A generalization of Euler numbers to finite Coxeter groups

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Abstract. It is known that Euler numbers, defined as the Taylor coefficients of the tangent and secant functions, count alternating permutations in the symmetric group. Springer defined a generalization of these numbers for each finite Coxeter group by considering the largest descent class, and computed the value in each case of the classification. We consider here another generalization of Euler numbers for finite Coxeter groups, building on Stanley's result about the number of orbits of maximal chains of set partitions. We present a method to compute these integers and obtain the value in each case of the classification. In the second part of this work, we consider maximal chains of noncrossing partitions, and how this set is divided into classes via the action of the group. We introduce a statistic related with the notion of interval partition, and show that the generating functions of classes, as well as the full generating function, are simple products. We recover Postnikov's hook-length formula in type A and obtain a variant in type B.

1. Introduction

It is known since long ago [1] that the Euler numbers T_n, defined by

$$\sec(z) + \tan(z) = \sum_{n\geq 0} T_n \frac{z^n}{n!}, \tag{1}$$

count alternating permutations in the symmetric group $\mathfrak{S}_n$ (σ is *alternating* if $\sigma(1) > \sigma(2) < \sigma(3) > \dots$). Since then, there has been a lot of interest to these numbers and permutations, as exposed in the recent survey of Stanley [18].

It can be shown that alternating permutations form the largest descent class in the symmetric group. Building on this, Springer [17] gave a characterization of the largest descent class of a finite Coxeter group, and computed its cardinality in each case of the classification. The analog of alternating permutations for other groups were studied by Arnol'd [2], who called these objects *snakes*. See also [13, Section 3] for an alternative proof of Springer's result.

In this article, we are interested in another construction that relates the number T_n with the symmetric group, and can also be generalized to finite Coxeter groups. Namely, there is an action of $\mathfrak{S}_n$ on the maximal chains in the lattice of set partitions of size n, and Stanley [19] showed that the number of orbits is T_{n-1}. It is now well established that set partitions can be realized as an intersection lattice generated by reflecting hyperplanes, so that the construction can be generalized and gives an integer $K(W)$ for each finite Coxeter group W, with $K(A_n) = T_n$. (Note that this differs from Springer's construction, where the the integer T_n is related with the group A_{n-1}.) We present a general method to compute $K(W)$

2000 *Mathematics Subject Classification.* 05A18, 05E18, 11B68, 20F55.

and apply it to obtain the value in each case of the classification. There is some similarity with a problem studied by Reading [16], which consists in the enumeration of maximal chains in the lattice of noncrossing partition (in both cases, there is a product formula for the reducible case and a recursion on maximal parabolic subgroup in the irreducible case).

In the second part of this article, we relate our initial problem with noncrossing partitions. We first introduce a statistic on chains of noncrossing partitions, whose value is 0 exactly for the chains of *interval partitions*. The enumeration of chains of noncrossing partitions with respect to this statistic turns out to be related with the Fuss-Catalan numbers, so that the result is a simple product in terms of the degrees of the group. The chains of noncrossing partitions are not stable under the group action, but we can still consider the equivalence classes induced by the action. The number of classes is $K(W)$, and we prove that the generating function of each class is also a simple product. We show that in type A and B, this leads to some hook-length formula for trees; more precisely, we recover Postnikov's hook formula in type A and obtain a variant in type B.

Acknowledgement

I thank Vic Reiner for his helpful comments about this work.

Part 1. Computation of $K(W)$

2. Definitions

Let V be an Euclidian space, and W a finite subgroup of $GL(V)$ generated by orthogonal reflections. Let n be the rank of W, i.e. $n = \dim V$. We call *reflecting hyperplane* an hyperplane $H \subset V$ which is the fixed point set of some reflection in W. The following definition is now well established, see for example [3, Chapter 4].

Definition 2.1. The *set partition lattice* $\mathcal{P}(W)$ is the set of linear subspaces of V that are an intersection of reflecting hyperplanes. It is ordered by reverse inclusion, i.e. $\pi \leq \rho$ if $\rho \subset \pi$.

Remark 2.2. We are mostly interested in the case where V is the standard geometric representation of a Coxeter group W. In this case, $\{0\} \in \mathcal{P}(W)$ and it is the maximal element. But in what follows, it will also be convenient to consider some reflection subgroup $U \subset W$. The definition is still valid and gives a subset $\mathcal{P}(U) \subset \mathcal{P}(W)$, and $\{0\} \notin \mathcal{P}(U)$ *a priori*.

In the case A_n of the classification, W is the symmetric group $\mathfrak{S}_{n+1}$ acting on $V = \{v \in \mathbb{R}^{n+1} : \sum v_i = 0\}$ by permuting coordinates. The reflecting hyperplanes are $H_{i,j} = \{v \in V : v_i = v_j\}$ where $i < j$. We recover the traditional definition of a set partition, for example if $n = 6$ then $H_{1,7} \cap H_{2,4} \cap H_{4,5} = \{v \in V : v_1 = v_7,\ v_2 = v_4 = v_5\} \in \mathcal{P}(A_6)$ corresponds to the set partition $17|245|3|6$.

Let $t \in W$ be a reflection, and $H = \mathrm{Fix}(t)$ be its fixed point set. Then $w(H) = \mathrm{Fix}(wtw^{-1})$ for $w \in W$. So the natural action of W on linear subspaces of V gives an action of W on the reflecting hyperplanes, and on $\mathcal{P}(W)$. Since inclusion and rank are preserved, this extends to an action on the maximal chains in $\mathcal{P}(W)$.

Definition 2.3. Let $\mathcal{M}(W)$ denote the set of maximal chains in $\mathcal{P}(W)$, i.e. sequences $C = (C_0, \dots, C_n) \in \mathcal{P}(W)^{n+1}$ where $C_0 < \cdots < C_n$ (this implies that C_i

has rank i). We define an integer $K(W)$ as the number of orbits for the W-action on $\mathcal{M}(W)$, i.e. $K(W) = \#(\mathcal{M}(W)/W)$.

An element of $\mathcal{M}(W)$ can be seen as a complete flag of V. Thus we can rephrase the definition: $K(W)$ is the number of W-orbits of complete flags in V where each element of the flag is a fixed point subspace of some $w \in W$.

Let us introduce further notations (see [6, 12]). We recall that the complement in V of the reflecting hyperplanes is divided into connected regions called *chambers*, and W acts simply transitively on the chambers. Let $H_1, \dots, H_n$ be the reflecting hyperplanes that enclose one particular chamber R_0, the fondamental chamber. Then the corresponding orthogonal reflections $s_1, \dots, s_n$ form a set S of simple generators for W. According to this choice, there is a Bruhat order on W and a longest element w_0. For any i, let $W_{(i)} \subset W$ be the (standard maximal parabolic) subgroup generated by the s_j with $j \neq i$. If $s \in S$, we also denote $W_{(s)} = W_{(i)}$ if $s = s_i$. An alternative description is that, if we define a line

$$L_i = \bigcap_{\substack{1 \leq j \leq n \\ j \neq i}} H_j, \tag{2}$$

then $w \in W_{(i)}$ if and only if $w(v) = v$ for any $v \in L_i$. The lines L_i are exactly those in $\mathcal{P}(W)$ that are incident to the fondamental chamber R_0.

For each line $L \in \mathcal{P}(W)$, we define two subgroups of W, respectively the stabilizer and the pointwise stabilizer:

$$\mathrm{Stab}(L) = \big\{ w \in W \,:\, w(L) = L \big\},$$
$$\mathrm{Stab}^*(L) = \big\{ w \in W \,:\, \forall x \in L,\, w(x) = x \big\}.$$

Note that $\mathrm{Stab}^*(L)$ is a subgroup of $\mathrm{Stab}(L)$ with index either 1 or 2. The group $\mathrm{Stab}^*(L)$ is generated by the reflections it contains and is itself a real reflection group, its reflecting hyperplanes being those of W containing L. So we can identify $\mathcal{P}(\mathrm{Stab}^*(L))$ with the interval $[V, L] \subset \mathcal{P}(W)$.

3. The general method

We describe how the integer $K(W)$ can be computed inductively. To begin, in the reducible case we have:

Proposition 3.1. *Let W_1 and W_2 be two Coxeter groups of respective ranks m and n, then*

$$K(W_1 \times W_2) = \binom{m+n}{m} K(W_1) K(W_2).$$

Proof. First, note that there is natural identification $\mathcal{P}(W_1) \times \mathcal{P}(W_2) = \mathcal{P}(W_1 \times W_2)$. Let $(x_0, \dots, x_m) \in \mathcal{M}(W_1)$ and $(y_0, \dots, y_n) \in \mathcal{M}(W_2)$. By elementary properties of the product order, we can form an element $C \in \mathcal{M}(W_1 \times W_2)$ by considering a sequence

$$C = ((x_{i_0}, y_{j_0}), \dots, (x_{i_{m+n}}, y_{j_{m+n}}))$$

where the indices are such that $i_0 = j_0 = 0$, $i_{m+n} = m$, $j_{m+n} = n$, and for $0 \leq k < m+n$:

- either $i_{k+1} = i_k$ and $j_{k+1} = j_k + 1$,
- or $i_{k+1} = i_k + 1$ and $j_{k+1} = j_k$.

If I denotes the set of possible choices for the indices i_k and j_k, this defines a bijection

$$I \times \mathcal{M}(W_1) \times \mathcal{M}(W_2) \to \mathcal{M}(W_1 \times W_2).$$

Since the bijection commutes with the action of $W_1 \times W_2$ and $\#I = \binom{m+n}{m}$, the result is proved. ☐

We suppose now that W is irreducible. A natural approach to find $K(W)$ is to distinguish the maximal chains according to the coatom they contain (in terms of complete flags, we distinguish them according to the line they contain). Doing the same thing at the level of orbits will lead to Proposition 3.2 below.

Recall that we can identify $\mathcal{P}(\mathrm{Stab}^*(L))$ with $[V, L] \subset \mathcal{P}(W)$. There is also a natural way to see $\mathcal{M}(\mathrm{Stab}^*(L))$ as a subset of $\mathcal{M}(W)$, namely $(C_0, \dots, C_{n-1}) \in \mathcal{M}(\mathrm{Stab}^*(L))$ is identified with $(C_0, \dots, C_{n-1}, \{0\})$. Clearly, $[V, L]$ is stable by the action of $\mathrm{Stab}(L)$ and this extends to an action of $\mathrm{Stab}(L)$ on $\mathcal{M}(\mathrm{Stab}^*(L))$. With this at hand, we have:

Proposition 3.2. *Let $\mathcal{L} \subset \mathcal{P}(W)$ be a set of orbit representatives for the action of W on lines in $\mathcal{P}(W)$, then:*

$$K(W) = \sum_{L \in \mathcal{L}} \#\big(\mathcal{M}(\mathrm{Stab}^*(L)) / \mathrm{Stab}(L)\big).$$

Proof. If $C = (C_0, \dots, C_n) \in \mathcal{M}(W)$, there is a unique $L \in \mathcal{L}$ such that the coatom C_{n-1} and L are in the same W-orbit. Moreover, L only depends on the W-orbit of C, so this defines a map $f : \mathcal{M}(W)/W \to \mathcal{L}$.

With the discussion above in mind, we identify $\mathcal{M}(\mathrm{Stab}^*(L))$ with the set of chains $C = (C_0, \dots, C_n) \in \mathcal{M}(W)$ satisfying $C_{n-1} = L$. Each element of $f^{-1}(L)$ is a W-orbit that can be represented by an element of $\mathcal{M}(\mathrm{Stab}^*(L))$, and two elements of $\mathcal{M}(\mathrm{Stab}^*(L))$ are in the same W-orbit if and only if they are in the same $\mathrm{Stab}(L)$-orbit. This permits to define a bijection between $f^{-1}(L)$ and $\mathcal{M}(\mathrm{Stab}^*(L)) / \mathrm{Stab}(L)$. Now, we can write:

$$K(W) = \#(\mathcal{M}(W)/W) = \sum_{L \in \mathcal{L}} \#(f^{-1}(L)) = \sum_{L \in \mathcal{L}} \#\big(\mathcal{M}(\mathrm{Stab}^*(L)) / \mathrm{Stab}(L)\big),$$

as announced. ☐

Now, let us describe how to find the set $\mathcal{L}$ of orbit representatives for the action of W on lines in $\mathcal{P}(W)$. We can use the lines L_i defined in Equation (2) from the previous section.

Proposition 3.3. *Each line $L \in \mathcal{P}(W)$ can be written $w(L_i)$ for some $w \in W$ and $1 \leq i \leq n$. Let $w \in W$ and $i \neq j$, then $w(L_i) = L_j$ implies $w_0(L_i) = L_j$.*

Similar considerations appeared in the work of Armstrong, Reiner and Rhoades [4], in the context of W-parking functions. Still, it is reasonable to include a short proof here.

Proof. Let us split the line L in two half-lines L^+ and L^-, and let R be a chamber incident to L^+. We also split L_i in two half-lines L_i^+ and L_i^-, where L_i^+ is the one incident to R_0. The group W acts simply transitively on the chambers, so there is $w \in W$ such that $w(R_0) = R$. Then $w^{-1}(L^+)$ is incident to R_0, so there is i such that $w^{-1}(L^+) = L_i^+$, and consequently $L^+ = w(L_i^+)$ and $L = w(L_i)$.

Now, suppose we have $i \neq j$ and $w(L_i) = L_j$. We have either $w(L_i^+) = L_j^+$ or $w(L_i^+) = L_j^-$ (where L_j^+ and L_j^- are defined in the same way as with L_i). In the first case, R_0 and $w(R_0)$ are both incident to L_j^+. This implies $w(L_j^+) = L_j^+$ (note that $W_{(j)}$ acts simply transitively on the set of chambers incident to L_j^+), but this is a contradiction with $i \neq j$ and $w(L_i) = L_j$. So we have $w(L_i^+) = L_j^-$. Since L_j^- is incident to both $-R_0$ and $w(R_0)$, there is $u \in W_{(j)}$ such that $uw(R_0) = -R_0$, i.e. $uw = w_0$. Then, we have $w_0(L_i^+) = uw(L_i^+) = u(L_j^-) = L_j^-$. So $w_0(L_i) = L_j$. □

From the definition of L_i in Equation (2), $w_0(L_i) = L_j$ is equivalent to $w_0(H_i) = H_j$, which is also equivalent to $w_0 s_i w_0 = s_j$. Elementary properties of the longest element show that the map defined on the simple generators by $s \mapsto w_0 s w_0$ is an involutive automorphism of the Coxeter graph. One can also show that this automorphism is the identity if and only if the exponents of the group are all odd, see [6, Exercise 4.10]. So the set $\mathcal{L}$ can be obtained by taking $\{L_1, \dots, L_n\}$, quotiented by the action of w_0 which can be described in a precise way.

We have $\mathrm{Stab}^*(L_i) = W_{(i)}$, the standard maximal parabolic subgroup. To identify the group $\mathrm{Stab}(L_i)$, we have the following:

Proposition 3.4. *Either* $\mathrm{Stab}(L_i) = W_{(i)}$, *or* $\mathrm{Stab}(L_i) = < W_{(i)}, w_0 >$.

Proof. Suppose there is $w \in \mathrm{Stab}(L_i)$ with $w \notin W_{(i)}$, which means that $w(L_i^+) = L_i^-$. So $w(R_0)$ is incident to L_i^-. Since $W_{(i)}$ acts transitively on the chambers incident to L_i^-, there is $u \in W_{(i)}$ with $uw(R_0) = -R_0$, i.e. $uw = w_0$. It follows $w_0 \in \mathrm{Stab}(L_i)$ with $w_0 \notin W_{(i)}$. □

Since $W_{(i)}$ has rank $n-1$, by induction we can assume we already know the integer $K(W_{(i)})$, which is useful in some situations.

Proposition 3.5. *With W, w_0, and $L_i \in \mathcal{L}$ as above, we have:*

- *If $w_0 s_i w_0 \neq s_i$, then*
$$\#(\mathcal{M}(W_{(i)})/\,\mathrm{Stab}(L_i)) = K(W_{(i)}).$$
- *If $w_0 s_i w_0 = s_i$, and there is $u \in W_{(i)}$ such that $w_0 s_j w_0 = u s_j u$ for any $j \neq i$, then*
$$\#(\mathcal{M}(W_{(i)})/\,\mathrm{Stab}(L_i)) = K(W_{(i)}).$$
- *If $w_0 s_i w_0 = s_i$, and the map $s \mapsto w_0 s w_0$ permutes nontrivially the connected components of the Coxeter graph of $W_{(i)}$, then:*
$$\#(\mathcal{M}(W_{(i)})/\,\mathrm{Stab}(L_i)) = \frac{1}{2} K(W_{(i)}).$$

Proof. If $w_0 s_i w_0 \neq s_i$, then $w_0 \notin \mathrm{Stab}(L_i)$, hence $\mathrm{Stab}(L_i) = W_{(i)}$ using Proposition 3.4. This proves the first point.

Suppose $w_0 s_i w_0 = s_i$ and there exists u as above. It means that the action of u on $\mathcal{M}(W_{(i)})$ is the same as the action of w_0. In either of the two cases given in Proposition 3.4, we find that the $\mathrm{Stab}(L_i)$-orbits are exactly the $W_{(i)}$-orbits. This proves the second point.

As for the third point, we suppose there are only two connected components in the Coxeter graph of $W_{(i)}$, the general case being similar. Let us write $W_{(i)} = W_1 \times W_2$. We have seen in the proof of Proposition 3.1 that the elements of $\mathcal{M}(W_{(i)})$ are obtained by "shuffling" two elements of $\mathcal{M}(W_1)$ and $\mathcal{M}(W_2)$. So if

$C = (C_0, \dots, C_{n-1}) \in \mathcal{M}(W_{(i)})$, the element C_1 is a pair $(C_1', C_1'') \in \mathcal{P}(W_1) \times \mathcal{P}(W_2)$ where the respective ranks of C_1' and C_1'' are either 0 and 1, or 1 and 0. These two conditions are preserved by the action of $W_{(i)}$, and are reversed by the action of w_0. So the action of w_0 on $\mathcal{M}(W_{(i)})/W_{(i)}$ has no fixed point and each orbit has cardinality 2. We can write:

$$\#(\mathcal{M}(W_{(i)})/\operatorname{Stab}(L_i)) = \#((\mathcal{M}(W_{(i)})/W_{(i)})/w_0)$$

and this proves the result. □

Let us summarize the situation. If w_0 is central in W, we can always apply the second case of Proposition 3.5, so that Proposition 3.2 gives

$$K(W) = \sum_{s \in S} K(W_{(s)}), \tag{3}$$

where each $W_{(s)}$ is a standard maximal parabolic subgroup of W. Furthermore, some of the terms are simplified using the product formula in Proposition 3.1. In particular, this equation can be directly obtained from the Coxeter graph.

When w_0 is not central, the map $s \mapsto w_0 s w_0$ is an involution on the set S of simple generators and we need to distinguish the two-element orbits and the fixed points. Indeed, we have:

$$K(W) = \sum_{\substack{\{s_i,s_j\} \subset S,\ s_i \neq s_j \\ w_0 s_i w_0 = s_j}} K(W_{(i)}) + \sum_{\substack{s_i \in S \\ w_0 s_i w_0 = s_i}} \#(\mathcal{M}(W_{(i)})/\operatorname{Stab}(L_i)). \tag{4}$$

Some terms in the first sum (respectively, the second sum) can be further simplified using Proposition 3.1 (respectively, Proposition 3.5).

Note that Proposition 3.5 does not exhaust all the possibilities, so we do not have a general solution to find all the terms $\#(\mathcal{M}(W_{(i)})/\operatorname{Stab}(L_i))$ in the second sum of Equation (4). As we will see in the next section, the only case that cannot be treated directly will appear when $W = D_n$ with n odd.

4. The case by case resolution

We follow the traditional notation for the classification of finite irreducible Coxeter groups, see [6]. We will denote $a_n = K(A_n)$, $b_n = K(B_n)$, $d_n = K(D_n)$. It will be convenient to take the conventions that $A_0 = B_0 = D_0$ (the trivial group with rank 0), $A_1 = B_1$, $D_2 = A_1 \times A_1$ and $D_3 = A_3$.

Proposition 4.1 (See [6], Exercise 4.10)**.** *In the groups $I_2(m)$ for m even, B_n, D_n for n even, G_2, H_3, H_4, E_7, and E_8, the longest element is central. In the other groups, i.e. $I_2(m)$ for m odd, A_n, D_n for n odd, and E_6, the map $s \mapsto w_0 s w_0$ is the unique nontrivial automorphism of the Coxeter graph.*

4.1. **Case of A_n.** We already know that $a_n = T_n$, but let us check how to prove it with our method. Here, w_0 is not central and $s \mapsto w_0 s w_0$ reverses the n vertices of the Coxeter graph. There is a fixed point only if n is odd, and it can be treated using the third case of Proposition 3.5. So Equation (4) gives, when $n \geq 2$:

$$a_n = \sum_{i=0}^{\lfloor n/2 \rfloor - 1} \binom{n-1}{i} a_i a_{n-1-i} + [n \bmod 2] \frac{1}{2} \binom{n-1}{(n-1)/2} a_{(n-1)/2}^2.$$

This can be rewritten as:

$$a_n = \frac{1}{2}\sum_{i=0}^{n-1}\binom{n-1}{i}a_i a_{n-1-i}. \tag{5}$$

Let us define

$$A(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!}.$$

Multiplying Equation (5) by $\frac{z^{n-1}}{(n-1)!}$ and summing over $n \geq 2$ gives

$$A'(z) - 1 = \frac{1}{2}(A(z)^2 - 1).$$

So $A(z)$ is the solution of the differential equation $A'(z) = \frac{1}{2}(A(z)^2+1)$ with the initial value $A(0) = 1$. It can be checked that $A(z) = \tan(z)+\sec(z)$ is the solution, so that $a_n = T_n$.

4.2. **Case of B_n.** In this group, the longest element is central. Equation (3) together with the product formula gives:

$$b_n = \sum_{i=0}^{n-1}\binom{n-1}{i}b_i a_{n-i-1}. \tag{6}$$

Now, let

$$B(z) = \sum_{n\geq 0} b_n \frac{z^n}{n!}.$$

Multiplying Equation (6) by $\frac{z^{n-1}}{(n-1)!}$ and summing over $n \geq 1$ gives

$$B'(z) = B(z)A(z).$$

So $B(z)$ is the solution of the differential equation $B'(z) = B(z)A(z)$ with initial value $B(0) = 1$. We can check that

$$B(z) = \frac{1}{1-\sin(z)}$$

is a solution. This function also satisfies $B(z) = A'(z)$, so that

$$b_n = T_{n+1}.$$

A bijective proof of this will be given at the end of Section 8.

4.3. **Case of D_n.** When n is even, the longest element of D_n is central and Equation (3) gives:

$$d_n = 2a_{n-1} + \sum_{2\leq i\leq n-1}\binom{n-1}{i}d_i a_{n-1-i}. \tag{7}$$

Now, suppose n is odd, writing the equation is not quite immediate in this case. The map $s \mapsto w_0 s w_0$ exchanges two vertices of the Coxeter graph, and this gives one term a_{n-1} coming from the first sum in Equation (4). As for the second sum, we are in the case where $s_i = w_0 s_i w_0$, and $W_{(i)} = D_i \times A_{n-1-i}$. If i is odd, we can apply the second case of Proposition 3.5 where u is chosen to be the longest element of the factor D_i. More care is needed when i is even, i.e. when we cannot directly apply Proposition 3.5. So we consider the set $\mathcal{M}(D_i \times A_{n-1-i})$, quotiented by $D_i \times A_{n-1-i}$, and further quotiented by the graph automorphism of the factor

D_i (the graph automorphism induces an action on $\mathcal{P}(D_i)$). An argument similar to the one in Proposition 3.1 shows that the number of orbits can be factorized. Eventually, we obtain:

$$d_n = a_{n-1} + \sum_{\substack{2\leq i\leq n-1\\ i \text{ odd}}} \binom{n-1}{i} d_i a_{n-1-i} + \sum_{\substack{2\leq i\leq n-1\\ i \text{ even}}} \binom{n-1}{i} \bar{d}_i a_{n-1-i}, \tag{8}$$

where $\bar{d}_i$ is defined as follows: it is the number of orbits for the action on $\mathcal{M}(D_i)$ generated by D_i together with the graph automorphism (except that if $i = 4$, the graph automorphism is not unique but we only consider the one that exchanges two vertices). Note that for odd i, we can define $\bar{d}_i$ similarly but it is clear that $\bar{d}_i = d_i$. We need to compute $\bar{d}_n$ before solving the recursion for d_n.

Proposition 4.2. *We have $\bar{d}_0 = 1$ and for any $n \geq 1$,*

$$\bar{d}_n = a_{n-1} + \sum_{i=2}^{n-1} \binom{n-1}{i} \bar{d}_i a_{n-1-i}. \tag{9}$$

Proof. Although we cannot directly apply Proposition 3.2 and Proposition 3.1, the argument is completely similar, so we omit details. Let Γ denote the graph automorphism of D_n.

Suppose L_1 and L_2 are the two coatoms that are exchanged by Γ. Counting orbits of maximal chains having L_1 or L_2 as coatom, we obtain the first term a_{n-1}.

If $i \neq 1, 2$, the number of orbits of maximal chains having L_i as coatom is the number of orbits in $\mathcal{M}(W_{(i)})/ < \mathrm{Stab}(L_i), \Gamma >$. This is also the number of orbits in $\mathcal{M}(W_{(i)})/ < W_{(i)}, \Gamma >$, since either $\mathrm{Stab}(L_i) = W_{(i)}$ or $\mathrm{Stab}(L_i) =< W_{(i)}, w_0 >$ where w_0 has the same action as Γ. We have a decomposition $W_{(i)} = D_i \times A_{n-1-i}$ and the graph automorphism only acts on the factor D_i. So the argument of Proposition 3.1 shows that this number is $\bar{d}_i a_{n-1-i}$. □

Proposition 4.3. *If $n \geq 2$, we have $\bar{d}_n = 2a_{n+1} - (n+1)a_n$.*

Proof. The recursion in the previous proposition shows that the generating function $\bar{D}(z) = \sum_{n\geq 0} \bar{d}_n \frac{z^n}{n!}$ satisfies the differential equation

$$\bar{D}'(z) = (\bar{D}(z) - z)A(z),$$

with the initial condition $\bar{D}(0) = 1$. This is solved by

$$\bar{D}(z) = \frac{2 - \cos(z) - z\sin(z)}{1 - \sin(z)}. \tag{10}$$

From this expression, we can get $\bar{D}(z) = (2 - z)A'(z) + z - A(z)$, and it follows that $\bar{d}_n = 2a_{n+1} - (n+1)a_n$ if $n \geq 2$. □

Proposition 4.4. *Let $n \geq 2$. We have $d_n - \bar{d}_n = a_n$ if n is even, and $d_n = \bar{d}_n$ otherwise.*

Proof. If $n \geq 2$, from (7), (8), and (9), we have:

$$d_n - \bar{d}_n = \chi[n \text{ even}]a_{n-1} + \sum_{\substack{2\leq i\leq n-1\\ n-i \text{ even}}} \binom{n-1}{i} (d_i - \bar{d}_i) a_{n-1-i}.$$

Here and in the sequel, χ means 1 or 0 depending on whether the condition within brackets is true or false. So the generating function

$$U(z) = 1 + \sum_{n\geq 2} (d_n - \bar{d}_n) \frac{z^n}{n!}$$

satisfies $U'(z) = U(z)\tan(z)$ and $U(0) = 1$. This is solved by $U(z) = \sec(z)$ and the result follows. ☐

From the previous two propositions, we get that for $n \geq 2$,

$$d_n = \begin{cases} 2T_{n+1} - nT_n & \text{if } n \text{ is even,} \\ 2T_{n+1} - (n+1)T_n & \text{if } n \text{ is odd.} \end{cases}$$

From (10), we can separate the odd and even parts of $\bar{D}(z)$ (multiply the numerator and denominator by $1 + \sin(z)$ and separate terms in the numerator). After some calculation, this leads to:

$$\sum_{n\geq 1} d_{2n} \frac{z^{2n}}{(2n)!} = \frac{\sin(z)(2\sin(z) - z)}{\cos(z)^2},$$

and

$$\sum_{n\geq 1} d_{2n+1} \frac{z^{2n+1}}{(2n+1)!} = \frac{\sin(z)(2 - \cos(z)) - z}{\cos(z)^2}.$$

We can take the sum of these two equations to obtain $\sum_{n\geq 2} d_n \frac{z^n}{n!}$, but there seems to be no particular simplification. The first values of $\bar{d}_n$ for $n \geq 2$ are as follows:

$$1, 2, 7, 26, 117, 594, 3407, 21682, 151853, 1160026, 9600567...$$

And the first values of d_n for $n \geq 2$ are:

$$2, 2, 12, 26, 178, 594, 4792, 21682, 202374, 1160026, 12303332, ...$$

4.4. **Remaining cases.** For the dihedral group, we have:

$$K(I_2(m)) = \begin{cases} 1 \text{ if } m \text{ is odd,} \\ 2 \text{ if } m \text{ is even.} \end{cases}$$

Among the exceptional groups, E_6 is the only one where the longest element is not central. We apply Equation (4) and the calculation is the following:

$$\begin{aligned} K(E_6) &= K(D_5) + K(A_4 \times A_1) + \frac{1}{2} K(A_2 \times A_1 \times A_2) + K(A_5) \\ &= 26 + 25 + 15 + 16 = 82. \end{aligned}$$

The first two terms correspond to the terms where $s_i \neq s_j$ and $w_0 s_i w_0 = s_j$. The third term corresponds to a fixed point of the graph automorphism, the vertex of degree 3. It is treated using the second part of Proposition 3.5. The fourth term corresponds to the other fixed point of the graph automorphism, it is treated using the first part of Proposition 3.5.

For all the remaining groups, the longest element is central and we can apply Equation (3). This gives:

$$\begin{aligned} K(H_3) &= K(I_2(5)) + K(A_1 \times A_1) + K(A_2) = 4, \\ K(H_4) &= K(H_3) + K(I_2(5) \times A_1) + K(A_2 \times A_1) + K(A_3) = 12, \\ K(F_4) &= K(B_3) + K(A_2 \times A_1) + K(A_1 \times A_2) + K(B_3) = 16. \end{aligned}$$

Eventually, we have:

$$\begin{aligned} K(E_7) = K(E_6) &+ K(D_5 \times A_1) + K(A_4 \times A_2) + \\ &K(A_3 \times A_1 \times A_2) + K(A_1 \times A_5) + K(D_6) + K(A_6) \\ &= 82 + 156 + 75 + 120 + 96 + 178 + 61 = 768, \end{aligned}$$

and

$$\begin{aligned} K(E_8) = K(E_7) &+ K(E_6 \times A_1) + K(D_5 \times A_2) + K(A_4 \times A_3) + \\ &K(A_2 \times A_1 \times A_4) + K(A_6 \times A_1) + K(D_7) + K(A_7) \\ &= 768 + 574 + 546 + 350 + 525 + 427 + 594 + 272 = 4056. \end{aligned}$$

5. A more general question

Let G be any subgroup of $GL(\mathbb{R}^n)$. We can define a set

$$\mathcal{P}(G) = \big\{ \pi \subset \mathbb{R}^n \,:\, \exists g \in G,\, \pi = \mathrm{Fix}(g) \big\},$$

and let $\mathcal{M}(G)$ denote the set of complete flags $\pi_0 \subset \pi_1 \subset \cdots \subset \pi_n$ where $\dim \pi_i = i$ and $\pi_i \in \mathcal{P}(G)$ for each i. The group G acts on $\mathcal{P}(G)$ and $\mathcal{M}(G)$, let $K(G) = \#(\mathcal{M}(G)/G)$. We have examined the case where G is a finite reflection group but we see that the definition is valid in this more general context.

Suppose for example that $G = T_n$, the set of invertible upper-triangular matrices. Then $\mathcal{P}(T_n)$ is the set of all linear subspaces of $\mathbb{R}^n$, as can be seen using the LU decomposition. So $\mathcal{M}(T_n)$ is the complete flag variety $GL(\mathbb{R}^n)/T_n$. Using the Bruhat decomposition, we see that $K(T_n) = n!$.

It might be of interest to examine the case of other groups.

Part 2. Noncrossing partitions, interval partitions, hook formulas

The results in this section relies on a property of standard Coxeter elements (Proposition 7.4) that we have not proved in full generality but can be checked in most cases. So we require here that W is not of type E_7, E_8, H_3 or H_4 (see Appendix A for details).

6. Definitions

In the classical case, an interval partition is a set partition where each block is a set of consecutive integers, for example $123|4|56$. When we a have a real reflection group $W \subset GL(V)$ together with a choice of simple generators $s_1, \dots, s_n$ and the associated simple hyperplanes $H_1, \dots, H_n$, there is a natural generalization (which might have been considered in previous work, with different terminology).

Definition 6.1. An element $\pi \in \mathcal{P}(W)$ is an *interval partition* if it is an intersection simple hyperplanes. Let $\mathcal{P}^I(W) \subset \mathcal{P}(W)$ denote the set of interval partitions, and $\mathcal{M}^I(W) \subset \mathcal{M}(W)$ denote the set of maximal chains in $\mathcal{P}^I(W)$.

The set $\mathcal{P}^I(W)$ is a sublattice of $\mathcal{P}(W)$ and is isomorphic to a boolean lattice. It follows that $\mathcal{M}^I(W)$ has cardinality $n!$. The coatoms of $\mathcal{P}^I(W)$ are exactly the lines $L_1, \dots, L_n$ defined in Equation (2). Besides, a consequence of Proposition 3.3 is the following:

Proposition 6.2. *Each orbit $O \in \mathcal{M}(W)/W$ contains an element of $\mathcal{M}^I(W)$.*

Proof. Let $C \in O$. Using Proposition 3.3, there exists $w \in W$ such that the coatom L in the chain $w(C)$ is an interval partition, i.e. L is one the L_i previously defined. We have seen that $\mathcal{M}(W_{(i)})$ is in bijection with the chains in $\mathcal{M}(W)$ having L_i as coatom. Clearly, this bijection sends $\mathcal{M}^I(W_{(i)})$ to the chains in $\mathcal{M}^I(W)$ having L_i as coatom. So we can make an induction on the rank and assume there is $u \in W_{(i)}$ such that $uw(C) \in \mathcal{M}^I(W)$, whence the result. $\square$

Let us motivate the next definition by some considerations in the "classical" case. Let π_1, π_2, π_3 be the noncrossing partitions represented in Figure 1 from left to right. Here, π is represented by drawing an arch between two consecutive elements of each block. Both π_2 and π_3 are covered by π_1, and more precisely they are obtained from π_1 by splitting the block $\{1,2,5,7\}$. But we can make one distinction: π_2 is obtained by removing one arch from π_1, and its two blocks $\{1,2\}$ and $\{5,7\}$ form an interval partition of the block $\{1,2,5,7\}$ of π_1. This is not the case for π_3.

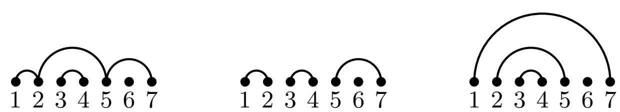

FIGURE 1. Noncrossing partitions.

To generalize this distinction, consider the group $\mathrm{Stab}^*(\pi_1) \subset \mathfrak{S}_7$. It has an irreducible factor $\mathfrak{S}_4$ acting on the block $\{1,2,5,7\}$. The simple roots of $\mathfrak{S}_7$ are $e_1 - e_2, \dots, e_6 - e_7$ where $(e_i)_{1 \le i \le 7}$ is the standard basis of $\mathbb{R}^7$. The ones of the irreducible factor $\mathfrak{S}_4$ of $\mathrm{Stab}^*(\pi_1)$ are $e_1 - e_2$, $e_2 - e_5$, $e_5 - e_7$. It can be seen that the simple roots of $\mathrm{Stab}^*(\pi_2)$ are included in the ones of $\mathrm{Stab}^*(\pi_1)$, but it is not the case for π_3.

Let us turn to the general case. Let Φ be a root system of W (in the sense of Coxeter groups, see [12]), and let Φ^+ be a choice of positive roots. For each $\pi \in \mathcal{P}(W)$, the group $\mathrm{Stab}^*(\pi)$ is a reflection subgroup of W, and its set of roots is $\Phi \cap \pi^\perp$. We will always take $\Phi^+ \cap \pi^\perp$ as a natural choice of positive root, and accordingly $\mathrm{Stab}^*(\pi)$ has a natural choice of simple roots and simple generators. In this setting, we have the following:

Definition 6.3. Let $\pi_1, \pi_2 \in \mathcal{P}(W)$, we denote $\pi_2 \sqsubseteq \pi_1$ and say that π_2 is an *interval refinement* of π_1 if the simple roots of $\mathrm{Stab}^*(\pi_2)$ are included in the simple roots of $\mathrm{Stab}^*(\pi_1)$.

Note that $\pi_2 \sqsubseteq \pi_1$ implies $\pi_1 \subset \pi_2$, i.e. $\pi_2 \le \pi_1$ in the lattice $\mathcal{P}(W)$. Also, interval partitions are exactly the interval refinements of the maximal partition.

Some preliminary definitions are needed before going to noncrossing partitions.

Definition 6.4. Let $T \subset W$ be the set of reflections. A *reduced T-word* of w is a factorization $w = t_1 \dots t_k$ where $t_1, \dots, t_k \in T$ and k is minimal. Let $u, v \in W$, the *absolute order* is defined by the condition that $u <_{abs} v$ if some reduced T-word of u is a subword of some reduced T-word of v.

Definition 6.5. If $\sigma \in \mathfrak{S}_n$, we call $c = s_{\sigma(1)} \dots s_{\sigma(n)}$ a *standard Coxeter element* of W with respect to S. Any element conjugated in W to a standard Coxeter element is called a *Coxeter element*.

This might differ from the terminology used in other references, but we need here some properties of the standard Coxeter elements that are not true in general. In what follows, we always assume that c is a standard Coxeter element.

Definition 6.6. A set partition $\pi \in \mathcal{P}(W)$ is *noncrossing* with respect to c if $\pi = \mathrm{Fix}(w)$ for some $w \in W$ such that $w <_{abs} c$. This w is actually unique and will be denoted $\underline{\pi}$. Let $\mathcal{P}^{NC}(W,c) \subset \mathcal{P}(W)$ denote the sublattice of noncrossing partitions with respect to c, and $\mathcal{M}^{NC}(W,c) \subset \mathcal{M}(W)$ denote the set of maximal chains of $\mathcal{P}^{NC}(W,c)$. If $\pi \in \mathcal{P}^{NC}(W,c)$, it is the Coxeter element of a unique parabolic subgroup of W that we denote $W_{(\underline{\pi})}$ or $W_{(\pi)}$ (although this interfers with the notation $W_{(s)}$ for maximal standard parabolic subgroup, there should be no confusion).

Note in particular that $\mathrm{Fix}(\underline{\pi}) = \pi$. We refer to [3] for more on the subject of noncrossing partitions. In general, $\mathcal{P}^{NC}(W,c)$ is not stable under the action of W. But from the invariance of the absolute order under conjugation, we can see that $\mathcal{P}^{NC}(W,c)$ is stable under the action of c.

Remark 6.7. Noncrossing partitions are usually defined as a subset of W, but here it is natural to have the inclusion $\mathcal{P}^{NC}(W,c) \subset \mathcal{P}(W)$. These two points of view are equivalent under the correspondence $\underline{\pi} \leftrightarrow \pi$ and we will also allow to identify noncrossing partitions with a subset of W. For example, if $u, v \in W$ are noncrossing, the notion of interval refinement $u \sqsubseteq v$ is well defined, and $u \in W$ is called an interval partition if it is so as a noncrossing partition.

Proposition 6.8. *We have $\mathcal{P}^I(W) \subset \mathcal{P}^{NC}(W,c)$. Let $\pi_1 \in \mathcal{P}^{NC}(W,c)$ and $\pi_2 \in \mathcal{P}(W)$ with $\pi_2 \sqsubseteq \pi_1$, then $\pi_2 \in \mathcal{P}^{NC}(W,c)$.*

Proof. The maximal partition is noncrossing since $\{0\} = \mathrm{Fix}(c)$, so the first point follows the second one.

To prove the second point, we need Proposition 7.4 from the next section. Let $r_1, \dots, r_k$ be the reflections associated with the simple roots of $\pi_1^\perp$, and we can assume there is $j \leq k$ such that $r_1, \dots, r_j$ are the reflections associated with the simple roots of $\pi_2^\perp$. Since π_1 is noncrossing, it means there is $u \in W$ with $u <_{abs} c$ and $\mathrm{Fix}(u) = \pi_1$. It is known that u is a Coxeter element of the subgroup $\mathrm{Stab}^*(\pi_1) \subset W$. But Proposition 7.4 shows more: it is a standard Coxeter element, so there is $\sigma \in \mathfrak{S}_k$ such that $u = r_{\sigma(1)} \dots r_{\sigma(k)}$. Let v be obtained from this factorization by keeping only the factors $r_1, \dots, r_j$. Then, we have $v <_{abs} u <_{abs} c$ and $\mathrm{Fix}(v) = \pi_2$, so π_2 is noncrossing. □

Remark 6.9. It is interesting to note that similar results hold for *nonnesting partitions* in the sense of Postnikov (defined only in the crystallographic case). A set partition $\pi \in \mathcal{P}(W)$ is nonnesting when the simple roots of $\mathrm{Stab}^*(\pi)$ form an antichain in the poset of positive roots. A subset of an antichain being itself an antichain, if $\pi_2 \sqsubseteq \pi_1$ and π_1 is nonnesting, then π_2 is nonnesting. Any interval partition is nonnesting, since the simple roots form an antichain. Note also that the intuition from the "classical" case is clear: it is impossible to create a crossing or a nesting by removing arches.

7. Chains of noncrossing partitions

Definition 7.1. For any chain $\Pi = (\pi_0, \dots, \pi_n) \in \mathcal{M}^{NC}(W,c)$, let $\mathrm{nir}(\Pi)$ be the number of i such that π_i is not an interval refinement of π_{i+1}. Let

$$M(W,q) = \sum_{\Pi \in \mathcal{M}^{NC}(W,c)} q^{\mathrm{nir}(\Pi)}.$$

It is not *a priori* obvious that $M(W,q)$ does not depend on the choice of the standard Coxeter element c. This will be proved below.

The coatoms of the lattice $\mathcal{P}^{NC}(W,c)$ are exactly the products ct for $t \in T$. Since T is stable by conjugation, the set cT of coatoms is stable by conjugation by c. An interesting property of standard Coxeter elements is that this action has good properties, similar to those of a bipartite Coxeter element.

In what follows, an orbit for the action of c will be called a *c-orbit*. Note that the action of c becomes conjugation when we see noncrossing partitions as elements of W, i.e. $\underline{c(\pi)} = c\underline{\pi}c^{-1}$ if $\pi \in \mathcal{P}^{NC}(W,c)$.

Proposition 7.2. *Let h be the Coxeter number of W (i.e. the order of c in W). For any $t \in T$, the c-orbit of ct satisfies one of the following condition:*

- *It contains h distinct elements, and exactly 2 interval partitions L_i and L_j, related by $L_i = w_0(L_j)$.*
- *Or it contains $\frac{h}{2}$ distinct elements, and exactly 1 interval partition L_i, satisfying $w_0(L_i) = L_i$. Moreover, $c^{h/2}$ restricted to L_i is -1 (i.e. $c^{h/2} \notin W_{(i)}$).*

The full proof is in Appendix A but let us give some comments. A standard Coxeter element $c = s_{\sigma(1)} \dots s_{\sigma(n)}$ is called *bipartite* if there is j such that $s_{\sigma(1)}, \dots, s_{\sigma(j)}$ are pairwise commuting, and $s_{\sigma(j+1)} \dots s_{\sigma(n)}$ too. Steinberg [21] proved that for a bipartite Coxeter element c, the c-orbit of a reflection contains either h elements and 2 simple reflections, or $\frac{h}{2}$ elements and 1 simple reflection. If h is even, another property of the bipartite Coxeter element is $c^{h/2} = w_0$. What we have is a variant that holds for any standard Coxeter element. It is natural to expect that our result can be seen as a consequence of Steinberg's but we have been unable to realize this in a uniform way.

Since the standard Coxeter element c is conjugated with a bipartite Coxeter element, and the bijection $t \mapsto ct$ from T to cT commutes with c-conjugation, we see that the c-orbit of ct contains either h or $\frac{h}{2}$ elements. In the case where w_0 is central, we can easily complete the proof of Proposition 7.2. It is known that in this case, h is even and $c^{h/2} = w_0 = -1$, which acts trivially on $\mathcal{P}(W)$ (see [12, Section 3.19]). So every orbit has $\frac{h}{2}$ elements. Proposition 3.3 shows that there is at most one interval partition in each orbit, and the equality $\#T = \frac{nh}{2}$ shows that there is exactly one interval partition in each orbit. See Appendix A for the other cases.

Remark 7.3. Suppose h is even and let L_i such that $c^{h/2}(L_i) = L_i$. As mentionned above, we have $c^{h/2} = w_0$ when c is a bipartite Coxeter element. In the general case, since w_0 and $c^{h/2}$ are both in $\mathrm{Stab}(L_i) - \mathrm{Stab}^*(L_i)$, we have $w_0 c^{h/2} \in W_{(i)}$.

From the properties of $x \mapsto w_0 x w_0$, one can deduce that the map $x \mapsto c^{h/2} x c^{h/2}$ permutes the irreducible factors of $W_{(i)}$ in the same way as $x \mapsto w_0 x w_0$. This will be needed in the sequel.

See also Appendix A for the proof of the following result.

Proposition 7.4. *For any $t \in T$, ct is a standard Coxeter element of the parabolic subgroup $W_{(ct)}$ for the natural choice of simple generators (except possibly for some cases in the exceptional groups, as mentioned at the beginning of this section).*

It is known that parabolic Coxeter elements can be characterized with the absolute order, see [5, Lemma 1.4.3], so that ct is a Coxeter element of $W_{(ct)}$. The point of the proposition is that it is actually a standard Coxeter element. Note that since the elements ct are the coatoms of $\mathcal{P}^{NC}(W,c)$, an immediate induction shows that any $\pi \in \mathcal{P}^{NC}(W,c)$ is a standard Coxeter element of $W_{(\pi)}$ for the natural choice of simple generators.

We are now ready to prove how $M(W,q)$ can be computed inductively, and in particular that it does not depend on the choice of a standard Coxeter element.

Proposition 7.5. *We have:*

$$M(W,q) = \frac{2+q(h-2)}{2} \sum_{s \in S} M(W_{(s)}, q). \tag{11}$$

Proof. For each $\Pi = (\pi_0, \dots, \pi_n) \in \mathcal{M}^{NC}(W,c)$, let $\Pi' = (\pi_0, \dots, \pi_{n-1})$. The coatom of Π is $\pi_{n-1} = ct$ for some $t \in T$, and the set of such Π with ct as coatom is in bijection with $\mathcal{M}^{NC}(W_{(ct)}, ct)$ via the map $\Pi \mapsto \Pi'$. Moreover, $\mathrm{nir}(\Pi) = \mathrm{nir}(\Pi')$ if $ct \sqsubseteq c$ (i.e. $ct \in \mathcal{P}^I(W)$) and $\mathrm{nir}(\Pi) = \mathrm{nir}(\Pi') + 1$ otherwise. So, distinguishing the chains in $\mathcal{M}^{NC}(W,c)$ according to their coatoms gives:

$$M(W,q) = \sum_{t \in T} q^{\chi[\, ct \notin \mathcal{P}^I(W)\,]} M(W_{(ct)}, q). \tag{12}$$

Note that to write this equation, we need to use Proposition 7.4. While it should be clear from the definition that the generating function of the chains $(\pi_0, \dots, \pi_{n-1}) \in \mathcal{M}^{NC}(W_{(ct)}, ct)$ with respect to the statistic nir is $M(W_{(ct)}, q)$, this quantity was only defined with respect to a standard Coxeter element. Since ct is indeed a standard Coxeter element of $W_{(ct)}$, we get the term $M(W_{(ct)}, q)$ which we assume we already know by induction.

Let $O \subset T$ be an orbit under conjugation by c. So if $t_1, t_2 \in O$, $W_{(ct_1)}$ and $W_{(ct_2)}$ are conjugated in W, so they are isomorphic and $M(W_{(ct_1)}, q) = M(W_{(ct_2)}, q)$. If $cO = \{co \,:\, o \in O\}$ contains $h/2$ elements and 1 interval partition L_i, we get

$$\sum_{t \in O} q^{\chi[\, ct \notin \mathcal{P}^I(W)\,]} M(W_{(ct)}, q) = (1 + q(\tfrac{h}{2} - 1)) M(W_{(i)}, q). \tag{13}$$

If it contains h elements and 2 interval partitions L_i and L_j, then

$$\sum_{t \in O} q^{\chi[\, ct \notin \mathcal{P}^I(W)\,]} M(W_{(ct)}, q) = (2 + q(h-2)) M(W_{(i)}, q),$$

and since the previous equation is true with i replaced with j, we also have

$$\sum_{t \in O} q^{\chi[\, ct \notin \mathcal{P}^I(W)\,]} M(W_{(ct)}, q) = \frac{2+q(h-2)}{2} (M(W_{(i)}, q) + M(W_{(j)}, q)). \tag{14}$$

Now, we can split the sum in the righ-hand side of (12) to group together the $t \in T$ that are in the same orbit, and using Equations (13) and (14), we get the desired formula for $M(W, q)$. $\square$

Equation (11) permits to make a link with the Fuss-Catalan numbers $\mathrm{Cat}^{(m)}(W)$ (see [3, Chapter 5]). These numbers can be defined in terms of the degrees of the group $d_1, \dots, d_n$ and the Coxeter number $h = d_n$ by

$$\mathrm{Cat}^{(m)}(W) = \frac{1}{|W|} \prod_{i=1}^{n} (hm + d_i).$$

Chapoton [7] showed that $\mathrm{Cat}^{(m)}(W)$ is the number of multichains $\pi_1 \leq \dots \leq \pi_m$ in $\mathcal{P}^{NC}(W, c)$, i.e. $\mathrm{Cat}^{(m)}(W) = Z(W, m+1)$ where $Z(W, m)$ is the zeta polynomial of $\mathcal{P}^{NC}(W, c)$. Fomin and Reading [11] introduced the so-called generalized cluster complex $\Delta^m(\Phi)$, and showed that its number of maximal simplices is $\mathrm{Cat}^{(m)}(W)$ (where Φ is the root system of W). Using this generalized cluster complex, they obtain in [11, Proposition 8.3] that

$$\mathrm{Cat}^{(m)}(W) = \frac{(m-1)h+2}{2n} \sum_{s \in S} \mathrm{Cat}^{(m)}(W_{(s)}). \tag{15}$$

Comparing the recursions (11) and (15) shows that

$$M(W, q) = n!(1-q)^n Z\big(W, \tfrac{1}{1-q}\big),$$

where we use the zeta polynomial rather than writing "$\mathrm{Cat}^{(\frac{q}{1-q})}(W)$" because it is generally assumed that $m \in \mathbb{N}$ when we write $\mathrm{Cat}^{(m)}(W)$. Then, the formula for $\mathrm{Cat}^{(m)}(W)$ in terms of the degrees proves:

Proposition 7.6.

$$M(W, q) = \frac{n!}{|W|} \prod_{i=1}^{n} \big(d_i + q(h - d_i)\big).$$

It is also possible to obtain this formula by solving the recursion (11) case by case. For the group A_n, we have:

$$M(A_n, q) = \frac{2 + q(n-1)}{2} \sum_{i=0}^{n-1} \binom{n-1}{i} M(A_i, q) M(A_{n-1-i}, q).$$

So the series $A = \sum_{n \geq 0} M(A_n, q) \frac{z^n}{n!}$ satisfies the differential equation

$$A' = A^2 + \tfrac{qz}{2}(A^2)'.$$

The prime symbol will always mean $\frac{\mathrm{d}}{\mathrm{d}z}$. If $q = 2$, the right-hand side is $(zA^2)'$. In general, after multiplying the equation by A^{q-2}, it can be rewritten

$$\left(\frac{A^{q-1}}{q-1}\right)' = (zA^q)'.$$

After checking the constant term, we arrive at the functional equation $A^{q-1} = 1 + (q-1)zA^q$. It would be possible to extract the coefficients of A with the Lagrange inversion formula. Another method is to use results about Fuss-Catalan numbers in type A. It is known that $\mathrm{Cat}^{(m-1)}(A_{n-1}) = \frac{1}{mn+1}\binom{mn+1}{n}$, which is the number of complete m-ary trees with n internal vertices, so that $F =$

$1+\sum_{n\geq 1}\mathrm{Cat}^{(m-1)}(A_{n-1})z^n$ satisfies $F=1+zF^m$. The equation for A can be rewritten

$$A^{1-q}=1+z(1-q)A$$

So, comparing the functional equations shows $F(z)=A(\frac{z}{1-q})^{1-q}$ if $m=\frac{1}{1-q}$. This is also $F(z)=1+zA(\frac{z}{1-q})$. Taking the coefficient of z^{n+1}, we obtain:

$$\frac{1}{\frac{n+1}{1-q}+1}\binom{\frac{n+1}{1-q}+1}{n+1}=\frac{1}{(1-q)^n n!}M(A_n,q),$$

hence

$$M(A_n,q)=\frac{n!(1-q)^n}{\frac{n+1}{1-q}+1}\binom{\frac{n+1}{1-q}+1}{n+1}=\prod_{i=1}^{n-1}(i+1+q(n-i)).$$

As for the group B_n, the recursion is:

$$M(B_n,q)=(1+q(n-1))\sum_{i=0}^{n-1}\binom{n-1}{i}M(B_i,q)M(A_{n-1-i},q).$$

It follows that the series $B=\sum_{n\geq 0}M(B_n,q)\frac{z^n}{n!}$ satisfies the differential equation

$$B'=AB+qz(AB)'.$$

We thus obtain

$$(1-qzA)B'=AB+qzA'B.$$

The differential equation for A is also $A'=\frac{A^2}{1-qzA}$. Thus we arrive at:

$$\frac{B'}{B}=\frac{A+qzA'}{1-qzA}=\frac{A'}{A}+\frac{qzA'}{1-qzA}.$$

which can be integrated as follows:

$$\log(B)=(1-q)\log(A)-\log(1-qzA).$$

Since $A^{1-q}=1+(1-q)zA$, we obtain:

$$B=\frac{1+(1-q)zA}{1-qzA}=1+\frac{zA}{1-qzA}=1+\frac{zA'}{A}.$$

By extracting the coefficients of $\frac{z^{n-1}}{(n-1)!}$ on both sides of $AB=A+zA'$, we readily obtain $(1+q(n-1))^{-1}M(B_n,q)=nM(A_{n-1},q)$, so that

$$M(B_n,q)=\prod_{i=1}^{n}(i+q(n-i)).$$

As for the group D_n, the formulas get more complicated and we will skip some details, but some simple arguments will show the result can be checked in a straightforward way. The recursion for $n\geq 2$ is:

$$M(D_n,q)=(1+q(n-2))\left(2M(A_{n-1},q)+\sum_{i=2}^{n-1}\binom{n-1}{i}M(D_i,q)M(A_{n-1-i},q)\right).$$

This suggests to define the series $D=2+\sum_{n\geq 2}M(D_n,q)\frac{z^n}{n!}$. After checking the constant term, we arrive at the differential equation

$$D'=(1-q)AD+qz(AD)'-2(1-q). \tag{16}$$

The result we want to prove is

$$M(D_n, q) = (n + q(n-2)) \prod_{i=1}^{n-1} (i + q(n-1-i)),$$

which can be reformulated as $M(D_n, q) = (n + q(n-2))M(B_{n-1}, q)$. This relation is equivalent to

$$D' = (1+q)(zB)' - 2qB + (q-1). \tag{17}$$

We proceed by necessary conditions and assume that both (16) and (17) are true. The calculations for B and B' above show that these two functions have an expression in terms of z and A. Knowing this, (17) can be used to have D' as a function of z and A. Then, (16) gives D in terms of A and D', so we can also have D as a function of z and A. Explicitly, after some manipulation we arrive at:

$$D = \frac{2 + (1-q)zA(zA+2)}{A(1-qzA)}.$$

It remains to check that if D is defined by this expression, then both (16) and (17) are true. In any of these equations, both sides can be rewritten as a function of z and A, which makes the identification lengthy but straightforward.

Checking the formula for the exceptional groups is immediate.

8. Generating functions of equivalence classes and hook formulas.

Definition 8.1. For any $\Pi \in \mathcal{M}^{NC}(W,c)$, let $[\Pi]$ denote its equivalence class for the W-action:

$$[\Pi] = \{w(\Pi) \ : \ w \in W\} \cap \mathcal{M}^{NC}(W,c).$$

We also define the class generating function:

$$M([\Pi], q) = \sum_{\Omega \in [\Pi]} q^{\mathrm{nir}(\Omega)}.$$

Theses classes partition the set $\mathcal{M}^{NC}(W,c)$, so that we have

$$M(W,q) = \sum_{[\Pi]} M([\Pi], q) \tag{18}$$

where we sum over all distinct equivalence classes.

We need some definitions before giving the formula for $M([\Pi], q)$.

Let $\tau \lessdot \pi$ be a cover relation in $\mathcal{P}^{NC}(W,c)$. The group $W_{(\pi)}$ can be decomposed into irreducible factors (that can be thought of as "blocks" of the set partition π). There is only one of these factors where $\underline{\tau}$ and $\underline{\pi}$ differ, as can be seen from the factorization of the poset $\mathcal{P}(W_{(\pi)})$ induced by the factorization of $W_{(\pi)}$.

Definition 8.2. With τ and π as above, let $h(\tau, \pi)$ be the Coxeter number of the irreducible factor of $W_{(\pi)}$ where $\underline{\tau}$ and $\underline{\pi}$ differ.

Definition 8.3. Let $g(\tau, \pi)$ be minimal $g > 0$ such that $\underline{\pi}^g \, \underline{\tau} \, \underline{\pi}^{-g} = \underline{\tau}$ and the map $x \to \underline{\pi}^g x \underline{\pi}^{-g}$ stabilizes each irreducible factor of $W_{(\tau)}$.

Note that by examining the irreducible factors of $W_{(\pi)}$, we can see that we have $\underline{\pi}^{h(\tau,\pi)} \, \underline{\tau} \, \underline{\pi}^{-h(\tau,\pi)} = \underline{\tau}$. From $\underline{\pi}^g \, \underline{\tau} \, \underline{\pi}^{-g} = \underline{\tau}$ and Proposition 7.4, we have either $g(\tau,\pi) = h(\tau,\pi)$ or $g(\tau,\pi) = \frac{1}{2}h(\tau,\pi)$. Note also that when $h(\tau,\pi)$ is even, as noted in Remark 7.3, we known that the map $x \to \underline{\pi}^{\frac{1}{2}h(\tau,\pi)} x \underline{\pi}^{-\frac{1}{2}h(\tau,\pi)}$ permutes the irreducible factors of $W_{(\tau)}$.

Proposition 8.4. *Let $\Pi = (\pi_0, \dots, \pi_n) \in \mathcal{M}^{NC}(W, c)$, let $h_i = h(\pi_{i-1}, \pi_i)$ and $g_i = g(\pi_{i-1}, \pi_i)$ for $2 \le i \le n$. Then we have:*

$$M([\Pi], q) = \prod_{i=2}^{n} \Big(\frac{2g_i}{h_i} + q\Big(g_i - \frac{2g_i}{h_i}\Big)\Big).$$

The proof is rather similar with that of Proposition 7.5. We need a few lemmas.

Lemma 8.5. *If $\Omega = (\omega_0, \dots, \omega_n) \in [\Pi]$, there is $k \ge 0$ such that $\omega_{n-1} = c^k(\pi_{n-1})$.*

Proof. Let L_i (respectively, L_j) be an interval partition in the c-orbit of ω_{n-1} (respectively, π_{n-1}). The fact that these exist follows Proposition 7.2. If $L_i = L_j$, the c-orbits are the same and this ends the proof.

Suppose now that $L_i \ne L_j$. Since $\Omega \in [\Pi]$, there is $w \in W$ such that $w(L_i) = L_j$, so Proposition 3.3 shows that $w_0(L_i) = L_j$. Then, Proposition 7.2 shows that L_i and L_j are in the same c-orbit. So ω_{n-1} and π_{n-1} are in the same c-orbit. □

Lemma 8.6. *Let $\Omega = (\omega_0, \dots, \omega_n) \in [\Pi]$, and assume inductively that Proposition 8.4 is true for the group $W_{(\omega_{n-1})}$. Let $\langle\Omega\rangle$ denote the class of Ω for the action of $W_{(\omega_{n-1})}$, i.e.*

$$\langle\Omega\rangle = \{ w(\Omega) \;:\; w \in W_{(\omega_{n-1})} \} \cap \mathcal{M}^{NC}(W, c).$$

Then the generating function of $\langle\Omega\rangle$ is:

$$M(\langle\Omega\rangle, q) = q^{\chi[\,\omega_{n-1} \notin \mathcal{P}^I(W)\,]} \prod_{i=2}^{n-1} \Big(\frac{2g_i}{h_i} + q\Big(g_i - \frac{2g_i}{h_i}\Big)\Big). \tag{19}$$

Proof. Let $\Omega' = (\omega_0, \dots, \omega_{n-1})$. Removing the last element of a chain gives a bijection between $\langle\Omega\rangle$ and

$$[\Omega'] = \{ w(\Omega') \;:\; w \in W_{(\omega_{n-1})} \} \cap \mathcal{M}^{NC}(W_{(\omega_{n-1})}, \underline{\omega_{n-1}}).$$

By induction, we can obtain $M([\Omega'], q)$. Since $\Omega \in [\Pi]$, it is straightforward to check that we have $g(\omega_{i-1}, \omega_i) = g(\pi_{i-1}, \pi_i)$ and $h(\omega_{i-1}, \omega_i) = h(\pi_{i-1}, \pi_i)$, although we see ω_{i-1}, ω_i as elements of $\mathcal{P}^{NC}(W_{(\omega_{n-1})}, \underline{\omega_{n-1}})$ and π_{i-1}, π_i as elements of $\mathcal{P}(W, c)$. We have $M(\langle\Omega\rangle, q) = q^{\chi[\,\omega_{n-1} \notin \mathcal{P}^I(W)\,]} M([\Omega'], q)$, and this gives the formula for $M(\langle\Omega\rangle, q)$. □

Lemma 8.7. *The minimal integer $g > 0$ such that $\langle\Pi\rangle = \langle c^g(\Pi)\rangle$ is g_n.*

Proof. This g satisfies $c^g(\pi_{n-1}) = \pi_{n-1}$, so that either $g = h_n$ or $g = \frac{h_n}{2}$. If we are not in the case where $c^{h_n/2}(\pi_{n-1}) = \pi_{n-1}$, we have $g = h_n = g_n$. So, suppose $c^{h_n/2}(\pi_{n-1}) = \pi_{n-1}$.

Consider the factorization of the poset $\mathcal{P}(W_{(\pi_{n-1})})$ induced by the factorization of $W_{(\pi_{n-1})}$ in irreducible factors. From the definition of g_n, the action of c^{g_n} stabilizes each factor of the poset, so it is the same action as some element $w \in W_{(\pi_{n-1})}$. So $\langle\Pi\rangle = \langle c^{g_n}(\Pi)\rangle$ and this proves $g \le g_n$.

Reciprocally, suppose that $c^g(\Pi) = w(\Pi)$ for some $w \in W_{(\pi_{n-1})}$. It follows that c^g stabilizes the irreducible factors of $W_{(\pi_{n-1})}$. The argument is similar to the one in the third part of Proposition 3.5: if the permutation on the factors is nontrivial, it would be possible to distinguish $c^g(\Pi)$ from $w(\Pi)$. So $g_n \geq g$, and eventually $g = g_n$. $\square$

Lemma 8.8. *The classes $\langle\Omega\rangle$ partition the set $[\Pi]$. A set of representatives is $\{\Pi, c(\Pi), \dots, c^{g_n-1}(\Pi)\}$.*

Proof. The first point is clear. From the previous lemma, the elements in the set $\{\Pi, c(\Pi), \dots, c^{g_n-1}(\Pi)\}$ are in distinct classes. It remains to show that the list is exhaustive.

Knowing Lemma 8.5, it remains to prove that if $\Omega \in [\Pi]$ is such that $\omega_{n-1} = \pi_{n-1}$, then there is k such that $\langle\Omega\rangle = \langle c^k(\Pi)\rangle$. Let $w \in W$ such that $\Omega = w(\Pi)$. In particular, $w(\pi_{n-1}) = \pi_{n-1}$.

If $w \in W_{(\pi_{n-1})}$, we have $\langle\Omega\rangle = \langle\Pi\rangle$. Otherwise, it means that $w \in \mathrm{Stab}(\pi_{n-1}) - \mathrm{Stab}^*(\pi_{n-1})$. Since the class $[\Pi]$ contains a chain of interval partitions, we might as well assume that π_{n-1} is an interval partition. It comes from Proposition 7.2 that $wc^{h/2} \in W_{(\pi_{n-1})}$. So we obtain $\langle\Omega\rangle = \langle c^{h/2}(\Pi)\rangle$. This completes the proof. $\square$

We can now prove Proposition 8.4.

Proof. Since the classes $\langle\Omega\rangle$ form a partition of $[\Pi]$, we have:

$$M([\Pi], q) = \sum_{\langle\Omega\rangle} M(\langle\Omega\rangle, q),$$

and $M([\Pi], q)$ can be obtained by summing Equation (19).

From the previous lemma, the number of distinct classes $\langle\Omega\rangle$ is g_n. As we have seen above (just before Proposition 8.4), either $g_n = h_n$ or $g_n = \frac{1}{2}h_n$, so that $\frac{2g_n}{h_n}$ is an integer. From Proposition 7.2, $\frac{2g_n}{h_n}$ among the distinct classes $\langle\Omega\rangle$ are such that their coatom is an interval partition. So, we get

$$\sum_{\langle\Omega\rangle} q^{\chi[\ \omega_{n-1} \notin \mathcal{P}^I(W)\ ]} = \Big(\frac{2g_n}{h_n} + q\Big(g_n - \frac{2g_n}{h_n}\Big)\Big).$$

So, summing Equation (19) over the classes $\langle\Omega\rangle$ gives the desired formula for $M([\Pi], q)$. $\square$

The rest of this section is devoted to explicit combinatorial description in type A and B, where Equation (18) can be interpreted as a hook-length formula for trees.

Definition 8.9. Let $\mathcal{A}_n$ denote the set of *André trees* on n vertices, i.e. trees such that:

- each internal node has either one son or two unordered sons,
- the vertices are labeled with integers from 1 to n, and the labels are decreasing from the root to the leaves.

The 5 elements of $\mathcal{A}_4$ are represented in Figure 2. These trees were introduced by Foata and Schützenberger [9, Chapter 5], who proved that $\#\mathcal{A}_n = T_n$. They were also used by Stanley [19] to prove $K(A_n) = T_n$.

Let us describe Stanley's bijection. We see it as a map $\mathcal{M}(A_{n-1}) \to \mathcal{A}_n$ that induces a bijection $\mathcal{M}(A_{n-1})/A_{n-1} \to \mathcal{A}_n$. We present an example on Figure 3 and refer to [19] for more details. Suppose that we start from the minimal partition 1|2|3|4|5|6|7 and at each step, two blocks merge into a larger block. We need 6 steps before arriving to the maximal partition 1234567. Each vertex v of the tree represents a subset b of $\{1, \dots, n\}$ of cardinality at least 2, that appears as a block

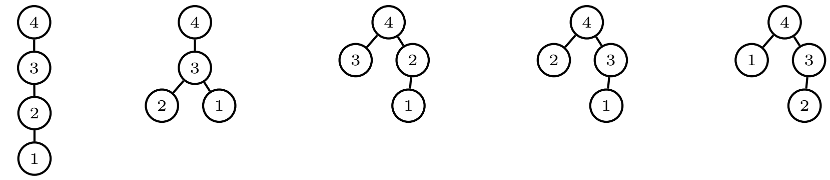

Figure 2. The André trees with 4 vertices.

of an element in the chain. This vertex v has label i if the block b appears after the ith merging. If v_1, v_2 are two vertices and b_1, b_2 the corresponding subsets of $\{1, \dots, n\}$ then v_1 is below v_2 in the tree if $b_1 \subset b_2$. In the example of Figure 3, the correspondence between blocks and labels is: $46 \to 1$, $15 \to 2$, $37 \to 3$, $3467 \to 4$, $125 \to 5$, $1234567 \to 6$.

1234567
125|3467
15|2|3467
15|2|37|46
15|2|3|46|7
1|2|3|46|5|7
1|2|3|4|5|6|7

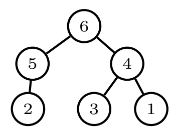

Figure 3. Stanley's bijection.

Proposition 8.10. *Let $\Pi \in \mathcal{M}^{NC}(A_{n-1})$, and $T \in \mathcal{A}_n$ its image under Stanley's bijection. Then we have*

$$M([\Pi], q) = \prod_{\substack{v \in T \\ h_v \neq 1}} (2 + q(h_v - 1)).$$

where h_v is the hook of the vertex v.

Proof. Let $2 \leq i \leq n$. There are $a > 0$ and $b > 0$ such that π_i is obtained from π_{i-1} by merging two blocks of size a and b into one block of size $a+b$. The integer h_i is the Coxeter number of $\mathfrak{S}_{a+b}$, i.e. $h_i = a+b$. If $a > 1$ or $b > 1$, i.e. one of the two blocks has cardinality at least 2, there is a nontrivial factor $\mathfrak{S}_a$ or $\mathfrak{S}_b$ that needs $a+b$ rotations through the cycle to go back to itself, so that $g_i = a+b$. But if $a = b = 1$, we have $g_i = 1 = \frac{h_i}{2}$.

Let v be the vertex of T with label i. From the properties of the bijection, the two sons of v contains $a-1$ and $b-1$ vertices, and $h_v = a+b-1$. So, we obtain:

$$\frac{2g_i}{h_i} + q(g_i - \frac{2g_i}{h_i}) = \begin{cases} 2 + q(h_v - 1) \text{ if } h_v > 1, \\ 1 \text{ otherwise.} \end{cases}$$

So Proposition 8.4 specializes as stated above. ∎

As a consequence, Equation (18) gives the following:

Theorem 8.11.

$$\prod_{i=1}^{n-1} (i + 1 + q(n-i)) = \sum_{T \in \mathcal{A}_n} \prod_{\substack{v \in T \\ h_v \neq 1}} (2 + q(h_v - 1)). \tag{20}$$

For example, for $n=4$, and taking the 5 trees as in Figure 2, we get:

$$(2+3q)(3+2q)(4+q) = (2+q)(2+2q)(2+3q) + (2+2q)(2+3q) + \\ (2+q)(2+3q) + (2+q)(2+3q) + (2+q)(2+3q).$$

We have to make the connection with previously-known results. Let $\mathcal{T}_n$ denote the set of binary plane trees on n vertices, and $\mathcal{T}_n^\ell$ denote the set of pairs (T,L) where $T\in\mathcal{T}_n$ and L is a decreasing labeling of the vertices. It is well-known that the number such labelings L for a given T is

$$\frac{n!}{\prod_{v\in T} h_v}.$$

Moreover, there is a map $\mathcal{T}_n^\ell \to \mathcal{A}_n$ which consists in "forgetting" the notion of left and right among the sons of each internal vertex. It is such that each $T\in\mathcal{A}_n$ has $2^{\mathrm{in}(T)}$ preimages, where $\mathrm{in}(T)$ is the number of internal vertices of T (i.e. $v\in T$ such that $h_v>1$). Then, we can rewrite the right-hand side of (20):

$$\sum_{T\in\mathcal{A}_n} \prod_{\substack{v\in T\\ h_v\neq 1}} (2+q(h_v-1)) = \frac{1}{2^n} \sum_{T\in\mathcal{A}_n} 2^{\mathrm{in}(T)} \prod_{v\in T} (2+q(h_v-1))$$
$$= \frac{1}{2^n} \sum_{T\in\mathcal{T}_n^\ell} \prod_{v\in T} (2+q(h_v-1)) = \frac{n!}{2^n} \sum_{T\in\mathcal{T}_n} \prod_{v\in T} \Big(\frac{2+q(h_v-1)}{h_v}\Big).$$

So we arrive at

$$\prod_{i=1}^{n-1} (i+1+q(n-i)) = \frac{n!}{2^n} \sum_{T\in\mathcal{T}_n} \prod_{v\in T} \Big(q + \frac{2-q}{h_v}\Big).$$

The particular case $q=1$ is Postnikov's hook-length formula [14, Corollary 17.3], proved in investigating the volume of generalized permutohedra. A one-parameter generalization was conjectured by Lascoux and proved by Du and Liu [8], it is exactly the previous equation up to the change of variable $(q,2-q)\to(q,1)$.

Let us turn to the type B analogue, where we can adapt Stanley's bijection. (Note that a type B analogue of André trees or permutations have been considered by Purtill, in relation with type B Springer numbers.)

For brevity, the integers $-1,-2$, etc. will be represented $\bar{1}$, $\bar{2}$, etc. A set partition of type B is a set partition of $\{\bar{n},\dots,\bar{1}\}\cup\{1,\dots,n\}$, unchanged under the map $x\to -x$, and such that there is at most one block b such that $b=-b$ (called the 0-block when it exists). For example, $1\bar{2}5|\bar{1}2\bar{5}|3\bar{3}6\bar{6}|4|\bar{4}\in\mathcal{P}(B_6)$.

Definition 8.12. A *pointed André tree* is an André tree with a distinguished vertex $v\in T$ having 0 or 1 son. Let $\mathcal{A}_n^*$ denote the set of pointed André trees on n vertices.

A tree $T\in\mathcal{A}_n^*$ is represented with the convention that the distinguished vertex has a starred label i^*. We can create a new tree as follows: increase the labels by 1, then add a new vertex with label 1 attached to the distinguished vertex. This is clearly a bijection between $\mathcal{A}_n^*$ and $\mathcal{A}_{n+1}$, showing that $\#\mathcal{A}_n^* = T_{n+1} = K(B_n)$. See Figure 4 for an example.

Let $\Pi=(\pi_0,\dots,\pi_n)\in\mathcal{M}(B_n)$. We build a tree $T\in\mathcal{A}_n^*$ by adapting Stanley's map. A vertex in T represents either the 0-block in some π_i, or a pair of distinct opposite blocks in some π_i where the elements of the pair have cardinality at least 2. This vertex has label i if this 0-block, or pair of opposite blocks, appears in

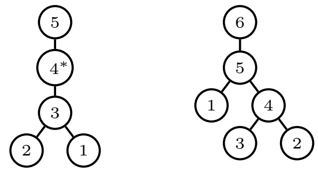

FIGURE 4. The bijection $\mathcal{A}_n^* \to \mathcal{A}_{n+1}$.

π_i but not in π_{i-1}. A vertex v_1 is below another vertex v_2 in the tree when the blocks represented by v_1 are included in the blocks represented by v_2. Eventually, we have the following rule: the distinguished vertex has label i if and only if π_i has a 0-block, and π_{i-1} has none. See Figure 5 for an example.

$1\bar{1}2\bar{2}3\bar{3}4\bar{4}5\bar{5}6\bar{6}$
$1\bar{1}3\bar{3}|2\bar{4}56|\bar{2}4\bar{5}\bar{6}$
$1\bar{1}3\bar{3}|25|\bar{2}\bar{5}|4\bar{6}|\bar{4}6$
$13|\bar{1}\bar{3}|25|\bar{2}\bar{5}|4\bar{6}|\bar{4}6$
$1|\bar{1}|3|\bar{3}|25|\bar{2}\bar{5}|4\bar{6}|\bar{4}6$
$1|\bar{1}|2|\bar{2}|3|\bar{3}|5|\bar{5}|4\bar{6}|\bar{4}6$
$1|\bar{1}|2|\bar{2}|3|\bar{3}|4|\bar{4}|5|\bar{5}|6|\bar{6}$

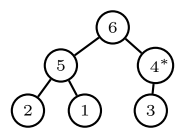

FIGURE 5. Stanley's bijection adapted to type B.

Proposition 8.13. *Let $\Pi \in \mathcal{M}(B_n)$ and $T \in \mathcal{A}_n^*$ its image under the bijection we have just defined. For any vertex v of the tree $T \in \mathcal{A}_n^*$, we define a factor $\beta(v)$ to be $1+q(h_v-1)$ if v belongs to the minimal path joining the root to the distinguished vertex, $2+q(h_v-1)$ otherwise. Then we have:*

$$M([\Pi],q) = \prod_{\substack{v\in T\\ h_v\neq 1}} \beta(v).$$

Proof. Let $2 \leq i \leq n$, let v be the vertex with label i.

Suppose first that π_i is obtained from π_{i-1} by merging two pairs of distinct opposite blocks into a pair of distinct opposite blocks (such as $25|\bar{2}\bar{5}$ and $4\bar{6}|\bar{4}6$ in the example). This is the case where v is not in the minimal path from the root to the distinguished vertex. This means that $W_{(\pi_i)}$ is obtained from $W_{(\pi_{i-1})}$ by replacing a factor $\mathfrak{S}_a \times \mathfrak{S}_b$ into $\mathfrak{S}_{a+b}$. As in the type A case, we get $g_i = h_i = a+b+1$, and $a-1$, $b-1$ are the number of vertices in the subtrees of v. This gives $\frac{2g_i}{h_i} + q(g_i - \frac{2g_i}{h_i}) = \beta(v)$.

Suppose then that π_i is obtained from π_{i-1} by merging two pairs of distinct opposite blocks into a 0-block (such as 13 and $\bar{1}\bar{3}$ in the example). This is the case where v is the distinguished vertex. This means that $W_{(\pi_i)}$ is obtained from $W_{(\pi_{i-1})}$ by replacing a factor $\mathfrak{S}_j = A_{j-1}$ into B_j where j is the size of the 0-block, and also the hook-length of v. We obtain $h_i = 2j$, and $g_i = j$. Also in this case, this gives $\frac{2g_i}{h_i} + q(g_i - \frac{2g_i}{h_i}) = \beta(v)$.

Eventually, suppose that π_i is obtained from π_{i-1} by merging a pair of distinct opposite blocks to the 0-block (such as $2\bar{4}56|\bar{2}4\bar{5}\bar{6}$ in the example). This is the case where v is in the minimal path from the root to the distinguished vertex (but is not the distinguished vertex). This means that $W_{(\pi_i)}$ is obtained from $W_{(\pi_{i-1})}$ by replacing a factor $A_{j-1} \times B_k$ into B_{j+k}. Here, $k > 0$ is the number of vertices in the subtree of v containing the distinguished vertex, and $j - 1 \geq 0$ is the number of vertices in the other subtree. We get $h_i = 2(j + k)$, $g_i = j + k = h_v$, and $\frac{2g_i}{h_i} + q(g_i - \frac{2g_i}{h_i}) = \beta(v)$.

So Proposition 8.4 specializes as stated above. □

So, in the type B case, Equation (18) gives:

Theorem 8.14.

$$\prod_{i=1}^{n}(i + q(n - i)) = \sum_{T \in \mathcal{A}_n^*} \prod_{\substack{v \in T \\ h_v \neq 1}} \beta(v).$$

For example, let $n = 3$. We take the 5 elements of $\mathcal{A}_n^*$ as they appear in Figure 2 after we apply the bijection $\mathcal{A}_{n+1} \to \mathcal{A}_n^*$, and we get:

$$\begin{aligned} 3(2 + q)(1 + 2q) = (1 + q)(1 + 2q) + (1 + q)(1 + 2q) + (1 + 2q) + \\ (1 + 2q) + (2 + q)(1 + 2q). \end{aligned}$$

Strictly-speaking, the identity in the previous theorem might be not considered as a hook-length formula since $\beta(v)$ does not depend only on the hook-length h_v. Still, it is on its own an interesting variant of the type A case.

Appendix A. Properties of the standard Coxeter elements

We sketch here a case-by-case proof of Propositions 7.2 and 7.4. In types A, B, D, the results become clear upon inspection, once we have explicit combinatorial descriptions of the objects. As for the exceptional groups, everything can be checked with a computer program.

We shall use the notion of cyclic order and cyclic intervals. Recall that a sequence $i_1, \dots, i_n$ is *unimodal* if there is k such that $i_1 \leq i_2 \leq \dots \leq i_k$ and $i_k \geq i_{k-1} \geq \dots \geq i_n$.

A.1. **Case of A_{n-1}.** Let $W = A_{n-1} = \mathfrak{S}_n$, $V = \{v \in \mathbb{R}^n : \sum v_i = 0\}$. Let $S = \{s_1, \dots, s_{n-1}\}$, where s_i acts by permuting the ith and $(i+1)$th coordinates. As a permutation, s_i is the simple transposition $(i, i+1)$. Let $c = s_{\sigma(1)} \dots s_{\sigma(n-1)}$ be a standard Coxeter element. By exchanging pairs of commuting generators, we can write c as a product of s_{n-1} with a standard Coxeter element of A_{n-2}. By an easy induction, we see that we can write c as the cycle $(i_1, \dots, i_n)$ where $i_1, \dots, i_n$ is a unimodal sequence (and a permutation of $1, \dots, n$).

Any coatom of $\mathcal{P}^{NC}(A_{n-1}, c)$ is a pair of cyclic intervals of the sequence $i_1, \dots, i_n$, complementary to each other, and the action of c is the “rotation” along the cycle. Two such coatoms are in the same c-orbit if and only if they have the same block sizes. So, for each k with $1 \leq k < \frac{n}{2}$, there is an orbit containing complementary cyclic intervals of size k and $n - k$. There are n such partitions, and the interval partitions among them are $1 \dots k|k+1 \dots n$ and $1 \dots n-k|n-k+1 \dots n$. Additionally, if n is even, there is an orbit containing two complementary cyclic intervals of size $\frac{n}{2}$. There are $\frac{n}{2}$ such partitions, and the only interval partition among them is $1 \dots \frac{n}{2}|\frac{n}{2} + 1 \dots n$. Proposition 7.2 follows.

We turn to Proposition 7.4. As simple roots of W, we take $e_i - e_{i+1}$ for $1 \leq i \leq n - 1$ where $(e_i)_{1 \leq i \leq n+1}$ is the canonical basis of $\mathbb{R}^n$. Let t be a reflection, then there are $1 \leq \ell < m \leq n$, such that t is the transposition (i_ℓ, i_m). The permutation ct is a product of two cycles:

$$ct = (i_1, \dots, i_\ell, i_{m+1}, \dots, i_n)(i_{\ell+1}, \dots, i_m).$$

The two sequences $i_1, \dots, i_\ell, i_{m+1}, \dots, i_n$ and $i_{\ell+1}, \dots, i_m$ are also unimodal, as subsequences of a unimodal sequence. The group $W_{(ct)}$ is a product of two symmetric groups, one acting on $i_1, \dots, i_\ell, i_{m+1}, \dots, i_n$ and the other on $i_{\ell+1}, \dots, i_m$. Its positive roots are $e_u - e_v$ where $u < v$ are two indices in one of the sequence $1, \dots, \ell, m+1, \dots, n$ or $\ell+1, \dots, m$, and taking two consecutive indices give the simple roots. As in the general case seen above, we can deduce that a standard Coxeter element of $W_{(ct)}$ is a product of two cycles given by unimodal sequences, i.e. exactly what we have obtained for ct.

A.2. **Case of B_n.** Proposition 7.2 was already proved in this case, since the longest element is central. We turn directly to Proposition 7.4. Let $W = B_n$ acting on $V = \mathbb{R}^n$. The group B_n is generated by $s_1, \dots, s_{n-1}$, i.e. generators of A_{n-1}, together with another generator s_0^B. The latter acts as $(v_1, \dots, v_n) \mapsto (-v_1, v_2, \dots, v_n)$. The simple roots are $-e_1$, together with $e_i - e_{i+1}$ for $1 \leq i < n$. We identify B_n with the group of signed permutations, and s_0^B is the transposition $(1, -1)$. We use the notation $((a_1, \dots, a_n)) = (a_1, \dots, a_n)(-a_1, \dots, -a_n)$ and $[[a_1, \dots, a_n]] = (a_1, \dots, a_n, -a_1, \dots, -a_n)$ for the cycles of signed permutations.

By exchanging pairs of commuting generators, we can see that a standard Coxeter element c of B_n is a product of s_0^B and a standard Coxeter element of A_{n-1}. So it is a cycle $c = [[i_1 \dots, i_n]]$ where the indices $i_1, \dots, i_n$ are a unimodal sequence, and a permutation of $1, \dots, n$.

Let t be a reflection. Suppose first that there are ℓ, m such that $\ell < m$, and t is the transposition $((i_\ell, i_m))$. Then $ct = c_1 c_2$, where:

$$c_1 = [[i_1, \dots, i_\ell, i_{m+1}, \dots, i_n]],$$
$$c_2 = ((i_{\ell+1}, \dots, i_m)).$$

We can see that $W_{(ct)}$ is a product of $B_u \times A_v$, where B_u acts on the elements that appear in c_1, and A_v acts on the elements that appear in c_2 (here $u = n + \ell - m$ and $v = m - \ell - 1$). The sequences $i_1, \dots, i_\ell, i_{m+1}, \dots, i_n$ and $i_{\ell+1}, \dots, i_m$ are unimodal, and we can deduce that each cycle is a standard Coxeter element of the factor B_u or A_v, as in the A_n case. So ct is a standard Coxeter element of $W_{(ct)}$.

In the second case, there are ℓ, m such that $\ell < m$, and $t = ((-i_\ell, i_m))$. Then $ct = c_1 c_2$, where:

$$c_1 = ((i_1, \dots, i_\ell, -i_{m+1}, \dots, -i_n)),$$
$$c_2 = [[i_{\ell+1}, \dots, i_m]].$$

In this case, $W_{(ct)}$ is a product of $A_u \times B_v$, where A_u acts on the elements that appear in c_1, and B_v acts on the elements that appear in c_2 (here $u = n + \ell - m - 1$ and $v = m - \ell$). Here, the structure of c_1 is not similar to what appeared in the first case, but Lemma A.1 below permits to see it as a standard Coxeter element of $W_{(c_1)}$.

In the third case, there is ℓ such that $t = (i_\ell, -i_\ell)$. Then:

$$ct = ((i_1, \dots, i_\ell, -i_{\ell+1}, \dots, -i_n)).$$

It has the same structure as c_1 defined in the previous case, and we can use Lemma A.1. We similarly obtain that $W_{(ct)}$ is a group A_{n-1}, and ct is indeed a standard Coxeter element.

Lemma A.1. *Let I and J be complementary subsets of $\{1,\dots,n\}$, let $i_1 < \dots < i_k$ be the elements of I and $j_1 < \dots < j_\ell$ be the elements of J. Let $W \subset B_n$ be the group of signed permutations w such that $w(i) \in I$ or $-w(i) \in J$ for all $i \in I$, and $w(j) \in J$ or $-w(j) \in I$ for all $j \in J$. Then W is a parabolic subgroup of B_n of type A_{n-1}, its simple generators (induced by our choice of positive roots for B_n) are $((i_u, i_{u+1}))$ for $1 \le u < k$, $((j_u, j_{u+1}))$ for $1 \le u < \ell$, and $((i_1, -j_1))$. The standard Coxeter elements of W have the form $((u_1, \dots, u_k, -v_1, \dots, -v_k))$ where $u_1, \dots, u_k$ form a unimodal sequence and a permutation of $i_1, \dots, i_k$, and $v_1, \dots, v_\ell$ form a unimodal sequence and a permutation of $j_1, \dots, j_\ell$.*

Proof. Omitted. □

A.3. **Case of D_n.** The group D_n is the subgroup of B_n generated by $s_1, \dots, s_{n-1}$ together with another generator s_0^D. The latter acts by the transformation $v = (v_1, \dots, v_n) \mapsto (-v_2, -v_1, v_3, \dots, v_n)$. As a signed permutation, it is the transposition $(-1,2)(1,-2)$. The simple roots are $-e_1 - e_2$, and $e_i - e_{i+1}$ for $1 \le i < n$. Note that this is the natural choice induced by our previous choice of positive roots for B_n when we see D_n as a subgroup of B_n. By exchanging pairs of commuting generators, we can see that a standard Coxeter element c of D_n is a product of s_0^D and a standard Coxeter element of A_{n-1}. So, either:

$$c = (1,-1)[[i_1, \dots, i_{n-1}]]$$

where $i_1, \dots, i_{n-1}$ are a unimodal sequence, and a permutation of $2, \dots, n$, or:

$$c = (2,-2)[[i_1, \dots, i_{n-1}]]$$

where $i_1, \dots, i_{n-1}$ are a unimodal sequence, and a permutation of $1, 3, \dots, n$. We only consider the first case, the other one being completely similar (it suffices to replace the 1's with 2's in the text).

We have four kinds of products ct where t is a reflection:

$$\begin{aligned}
c((1, i_m)) &= ((1, i_{m+1}, \dots, i_{n-1}, -i_1, \dots, -i_m)),\\
c((-1, i_m)) &= ((1, -i_{m+1}, \dots, -i_{n-1}, i_1, \dots, i_m)),\\
c((i_\ell, i_m)) &= (1,-1)[[i_1, \dots, i_\ell, i_{m+1}, \dots, i_{n-1}]]((i_{\ell+1}, \dots, i_m)),\\
c((-i_\ell, i_m)) &= (1,-1)[[i_{\ell+1}, \dots, i_m]]((i_1, \dots, i_\ell, -i_{m+1}, \dots, -i_{n-1})).
\end{aligned}$$

Using the notation for type B set partitions, we obtain from the list above that the coatoms of $\mathcal{P}^{NC}(D_n, c)$ are:

- $1 i_{m+1} \dots i_{n-1} \bar{i}_1 \dots \bar{i}_m | \bar{1} \bar{i}_{m+1} \dots \bar{i}_{n-1} i_1 \dots i_m$,
- $\bar{1} i_{m+1} \dots i_{n-1} \bar{i}_1 \dots \bar{i}_m | 1 \bar{i}_{m+1} \dots \bar{i}_{n-1} i_1 \dots i_m$,
- $1 i_1 \dots i_\ell i_{m+1} \dots i_{n-1} \bar{1} \bar{i}_1 \dots \bar{i}_\ell \bar{i}_{m+1} \dots \bar{i}_{n-1} | i_{\ell+1} \dots i_m | \bar{i}_{\ell+1} \dots \bar{i}_m$,
- $1 i_{\ell+1} \dots i_m \bar{1} \bar{i}_{\ell+1} \dots \bar{i}_m | i_1 \dots i_\ell \bar{i}_{m+1} \dots \bar{i}_{n-1} | \bar{i}_1 \dots \bar{i}_\ell i_{m+1} \dots i_{n-1}$.

And the interval partitions among them are $1 \dots n | \bar{1} \dots \bar{n}$, $1\bar{2} \dots \bar{n} | \bar{1} 2 \dots n$, and

$$1 \dots i \bar{1} \dots \bar{i} | i+1 \dots n | \overline{i+1} \dots \bar{n},$$

where $2 \le i < n$. From these explicit description, we can check Proposition 7.2. We find that all orbits have size $\frac{h}{2}$ (here $h = 2n-2$), except that $1 \dots n | \bar{1} \dots \bar{n}$ and $1\bar{2} \dots \bar{n} | \bar{1} 2 \dots n$ are in a same orbit of size h if n is even.

It remains to check Proposition 7.4. For the first and second kind of products ct, we can use Lemma A.1 to find that it is a standard Coxeter element for a subgroup of type A_{n-1}. For the third kind, we can directly recognize a standard Coxeter element of type $D_{n-m+\ell} \times A_{m-\ell-1}$. For the fourth kind, we can recognize a standard Coxeter element of type $D_{m-\ell} \times A_{n-m+\ell-1}$ (using Lemma A.1 for the second factor).

A.4. **Remaining cases.** The only case of Proposition 7.2 that remains to be checked is the one of E_6. This can be done with the following Sage program [20] (tested with Sage 5.4).

```
W = WeylGroup(['E',6])
n = 6
h = 12

S = W.simple_reflections()
w0 = W.long_element()

def checkorbits(l):
  c = prod( S[i] for i in l )
  inte = []
  for i in range(1,n+1):
    inte.append( prod( S[j] for j in l if j!=i ) )
  for ct in inte:
    i=1; j=1; k= c * ct * c**(-1) ;
    while k != ct :
      i+=1
      if k in inte:
        j+=1
        ct2 = k
      k = c * k * c**(-1)
    if not (((j==2) and (i==h)) or ((mod(h,2)==0) and (i==h/2) and (j==1))):
      raise TypeError('ERROR!!!')
    if not (((j==2) and (ct2==w0*ct*w0)) or ((j==1) and (ct == w0*ct*w0))):
      raise TypeError('ERROR!!!')

for l in Permutations(n):
  checkorbits(l)
```

As for Proposition 7.4, we can write a Sage program for the case of crystallographic groups (where it is rather simple to compute the simple roots of a subgroup). This is why we exclude H_3 and H_4 here. The code we have written is very naive, on the algorithmic point of vue. It would need an unreasonable amount of time to run it on E_7 and E_8, but we used it to checked the result for E_6 and F_4.

```
W = WeylGroup(['E',6])
n = 6
h = 12
S = W.simple_reflections()
V = W.domain()
P = V.positive_roots()

# compute the simple roots of the parabolic subgroup W_{(w)}
def simp(w):
```

```
  bas = kernel(  w - W.one().matrix() ).basis()
  ll=[]
  for i in P:
    b = true
    for v in bas:
      if v.inner_product( vector(i) ) != 0:
        b = false
    if b==true:
      ll.append(i)
  kk = []
  for i in ll:
    for j in ll:
      for k in ll:
        if vector(i)+vector(j)==vector(k):
          kk.append(k)
  for i in kk:
    if i in ll:
      ll.remove(i)
  return ll

# matrix of the reflection with respect to a vector v
def ref(v):
  n = len(v)
  m = matrix(v)
  t = matrix(n,n,1) - 2 / norm(v)**2 * m.transpose() * m
  return t

# list of reflections
T = map( ref , map( vector , P ) )

def checkcoatoms(l):
  c = prod( S[i].matrix() for i in l )
  perm = Permutations(n-1)
  for t in T:
    ll = map( ref , map( vector , simp( c*t ) ) )
    bb = false;
    for i in perm:
      if c*t == prod( ll[j-1] for j in i ):
        bb = true; break
    if not bb:
      raise TypeError('ERROR!!')

for l in permutations(n):
  checkcoatoms(l)
```

References

[1] D. André: Développement de sec x and tg x, C. R. Math. Acad. Sci. Paris 88 (1879), 965–979.

[2] V.I. Arnol'd: The calculus of snakes and the combinatorics of Bernoulli, Euler, and Springer numbers for Coxeter groups. Russian Math. Surveys 47 (1992), 1–51.

[3] D. Armstrong: Generalized Noncrossing Partitions and Combinatorics of Coxeter Groups. Mem. Amer. Math. Soc. 202, 2009.

[4] D. Armstrong, V. Reiner and B. Rhoades: Parking spaces. Preprint (2012), available at arXiv:1204.1760.
[5] D. Bessis: The dual braid monoid, Ann. Sci. Ecole Norm. Sup. 36 (2003), 647–683.
[6] A. Björner and F. Brenti: Combinatorics of Coxeter groups. Graduate Texts in Math., Vol. 231, Springer, New York, 2005.
[7] F. Chapoton: Enumerative properties of generalized associahedra, Séminaire Lotharingien de Combinatoire 51 (2004), Article B51b.
[8] R.R.X. Du and F. Liu: (k, m)-Catalan numbers and hook length polynomials for plane trees. European J. Combin. 28(4) (2007) 1312–1321.
[9] D. Foata and V. Strehl: Rearrangements of the symmetric group and enumerative properties of the tangent and secant numbers. Math. Z. 137 (1974), 257–264.
[10] D. Foata and M.-P. Schützenberger: Nombres d'Euler et permutations alternantes (unabridged version). Technical Report, University of Florida, 1971, 71 pages. http://www.emis.de/journals/SLC/books/foaschuetz1.html
[11] S. Fomin and N. Reading: Generalized cluster complexes and Coxeter combinatorics. Int. Math. Res. Not. 2005, no. 44, 2709–2757.
[12] J.E. Humphreys: Reflection Groups and Coxeter Groups. Cambridge University Press, Cambridge, 1990.
[13] M. Josuat-Vergès, J.-C. Novelli, and J.-Y. Thibon: Algebraic combinatorics on snakes. J. Combin. Theory Ser. A. 119(8) (2012), 1613–1638.
[14] A. Postnikov: Permutohedra, Associahedra, and Beyond. Internat. Math. Res. Notices 2009(6) 1026–1106.
[15] M. Purtill: André permutations, lexicographic shellability and the cd-index of a convex polytope. Trans. Amer. Math. Soc. 338(1) (1993), 77–104.
[16] N. Reading: Chains in the noncrossing partition lattice. SIAM J. Discrete Math. 22(3) (2008), 875–886.
[17] T.A. Springer: Remarks on a combinatorial problem. Nieuw. Arch. Wisk. 19(3) (1971), 30–36.
[18] R.P. Stanley: Some aspects of groups acting on finite posets. J. Combin. Theory Ser. A 32 (1982), 132–161.
[19] R.P. Stanley: A survey of alternating permutations. Contemp. Math. 531 (2010), 165–196.
[20] W. A. Stein et al.: Sage Mathematics Software (Version 5.4), The Sage Development Team, 2013, http://www.sagemath.org.
[21] R. Steinberg: Finite reflection groups. Trans. Amer. Math. Soc. 91 (1959), 493–504.

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