# Envy-free two-player m-cake and three-player two-cake divisions

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# ENVY-FREE TWO-PLAYER $m$-CAKE AND THREE-PLAYER TWO-CAKE DIVISIONS

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Abstract. Cloutier, Nyman, and Su (*Mathematical Social Sciences* **59** (2005), 26–37) initiated the study of envy-free cake-cutting problems involving several cakes. The classical result in this area is that when there are $q$ players and one cake, an envy-free cake-division requiring only $q-1$ cuts exists under weak and natural assumptions. Among other results, Cloutier, Nyman, and Su showed that when there are two players and two or three cakes it is again possible to find envy-free cake-divisions requiring few cuts, under same assumptions.

In the present note, we prove that such a result also exists when there are two players and any number of cakes and when there are three players and two cakes. The proof relies on a theorem by Gyárfás linking the matching number and the fractional matching number in $m$-partite hypergraphs.

## 1. Introduction

1.1. **Context.** Cake-cutting problems ask whether it is possible to divide a cake among players in such a way each of them believes the division is fair. These problems go back to Steinhaus [6] and have received a lot of attention. Many variations are possible depending on whether the pieces may be disconnected or not and on how "fair" is understood. In this note, we consider a division to be fair if all players consider their own pieces to be at least as valuable as any of the others. Dubins and Spanier [3] were able to prove that such a division exists provided that each player's preference are defined by a nonatomic measure over the cake. Unfortunately, a piece of cake in their result may be a collection of many (possibly infinite) disjoint connected subsets. Stromquist [7] improved their result by showing that such a division can be obtained by cutting the cake by $q-1$ planes, each parallel to a given plane, where $q$ is the number of player. Moreover, he showed that such divisions exist for a larger class of "preferences". Unformally, given a partition of the cake into pieces, we require simply that each player is able to say which pieces he prefers. We consider this kind of preferences in the present note and there are formally defined hereafter. Su [8] found then a constructive proof of Stromquist's theorem based on Sperner's lemma [5], the combinatorial counterpart of Brouwer's fixed point theorem.

In 2010, Cloutier, Nyman, and Su [1] asked whether extensions of this theorem are possible when there are more than one cake. Given $m$ cakes, is it possible to divide each cake into a given number of connected pieces and assign one piece from each cake to each player in such a way that each player believes his assignment at least as good as any other assignment?

1.2. **Model.** Each cake is identified with the interval $[0,1]$. A *division* of the cake $i$ into $r_i$ pieces is an $r_i$-tuple $\boldsymbol{x}_i = (x_{i1},\ldots,x_{ir_i})$, with $x_{ij} \geq 0$ for all $j \in [r_i]$ and $\sum_{j=1}^{r_i} x_{ij} = 1$, where $x_{ij}$ is the size of the $j$th piece (ordered from left to right) of the cake $i$. Given a division $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m$ of the $m$ cakes, a *piece selection* is the selection of one piece in each cake, i.e. it is an $m$-tuple $(j_1,\ldots,j_m) \in [r_1]\times\ldots\times[r_m]$. A player *prefers* a certain piece selection if that player does not think that any other piece selection is strictly better. For some divisions a player may be indifferent to two or more "preferred" piece selections.

*Date*: May 3, 2013.

We make the following assumptions on the preferences, which are the ones considered by Stromquist in the one-cake case.

(1) *Independence of preferences:* The preferences of one player do not depend of the choices made by the other players.
(2) *The players are hungry:* A player will never choose an empty piece.
(3) *Preference sets are closed:* If one player prefer the same piece selection for a convergent sequence of division, then that piece selection will be preferred at the limit.

A division of $m$ cakes between $q$ players is *envy-free* if there exist $q$ disjoint piece selections and an assignment of the piece selections to the players such that each player prefers the piece selection he gets to any other piece selection among all possible piece selections for this division. The question studied by Cloutier, Nyman, and Su is whether there exists an integer $r(q,m)$, independent of the preferences, such that there exists an envy-free division of the $m$ cakes not requiring to divide each cake into more than $r(q,m)$ pieces. Note that Stromquist's theorem asserts the existence of $r(q,1)$ for any $q$ and that $r(q,1)=q$. Using a polytopal version of Sperner's lemma [2], Cloutier, Nyman, and Su proved the existence of $r(2,2)$ and $r(2,3)$ and that $r(2,2)=3$ and $r(2,3)\leq 4$. They asked moreover whether $r(2,m)\leq m+1$.

1.3. **Main results.** We contribute to the questions by proving the following theorem.

**Theorem 1.** *The integer $r(2,m)$ exists for any $m\geq 2$ and is such that $r(2,m)\leq m(m-1)+1$.*

In addition to the polytopal version of Sperner's lemma, the proof uses an inequality (Gyárfás's theorem) between the matching number and the fractionnal matching number in $m$-graphs.

We are also able to prove a first result involving three players.

**Theorem 2.** *The integer $r(3,2)$ exists and is such that $r(3,2)\leq 5$.*

1.4. **Plan.** In Section 2, we give the main tools used in the proofs, such as a polytopal version Sperner's lemma or Gyárfás's theorem. Section 3 is devoted to the proof of Theorem 1 and Section 4 to the proof of Theorem 2.

## 2. Tools

2.1. **Sperner's labeling.** Given a triangulation $\mathsf{T}$ of a polytope $P$, a *Sperner labeling* is a map $\lambda : V(\mathsf{T})\rightarrow V(P)$, where $V(\mathsf{T})$ and $V(P)$ are respectively the vertex sets of $\mathsf{T}$ and $P$, such that $\lambda(v)$ is a vertex of the minimal face of $P$ containing $v$. The following theorem is proved in [2]. Given a simplex $\sigma$, we denote its vertex set $V(\sigma)$. The polytopal version of Sperner's lemma already mentioned is the following theorem.

**Theorem 3.** *Let $\mathsf{T}$ be a triangulation of a polytope $P$. If $\lambda$ is a Sperner labeling of $\mathsf{T}$, then $\bigcup_{\sigma\in\mathsf{T}}\operatorname{conv}(\lambda(V(\sigma)))=P$.*

Note that this theorem implies that for any point $x$ of $P$, there is a $\sigma\in\mathsf{T}$ with $\dim\sigma=\dim P$ such that $x\in\operatorname{conv}(\lambda(V(\sigma)))$ and such that $\operatorname{conv}(\lambda(V(\sigma)))$ is non-degenerate, i.e. of dimension $\dim P$.

2.2. **Divisions, polytopes, owner-labeling, and preference-labeling.** The divisions of the $m$ cakes with $r_i$ pieces in cake $i$ are exactly the points of the polytope $P=\triangle_1\times\ldots\times\triangle_m$, where $\triangle_i$ is the $(r_i-1)$-simplex $\{(x_{i1},\ldots,x_{ir_i})\in\mathbb{R}_+^{r_i}:\sum_{j=1}^{r_i}x_{ij}=1\}$. The polytope $P$ – called the *polytope of divisions* – has $\Pi_{i=1}^m r_i$ vertices and is of dimension $\sum_{i=1}^m r_i-m$.

Following [1], we explain how to locate the envy-free divisions on $P$. We assume given a triangulation $\mathsf{T}$ of $P$. We label the vertices of $\mathsf{T}$ with an *owner-labeling*, which is a map $o:V(\mathsf{T})\rightarrow[q]$ assigning a player to each vertex of the triangulation. We require moreover this owner-labeling to be *uniform*: on each simplex, the number of times each player appears as a label differs by at most one from any other player. In other words, given any simplex $\sigma \in \mathsf{T}$ and its vertex set $V(\sigma)$, we have $\big||o^{-1}(k) \cap V(\sigma)| - |o^{-1}(k') \cap V(\sigma)|\big| \leq 1$ for all $k, k' \in [q]$.

The following proposition is proved in [1]. The *mesh-size* of a triangulation is the maximum diameter of its simplices.

**Proposition 1.** *There exists a triangulation* $\mathsf{T}$ *admitting a uniform owner-labeling for any polytope, any number* $q$ *of players, and of arbitrary small mesh-size.*

Given a triangulation $\mathsf{T}$ of $P$ with a uniform owner-labeling $o$, we define a new labeling $\lambda : V(\mathsf{T}) \to V(P)$ of the vertices of $\mathsf{T}$ with the vertices of $P$: the *preference-labeling*. Each vertex $v$ of $\mathsf{T}$ is a point in $P$ and as such corresponds to a division of the $m$ cakes. The player $o(v)$ prefers some piece selection $(j_1, \ldots, j_m)$ for this division (in case of a tie, make an arbitrary choice). We define then $\lambda(v)$ to be the vertex of $P$ with coordinates $(\lambda_{ij}(v))$ with $\lambda_{ij}(v) = 0$ except for the pairs $(i, j_i)$ for which $\lambda_{ij_i}(v) = 1$. Since "the players are hungry", the map $\lambda$ is a Sperner labeling.

2.3. **$m$-graphs and matchings.** A *hypergraph* is a pair $H = (V, E)$ where $V$ is a finite set and $E$ a family of subsets of $V$. The elements of $V$ are called the *vertices* and the elements of $E$ are called the *edges*. We denote by $\delta(v)$ the set of edges containing a vertex $v$. A *matching* is a collection of pairwise disjoint edges. A *fractional matching* is a vector $\boldsymbol{w} \in \mathbb{R}_+^E$ such that $\sum_{e \in \delta(v)} w_e \leq 1$ for all $v \in V$. Note that a matching is a fractional matching with $w_e \in \{0, 1\}$ for all edges $e \in E$. The maximum cardinality of a matching, called the *matching number*, is denoted $\nu(H)$ and the maximum possible value of $\sum_{e \in E} w_e$ for a fractional matching $\boldsymbol{w}$, called the *fractional matching number*, is denoted $\nu^*(H)$. Since a matching is a fractional matching, we have $\nu^*(H) \geq \nu(H)$.

An *$m$-partite hypergraph*, or *$m$-graph*, is a hypergraph $H = (V, E)$ whose vertex set is the disjoint union of $m$ sets $V_1, \ldots, V_m$ and such that each edge intersects each $V_i$ in exactly one vertex. The following theorem, proved by Gyárfás (according to Füredi [4]), shows that the gap between the fractional matching number and the matching number is not too large for $m$-graphs.

**Theorem 4** (Gyárfás's theorem)**.** *If* $H$ *is an* $m$*-graph, then* $\nu^*(H) \leq (m-1)\nu(H)$.

2.4. **A hypergraphic condition of existence of disjoint envy-free piece selections.** We consider a triangulation $\mathsf{T}$ of the polytope of division $P$ and assume that we have for $\mathsf{T}$ an owner-labeling $o$ and a preference-labeling $\lambda$. As in [1], we will use Theorem 3 to get the existence of a simplex $\sigma \in \mathsf{T}$ with special features to obtain eventually the existence of an envy-free division of the cakes with no more than $r_i$ pieces per cake $i$. However, we introduce an additional combinatorial criterion based on some hypergraph properties in an attempt to systematize the reasoning.

Given a simplex $\sigma \in \mathsf{T}$, we define $H(\sigma)$ to be the $m$-graph with $V_i = \{(i, j) : j \in [r_i]\}$ for $i = 1, \ldots, m$. The edges of $H(\sigma)$ are given by the vertices of $\sigma$ as follows. For a vertex $v$ of $\sigma$, the piece selection $(j_1, \ldots, j_m)$ preferred by $o(v)$ and corresponding to $\lambda(v)$ gives the edge $\{(1, j_1), \ldots, (m, j_m)\}$, which we denote $e_v$. In other words, the edge $\{(1, j_1), \ldots, (m, j_m)\}$ exists in $H(\sigma)$ if and only if a vertex $v$ of $\sigma$ is such that $\lambda_{ij}(v) = 1$ if $j = j_i$ and $\lambda_{ij}(v) = 0$ otherwise. Note that it implies that $H(\sigma)$ has $|V(\sigma)|$ edges.

The following lemma is a key tool. A coloring of the edges on a hypergraph is said to be *balanced* if the number of times a color appears differs by at most one from any other color. A *rainbow matching* in a hypergraph is a matching in which each color is present exactly once. If we see each player as a color, we get a natural balanced edge coloring of $H(\sigma)$: for each edge $e_v$ of $H(\sigma)$, we color it with $o(v)$. The coloring is balanced because $o$ is an owner-labeling. In this framework, a rainbow matching provides "almost disjoint envy-free piece selections": each player selects distincts pieces in the cakes and each of them prefer his selection to any other. However, the cuts used for each player are not identical: each player considers cuts given by distinct vertices of $\sigma$.

**Lemma 1.** *If every triangulation* $\mathsf{T}$ *admitting a uniform owner-labeling has a simplex* $\sigma$ *such that every balanced edge coloring of* $H(\sigma)$ *with* $q$ *colors gets a rainbow matching, then there exists an envy-free division of the cakes with no more than* $r_i$ *pieces per cake* $i$*.*

We denote by $\operatorname{diam}\sigma$ the diameter of a simplex $\sigma$.

*Proof of Lemma 1.* We consider a sequence of triangulations admitting an owner-labeling with a mesh-size tending towards zero. This sequence exists according to Proposition 1. By compactness, there is a sequence of simplices $(\sigma^{(n)})_{n\in\mathbb{Z}_+}$ with $\lim_{n\to+\infty}\operatorname{diam}\sigma^{(n)}=0$ such that every balanced edge coloring of $H(\sigma^{(n)})$ for $n=0,1,\ldots$ gets the same rainbow matching. With the colors defined as in the paragraph above, the rainbow matching provides distinct piece-selections for the players. For the division given by the limit point, we can therefore find disjoint envy-free piece-selections (here, we use the closedness of the preference sets). ☐

The following lemma helps for finding a $\sigma$ such that every balanced edge coloring of $H(\sigma)$ gets a rainbow matching.

**Lemma 2.** *Let* $\sigma$ *be a simplex of* $\mathsf{T}$*. If* $\operatorname{conv}(\lambda(V(\sigma)))$ *contains the central point of* $P$*, that is the point* $(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m)$ *with* $x_{ij}=1/r_i$ *for* $i\in[m]$ *and* $j\in[r_i]$*, then* $H(\sigma)$ *contains a matching of size at least* $\lceil r/(m-1)\rceil$*, where* $r=\min_{i\in[m]}r_i$*.*

*Proof.* Assume that $\operatorname{conv}(\lambda(V(\sigma)))$ contains the central point. Denote $\lambda(v)=(\lambda_{ij}(v))$. Note that $\lambda_{ij}(v)\in\{0,1\}$ and $\lambda_{ij}(v)=1$ exactly when $(i,j)\in e_v$ (with the notation of Section 2.4). We have then $\alpha_v\in\mathbb{R}_+$ for each $v\in V(\sigma)$ such that $\sum_{v\in V(\sigma)}\alpha_v\lambda_{ij}(v)=1/r_i$ for all $i$ and $j$, and $\sum_{v\in V(\sigma)}\alpha_v=1$. For a vertex $v$ of $\sigma$, we consider the edge $e_v$ of $H(\sigma)$ and we define $w_{e_v}$ to be $r\alpha_v$.

Now, take $i\in[m]$ and consider a vertex $(i,j)$ in $V_i$. We have $\sum_{v\in V(\sigma)}w_{e_v}\lambda_{ij}(v)=r/r_i$, i.e.

$$\sum_{v\in V(\sigma)\text{ and }(i,j)\in e_v}w_{e_v}=\frac{r}{r_i},$$

which can finally be rewritten

$$\sum_{e\in\delta(i,j)}w_e\leq 1,$$

with equality when $r_i=r$. It shows that $\boldsymbol{w}$ is a fractional matching of $H(\sigma)$.

Moreover, we have $\sum_{e\in E(H(\sigma))}w_e=\sum_{j\in[r_i]}\sum_{e\in\delta(i,j)}w_e$ for any $i$. In particular, using $i$ such that $r_i=r$, we get that $\sum_{e\in E(H(\sigma))}w_e=r$. Using Gyárfás's theorem (Theorem 4), we get that $\nu(H(\sigma))\geq r/(m-1)$. The integrality of the matching number allows to conclude. ☐

We have an additional property for $H(\sigma)$. A hypergraph is *simple* if it has no *parallel edges*, that is no two edges have the same vertex set.

**Lemma 3.** *Let* $\sigma$ *be a simplex of* $\mathsf{T}$*. If* $\operatorname{conv}(\lambda(V(\sigma)))$ *is non-degenerate, i.e. of dimension* $|V(\sigma)|-1$*, then* $H(\sigma)$ *is simple.*

*Proof.* If $H(\sigma)$ is not simple, then it means that we have two edges $e_u=e_v$ with distinct vertices $u$ and $v$ from $\sigma$. It means then that $\lambda(u)=\lambda(v)$ which prevents $\operatorname{conv}(\lambda(V(\sigma)))$ of being non-degenerate. ☐

## 3. Proof of Theorem 1

*Proof of Theorem 1.* We have $m$ cakes. Each cake is cut into $m(m-1)+1$ pieces, which means $r_i=r$ for all $i=1,\ldots,m$, with $r=m(m-1)+1$.

For a triangulation $\mathsf{T}$ of the polytope of divisions $P$ admitting a uniform owner-labeling, we consider a preference-labeling $\lambda : V(\mathsf{T})\to V(P)$ defined according to Section 2.2. Theorem 3 ensures that a simplex $\sigma$ of $\mathsf{T}$ is such that $\operatorname{conv}(\lambda(V(\sigma)))$ contains the central point of $P$, that is the point $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)$ with $x_{ij} = 1/r$ for $i \in [m]$ and $j \in [r]$. We prove now that every balanced coloring of the hypergraph $H(\sigma)$ with two colors has a rainbow matching, i.e. $H(\sigma)$ has two disjoint edges of distinct colors. The conclusion follows from Lemma 1.

We consider $H(\sigma)$ and assume that it gets a balanced edge coloring with two colors, say blue and red. According to Lemma 2, the hypergraph $H(\sigma)$ contains a matching of size at least $m+1$. If both colors are present in the matching, we have our rainbow matching. If not, suppose w.l.o.g. that all edges of the matching are blue. Consider some red edge (which exists because the coloring is balanced). This edge cannot intersect all $m+1$ disjoint blue edges of the matching, since each edge of $H(\sigma)$ has $m$ vertices. Thus there are disjoint red and blue edges. □

## 4. Proof of Theorem 2

*Proof of Theorem 2.* We have $m = 2$ cakes. Each cake is cut into 5 pieces, which means $r_i = r = 5$ for $i = 1, 2$.

For a triangulation $\mathsf{T}$ of the polytope of divisions $P$ admitting a uniform owner-labeling, we consider a preference-labeling $\lambda : V(\mathsf{T}) \to V(P)$ defined according to Section 2.2. Theorem 3 ensures that a simplex $\sigma$ of $\mathsf{T}$ is such that $\operatorname{conv}(\lambda(V(\sigma)))$ contains the central point of $P$, that is the point $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ with $x_{ij} = 1/5$ for $i = 1, 2$ and $j = 1, \ldots, 5$. We can require moreover that $\operatorname{conv}(\lambda(V(\sigma)))$ is non-degenerate and of dimension $\dim P = 8$. We prove now that every balanced coloring of the hypergraph $H(\sigma)$ with three colors (there are three players) has a rainbow matching, i.e. $H(\sigma)$ has three disjoint edges of distinct colors. The conclusion follows again from Lemma 1.

We consider $H(\sigma)$. Note that it is a bipartite graph since $m = 2$. Assume that it gets a balanced edge coloring with three colors, say blue, red, and yellow. Since $H(\sigma)$ has 9 edges, there are three edges of each color. According to Lemma 2, the graph $H(\sigma)$ contains a matching of $\lceil r/(m-1) \rceil = 5$ edges, that is, a perfect matching. If the three colors are present in the matching, we have our rainbow matching. If not, suppose w.l.o.g. that there are three blues edges and two red edges in the matching. Consider a yellow edge. If it intersects one of these blue edges, it misses at least one of the red edges and one of the blue edges of the matching, and we are done. Thus, we have to deal with the case when all three yellow edges intersect the two red edges of the matching. But in this case, we would have two parallel edges in $H(\sigma)$, which is impossible according to Lemma 3. □

This proof does not seem to extend to more general cases.

As an example, we may consider the case $m = 2$ cakes, $q = 4$ players, and $r$ pieces for each cake. We would try to find a rainbow matching in a bipartite graph with $r$ vertices on each side, $2r - 1$ distinct edges colored in 4 colors, $\simeq r/2$ edges of each color, and with a perfect matching. It is easy to see that unfortunately such a bipartite graph does not contain necessarily a rainbow matching. For instance, the edges of two colors can provide the edges of the perfect matching whereas the other edges share all a same vertex.

Another example is with $m = 3$ cakes, $q = 3$ players, and $r$ pieces for each cake. We would try to find a rainbow matching in a 3-graph with $r$ vertices on each side, $3r - 1$ distinct edges colored in 3 colors, $\simeq r$ edges of each color, and with a matching of size $r/2$. Again, we can force all edges of two colors to intersect at the same vertex.

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