s, Y_s^{q,p,N,M}, Z_s^{q,p,N,M}\right)$.

3.2.2 Pseudo-code of the Algorithm

In this Section, we describe in details the algorithm. We aim at computing M trajectories of an approximation of (Y,Z) on the grid $\mathcal{T} = \{\overline{t}_i = i\frac{T}{N}, i = 0,\cdots,N\}$. Starting from $(Y^{0,p,N,M}, Z^{0,p,N,M}) = (0,0)$, (3.11) enables to get $(Y^{q,p,N,M}, Z^{q,p,N,M})$ for each Picard's iteration q on $\mathcal{T}$. Practically, we discretize the integral $\int_0^t f\left(\theta_s^{q,p,N,M}\right)ds$ which leads to approximated values of $(Y^{q,p,N,M}, Z^{q,p,N,M})$ computed on a grid.

Let us introduce $(\overline{Y}_{\overline{t}_i}^{q+1,p,N,M}, \overline{Z}_{\overline{t}_i}^{q+1,p,N,M})_{1\le i\le N}$, defined by $(\overline{Y}^{0,p,N,M}, \overline{Z}^{0,p,N,M}) = (0,0)$ and for all $q\ge 0$

$$\begin{aligned}\overline{Y}_{\overline{t}_i}^{q+1,p,N,M} &= \mathbb{E}_{\overline{t}_i}(\mathcal{C}_p^{N,M}(\overline{F}^{q,p,N,M})) - h\sum_{j=1}^{i} f\left(\overline{t}_j, \overline{Y}_{\overline{t}_j}^{q,p,N,M}, \overline{Z}_{\overline{t}_j}^{q,p,N,M}\right),\\ \overline{Z}_{\overline{t}_i}^{q+1,p,N,M} &= D_{\overline{t}_i}(\mathbb{E}_{\overline{t}_i}(\mathcal{C}_p^{N,M}(\overline{F}^{q,p,N,M}))),\end{aligned} \tag{3.12}$$

where $\overline{F}^{q,p,N,M} := \xi + h\sum_{i=1}^N f(\overline{t}_i, \overline{Y}_{\overline{t}_i}^{q,p,N,M}, \overline{Z}_{\overline{t}_i}^{q,p,N,M})$. Here are the notations we use in the algorithm.

- d: dimension of the Brownian motion
- q: index of Picard's iteration
- K_{it}: number of Picard's iterations

- M: number of Monte–Carlo samples
- N: number of time steps used for the discretization of Y and Z
- p: order of the chaos decomposition
- $\boldsymbol{Y}^q \in \mathcal{M}_{N+1,M}(\mathbb{R})$ represents M paths of $\overline{Y}^{q,p,N,M}$ computed on the grid $\mathcal{T}$.
- For all $l \in \{1, \cdots, d\}$, $(\boldsymbol{Z}^q)_l \in \mathcal{M}_{N+1,M}(\mathbb{R})$ represents M paths of $(\overline{Z}^{q,p,N,M})_l$ computed on the grid $\mathcal{T}$.

Since $\xi \in \mathrm{L}^2(\mathcal{F}_T)$, ξ can be written as a measurable function of the Brownian path. Then, one gets one sample of ξ from one sample of $(G_1, \cdots, G_N)$ (where G_i represents $\frac{B_{\bar{t}_i} - B_{\bar{t}_{i-1}}}{\sqrt{h}}$).

For the sake of clearness, we detail the algorithm for $d = 1$.

Algorithm 1 Iterative algorithm

1: Pick at random $N \times M$ values of standard Gaussian r.v. stored in $\boldsymbol{G}$.
2: Using $\boldsymbol{G}$, compute $(\xi^m)_{0 \leq m \leq M-1}$.
3: $\boldsymbol{Y}^0 \equiv 0$, $\boldsymbol{Z}^0 \equiv 0$.
4: **for** $q = 0 : K_{it} - 1$ **do**
5:     **for** $m = 0 : M - 1$ **do**
6:         Compute $(F^q)^m = \xi^m + h \sum_{i=1}^N f(\bar{t}_i, (\boldsymbol{Y}^q)_{i,m}, (\boldsymbol{Z}^q)_{i,m})$
7:     **end for**
8:     Compute the vector $\mathbf{d} = (\widehat{d_0}, (\widehat{d_k^n})_{1 \leq k \leq p, |n|=k})$ of the chaos decomposition of F^q
9:     $\widehat{d_0} := \frac{1}{M} \sum_{m=0}^{M-1} (F^q)^m$, $\widehat{d_k^n} = \frac{n!}{M} \sum_{m=0}^{M-1} (F^q)^m K_{n_1}(G_1^m) \cdots K_{n_N}(G_N^m)$
10:     **for** $j = 1 : N$ **do**
11:         **for** $m = 0 : M - 1$ **do**
12:             Compute $(\mathbb{E}_{\bar{t}_j}(\mathcal{C}_p^{N,M} F^q))^m$, $(D_{\bar{t}_j}(\mathbb{E}_{\bar{t}_j}(\mathcal{C}_p^{N,M} F^q)))^m$
13:             $(\boldsymbol{Y}^{q+1})_{j,m} = (\mathbb{E}_{\bar{t}_j}(\mathcal{C}_p^{N,M} F^q))^m - h \sum_{i=1}^j f(\bar{t}_i, (\boldsymbol{Y}^q)_{i,m}, (\boldsymbol{Z}^q)_{i,m})$
14:             $(\boldsymbol{Z}^{q+1})_{j,m} = (D_{\bar{t}_j}(\mathbb{E}_{\bar{t}_j}(\mathcal{C}_p^{N,M} F^q)))^m$
15:         **end for**
16:     **end for**
17: **end for**
18: Return $(\boldsymbol{Y}^{K_{it}})_{0,:} = \hat{d}_0$ and $(\boldsymbol{Z}^{K_{it}})_{0,:} = \frac{1}{\sqrt{h}} \hat{d}_1^{e_1}$

Let us now deal with the complexity of the algorithm :
For each q:

- the computation of the vector F^q (loop line 5) requires $O(M \times N)$ computations,
- the computation of the vector $\mathbf{d}$ (line 8) requires $O(M \times p \times (N \times d)^p)$ computations, (in dimension d we have $O((N \times d)^p)$ coefficients, and the computation of each coefficient requires $O(M \times p)$ computations (see Remark 3.1)),
- for each N and M (lines 10-11)
  - the computation of $(\mathbb{E}_{\bar{t}_j}(\mathcal{C}_p^{N,M} F^q))^m$ and of $(D_{\bar{t}_j}^l(\mathbb{E}_{\bar{t}_j}(\mathcal{C}_p^{N,M} F^q)))_{1 \leq l \leq d}^m$ (line 12) requires $O(d \times p \times (N \times d)^p)$ computations
  - the computation of $(\boldsymbol{Y}^{q+1})_{j,m}$ (loop line 13) requires $O(N)$ computations and the computation of $((\boldsymbol{Z}^{q+1})_{j,m}^l)_{1 \leq l \leq d}$ requires $O(d)$ computations.

The complexity of the algorithm is then $O(K_{it} \times M \times p \times (N \times d)^{p+1})$.

4 Convergence results

We aim at bounding the error between (Y, Z) — the solution of (1.1) — and $(Y^{q,p,N,M}, Z^{q,p,N,M})$ defined by (3.11). Before stating the main result of the paper, we introduce some hypotheses.

In the following, $(t_1, \cdots, t_n)$ and $(s_1, \cdots, s_n)$ denote two vectors such that

$$0 \leq t_1 \leq \cdots \leq t_n \leq T,\ 0 \leq s_1 \leq \cdots \leq s_n \leq T \text{ and } \forall i,\ s_i \leq t_i.$$

Hypothesis 4.1 (Hypothesis $\mathcal{H}_m$)**.** *Let $m \in \mathbb{N}^*$. We say that F satisfies Hypothesis $\mathcal{H}_m$ if F satisfies the two following hypotheses*

- $\mathcal{H}_m^1$ *: $\forall j \geq 2$ $F \in \mathcal{D}^{m,j}$, i.e. $\|F\|_{m,j}^j < \infty$*
- $\mathcal{H}_m^2$ *: $\forall j \geq 2$, $\forall i \in \{1, \cdots, m\}$, $\forall l_0 \leq i-1$, $\forall l_1 \leq m-i$, $\forall l \in \{1, \cdots, d\}$ and for all multi-indices α_0 and α_1 such that $|\alpha_0| = l_0$ and $|\alpha_1| = l_1 + 1$, there exist two positive constants β_F and k_l^F such that*

$$\sup_{t_1 \leq \cdots \leq t_{l_0}} \sup_{s_{i+1} \leq \cdots \leq s_{i+l_1}} \mathbb{E}[|D^{\alpha_0}_{t_1,\cdots,t_{l_0}}(D^{\alpha_1}_{t_i,s_{i+1},\cdots,s_{i+l_1}}F - D^{\alpha_1}_{s_i,\cdots,s_{i+l_1}}F)|^j] \leq k_l^F(j)(t_i - s_i)^{j\beta_F},$$

where $l = l_0 + l_1 + 1$. In the following, we denote $K_m^F(j) = \sup_{l \leq m} k_l^F(j)$.

Remark 4.2. *If F satisfies $\mathcal{H}_m^2$, for all multi-index α such that $|\alpha| = l$ we have*

$$|\mathbb{E}(D^{\alpha}_{t_1,\cdots,t_l}F) - \mathbb{E}(D^{\alpha}_{s_1,\cdots,s_l}F)| \leq K_l^F((t_1 - s_1)^{\beta_F} + \cdots + (t_l - s_l)^{\beta_F}), \tag{4.1}$$

where K_l^F is a constant.

Hypothesis 4.3 (Hypothesis $\mathcal{H}^3_{p,N}$)**.** *Let $(p, N) \in \mathbb{N}^2$. We say that a r.v. F satisfies $\mathcal{H}^3_{p,N}$ if*

$$V_{p,N}(F) := \mathbb{V}(F) + \sum_{k=1}^{p} \sum_{|n|=k} n!\mathbb{V}\left(F \prod_{i=1}^{N} K_{n_i}(G_i)\right) < \infty.$$

Remark 4.4. *If F is bounded by K, we get $V_{p,N}(F) \leq K^2 \sum_{k=0}^{p} \binom{N}{k}$. Then, every bounded r.v. satisfies $\mathcal{H}^3_{p,N}$.*

This Remark ensues from $\mathbb{E}\left(\prod_{i=1}^{N} K^2_{n_i}(G_i)\right) = \frac{1}{n!}$.

Remark 4.5. *Let X be the $\mathbb{R}^n$-valued process solution of*

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s,$$

where B is a d-dimensional Brownian motion and $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ are two $C^{0,m}$ functions uniformly lipschitz w.r.t. x and Hölder continuous of parameter $\frac{1}{2}$ w.r.t. t, with linear growth in x and with bounded derivatives. Then, every random variable ξ of type $g(X_T)$ or $g(\int_0^T X_s ds)$ with $g : \mathbb{R}^n \to \mathbb{R}$ in C_p^∞ satisfies $\mathcal{H}_m$ and $\mathcal{H}^3_{p,N}$, for all p and N.

We refer to Section A.1 for the proof of Remark 4.5.

Theorem 4.6. *Let k be an integer s.t. $k \leq p$. Assume that ξ satisfies $\mathcal{H}_{p+q}$ and $\mathcal{H}^3_{p,N}$ and $f \in C_b^{0,p+q-1,p+q-1}$. We have*

$$\|(Y - Y^{q,p,N,M}, Z - Z^{q,p,N,M})\|^2_{\mathrm{L}^2} \leq \frac{A_0}{2^q} + \frac{A_1(q,k)}{(p+1)^k} + A_2(q,p)\left(\frac{T}{N}\right)^{2\beta_\xi \wedge 1} + \frac{A_3(q,p,N)}{M},$$

where A_0 is given in Section 4.1, A_1 is given in Proposition 4.11, A_2 is given in Proposition 4.15, and A_3 is given in Proposition 4.17.
If $f \in C_b^{0,\infty,\infty}$ and ξ satisfies $\mathcal{H}_\infty$ and $\mathcal{H}^3_{\infty,\infty}$, we get

$$\lim_{q\to\infty} \lim_{p\to\infty} \lim_{N\to\infty} \lim_{M\to\infty} \|(Y - Y^{q,p,N,M}, Z - Z^{q,p,N,M})\|^2_{\mathrm{L}^2} = 0.$$

Remark 4.7. *If f is a path-dependent generator, theorem 4.6 still holds true under the following hypotheses : $\forall l \leq p$, $\forall j \geq 2$, for all multi-index α in $\{1 \cdots, d+1\}^l$ (d is the dimension of the Brownian motion) s.t. $a(i) = d+1$ means that the Malliavin derivative w.r.t. t_i concerns the path-dependent component, we assume*

$$\int_0^T \|D^\alpha_{t_1,\cdots,t_l} f(s, Y^q_s, Z^q_s)\|^2_{\mathrm{L}^2(\Omega\times[0,T]^m)} ds < \infty,$$
$$\int_0^T \mathbb{E}[|D^\alpha_{t_1,\cdots,t_l} f(s, Y^q_s, Z^q_s)|^j] ds < \infty, \int_0^T \mathbb{E}[|D^\alpha_{t_1,\cdots,t_l} f(s, Y^{q,p}_s, Z^{q,p}_s)|^j] ds < \infty, \text{ and}$$
$$|\mathbb{E}(D^\alpha_{t_1,\cdots,t_l} I_{q,p}) - \mathbb{E}(D^\alpha_{s_1,\cdots,s_l} I_{q,p})| \leq K_l^{I_{q,p}}((t_1-s_1)^{\beta_{I_{q,p}}} + \cdots + (t_l - s_l)^{\beta_{I_{q,p}}}),$$

where $I_{q,p} = \int_0^T f(\theta^{q,p}_r) dr$, and $K_l^{I_{q,p}}$ and $\beta_{I_{q,p}}$ are two positive constants.

Remark 4.8. *Given the complexity C_0 of the algorithm (and a given value of d), we can choose the parameters p, q, N and M such that they minimize the error $\frac{A_0}{2^q} + \frac{A_1(q,p)}{(p+1)^p} + A_2(q,p)\left(\frac{T}{N}\right)^a + \frac{A_3(q,p,N)}{M}$, where $a := 2\beta_\xi \wedge 1$. This boilds down to solving the following constrained minimization problem*

$$\min_{q,p,N,M \text{ s.t. } qpMN^{p+1}=C_0} \left(\frac{1}{2^q} + \frac{C^q}{(p+1)^p} + \frac{C^q}{N^a} + \frac{C^q N^p}{M}\right).$$

The Karush-Kuhn-Tucker theorem gives $M \sim \frac{2p}{a}(p+1)^{p+\frac{p^2}{a}}$, $N \sim (p+1)^{\frac{p}{a}}$, $q \sim \frac{1}{\ln(2C)} p \ln(p+1)$ and p such that $(p+1)^{2p(1+\frac{p}{a})} p^3 \ln(p+1) \sim a \log(2C) C_0$.

Proof of Theorem 4.6. We split the error in 4 terms :

1. Picard's iterations : $\mathcal{E}^q = \|(Y - Y^q, Z - Z^q)\|^2_{\mathrm{L}^2}$, where (Y^q, Z^q) is defined by (3.1),
2. the truncation of the chaos decomposition : $\mathcal{E}^{q,p} = \|(Y^q - Y^{q,p}, Z^q - Z^{q,p})\|^2_{\mathrm{L}^2}$, where $(Y^{q,p}, Z^{q,p})$ is defined by (3.5),
3. the truncation of the $\mathrm{L}^2(0,T)$ basis : $\mathcal{E}^{q,p,N} = \|(Y^{q,p} - Y^{q,p,N}, Z^{q,p} - Z^{q,p,N})\|^2_{\mathrm{L}^2}$, where $(Y^{q,p,N}, Z^{q,p,N})$ is defined by (3.7).
4. the Monte-Carlo approximation to compute the expectations : $\mathcal{E}^{q,p,N,M} = \|(Y^{q,p,N} - Y^{q,p,N,M}, Z^{q,p,N} - Z^{q,p,N,M})\|^2_{\mathrm{L}^2}$, where $(Y^{q,p,N,M}, Z^{q,p,N,M})$ is defined by (3.11).

We have

$$\|(Y - Y^{q,p,N,M}, Z - Z^{q,p,N,M})\|^2_{\mathrm{L}^2} \leq 4(\mathcal{E}^q + \mathcal{E}^{q,p} + \mathcal{E}^{q,p,N} + \mathcal{E}^{q,p,N,M}).$$

It remains to combine (4.2), Proposition 4.11, Proposition 4.15 and Proposition 4.17 to get the first result. □

4.1 Picard's iterations

The first type of error has already been studied in [PP92] and [EPQ97], we only recall the main result.

Hypothesis 4.9. *We assume*

- *the generator $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is Lipschitz continuous: there exists a constant L_f such that for all $t \in \mathbb{R}^+$, $y_1, y_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{R}^d$*

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq L_f (|y_1 - y_2| + |z_1 - z_2|),$$

- $\mathbb{E}[|\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds] < \infty$.

From [EPQ97, Corollary 2.1], we know that under Hypothesis 4.9, the sequence $(Y^q, Z^q)_q$ defined by (3.1) converges to (Y, Z) $d\mathbb{P} \times dt$ a.s. and in $\mathrm{S}^2_T(\mathbb{R}) \times \mathrm{H}^2_T(\mathbb{R}^d)$. Moreover, we have

$$\mathcal{E}^q := \|(Y - Y^q, Z - Z^q)\|^2_{\mathrm{L}^2} \leq \frac{A_0}{2^q}, \tag{4.2}$$

where A_0 depends on T, $\|\xi\|^2$ and on $\|f(\cdot, 0, 0)\|^2_{\mathrm{L}^2_{(0,T)}}$.

4.2 Error due to the truncation of the chaos decomposition

We assume that the integrals are computed exactly, as well as expectations. The error is only due to the truncation of the chaos decomposition $\mathcal{C}_p$ introduced in (2.3).

For the sequel, we also need the following Lemma. We postpone its proof to the Appendix A.2.

Lemma 4.10. *Assume that ξ satisfies $\mathcal{H}^1_{m+q}$ and $f \in \mathcal{C}^{0,m+q-1,m+q-1}_b$. Then $\forall q' \leq q$, $\forall p \in \mathbb{N}$, $(Y^{q'}, Z^{q'})$ and $(Y^{q',p}, Z^{q',p})$ belong to $\mathcal{S}^{m,\infty}$. Moreover*

$$\|(Y^q, Z^q)\|^j_{m,j} + \|(Y^{q,p}, Z^{q,p})\|^j_{m,j} \leq C(\|\xi\|_{m+q, \frac{(m+q-1)!}{m!}j}, (\|\partial^k_{sp} f\|_\infty)_{k \leq m+q-1}),$$

where C is a constant depending on $\|\xi\|_{m+q, \frac{(m+q-1)!}{m!}j}$ and on $(\|\partial^k_{sp} f\|_\infty)_{k \leq m+q-1}$.

Proposition 4.11. *Let $m \in \mathbb{N}^\star$. Assume that ξ satisfies $\mathcal{H}^1_{m+q}$ and $f \in C^{0,m+q-1,m+q-1}_b$. We recall $\mathcal{E}^{q,p} = \|(Y^q - Y^{q,p}, Z^q - Z^{q,p})\|^2_{\mathrm{L}^2}$. We get*

$$\mathcal{E}^{q+1,p} \leq C_1 T(T+1) L^2_f \mathcal{E}^{q,p} + \frac{K_1(q,m)}{(p+1)\cdots(p+m)} \tag{4.3}$$

where C_1 is a scalar and $K_1(q,m)$ depends on T, m, $\|\xi\|_{m+q, 2\frac{(m+q-1)!}{(m-1)!}}$ and on $(\|\partial^k_{sp} f\|_\infty)_{1 \leq k \leq m+q-1}$.

Since $\mathcal{E}^{0,p} = 0$, we deduce from (4.3) that $\mathcal{E}^{q,p} \leq \frac{A_1(q,m)}{(p+1)^m}$ where $A_1(q,m) := \frac{(C_1 T(T+1) L^2_f)^q - 1}{C_1 T(T+1) L^2_f - 1} K_1(q,m)$. Then, $(Y^{p,q}, Z^{p,q})$ converges to (Y^q, Z^q) when p tends to ∞ in $\|(\cdot,\cdot)\|_{\mathrm{L}^2}$ (see (2.1) for the Definition of the norm).

Remark 4.12. *We deduce from Proposition 4.11 that for all T and L_f, we have $lim_{p \to \infty} \mathcal{E}^{q,p} = 0$. When $C_1 T(T+1) L^2_f < 1$, i.e. for T small enough, we also get $lim_{p\to\infty} lim_{q\to\infty} \mathcal{E}^{q,p} = 0$.*

Proof of Proposition 4.11. For the sake of clearness, we assume $d = 1$. In the following, one notes $\Delta Y^{q,p}_t := Y^{q,p}_t - Y^q_t$, $\Delta Z^{q,p}_t := Z^{q,p}_t - Z^q_t$ and $\Delta f^{q,p}_t := f(t, Y^{q,p}_t, Z^{q,p}_t) - f(t, Y^q_t, Z^q_t)$. Firstly, we deal with $\mathbb{E}[\sup_{0 \leq t \leq T} |\Delta Y^{q+1,p}_t|^2]$. From (3.4) and (3.5) we get

$$\begin{aligned}
\Delta Y^{q+1,p}_t =& \mathbb{E}_t[\mathcal{C}_p(F^{q,p}) - F^q] - \int_0^t \Delta f^{q,p}_s ds, \\
=& \mathbb{E}_t[\mathcal{C}_p(\xi) - \xi] + \mathbb{E}_t\left[\mathcal{C}_p\left(\int_0^T f(s, Y^{q,p}_s, Z^{q,p}_s) ds\right) - \int_0^T f(s, Y^q_s, Z^q_s) ds\right] - \int_0^t \Delta f^{q,p}_s ds.
\end{aligned}$$

We introduce $\pm \mathcal{C}_p\left(\int_0^T f(s, Y^q_s, Z^q_s) ds\right)$ in the second conditional expectation. This leads to

$$\begin{aligned}
\Delta Y^{q+1,p}_t =& \mathbb{E}_t[\mathcal{C}_p(\xi) - \xi] + \mathbb{E}_t\left[\mathcal{C}_p\left(\int_0^T \Delta f^{q,p}_s ds\right)\right] + \mathbb{E}_t\left[\int_0^T \mathcal{C}_p(f(s, Y^q_s, Z^q_s)) - f(s, Y^q_s, Z^q_s) ds\right] \\
& - \int_0^t \Delta f^{q,p}_s ds,
\end{aligned}$$

where we have used the second property of Lemma 2.4 to rewrite the third term.

From the previous equation, we bound $\mathbb{E}[\sup_{0\le t\le T}|\Delta Y_t^{q+1,p}|^2]$ by using Doob's inequality and the Lipschitz property of f

$$\mathbb{E}[\sup_{0\le t\le T}|\Delta Y_t^{q+1,p}|^2] \le 16\mathbb{E}[|\mathcal{C}_p(\xi)-\xi|^2] + 16\mathbb{E}\left[\left|\mathcal{C}_p\left(\int_0^T \Delta f_s^{q,p} ds\right)\right|^2\right]$$
$$+ 16T\int_0^T \mathbb{E}\left[|\mathcal{C}_p(f(s,Y_s^q,Z_s^q)) - f(s,Y_s^q,Z_s^q)|^2\right] ds + 8TL_f^2\int_0^T \mathbb{E}[|\Delta Y_s^{q,p}|^2 + |\Delta Z_s^{q,p}|^2]ds.$$

To bound the second expectation of the previous inequality, we use the first property of Lemma 2.4 and the Lispchitz property of f. Then, we bring together this term with the last one to get

$$\mathbb{E}[\sup_{0\le t\le T}|\Delta Y_t^{q+1,p}|^2] \le 16\mathbb{E}[|\mathcal{C}_p(\xi)-\xi|^2] + 16T\int_0^T \mathbb{E}\left[|\mathcal{C}_p(f(s,Y_s^q,Z_s^q)) - f(s,Y_s^q,Z_s^q)|^2\right] ds$$
$$+ 40TL_f^2\int_0^T \mathbb{E}[|\Delta Y_s^{q,p}|^2 + |\Delta Z_s^{q,p}|^2]ds. \tag{4.4}$$

Let us now upper bound $\mathbb{E}[\int_0^T |\Delta Z_s^{q+1,p}|^2 ds]$. To do so, we use the Itô isometry $\mathbb{E}[\int_0^T |\Delta Z_s^{q+1,p}|^2 ds] = \mathbb{E}[(\int_0^T \Delta Z_s^{q+1,p} dB_s)^2]$. Using the Definitions (3.4)-(3.6) of Z_t^{q+1} and $Z_t^{q+1,p}$ and the Clark-Ocone Theorem leads to

$$\int_0^T \Delta Z_s^{q+1,p} dB_s = F^q - \mathbb{E}(F^q) - (\mathcal{C}_p(F^{q,p}) - \mathbb{E}(\mathcal{C}_p(F^{q,p}))),$$
$$= Y_T^{q+1} + \int_0^T f(s,Y_s^q,Z_s^q)ds - Y_0^{q+1} - \left(Y_T^{q+1,p} + \int_0^T f(s,Y_s^{q,p},Z_s^{q,p})ds - Y_0^{q+1,p}\right)$$

Rearranging this summation makes appear $\Delta Y_T^{q+1,p} - (\Delta Y_0^{q+1,p})$. We get

$$\mathbb{E}\left[\int_0^T |\Delta Z_s^{q+1,p}|^2 ds\right] \le 6\mathbb{E}[\sup_{0\le t\le T}|\Delta Y_t^{q+1,p}|^2] + 6TL_f^2\int_0^T \mathbb{E}[|\Delta Y_s^{q,p}|^2 + |\Delta Z_s^{q,p}|^2]ds. \tag{4.5}$$

Since $\int_0^T \mathbb{E}[|\Delta Y_s^{q,p}|^2 + |\Delta Z_s^{q,p}|^2]ds \le (T+1)\mathcal{E}^{q,p}$, by computing $7\times$(4.4)+(4.5) we obtain

$$\mathcal{E}^{q+1,p} \le 112\mathbb{E}[|\mathcal{C}_p(\xi)-\xi|^2] + 112T\int_0^T \mathbb{E}\left[|\mathcal{C}_p(f(s,Y_s^q,Z_s^q)) - f(s,Y_s^q,Z_s^q)|^2\right] ds + 286T(T+1)L_f^2\mathcal{E}^{q,p}.$$

Since ξ and $f(s,Y_s^q,Z_s^q)$ belong to $\mathbb{D}^{m,2}$ (ξ satisfies $\mathcal{H}^1_{m+q}$, $f\in C_b^{0,m+q-1,m+q-1}$ and $(Y^q,Z^q)\in \mathcal{S}^{m,\infty}$ (see Lemma 4.10)), Lemma 2.3 gives

$$\mathcal{E}^{q+1,p} \le \frac{112}{(p+1)\cdots(p+m)}\|D^m\xi\|^2_{\mathrm{L}^2(\Omega\times[0,T]^m)}$$
$$+ \frac{112T}{(p+1)\cdots(p+m)}\left(\int_0^T \|D^m f(s,Y_s^q,Z_s^q)\|^2_{\mathrm{L}^2(\Omega\times[0,T]^m)}ds\right) + 286T(T+1)L_f^2\mathcal{E}^{q,p}.$$

Since $\int_0^T \|D^m f(s,Y_s^q,Z_s^q)\|^2_{\mathrm{L}^2(\Omega\times[0,T]^m)}ds$ is bounded by $C(T,m,(\|\partial^k_{sp}f\|_\infty)_{k\le m},\|(Y^q,Z^q)\|^{2m}_{m,2m})$, Lemma 4.10 gives the result.

□

4.3 Error due to the truncation of the basis

We are now interested in bounding the error between $(Y^{q,p},Z^{q,p})$ (defined by (3.5)) and $(Y^{q,p,N},Z^{q,p,N})$ (defined by (3.7)).

Before giving an upper bound for the error, we measure the error between $\mathcal{C}_p$ and $\mathcal{C}_p^N$ for a r.v. satisfying (4.1) when $r=p$.

Remark 4.13. *Let $r \in \mathbb{N}^\star$, ξ satisfies $\mathcal{H}_{r+q}$ and $f \in C_b^{0,r+q-1,r+q-1}$. Then, for all integers p and q, $I_{q,p} := \int_0^T f(s, Y_s^{q,p}, Z_s^{q,p})ds$ satisfies (4.1), i.e. for all multi-index α such that $|\alpha| = r$ we have*

$$|\mathbb{E}(D^\alpha_{t_1,\cdots,t_r} I_{q,p}) - \mathbb{E}(D^\alpha_{s_1,\cdots,s_r} I_{q,p})| \le K_r^{I_{q,p}}((t_1 - s_1)^{\beta_{I_{q,p}}} + \cdots + (t_r - s_r)^{\beta_{I_{q,p}}}),$$

where $\beta_{I_{q,p}} = \frac{1}{2} \wedge \beta_\xi$ and $K_r^{I_{q,p}}$ depends on K_r^ξ, $\|\xi\|_{r+q,2\frac{(r+q-1)!}{(r-1)!}}$, T and on $(\|\partial^k_{sp} f\|_\infty)_{1\le k\le r+q-1}$.

We refer to Section A.3 for the proof of Remark 4.13.

Lemma 4.14. *Let F denote a r.v. in $\mathrm{L}^2(\mathcal{F}_T)$ satisfying (4.1) for $r = p$. We have*

$$\mathbb{E}(|(\mathcal{C}_p^N - \mathcal{C}_p)(F)|^2) \le (K_p^F)^2 \left(\frac{T}{N}\right)^{2\beta_F} \sum_{i=1}^p i^2 \frac{T^i}{i!} \le (K_p^F)^2 \left(\frac{T}{N}\right)^{2\beta_F} T(1+T)e^T,$$

where K_p^F and β_F are defined in Hypothesis 4.1.

We refer to Section A.4 for the proof of the Lemma.

Proposition 4.15. *Assume that ξ satisfies $\mathcal{H}_{p+q}$ and $f \in C_b^{0,p+q-1,p+q-1}$. We recall $\mathcal{E}^{q,p,N} := \|(Y^{q,p} - Y^{q,p,N}, Z^{q,p} - Z^{q,p,N})\|^2_{\mathrm{L}^2}$. We get*

$$\mathcal{E}^{q+1,p,N} \le C_2 T(T+1) L_f^2 \mathcal{E}^{q,p,N} + K_2(q,p) \left(\frac{T}{N}\right)^{1\wedge 2\beta_\xi} \tag{4.6}$$

where C_2 is a scalar and $K_2(q,p)$ depends on K_p^ξ, T, $\|\xi\|_{p+q,2\frac{(p+q-1)!}{(p-1)!}}$ and on $(\|\partial^k_{sp} f\|_\infty)_{1\le k\le p+q-1}$. Since $\mathcal{E}^{0,p,N} = 0$, we deduce from (4.6) that $\mathcal{E}^{q,p,N} \le A_2(q,p)\left(\frac{T}{N}\right)^{1\wedge 2\beta_\xi}$, where $A_2(q,p) := K_2(q,p)T(T+1)e^T \frac{(C_2T(T+1)L_f^2)^q - 1}{C_2T(T+1)L_f^2 - 1}$. Then, $(Y^{p,q,N}, Z^{p,q,N})$ converges to $(Y^{q,p}, Z^{q,p})$ when N tends to ∞ in $\|(\cdot,\cdot)\|_{\mathrm{L}^2}$.

Proof of Proposition 4.15. For the sake of clearness, we assume $d = 1$. In the following, one notes $\Delta Y_t^{q,p,N} := Y_t^{q,p,N} - Y_t^{q,p}$, $\Delta Z_t^{q,p,N} := Z_t^{q,p,N} - Z_t^{q,p}$ and $\Delta f_t^{q,p,N} := f(t, Y_t^{q,p,N}, Z_t^{q,p,N}) - f(t, Y_t^{q,p}, Z_t^{q,p})$. Firstly, we deal with $\mathbb{E}[\sup_{0\le t\le T} |\Delta Y_t^{q+1,p,N}|^2]$. From (3.5) and (3.7) we get

$$\Delta Y_t^{q+1,p,N} = \mathbb{E}_t[\mathcal{C}_p^N(F^{q,p,N}) - \mathcal{C}_p(F^{q,p})] + \int_0^t \Delta f_s^{q,p,N} ds.$$

Following the same steps as in the proof of Proposition 4.11, one gets

$$\begin{aligned}\mathbb{E}[\sup_{0\le t\le T} |\Delta Y_t^{q+1,p,N}|^2] \le & 16\mathbb{E}[|\mathcal{C}_p^N(\xi) - \mathcal{C}_p(\xi)|^2] + 16\mathbb{E}\left[\left|(\mathcal{C}_p^N - \mathcal{C}_p)\left(\int_0^T f(s, Y_s^{q,p}, Z_s^{q,p})ds\right)\right|^2\right] \\ & + 40TL_f^2 \int_0^T \mathbb{E}[|\Delta Y_s^{q,p,N}|^2 + |\Delta Z_s^{q,p,N}|^2]ds. \end{aligned} \tag{4.7}$$

Let us now upper bound $\mathbb{E}[\int_0^T |\Delta Z_s^{q+1,p,N}|^2 ds]$. Following the same steps as in the proof of Proposition 4.11, one gets

$$\mathbb{E}\left[\int_0^T |\Delta Z_s^{q+1,p,N}|^2 ds\right] \le 6\mathbb{E}[\sup_{0\le t\le T} |\Delta Y_t^{q+1,p,N}|^2] + 6TL_f^2 \int_0^T \mathbb{E}[|\Delta Y_s^{q,p,N}|^2 + |\Delta Z_s^{q,p,N}|^2]ds. \tag{4.8}$$

Adding $7\times$(4.7) and (4.8) gives

$$\mathcal{E}^{q+1,p,N} \leq 112\mathbb{E}[|(\mathcal{C}_p^N - \mathcal{C}_p)(\xi)|^2] + 112\mathbb{E}\left[\left|(\mathcal{C}_p^N - \mathcal{C}_p)\left(\int_0^T f(s, Y_s^{q,p}, Z_s^{q,p})ds\right)\right|^2\right]$$
$$+ 286T(T+1)L_f^2\mathcal{E}^{q,p,N}.$$

Since ξ and $I_{q,p}$ satisfy (4.1) (see Remarks 4.4 and 4.13), Lemma 4.14 gives

$$\mathcal{E}^{q+1,p,N} \leq 112\left(\frac{T}{N}\right)^{2\alpha_\xi\wedge 1} T(T+1)e^T\left((K_p^\xi)^2 + (K_p^{I_{q,p}})^2\right) + 286T(T+1)L_f^2\mathcal{E}^{q,p,N},$$

and (4.6) follows. □

4.4 Error due to the Monte-Carlo approximation

We are now interested in bounding the error between $(Y^{q,p,N}, Z^{q,p,N})$ defined by (3.7) and $(Y^{q,p,N,M}, Z^{q,p,N,M})$ defined by (3.11). $\mathcal{C}_p^{N,M}$ is defined by (3.9) and (3.10). In this Section, we assume that the coefficients $\hat{d_k^n}$ are independent of the vector $(G_1, \cdots, G_N)$, which corresponds to the second approach proposed in Remark 3.2.

Before giving an upper bound for the error, we measure the error between $\mathcal{C}_p^N$ and $\mathcal{C}_p^{N,M}$ for a r.v. satisfying $\mathcal{H}_{p,N}^3$ (see Hypothesis 4.3).

Lemma 4.16. *Let F be a r.v. satisfying Hypothesis $\mathcal{H}_{p,N}^3$. We have*

$$\mathbb{E}(|(\mathcal{C}_p^N - \mathcal{C}_p^{N,M})(F)|^2) = \frac{1}{M}V_{p,N}(F).$$

Moreover, we have $\mathbb{E}(|\mathcal{C}_p^{N,M}(F)|^2) \leq \mathbb{E}(|F|^2) + \frac{1}{M}V_{p,N}(F)$.

We refer to Section A.5 for the proof of the Lemma.

Proposition 4.17. *Let ξ satisfy Hypothesis $\mathcal{H}_{p,N}^3$ and f be a bounded function. Let $\mathcal{E}^{q,p,N,M} := \|(Y^{q,p,N} - Y^{q,p,N,M}, Z^{q,p,N} - Z^{q,p,N,M})\|_{\mathrm{L}^2}^2$. We get*

$$\mathcal{E}^{q+1,p,N,M} \leq C_3T(T+1)L_f^2\mathcal{E}^{q,p,N,M} + \frac{K_3(q,p,N)}{M},$$

where C_3 is a scalar and $K_3(q,p,N) := 168\left(V_{p,N}(\xi) + T^2\|f\|_\infty^2\sum_{k=0}^p \binom{N}{k}\right)$.
Since $\mathcal{E}^{0,p,N,M} = 0$, we deduce from the previous inequality that $\mathcal{E}^{q,p,N,M} \leq \frac{A_3(q,p,N)}{M}$, where $A_3(q,p,N) := K_3(q,p,N)\frac{(C_3T(T+1)L_f^2)^q - 1}{C_3T(T+1)L_f^2 - 1}$. Then, $(Y^{p,q,N,M}, Z^{p,q,N,M})$ converges to $(Y^{q,p,N}, Z^{q,p,N})$ when M tends to ∞ in $\|(\cdot,\cdot)\|_{\mathrm{L}^2}$.

Proof of Proposition 4.17. For the sake of clearness, we assume $d = 1$. In the following, one notes $\Delta Y_t^{q,p,N,M} := Y_t^{q,p,N,M} - Y_t^{q,p,N}$, $\Delta Z_t^{q,p,N,M} := Z_t^{q,p,N,M} - Z_t^{q,p,N}$ and $\Delta f_t^{q,p,N,M} := f(t, Y_t^{q,p,N,M}, Z_t^{q,p,N,M}) - f(t, Y_t^{q,p,N}, Z_t^{q,p,N})$. Firstly, we deal with $\mathbb{E}[\sup_{0\leq t\leq T}|\Delta Y_t^{q+1,p,N,M}|^2]$. From (3.7) and (3.11) we get

$$\Delta Y_t^{q+1,p,N,M} = \mathbb{E}_t[\mathcal{C}_p^{N,M}(F^{q,p,N,M}) - \mathcal{C}_p^N(F^{q,p,N})] + \int_0^t \Delta f_s^{q,p,N,M}ds.$$

By introducing $\pm\mathcal{C}_p^N(F^{q,p,N,M})$ and by using Lemma 2.6, we obtain

$$\mathbb{E}[\sup_{0\leq t\leq T}|\Delta Y_t^{q+1,p,N,M}|^2] \leq 12\mathbb{E}[|(\mathcal{C}_p^{N,M} - \mathcal{C}_p^N)(F^{q,p,N,M})|^2] + 12\mathbb{E}\left(|F^{q,p,N,M} - F^{q,p,N}|^2\right)$$
$$+ 6TL_f^2\int_0^T \mathbb{E}[|\Delta Y_s^{q,p,N,M}|^2 + |\Delta Z_s^{q,p,N,M}|^2]ds.$$

From Lemma 4.16, we get $\mathbb{E}[|(\mathcal{C}_p^{N,M}-\mathcal{C}_p^N)(F^{q,p,N,M})|^2] \leq \frac{2}{M}\left(V_{p,N}(\xi)+V_{p,N}\left(\int_0^T f(\theta_s^{q,p,N,M})ds\right)\right)$. Then

$$\mathbb{E}[\sup_{0\leq t\leq T}|\Delta Y_t^{q+1,p,N,M}|^2] \leq \frac{24}{M}\left(V_{p,N}(\xi)+T^2\|f\|_\infty^2\sum_{k=0}^{p}\binom{N}{k}\right) + 30TL_f^2\int_0^T \mathbb{E}[|\Delta Y_s^{q,p,N,M}|^2+|\Delta Z_s^{q,p,N,M}|^2]ds. \tag{4.9}$$

Let us now upper bound $\mathbb{E}[\int_0^T|\Delta Z_s^{q+1,p,N,M}|^2ds]$. Following the same steps as in the proof of Proposition 4.11, one gets

$$\mathbb{E}\left[\int_0^T|\Delta Z_s^{q+1,p,N,M}|^2ds\right] \leq 6\mathbb{E}[\sup_{0\leq t\leq T}|\Delta Y_t^{q+1,p,N,M}|^2]+6TL_f^2\int_0^T\mathbb{E}[|\Delta Y_s^{q,p,N,M}|^2+|\Delta Z_s^{q,p,N,M}|^2]ds. \tag{4.10}$$

Adding 7×(4.9) and (4.10) gives the result. □

5 Numerical Examples

The computations have been done on a PC INTEL Core 2 Duo P9600 2.53 GHz with 4Gb of RAM.

5.1 Non linear driver and path-dependent terminal condition

We consider the case $d=1$, $f(t,y,z)=\cos(y)$ and $\xi=\sup_{0\leq t\leq 1}B_t$.

- **Convergence in p.** Table 1 represents the evolution of $\overline{Y}_0^{q,p,N,M}$ and $\overline{Z}_0^{q,p,N,M}$ w.r.t q (Picard's iteration index), when $p=2$ and $p=3$. We also give the CPU time needed to get $\overline{Y}_0^{6,p,N,M}$ and $\overline{Z}_0^{6,p,N,M}$. We fix $M=10^5$ and $N=20$. The seed of the generator is also fixed.

iterations	1	2	3	4	5	6	CPU time
$p=2$	1.656357	1.017117	1.237135	1.186691	1.195462	1.194256	14.06
$p=3$	1.656357	1.012091	1.234398	1.183544	1.192367	1.191173	174.09

Table 1: Evolution of $\overline{Y}_0^{q,p,N,M}$ w.r.t. Picard's iterations, $M=10^5$, $N=20$ and CPU time

iterations	1	2	3	4	5	6	CPU time
$p=2$	0.969128	0.249148	0.525273	0.459326	0.470069	0.469117	14.06
$p=3$	0.969128	0.242977	0.523846	0.455827	0.466903	0.465939	174.09

Table 2: Evolution of $\overline{Z}_0^{q,p,N,M}$ w.r.t. Picard's iterations, $M=10^5$, $N=20$ and CPU time

One notes that the difference between the values of $\overline{Y}_0^{q,2,N,M}$ and $\overline{Y}_0^{q,3,N,M}$ (resp. $\overline{Z}_0^{q,2,N,M}$ and $\overline{Z}_0^{q,3,N,M}$) doesn't exceed 0.2% (resp. 0.6%). This is due to the fast convergence of the algorithm in p. The CPU time is 12 times higher when $p=3$ than when $p=2$. Then, the use of order 3 in the chaos decomposition is not necessary. In the following, we take $p=2$.

- **Convergence in M.** Figure 1 illustrates the evolution of $\overline{Y}_0^{q,p,N,M}$ and $\overline{Z}_0^{q,p,N,M}$ w.r.t. q when $p=2$ and $N=20$ for different values of M. The seed of the generator is random. When M equals 10^4 and 10^5 the algorithm stabilizes after very few iterations. When $M=10^3$, there is no convergence.

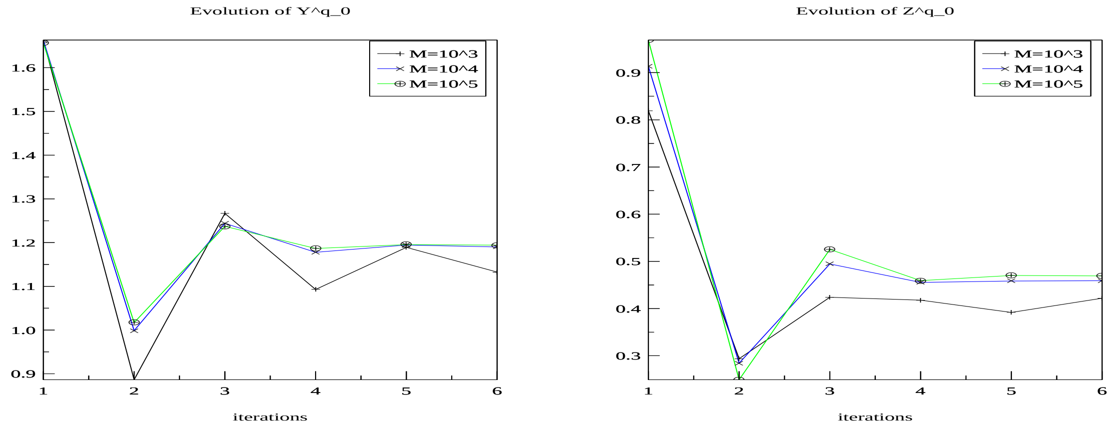

Figure 1: Evolution of $\overline{Y}_0^{q,p,N,M}$ and $\overline{Z}_0^{q,p,N,M}$ w.r.t. q and M when $N = 20$, $p = 2$ - $\xi = \sup_{0\le t\le 1} B_t$, $f(t, y, z) = \cos(y)$.

- **Convergence in** N**.** Figure 2 illustrates the evolution of $\overline{Y}_0^{q,p,N,M}$ and $\overline{Z}_0^{q,p,N,M}$ w.r.t. q when $p = 2$ and $M = 10^5$ for different values of N. The seed of the generator is random. The algorithm converges even when $N = 10$, but $\overline{Y}_0^{6,p,10,M}$ is quite below $\overline{Y}_0^{6,p,40,M}$.

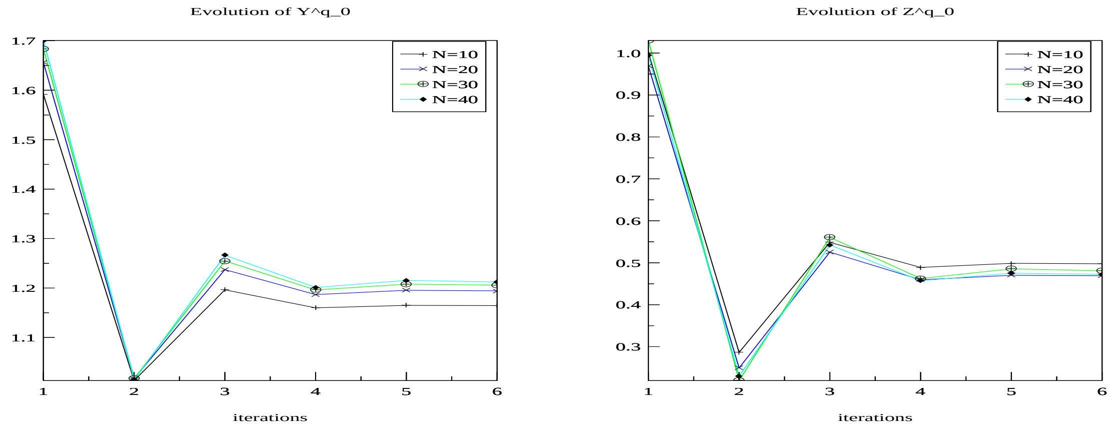

Figure 2: Evolution of $\overline{Y}_0^{q,p,N,M}$ and $\overline{Z}_0^{q,p,N,M}$ w.r.t. N when $M = 10^5$, $p = 2$ - $\xi = \sup_{0\le t\le 1} B_t$, $f(t, y, z) = \cos(y)$

5.2 Linear Driver - Financial Benchmark

We consider the case of pricing and hedging a Discrete Down and Out Barrier Call option, i.e. $f(t, y, z) = -ry$ and $\xi := (S_T - K)_+ \mathbf{1}_{\forall n\in[0,N] S_{t_n}\ge L}$, where S represents the Black-Scholes diffusion

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}, \quad \forall t \in [0, T].$$

The option parameters are $r = 0.01$, $\sigma = 0.2$, $T = 1$, $K = 0.9$, $L = 0.85$, $S_0 = 1$ and $N = 20$ (N is also the number of time discretizations of the chaos decomposition).

We can get a benchmark for Y_0 and Z_0 by using a variance reduction Monte Carlo method. For this set of parameters, the reference values are $Y_0 = 0.134267$ with a confidence interval

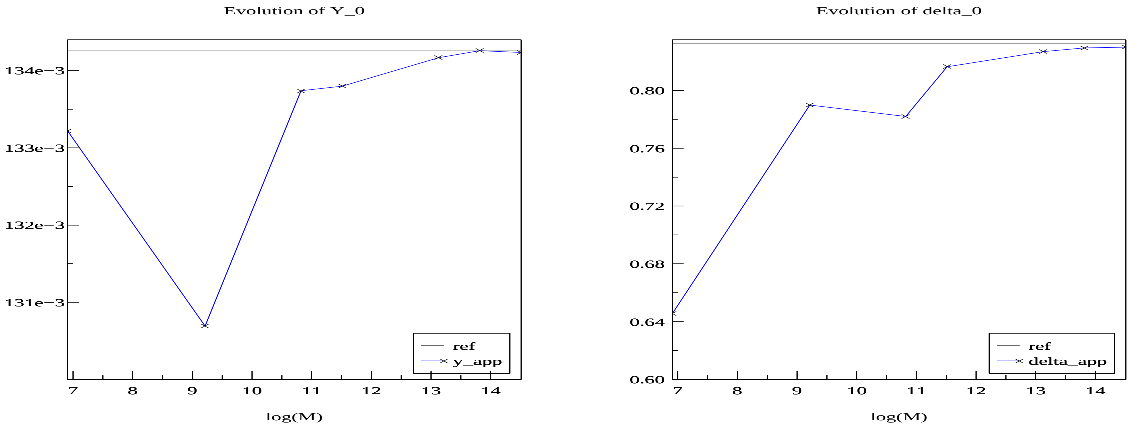

Figure 3: Evolution of $\overline{Y}_0^{q,p,N,M}$ and $\delta_0 := \frac{\overline{Z}_0^{q,p,N,M}}{\sigma S_0}$ w.r.t. $log(M)$ when $N = 20$, $p = 2$, $q = 5$ -Discrete Down and Out Barrier Call option

$7.9468e-05$ and $\delta_0 = \frac{Z_0}{\sigma S_0} = 0.8327$. We compare them with $\overline{Y}_0^{q,p,N,M}$ and $\frac{\overline{Z}_0^{q,p,N,M}}{\sigma S_0}$ when $N = 20$, $p = 2$, $q = 5$ (we choose the first value of q from which the algorithm has converged) for different values of M. Figure 3 represents the evolution of $\overline{Y}_0^{5,p,N,M}$ and $\delta_0^{5,p,N,M}$ w.r.t. $\log(M)$. One notices that for $M = 10^6$ the computed values are very close to the reference ones.

5.3 Non linear driver in dimension 5 - Financial Benchmark

We consider the pricing and hedging of a Put Basket option in dimension 5, i.e. $\xi = (K - \frac{1}{5}\sum_{i=1}^5 S_T^i)_+$, where

$$\forall i = 1, \cdots, 5 \;\; S_t^i = S_0^i e^{(\mu^i - \frac{(\sigma^i)^2}{2})t + \sigma^i B_t^i}.$$

μ^i (resp. σ^i) represents the trend (resp. the volatility) of the i^{th} asset. $B = (B^1, \cdots, B^5)$ is a 5-dimensional Brownian motion such that $\langle B^i, B^j\rangle_t = \rho t \mathbf{1}_{i\neq j} + t\mathbf{1}_{i=j}$. We suppose that $\rho \in (-\frac{1}{4}, 1)$, which ensures that the matrix $C = (\rho \mathbf{1}_{i\neq j} + \mathbf{1}_{i=j})_{1\leq i,j\leq 5}$ is positive definite. We also assume that the borrowing rate R is higher than the bond one r. In such a case, pricing and hedging the Put Basket option is equivalent to solving a BSDE with terminal condition ξ and with driver f defined by $f(t, y, z) = -ry - \theta \cdot z + (R - r)(y - \sum_{i=1}^5 (\Sigma^{-1} z)_i)^-$, where $\theta := \Sigma^{-1}(\mu - r\mathbf{1})$ ($\mathbf{1}$ is the vector whose every component is one) and Σ is the matrix defined by $\Sigma_{ij} = \sigma^i L_{ij}$ (L denote the lower triangular matrix involved in the Cholesky decomposition $C = LL^*$). We refer to [EPQ97][Example 1.1] for more details.
The option parameters are $r = 0.02$, $R = 0.1$, $T = 1$, $K = 95$, $\rho = 0.1$, and for all $i = 1, \cdots, 5$, $S_0^i = 100$, $\mu_0^i = 0.05$ and $\sigma_0^i = 0.2$. Figure 4 represents the evolution of $\overline{Y}_0^{5,p,N,M}$, the approximated price at time 0, and the relative error on $\delta_0^1 := \frac{(\Sigma^{-1}\overline{Z}_0^{5,p,N,M})^1}{S_0^1}$ — the quantity of asset 1 to possess at time 0 — w.r.t. $\log(M)$. We compare our results with the ones obtained using the Algorithm proposed in [GL10] (cited here as reference values). The CPU time needed to compute price and delta when $M = 50000$ and $N = 20$ is 161s. One notices that the convergence is very fast and quite accurate for $M = 50000$.

Conclusion. In this paper, we use Wiener chaos expansions together with the Picard procedure to compute the solution to (1.1). Once computed the chaos decomposition of F^q, we get explicit formulas for both conditional expectations and the Malliavin derivative of conditional expectations. This enables to easily compute (Y^q, Z^q). Numerically, we obtain fast and accurate results, which encourage us to extend these results to other type of BSDEs, like 2-BSDEs. It is

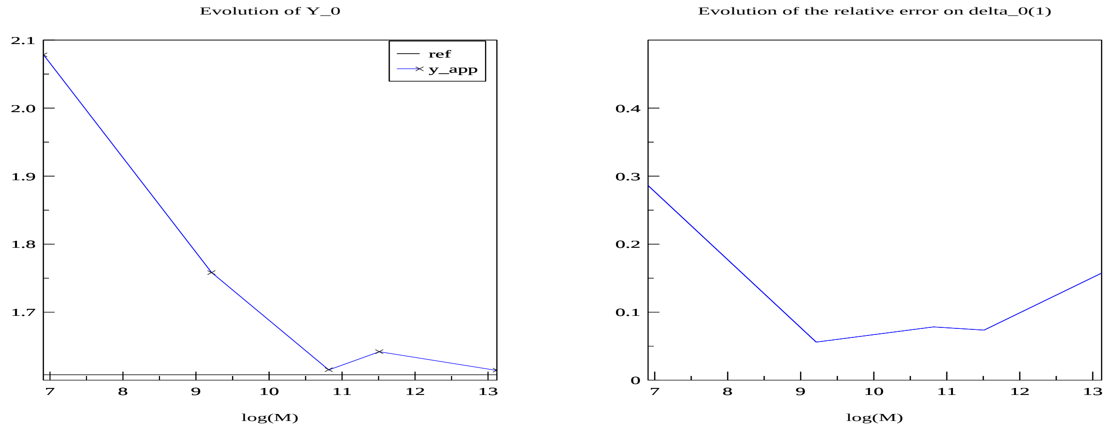

Figure 4: Evolution of $\overline{Y}_0^{q,p,N,M}$ and $\delta_0(1)$ w.r.t. $\log(M)$ when $N = 20$, $p = 2$, $q = 5$, $d = 5$ - Basket Put option with different interest and borrowing rates

also possible to couple these Wiener chaos expansions together with the dynamic programming approach. This will be the subject of a forthcoming publication.

A Technical results of Section 4

In the following, for any regular r.v. $F \in \mathcal{F}_T$, $D_t^{(l_0)}\Delta_i D_s^{(l_1)}F$ denotes $D_{t_1,\cdots,t_{l_0}}^{(l_0)}(D_{t_i,s_{i+1},\cdots,s_{i+l_1}}^{(l_1+1)}F - D_{s_i,\cdots,s_{i+l_1}}^{(l_1+1)}F)$.

A.1 Proof of Remark 4.5

Before proving Remark 4.5, we prove the following Lemma.

Lemma A.1. *Let X be the $\mathbb{R}^n$-valued process solution of*

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s,$$

where B is a d-dimensional Brownian motion and $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ are two $C^{0,m}$ functions uniformly lipschitz w.r.t. x and Hölder continuous of parameter $\frac{1}{2}$ w.r.t. t, with linear growth in x (of constant K) and with bounded derivatives. Then

- *$\forall l \leq m$, $\forall j \geq 2$ we have*

$$M_l^j := \sup_{t_1 \leq \cdots \leq t_l} \mathbb{E}(\sup_{r \in [t_l, T]} |D_{t_1,\cdots,t_l}^{(l)} X_r|^j) < \infty, \tag{A.1}$$

  the upper bound depends on $(\|b^{(l')}\|_\infty)_{l' \leq l}$, $(\|\sigma^{(l')}\|_\infty)_{l' \leq l}$, x and K,

- *$\forall j \geq 2$, $\forall i \in \{1, \cdots, m\}$, $\forall l_0 \leq i-1$, $\forall l_1 \leq m-i$, we have*

$$\sup_{t_1 \leq \cdots \leq t_{l_0}} \sup_{s_{i+1} \leq \cdots \leq s_{i+l_1}} \mathbb{E}(\sup_{r \in [s_{i+l_1}, T]} |D_t^{(l_0)}\Delta_i D_s^{(l_1)} X_r|^j) < k_l^X(j)(t_i - s_i)^{\frac{j}{2}}, \tag{A.2}$$

  where $l := l_0 + l_1 + 1$ and k_l^X depends on T, $(M_{l'}^{j'})_{l' \leq l, j' \leq lj}$, $(\|b^{(l')}\|_\infty)_{l' \leq l}$, and on $(\|\sigma^{(l')}\|_\infty)_{l' \leq l}$.

Proof of Lemma A.1. The first point is proved in [Nua06, Theorem 2.2.2]. For the sake of clearness, we prove the second result for $d = 1$. We also assume that the vectors $(t_1, \cdots, t_n)$ and $(s_1, \cdots, s_n)$ are such that $0 \le s_1 \le t_1 \le s_2 \le \cdots \le s_n \le t_n \le T$. We do it by induction on l_0 and l_1. We detail the case b and σ only depending on x and do the proof for $l_0 = l_1 = 0$ and $l_0 = 0$, $l_1 = 1$. We recall that under these hypotheses on b and σ, we have $\forall l \le m$

$$\sup_{t_1 \le \cdots \le t_l} \mathbb{E}[|D^{(l)}_{t_1,\cdots,t_l}(X_{t_{l+1}} - X_{s_{l+1}})|^j] \le C(t_{l+1} - s_{l+1})^{\frac{j}{2}},$$

where C depends on T, j, $(M^{j'}_{l'})_{l' \le l, j' \le lj}$ and on $(\|b^{(j')}\|_\infty)_{j' \le j}$, and on $(\|\sigma^{(j')}\|_\infty)_{j' \le j}$.

Case $l_0 = l_1 = 0$. We have

$$D_{t_n} X_r = \int_{t_n}^r b'(X_u) D_{t_n} X_u du + \sigma(X_{t_n}) + \int_{t_n}^r \sigma'(X_u) D_{t_n} X_u dBu.$$

Then

$$\begin{aligned}\Delta_n X_r := D_{t_n} X_r - D_{s_n} X_r = &\int_{t_n}^r b'(X_u)\Delta_n X_u du - \int_{s_n}^{t_n} b(X_u) D_{s_n}(X_u) du \\ &+ \sigma(X_{t_n}) - \sigma(X_{s_n}) + \int_{t_n}^r \sigma'(X_u)\Delta_n X_u dBu - \int_{s_n}^{t_n} \sigma'(X_u) D_{s_n}(X_u) dBu.\end{aligned}$$

In the following, C denotes a generic constant depending only on T and j and L_σ denotes the Lipschitz contant of σ.

$$\begin{aligned}|\Delta_n X_r|^j \le & C\left(\|b'\|^j_\infty \int_{t_n}^r |\Delta_n X_u|^j du + (t_n - s_n)^{j-1}\|b'\|^j_\infty \int_{s_n}^{t_n} |D_{s_n}(X_u)|^j du\right. \\ &\left. + L^j_\sigma |X_{t_n} - X_{s_n}|^j + \left|\int_{t_n}^r \sigma'(X_u)\Delta_n X_u dBu\right|^j + \left|\int_{s_n}^{t_n} \sigma'(X_u) D_{s_n}(X_u) dBu\right|^j\right)\end{aligned}$$

We introduce $\Psi^{0,j}_n(T) := \mathbb{E}[\sup_{r\in[t_n,T]} |\Delta_n X_r|^j]$. Doob's inequality and Burkhölder-Davis-Gundy inequality lead to

$$\Psi^{0,j}_n(T) \le C\left((\|b'\|^j_\infty + \|\sigma'\|^j_\infty)\int_{t_n}^T \Psi^{0,j}_n(u) du + \|b'\|^j_\infty M^j_1 (t_n - s_n)^j + (L^j_\sigma + \|\sigma'\|^j_\infty M^j_1)|t_n - s_n|^{\frac{j}{2}}\right)$$

Gronwall's lemma yields the result.

Case $l_0 = 0, l_1 = 1$. We consider $\Delta_{n-1} D_{t_n} X_r = D_{t_{n-1},t_n} X_r - D_{s_{n-1},t_n} X_r$. We have

$$\begin{aligned}D_{t_{n-1},t_n} X_r = &\int_{t_n}^r b''(X_u) D_{t_{n-1}} X_u D_{t_n} X_u + b'(X_u) D_{t_{n-1},t_n} X_u du + \sigma'(X_{t_n}) D_{t_{n-1}} X_{t_n} \\ &+ \int_{t_n}^r \sigma''(X_u) D_{t_{n-1}} X_u D_{t_n} X_u + \sigma'(X_u) D_{t_{n-1},t_n} X_u dBu.\end{aligned}$$

Then,

$$\begin{aligned}\Delta_{n-1} D_{t_n} X_r = &\int_{t_n}^r b''(X_u)\Delta_{n-1} X_u D_{t_n} X_u + b'(X_u)\Delta_{n-1} D_{t_n} X_u du + \sigma'(X_{t_n})\Delta_{n-1} X_{t_n} \\ &+ \int_{t_n}^r \sigma''(X_u)\Delta_{n-1} X_u D_{t_n} X_u + \sigma'(X_u)\Delta_{n-1} D_{t_n} X_u dBu.\end{aligned}$$

Doob's inequality and Burkhölder-Davis-Gundy inequality lead to

$$\begin{aligned}\mathbb{E}[\sup_{r\in[t_n,T]} |\Delta_{n-1} D_{t_n} X_r|^j] \le C&\left(\int_{t_n}^T \|b''\|^j_\infty \mathbb{E}[|\Delta_{n-1} X_u|^j |D_{t_n} X_u|^j] + \|b'\|^j_\infty \mathbb{E}[|\Delta_{n-1} D_{t_n} X_u|^j] du\right. \\ &\left. + \|\sigma'\|^j_\infty \mathbb{E}[|\Delta_{n-1} X_{t_n}|^j] + \int_{t_n}^T \|\sigma''\|^j_\infty \mathbb{E}[|\Delta_{n-1} X_u|^j |D_{t_n} X_u|^j] + \|\sigma'\|^j_\infty \mathbb{E}[|\Delta_{n-1} D_{t_n} X_u|^j] du\right).\end{aligned}$$

We introduce $\Psi^{1,j}_{n-1}(T) := \sup_{t_n \le T} \mathbb{E}[\sup_{r\in[t_n,T]} |\Delta_{n-1} D_{t_n} X_r|^j]$. Cauchy-Schwarz inequality yields

$$\Psi^{1,j}_{n-1}(T) \le C\left((\|b'\|^j_\infty + \|\sigma'\|^j_\infty) \int_{t_n}^T \Psi^{1,j}_{n-1}(u)du + (\|b''\|^j_\infty + \|\sigma''\|^j_\infty)(M_1^{2j})^{\frac12}(\Psi^{0,2j}_{n-1}(T))^{\frac12} + \|\sigma'\|^j_\infty \Psi^{0,j}_{n-1}(T)\right).$$

Since $\Psi^{0,2j}_{n-1}(T) \le K(t_{n-1}-s_{n-1})^j$, and $\Psi^{0,j}_{n-1}(T) \le K(t_{n-1}-s_{n-1})^{\frac{j}{2}}$, Gronwall's Lemma ends the proof. □

Proof of Remark 4.5. We prove the result for $d=1$. We first prove that $g(X_T)$ belongs to $\mathcal{D}^{m,j}$ for all $j \ge 2$, i.e. $\|g(X_T)\|^j_{m,j} = \sum_{l\le m}\sum_{t_1,\cdots,t_l} \mathbb{E}[|D^{(l)}_{t_1,\cdots,t_l} g(X_T)|^j] < \infty$. $D^{(l)}_{t_1,\cdots,t_l} g(X_T)$ contains a sum of terms of type $g^{(k)}(X_T)\prod_{i=1}^k D^{(j_i)}_t X_T$, where k varies in $\{1,\cdots,l\}$, $|j|_1 = l$ and $a(j)=k$ ($a(j)$ denotes the number of non zero components of j). Since $g \in C_p^\infty$ and X satisfies (A.1), we get the result.
Let us now prove that $g(X_T)$ satisfies $\mathcal{H}^2_m$. $D^{(l_0)}_t \Delta_{t_i,s_i} D^{(l_1)}_s g(X_T)$ contains a sum of terms of type $g^{(k)}(X_T)\prod_{i=1}^{k-1}(D^{(j_i)}_t X_T) D^{(l_0')}_t \Delta_{t_i,s_i} D^{(l_1')}_s X_T$, where k varies in $\{1,\cdots,l\}$, $|j|_1 = l-1-l_0'-l_1'$, $a(j)=k-1$, $l_0' \le l_0$ and $l_1' \le l_1$. Then, since $g \in C_p^\infty$, X satisfies (A.1) and (A.2), we get $g(X_T)$ satisfies $\mathcal{H}^2_m$, with $\beta_{g(X_T)} = \frac12$ and $k_l^{g(X_T)}$ depends on $(\|g^{(l')}\|_\infty)_{l'\le l}$, on $(M_{l'}^{j'})_{l'\le l, j'\le lj}$ and on K_l^X.
It remains to prove that $g(X_T)$ satisfies $\mathcal{H}^3_{p,N}$. $\mathbb{V}(g(X_T))$ is bounded by $\mathbb{E}((g(X_T))^2)$. Since $g \in C_p^\infty$ and X satisfies $\mathbb{E}(|X_T|^j) < \infty$ for all j, we get that $\mathbb{V}(g(X_T))$ is bounded. We prove that $\mathbb{V}\left(g(X_T)\prod_{i=1}^N K_{n_i}(G_i)\right)$ is bounded by the same way. □

A.2 Proof of Lemma 4.10

We do the proof for $d=1$. We prove by induction that $\forall q' \le q$, $(Y^{q'}, Z^{q'})$ belongs to $\mathcal{S}^{m,\infty}$, i.e. $\forall j \ge 2$

$$\|(Y^{q'},Z^{q'})\|^j_{m,j} = \sum_{1\le l\le m} \sup_{t_1\le\cdots\le t_l}\left\{ \mathbb{E}[\sup_{t_l\le r\le T} |D^{(l)}_{t_1,\cdots,t_l} Y^{q'}_r|^j] + \int_{t_l}^T \mathbb{E}[|D^{(l)}_{t_1,\cdots,t_l} Z^{q'}_r|^j]dr \right\} < \infty.$$

Using (3.4) gives

$$D^{(l)}_{t_1,\cdots,t_l} Y^{q'}_r = \mathbb{E}_r[D^{(l)}_{t_1,\cdots,t_l} F^{q'-1}] - \int_{t_l}^r D^{(l)}_{t_1,\cdots,t_l} f(\theta^{q'-1}_u)du, \text{ where } \theta^{q'-1}_u := (u, Y^{q'-1}_u, Z^{q'-1}_u).$$

Using the Definition of $F^{q'-1}$ and applying Doob's inequality leads to

$$\mathbb{E}[\sup_{t_l\le r\le T} |D^{(l)}_{t_1,\cdots,t_l} Y^{q'}_r|^j] \le C\left(\mathbb{E}[|D^{(l)}_{t_1,\cdots,t_l}\xi|^j] + \mathbb{E}\left(\int_{t_l}^T |D^{(l)}_{t_1,\cdots,t_l} f(\theta^{q'-1}_u)|^j du\right)\right),$$

where C is a generic constant depending on T and j.

$D^{(l)}_{t_1,\cdots,t_l} f(\theta^{q'-1}_u)$ contains a sum of terms of type $\partial_y^{l_0}\partial_z^{l_1} f(\theta^{q'-1}_u)\prod_{i=1}^{l_0} D^{j_i}_t Y^{q'-1}_u \prod_{i=1}^{l_1} D^{k_i}_t Z^{q'-1}_u$, where $|j|_1 + |k|_1 = l$, $a(j)=l_0$, $a(k)=l_1$ and $l_0+l_1 \le l$. Then, Hölder's inequality gives

$$\mathbb{E}\left(\int_{t_l}^T |D^{(l)}_{t_1,\cdots,t_l} f(\theta^{q'-1}_u)|^j du\right) \le C(\sum_{k=1}^l \|\partial^k_{sp} f\|^j_\infty)\|(Y^{q'-1}, Z^{q'-1})\|^{lj}_{l,lj} \tag{A.3}$$

and

$$\sum_{1\le l\le m}\sup_{t_1\le\cdots\le t_l} \mathbb{E}[\sup_{t_l\le r\le T} |D^{(l)}_{t_1,\cdots,t_l} Y^{q'}_r|^j] \le C\left(\|\xi\|^j_{m,j} + \sum_{l=1}^m (\sum_{k=1}^l \|\partial^k_{sp} f\|^j_\infty)\|(Y^{q'-1},Z^{q'-1})\|^{lj}_{l,lj}\right). \tag{A.4}$$

From (3.4), we get $D^{(l)}_{t_1,\cdots,t_l}Z^{q'}_r = \mathbb{E}_r[D^{(l+1)}_{t_1,\cdots,t_l,r}\xi + \int_r^T D^{(l+1)}_{t_1,\cdots,t_l,r}f(\theta^{q'-1}_u)du]$. Then

$$\int_{t_l}^T \mathbb{E}[|D^{(l)}_{t_1,\cdots,t_l}Z^{q'}_r|^j]dr \leq C\left(\int_{t_l}^T \mathbb{E}[|D^{(l+1)}_{t_1,\cdots,t_l,r}\xi|^j]dr + \int_{t_l}^T \mathbb{E}\left(\left|\int_r^T D^{(l+1)}_{t_1,\cdots,t_l,r}f(\theta^{q'-1}_u)du\right|^j\right)dr\right)$$

Using (A.3) yields

$$\sum_{1\leq l\leq m}\sup_{t_1\leq\cdots\leq t_l}\int_{t_l}^T \mathbb{E}[|D^{(l)}_{t_1,\cdots,t_l}Z^{q'}_r|^j]dr \leq C\left(\|\xi\|^j_{m+1,j} + \sum_{l=1}^m(\sum_{k=1}^l \|\partial^k_{sp}f\|^j_\infty)\|(Y^{q'-1},Z^{q'-1})\|^{(l+1)j}_{(l+1),(l+1)j}\right).$$

Combining this equation with (A.4) gives

$$\|(Y^{q'},Z^{q'})\|^j_{m,j} \leq C(\|\xi\|^j_{m+1,j} + (\sum_{k=1}^m \|\partial^k_{sp}f\|^j_\infty)\sum_{l=1}^m \|(Y^{q'-1},Z^{q'-1})\|^{(l+1)j}_{(l+1),(l+1)j}).$$

Iterating this inequality yields the result. We prove that $\forall q' \leq q$, $(Y^{q',p},Z^{q',p})$ belongs to $\mathcal{S}^{m,\infty}$ in the same way. In this case, the generic constant C depends on T, j and p, since we need to use the first part of Lemma 2.4 to upper bound $\mathbb{E}(|\mathcal{C}_{p-l}(D^{(l)}_t F^{(q-1,p)})|^j)$.

A.3 Proof of Remark 4.13

For the sake of clearness, we assume that $\forall i \leq r$, $t_{i-1} \leq s_i \leq t_i$ and $d = 1$. Then, we show that if ξ satisfies $\mathcal{H}_{r+q}$ and $f \in C_b^{0,r+q-1,r+q-1}$, then $I_{q,p} := \int_0^T f(s,Y^{q,p}_s,Z^{q,p}_s)ds$ satisfies

$$|\mathbb{E}(D^{(r)}_{t_1,\cdots,t_r}I_{q,p}) - \mathbb{E}(D^{(r)}_{s_1,\cdots,s_r}I_{q,p})| \leq K_r^{I_{q,p}}((t_1-s_1)^{\beta_{I_{q,p}}} + \cdots + (t_r-s_r)^{\beta_{I_{q,p}}}).$$

Since $I_{0,p} = 0$, we deal with the case $q \geq 1$. Since we have $D^{(r)}_{t_1,\cdots,t_r}I_{q,p} - D^{(r)}_{s_1,\cdots,s_r}I_{q,p} = \sum_{i=1}^r D^{(i-1)}_t \Delta_i D^{(r-i)}_s I_{q,p}$, it is enough to prove that $\mathbb{E}(D^{(i-1)}_t\Delta_i D^{(r-i)}_s I_{q,p}) \leq K_i(t_i-s_i)^{\beta_{I_{q,p}}}$ (we refer to the beginning of Section A for the definition of $D^{(i-1)}_t\Delta_i D^{(r-i)}_s F$).

We introduce $\theta^{q,p}_u = (u,Y^{q,p}_u,Z^{q,p}_u)$, two vectors j and m, and four integers k_0, k_1, l_0 and l_1 such that $l_0 \leq i-1$, $l_1 \leq r-i$, $|j|_1+|m|_1 = r-1-l_0-l_1$ and $k_0+k_1 \leq r$. If $i < r$, $D^{(i-1)}_t\Delta_i D^{(r-i)}_s I_{q,p}$ contains a sum of terms of type

$$\int_{s_r}^T \partial^{k_0}_y\partial^{k_1}_z f(\theta^{q,p}_u)\prod_{i=1}^{k_0-1} D^{j_i}_{ts}Y^{q,p}_u \prod_{i=1}^{k_1} D^{m_i}_{ts}Z^{q,p}_u(D^{(l_0)}_t\Delta_i D^{(l_1)}_s Y^{q,p}_u)du$$

where $a(j) = k_0-1$ ($a(j)$ denotes the number of non zero components of j) and $a(m) = k_1$ and of type

$$\int_{s_r}^T \partial^{k_0}_y\partial^{k_1}_z f(\theta^{q,p}_u)\prod_{i=1}^{k_0} D^{j_i}_{ts}Y^{q,p}_u \prod_{i=1}^{k_1-1} D^{m_i}_{ts}Z^{q,p}_u(D^{(l_0)}_t\Delta_i D^{(l_1)}_s Z^{q,p}_u)du,$$

where $a(j) = k_0$, $a(m) = k_1-1$. By using Cauchy-Schwarz inequality, we get that $\mathbb{E}[D^{(i-1)}_t\Delta_i D^{(r-i)}_s I_{q,p}]$ is bounded by

$$\|\partial^{k_0+k_1}_{sp}f\|_\infty \mathbb{E}\left(\int_{s_r}^T \prod_{i=1}^{k_0-1}(D^{j_i}_{ts}Y^{q,p}_u)^2\prod_{i=1}^{k_1}(D^{m_i}_{ts}Z^{q,p}_u)^2du\int_{s_r}^T(D^{(l_0)}_t\Delta_i D^{(l_1)}_s Y^{q,p}_u)^2du\right)^{\frac{1}{2}}$$

(and the same type of term in $D^{l_0}_t\Delta_i D^{(l_1)}_s Z^{q,p}_u$) which leads to

$$\mathbb{E}[D^{(i-1)}_t\Delta_i D^{(r-i)}_s I_{q,p}] \leq C(T,(\|\partial^k_{sp}f\|_\infty)_{k\leq r},\|(Y^{q,p},Z^{q,p})\|_{r-1,2(r-1)})\sum_{l_0=0}^{i-1}\sum_{l_1=0}^{r-i}\sqrt{\left(D^{(l_0)}_t\Delta^{q,p}_i D^{(l_1)}_s\right)_2}, \tag{A.5}$$

where $(D_t^{(l_0)}\Delta_i^{q,p}D_s^{(l_1)})_j := \mathbb{E}[\sup_{s_r\le u\le T}|D_t^{(l_0)}\Delta_i D_s^{(l_1)}Y_u^{q,p}|^j] + \mathbb{E}\left(\int_{s_r}^T |D_t^{(l_0)}\Delta_i D_s^{(l_1)}Z_u^{q,p}|^2du\right)^{\frac{j}{2}}$. If $i = r$, $D_t^{(r-1)}\Delta_i I_{q,p}$ contains the same type of integrals between s_r and T plus an integral between s_r and t_r, which is bounded by $C(T,(\|\partial^k_{sp}f\|_\infty)_{k\le r},\|(Y^{q,p},Z^{q,p})\|_{r,2r})(t_r-s_r)$. Then, since $(Y^{q,p},Z^{q,p})\in\mathcal{S}^{r,\infty}$ and $f\in C_b^{0,r+q-1,r+q-1}$, it remains to take the supremum over $t_1,\cdots,t_{l_0},s_{i+1},\cdots,s_{i+l_1}$ in (A.5) and to apply Lemma A.2 to end the proof. K_i depends on $\|\xi\|_{r+q,2\frac{(r+q-1)!}{(r-1)!}}$, $(\|\partial^k_{sp}f\|_\infty)_{1\le k\le r+q-1}$, T and $K_r^{q,p} := \sup_{l\le r} k_l^{q,p}$ (where $k_l^{q,p}$ is defined in Lemma A.2).

Lemma A.2. *Assume ξ satisfies $\mathcal{H}^2_{r+q}$ and $f\in\mathcal{C}_b^{0,r+q-1,r+q-1}$. Then $\forall i\in\{1,\cdots,r\}$, $\forall l_0\le i-1$, $\forall l_1\le r-i$ and $\forall j\ge 2$*

$$\sup_{t_1\le\cdots\le t_{l_0}}\ \sup_{s_{i+1}\le\cdots\le s_{i+l_1}} \mathbb{E}[(D_t^{(l_0)}\Delta_i^{q,p}D_s^{(l_1)})_j] \le k_l^{q,p}(t_i-s_i)^{j(\frac12\wedge\beta_\xi)}$$

where $l=l_0+l_1+1$ and $k_l^{q,p}$ depends on k_l^ξ, T,$\|\xi\|_{l+q-1,\frac{(l+q-2)!}{(l-1)!}j}$ and on $(\|\partial^k_{sp}f\|_\infty)_{1\le k\le l+q-2}$.

Proof of Lemma A.2. We do the proof by induction on q. We distinguish cases $l_1>0$ and $l_1=0$. We first consider $l_1>0$. Let u be in $[s_r,T]$ and $l\le p$ (if $l>p$, the first term of the right hand side of the following equality vanishes). From (3.5) and Lemma 2.4, we get $D_t^{(l_0)}\Delta_i D_s^{(l_1)}Y_u^{q,p} = \mathbb{E}_u[\mathcal{C}_{p-r}(D_t^{(l_0)}\Delta_i D_s^{(l_1)}F^{q-1,p})] - \int_{s_{i+l_1}}^u D_t^{(l_0)}\Delta_i D_s^{(l_1)}f(\theta_v^{q-1})dv$. Using the definition of $F^{q-1,p}$ (see (3.5)), Doob's inequality and Lemma 2.4 yields

$$\mathbb{E}[\sup_{u\in[s_r,T]}(D_t^{(l_0)}\Delta_i D_s^{(l_1)}Y_u^{q,p})^j] \le C\left(\mathbb{E}[|D_t^{(l_0)}\Delta_i D_s^{(l_1)}\xi|^j] + \mathbb{E}[\int_{s_{i+l_1}}^T |D_t^{(l_0)}\Delta_i D_s^{(l_1)}f(\theta_v^{q-1,p})|dv]^j\right). \tag{A.6}$$

where C denotes a generic constant depending on T, j and p.

Let us now upper bound $\mathbb{E}\left(\int_{s_r}^T|D_t^{(l_0)}\Delta_i D_s^{(l_1)}Z_u^{q,p}|^2du\right)^{\frac{j}{2}}$. Using (3.6) and the Clark-Ocone formula gives $\int_0^T Z_u^{q,p}dB_u = \mathcal{C}_p(F^{q-1,p}) - \mathbb{E}(\mathcal{C}_p(F^{q-1,p}))$. Hence, for $v\in[s_r,T]$, we have $\int_{s_r}^v Z_u^{q,p}dB_u = \mathbb{E}_v(\mathcal{C}_p(F^{q-1,p})) - \mathbb{E}_{s_r}(\mathcal{C}_p(F^{q-1,p})) = Y_v^{q,p} + \int_{s_r}^v f(\theta_u^{q-1,p})du - Y_{s_r}^{q,p}$. Then, we get

$$\int_{s_r}^v D_t^{(l_0)}\Delta_i D_s^{(l_1)}Z_u^{q,p}dB_u = D_t^{(l_0)}\Delta_i D_s^{(l_1)}Y_v^{q,p} - D_t^{(l_0)}\Delta_i D_s^{(l_1)}Y_{s_r}^{q,p} + \int_{s_r}^v D_t^{(l_0)}\Delta_i D_s^{(l_1)}f(\theta_u^{q-1,p})du.$$

The left hand side of the Burkholder-Davis-Gundy inequality gives

$$\begin{aligned}\mathbb{E}\left(\int_{s_r}^T|D_t^{(l_0)}\Delta_i D_s^{(l_1)}Z_u^{q,p}|^2du\right)^{\frac{j}{2}} \le & C'\left(\mathbb{E}[\sup_{u\in[s_r,T]}|D_t^{(l_0)}\Delta_i D_s^{(l_1)}Y_u^{q,p}|^j]\right.\\ & \left.+\mathbb{E}[\int_{s_r}^T|D_t^{(l_0)}\Delta_i D_s^{(l_1)}f(\theta_u^{q-1,p})|du]^j\right),\end{aligned}$$

where C' denotes a generic constant depending on T and j. Adding $(C'+1)\times$ (A.6) to the previous equation leads to

$$(D_t^{(l_0)}\Delta_i^{q,p}D_s^{(l_1)})_j \le C\left(\mathbb{E}[|D_t^{(l_0)}\Delta_i D_s^{(l_1)}\xi|^j] + \mathbb{E}[\int_{s_{i+l_1}}^T|D_t^{(l_0)}\Delta_i D_s^{(l_1)}f(\theta_u^{q-1,p})|du]^j\right). \tag{A.7}$$

We introduce two vectors j and m, and four integers k_0, k_1, l_0' and l_1' such that $l_0'\le l_0$, $l_1'\le l_1$, $|j|_1+|m|_1 = l-1-l_0'-l_1'$ and $k_0+k_1\le l$. $D_t^{(l_0)}\Delta_i D_s^{(l_1)}f(\theta_u^{q-1,p})$ contains a sum of terms of type

$$\partial_y^{k_0}\partial_z^{k_1}f(\theta_u^{q-1,p})\prod_{i=1}^{k_0-1}D_{ts}^{j_i}Y_u^{q-1,p}\prod_{i=1}^{k_1}D_{ts}^{m_i}Z_u^{q-1,p}(D_t^{(l_0')}\Delta_i D_s^{(l_1')}Y_u^{q-1,p})$$

where $a(j) = k_0 - 1$ and $a(m) = k_1$ and of type

$$\partial_y^{k_0}\partial_z^{k_1} f(\theta_u^{q-1,p}) \prod_{i=1}^{k_0} D_{ts}^{j_i} Y_u^{q-1,p} \prod_{i=1}^{k_1-1} D_{ts}^{m_i} Z_u^{q-1,p} (D_t^{(l_0')}\Delta_i D_s^{(l_1')} Z_u^{q-1,p}),$$

where $a(j) = k_0$, $a(m) = k_1 - 1$.

By using Cauchy-Schwarz inequality, we get that $\mathbb{E}[\int_{s_{i+l_1}}^T |D_t^{(l_0)}\Delta_i D_s^{(l_1)} f(\theta_u^{q-1,p})|du]^j$ is bounded by

$$\|\partial_{sp}^{k_0+k_1} f\|_\infty^j \mathbb{E}\left((\int_{s_{i+l_1}}^T \prod_{i=1}^{k_0-1} (D_{ts}^{j_i} Y_u^{q-1,p})^2 \prod_{i=1}^{k_1} (D_{ts}^{m_i} Z_u^{q-1,p})^2 du)^{\frac{j}{2}} (\int_{s_{l_1+i}}^T (D_t^{(l_0')}\Delta_i D_s^{(l_1')} Y_u^{q-1,p})^2 du)^{\frac{j}{2}} \right)$$

(and the same type of term in $D_t^{l_0'}\Delta_i D_s^{(l_1')} Z_u^{q-1,p}$) which leads to

$$\mathbb{E}[\int_{s_{i+l_1}}^T |D_t^{(l_0)}\Delta_i D_s^{(l_1)} f(\theta_u^{q-1,p})|du]^j$$
$$\leq C((\|\partial_{sp}^k f\|_\infty)_{k\leq l}, \|(Y^{q-1,p}, Z^{q-1,p})\|_{l-1,(l-1)j}) \sum_{l_0'=0}^{l_0}\sum_{l_1'=0}^{l_1} \sqrt{\left(D_t^{(l_0')}\Delta_i^{q-1,p} D_s^{(l_1')}\right)_{2j}},$$

It remains to plug this result in (A.7), to take the supremum in $t_1, \cdots, t_{l_0}, s_{i+1}, \cdots, s_{i+l_1}$ and to apply the induction hypothesis to obtain

$$\sup_{t_1\leq\cdots\leq t_{l_0}} \sup_{s_{i+1}\leq\cdots\leq s_{i+l_1}} \mathbb{E}[(D_t^{(l_0)}\Delta_i^{q,p} D_s^{(l_1)})_j] \leq k_l^\xi (t_i - s_i)^{j\beta_\xi} \tag{A.8}$$
$$+ C((\|\partial_{sp}^k f\|_\infty)_{1\leq k\leq l}, \|(Y^{q-1,p}, Z^{q-1,p})\|_{l-1,(l-1)j}) k_l^{q-1,p} (t_i - s_i)^{j(\frac{1}{2}\wedge\beta_\xi)} \tag{A.9}$$

and the result follows. If $l_1 = 0$, we get

$$D_t^{(l_0)}\Delta_i Y_u^{q,p} = \mathbb{E}_r[\mathcal{C}_{p-r}(D_t^{(l_0)}\Delta_i F^{q-1,p})] - \int_{s_i}^u D_t^{(l_0)}\Delta_i D_s^{(l_1)} f(\theta_v^{q-1}) dv + \int_{s_i}^{t_i} D_t^{(l_0)}\Delta_i D_s^{(l_1)} f(\theta_v^{q-1}) dv.$$

When bounding $\mathbb{E}[\sup_{u\in[s_r,T]} |D_t^{(l_0)}\Delta_i Y_u^{q,p}|^j]$, we deal with the first two terms as we did before, we bound the term $\mathbb{E}[\int_{s_i}^{t_i} |D_t^{(l_0)}\Delta_i f(\theta_v^{q-1,p})|dv]^j$ by

$$C((\|\partial_{sp}^k f\|_\infty)_{1\leq k\leq l}, \|(Y^{q-1,p}, Z^{q-1,p})\|_{l,lj})(t_i - s_i)^j,$$

which ends the proof. □

A.4 Proof of Lemma 4.14

We prove the result by induction. Lemma 4.14 is true for $p = 0$, since $\mathcal{C}_0^N(F) = \mathcal{C}_0(F)$. Assume that $\mathbb{E}(|(\mathcal{C}_{p-1}^N - \mathcal{C}_{p-1})(F)|^2) \leq (K_{p-1}^F)^2 \left(\frac{T}{N}\right)^{2\alpha_F} \sum_{i=1}^{p-1} i^2 \frac{T^i}{i!}$. Since we have

$$(\mathcal{C}_p^N - \mathcal{C}_p)(F) = (\mathcal{C}_{p-1}^N - \mathcal{C}_{p-1})(F) + (P_p^N - P_p)(F),$$

it remains to show that $\mathbb{E}(|(P_p^N - P_p)(F)|^2) \leq (k_p^F)^2 \left(\frac{T}{N}\right)^{2\alpha_F} p^2 \frac{T^p}{p!}$. We recall

$$P_p(F) = \int_0^T \int_0^{s_p} \cdots \int_0^{s_2} u_p(s_p, \cdots, s_1) dB_{s_1} \cdots dB_{s_p}, \quad \text{where } u_p : s_p, \cdots, s_1 \longmapsto \mathbb{E}(D_{s_1\cdots s_p}^{(p)} F), \tag{A.10}$$

$$P_p^N(F) = \sum_{|n|=p} d_p^n \prod_{1\leq i\leq N} K_{n_i}(G_i), \quad \text{where } d_p^n = n!\mathbb{E}\left(F \prod_{1\leq i\leq N} K_{n_i}(G_i)\right). \tag{A.11}$$

Let us rewrite $P_p^N(F)$ as a sum of stochastic integrals. Let $r \in \mathbb{N}$. Applying Lemma 2.5 to $g : t \longmapsto \mathbf{1}_{]\bar{t}_{i-1},\bar{t}_i]}(t)$ yields $M_t^r := h^{r/2} K_r\left(\frac{B_t - B_{\bar{t}_{i-1}}}{\sqrt{h}}\right)$ is a martingale and $M_t^r = \int_{\bar{t}_{i-1}}^t M_s^{r-1} dB_s$. Then, $M_t^r = \int_{\bar{t}_{i-1}}^t \int_{\bar{t}_{i-1}}^{s_r} \cdots \int_{\bar{t}_{i-1}}^{s_2} M_{s_1}^0 dB_{s_1} \cdots dB_{s_r}$. For $r = n_i$ and $t = \bar{t}_i$, we get

$$K_{n_i}(G_i) = \frac{1}{h^{\frac{n_i}{2}}} \int_{\bar{t}_{i-1}}^{\bar{t}_i} \int_{\bar{t}_{i-1}}^{s_{n_i}} \cdots \int_{\bar{t}_{i-1}}^{s_2} dB_{s_1} \cdots dB_{s_{n_i}}.$$

For $|n| := n_1 + \cdots + n_N = p$, we obtain

$$\prod_{1 \le i \le N} K_{n_i}(G_i) = \frac{1}{h^{\frac{p}{2}}} \underbrace{\int_{\bar{t}_{N-1}}^T \cdots \int_{\bar{t}_{N-1}}^{s_{|n(N-1)|+2}}}_{n_N \text{ integrals}} \cdots \underbrace{\int_{\bar{t}_1}^{\bar{t}_2} \cdots \int_{\bar{t}_1}^{s_{|n(1)|+2}}}_{n_2 \text{ integrals}} \underbrace{\int_0^{\bar{t}_1} \cdots \int_0^{s_2}}_{n_1 \text{ integrals}} dB_{s_1} \cdots dB_{s_p}, \tag{A.12}$$

$$d_p^n = n! \frac{1}{h^{\frac{p}{2}}} \underbrace{\int_{\bar{t}_{N-1}}^T \cdots \int_{\bar{t}_{N-1}}^{l_{|n(N-1)|+2}}}_{n_N \text{ integrals}} \cdots \underbrace{\int_{\bar{t}_1}^{\bar{t}_2} \cdots \int_{\bar{t}_1}^{l_{|n(1)|+2}}}_{n_2 \text{ integrals}} \underbrace{\int_0^{\bar{t}_1} \cdots \int_0^{l_2}}_{n_1 \text{ integrals}} u_p(l_p, \cdots, l_1) dl_1 \cdots dl_p. \tag{A.13}$$

To compare $P_p(F)$ and $P_p^N(F)$, we split the integrals in (A.10)

$$P_p(F) = \sum_{|n|=p} \underbrace{\int_{\bar{t}_{N-1}}^T \cdots \int_{\bar{t}_{N-1}}^{s_{|n(N-1)|+2}}}_{n_N \text{ integrals}} \cdots \underbrace{\int_{\bar{t}_1}^{\bar{t}_2} \cdots \int_{\bar{t}_1}^{s_{|n(1)|+2}}}_{n_2 \text{ integrals}} \underbrace{\int_0^{\bar{t}_1} \cdots \int_0^{s_2}}_{n_1 \text{ integrals}} u_p(s_p, \cdots, s_1) dB_{s_1} \cdots dB_{s_p}. \tag{A.14}$$

Combining (A.11)-(A.12)-(A.13) and (A.14) yields $\mathbb{E}(|(P_p^N - P_p)(F)|^2) =$

$$\sum_{|n|=p} \underbrace{\int_{\bar{t}_{N-1}}^T \cdots \int_{\bar{t}_{N-1}}^{s_{|n(N-1)|+2}}}_{n_N \text{ integrals}} \cdots \underbrace{\int_{\bar{t}_1}^{\bar{t}_2} \cdots \int_{\bar{t}_1}^{s_{|n(1)|+2}}}_{n_2 \text{ integrals}} \underbrace{\int_0^{\bar{t}_1} \cdots \int_0^{s_2}}_{n_1 \text{ integrals}} \left| \frac{d_p^n}{h^{\frac{p}{2}}} - u_p(s_p, \cdots, s_1) \right|^2 ds_1 \cdots ds_p, \tag{A.15}$$

Moreover, $\frac{d_p^n}{h^{\frac{p}{2}}} - u_p(s_p, \cdots, s_1) =$

$$\frac{n!}{h^p} \underbrace{\int_{\bar{t}_{N-1}}^T \cdots \int_{\bar{t}_{N-1}}^{l_{N-1}+1}}_{n_N \text{ integrals}} \cdots \underbrace{\int_{\bar{t}_1}^{\bar{t}_2} \cdots \int_{\bar{t}_1}^{l_{n_1}+1}}_{n_2 \text{ integrals}} \underbrace{\int_0^{\bar{t}_1} \cdots \int_0^{l_2}}_{n_1 \text{ integrals}} (u_p(l_p, \cdots, l_1) - u_p(s_p, \cdots, s_1)) dl_1 \cdots dl_p.$$

Since u_p satisfies Hypothesis 4.1, we get $|u_p(l_p, \cdots, l_1) - u_p(s_p, \cdots, s_1)| \le k_p^F(|l_p - s_p|^{\beta_F} + \cdots + |l_1 - s_1|^{\beta_F}) \le p k_p^F h^{\beta_F}$. Then $\left| \frac{d_p^n}{h^{\frac{p}{2}}} - u_p(s_p, \cdots, s_1) \right| \le p k_p^F h^{\beta_F}$. Plugging this result in (A.15) ends the proof.

A.5 Proof of Lemma 4.16

Using the definitions 2.8 and 3.10 leads to

$$(\mathcal{C}_p^N - \mathcal{C}_p^{N,M})(F) = d_0 - \hat{d}_0 + \sum_{k=1}^p \sum_{|n|=k} (d_k^n - \hat{d}_k^n) \prod_{i=1}^N K_{n_i}(G_i).$$

Since $\hat{d}_k^n$ is independent of $(G_i)_i$

$$\mathbb{E}(|(\mathcal{C}_p^N - \mathcal{C}_p^{N,M})(F)|^2) = \mathbb{E}(|d_0 - \hat{d}_0|^2) + \sum_{k=1}^{p} \sum_{|n|=k} \frac{1}{n!} \mathbb{E}(|d_k^n - \hat{d}_k^n|^2)$$

The definition of the coefficients d_0 and d_k^n given in (2.9) leads to

$$\mathbb{E}(|(\mathcal{C}_p^N - \mathcal{C}_p^{N,M})(F)|^2) = \mathbb{V}(\hat{d}_0) + \sum_{k=1}^{p} \sum_{|n|=k} \frac{1}{n!} \mathbb{V}(\hat{d}_k^n),$$

and the first result follows. To get the second result, we write $\mathcal{C}_p^{N,M}(F) = (\mathcal{C}_p^{N,M} - \mathcal{C}_p^N)(F) + \mathcal{C}_p^N(F)$. Since $\mathbb{E}\left((\mathcal{C}_p^{N,M} - \mathcal{C}_p^N)(F)\mathcal{C}_p^N(F)\right) = 0$, we get

$$\mathbb{E}(|\mathcal{C}_p^{N,M}(F)|^2) = \mathbb{E}(|(\mathcal{C}_p^{N,M} - \mathcal{C}_p^N)(F)|^2) + \mathbb{E}(|\mathcal{C}_p^N(F)|^2).$$

Lemma 2.6 ends the proof.

B Wiener chaos expansion formulas

B.1 Proof of Proposition 2.7

Firstly, we compute $\mathbb{E}_t(\mathcal{C}_p^N(F))$ for $t \in]\bar{t}_{r-1}, \bar{t}_r]$. From (2.8), we get

$$\mathbb{E}_t(\mathcal{C}_p^N F) = d_0 + \sum_{k=1}^{p} \sum_{|n|=k} d_k^n \prod_{i<r} K_{n_i}(G_i) \times \mathbb{E}_t\left(\prod_{i \geq r} K_{n_i}(G_i)\right).$$

Since Brownian increments are independent, we get $\mathbb{E}_{\bar{t}_r}(\prod_{i \geq r} K_{n_i}(G_i)) = K_{n_r}(G_r) \prod_{i>r} \mathbb{E}[K_{n_i}(G_i)]$, which is null as soon as $n_{r+1} + \cdots + n_N > 0$. Then, nested conditional expectations give

$$\mathbb{E}_t(\mathcal{C}_p^N F) = d_0 + \sum_{k=1}^{p} \sum_{|n(r)|=k} d_k^n \prod_{i<r} K_{n_i}(G_i) \times \mathbb{E}_t\left(K_{n_r}(G_r)\right).$$

By applying Lemma 2.5 when $g : t \longmapsto \mathbf{1}_{]\bar{t}_{r-1}, \bar{t}_r]}(t)$, we get $\mathbb{E}_t\left(K_{n_r}(G_r)\right) = \left(\frac{t - \bar{t}_{r-1}}{h}\right)^{n_r/2} K_{n_r}\left(\frac{B_t - B_{\bar{t}_{r-1}}}{\sqrt{t - \bar{t}_{r-1}}}\right)$, which yields the first result. Since $K_n'(x) = K_{n-1}(x)$, the second result follows.

B.2 Wiener chaos expansion formulas in $\mathbb{R}^d$

We want to approximate $F \in \mathrm{L}^2(\mathcal{F}_T)$ using its chaos decomposition up to order p. We assume $N \geq dp$. We consider the following truncated basis of $\mathrm{L}^2\left([0,T]; \mathbb{R}^d\right)$

$$\frac{\mathbf{1}_{]\bar{t}_{i-1}, \bar{t}_i]}(t)}{\sqrt{h}} e_j, \quad i = 1, \ldots, N, \quad j = 1, \ldots, d, \text{ where } h = \frac{T}{N}$$

where $\{\bar{t}_i := ih, i = 0, \cdots, N\}$ is a regular mesh grid and $(e_j)_{1 \leq j \leq d}$ represents the canonical basis of $\mathbb{R}^d$. P_k, the k^{th} chaos, is generated by

$$\left\{\prod_{j=1}^{d} \prod_{i=1}^{N} K_{n_i^j}\left(G_i^j\right) : \sum_{j=1}^{d} \sum_{i=1}^{N} n_i^j = k\right\}, \quad G_i^j = \frac{\Delta_i^j}{\sqrt{h}}, \quad \Delta_i^j = B_{\bar{t}_i}^j - B_{\bar{t}_{i-1}}^j.$$

For $j = 1, \ldots, d$, $n^j = (n_1^j, \ldots, n_N^j)$, one notes $|n^j| = n_1^j + \ldots + n_N^j$, $n^j! = n_1^j! \ldots n_N^j!$ and for $r \leq N$, $n^j(r) = (n_1^j, \ldots, n_r^j)$. $n = (n^1, \ldots, n^d)^*$, $|n| = |n^1| + \cdots + |n^d|$, $n! = n^1! \ldots n^d!$ and

$n(r) = (n^1(r), \dots, n^d(r))^*$. Since the r.v. $\left(\prod_{1\le j\le d}\prod_{1\le i\le N} K_{n_i^j}\left(G_i^j\right)\right)_n$ are orthogonal ones, the projection of F is given by

$$\mathcal{C}_p^N(F) = d_0 + \sum_{k=1}^{p}\sum_{|n|=k} d_k^n \prod_{1\le j\le d}\prod_{1\le i\le N} K_{n_i^j}\left(G_i^j\right),$$

where the coefficients d_k^n are given by

$$d_k^n = n!\,\mathbb{E}\left[F\prod_{1\le j\le d}\prod_{1\le i\le N} K_{n_i^j}\left(G_i^j\right)\right].$$

Proposition B.1. *For $\bar{t}_{r-1} < t \le \bar{t}_r$, we have*

$$\mathbb{E}_t(\mathcal{C}_p^N F) = d_0 + \sum_{k=1}^{p}\sum_{|n(r)|=k} d_k^n \prod_{i<r}\prod_{1\le j\le d} K_{n_i^j}\left(G_i^j\right) \times \prod_{1\le j\le d}\left(\frac{t-\bar{t}_{r-1}}{h}\right)^{\frac{n_r^j}{2}} K_{n_r^j}\left(\frac{B_t^j - B_{\bar{t}_{r-1}}^j}{\sqrt{t-\bar{t}_{r-1}}}\right).$$

and for $l = 1, \dots, d$,

$$D_t^l(\mathbb{E}_t(\mathcal{C}_p^N F)) = \sum_{k=1}^{p}\sum_{\substack{|n(r)|=k\\ n_r^l>0}} d_k^n h^{-1/2}\prod_{i<r}\prod_{1\le j\le d} K_{n_i^j}\left(G_i^j\right)\times$$
$$\left(\frac{t-\bar{t}_{r-1}}{h}\right)^{\frac{n_r^l-1}{2}} K_{n_r^l-1}\left(\frac{B_t^l - B_{\bar{t}_{r-1}}^l}{\sqrt{t-\bar{t}_{r-1}}}\right)\prod_{j\ne l}\left(\frac{t-\bar{t}_{r-1}}{h}\right)^{\frac{n_r^j}{2}} K_{n_r^j}\left(\frac{B_t^j - B_{\bar{t}_{r-1}}^j}{\sqrt{t-\bar{t}_{r-1}}}\right).$$

Remark B.2. *In particular, for $t = \bar{t}_r$, $r \ge 1$ and $l = 1, \dots, d$,*

$$\mathbb{E}_{\bar{t}_r}(\mathcal{C}_p^N F) = d_0 + \sum_{k=1}^{p}\sum_{|n(r)|=k} d_k^n \prod_{i\le r}\prod_{1\le j\le d} K_{n_i^j}\left(G_i^j\right)$$
$$D_{\bar{t}_r}^l(\mathbb{E}_{\bar{t}_r}(\mathcal{C}_p^N F)) = \sum_{k=1}^{p}\sum_{\substack{|n(r)|=k\\ n_r^l>0}} d_k^n h^{-1/2}\prod_{i<r}\prod_{1\le j\le d} K_{n_i^j}\left(G_i^j\right)\times K_{n_r^l-1}\left(G_r^l\right)\prod_{j\ne l} K_{n_r^j}\left(G_r^j\right).$$

When $r = 0$, we get $\mathbb{E}_{\bar{t}_0}(\mathcal{C}_p^N F) = d_0$ and we define $D_{\bar{t}_0}^l(\mathbb{E}_{\bar{t}_0}(\mathcal{C}_p^N F)) = \frac{1}{\sqrt{h}} d_1^{e_1^l}$, where (e_j^i) is a matrix of size $d \times N$ whose component (i,j) equals 1 and the other ones are null.

Proof of Proposition B.1. We first compute $\mathbb{E}_t(\mathcal{C}_p^N F)$ for $t \in]\bar{t}_{r-1}, \bar{t}_r]$. We have

$$\mathbb{E}_t(\mathcal{C}_p^N F) = d_0 + \sum_{k=1}^{p}\sum_{|n|=k} d_k^n \prod_{i<r}\prod_{1\le j\le d} K_{n_i^j}\left(G_i^j\right)\times\mathbb{E}_t\left(\prod_{i\ge r}\prod_{1\le j\le d} G_{n_i^j}\left(W_i^j\right)\right)$$

Since Brownian motions and their increments are independents, we get

$$\mathbb{E}_{\bar{t}_r}\left(\prod_{i\ge r}\prod_{1\le j\le d} K_{n_i^j}\left(G_i^j\right)\right) = \prod_{1\le j\le d} K_{n_r^j}\left(G_r^j\right)\prod_{i>r}\prod_{1\le j\le d}\mathbb{E}\left[K_{n_i^j}\left(G_i^j\right)\right];$$

which is null as soon as $n_{r+1}^1 + \dots + n_N^1 + \dots + n_{r+1}^d + \dots + n_N^d > 0$. Then, nested conditional expectations give

$$\mathbb{E}_t(F) = d_0 + \sum_{k=1}^{p}\sum_{|n(r)|=k} d_k^n \prod_{i<r}\prod_{1\le j\le d} K_{n_i^j}\left(G_i^j\right)\times\mathbb{E}_t\left(\prod_{1\le j\le d} K_{n_r^j}\left(G_r^j\right)\right).$$

From Lemma 2.5, for $j = 1, \dots, d$ $M_t^{n_r^j} := \left(t - \overline{t}_{r-1}\right)^{n_r^j/2} K_{n_r^j}\left(\frac{B_t^j - B_{\overline{t}_{r-1}}^j}{\sqrt{t-\overline{t}_{r-1}}}\right)$ is a martingale and $dM_t^{n_r^j} = M_t^{n_r^j-1} \mathbf{1}_{]\overline{t}_{r-1},\overline{t}_r]}(t)\, dB_t^j$. Then, $\prod_{1\leq j\leq d} \left(t - \overline{t}_{r-1}\right)^{n_r^j/2} K_{n_r^j}\left(\frac{B_t^j - B_{\overline{t}_{r-1}}^j}{\sqrt{t-\overline{t}_{r-1}}}\right)$ is also a martingale and the first result follows. Since $K'_{n_r^l}(x) = K_{n_r^l-1}(x)$, we get the second result.

□

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