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Polynomial representations of the Hecke algebra of the symmetric group

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Abstract

We give a polynomial basis of each irreducible representation of the Hecke algebra, as well as an adjoint basis. Decompositions in these bases are obtained by mere specializations.

1   Introduction

Many properties of the algebra of the symmetric group $\mathfrak{S}_n$ in characteristic 0 are clarified by having recourse to a q-deformation of it, called the *Hecke algebra*, and denoted $\mathcal{H}_n$. Essentially, this algebra is obtained by replacing the quadratic relations $s_i^2 = 1$ satisfied by the simple transpositions s_i, by the relations $(s_i - q)(s_i + 1) = 0$, or more symmetrically, by $(s_i - q_1)(s_i - q_2) = 0$.

Finding the correct generalizations of Young matrices representing $\mathfrak{S}_2$:

$$\begin{bmatrix} g^{-1} & 1 - g^{-2} \\ 1 & -g^{-1} \end{bmatrix} \tag{1}$$

was done by Hoefsmit [9], who proposed, for $q_1 = q, q_2 = -1$,

$$\begin{bmatrix} -\frac{(q_1+q_2)}{(\gamma-1)} & -\frac{(q_1\gamma+q_2)(q_1+\gamma q_2)}{(\gamma-1)^2} \\ 1 & \frac{(q_1+q_2)}{(\gamma^{-1}-1)} \end{bmatrix}, \tag{2}$$

both parameters g, γ being different from 1.

Young's generators for an irreducible representation of $\mathfrak{S}_n$ are made of 2×2 blocks of the type(1), and similarly matrices made of blocks of the type (2) represent the Hecke algebra.

A simpler approach to representations of the symmetric group is to find spaces of polynomials in $x_1, \ldots, x_n$, the simple transpositions s_i acting by the exchange of x_i, x_{i+1}. For example, the linear span of

$$\{(x_1{-}x_2)(x_3{-}x_4) \ , \ (x_1{-}x_3)(x_2{-}x_4)\}$$

is a 2-dimensional representation of $\mathfrak{S}_4$, called *Specht representation*. However, it is not straightforward that the corresponding space for $\mathcal{H}_4$ is the span of

$$(q_2 x_1 + q_1 x_2)(q_1 x_4 + q_2 x_3) \ \& \ (q_2 x_2 + q_1 x_3)\left({q_1}^2 x_4 - {q_2}^2 x_1\right) \ .$$

In fact, it is not possible in general to find a basis of polynomials factorizing into linear factors. Moreover, the knowledge of a linear basis is insufficient for most purposes. One usually needs also to be able to decompose any element of the space in this basis. In the case of Specht representations [27], this is insured by the recursive use of *straightening relations* so much advocated by Gian Carlo Rota [24, 7, 8, 2].

Since straightening relations are a direct consequence of the quadratic Plücker relations between minors of a matrix, one could think of defining a q-straightening, and of using it to study representations of the Hecke algebra. However this leads to quantum considerations of much higher complexity than the original work of Young, considerations which cannot be considered as a concrete tool in the theory of representations.

We show in this text that there exist bases of irreducible representations of $\mathcal{H}_n$ satisfying easy vanishing properties. In particular, straightening is replaced by mere specializations. We refer to [1] for general considerations on representations, and to [29, 23] and the book [3] for a modern description of the representations of the symmetric group.

2 Hecke algebra and Yang-Baxter relations

The *Hecke algebra* $\mathcal{H}_n$ is the algebra generated by elements $T_1, \ldots, T_{n-1}$ satisfying the *braid relations*

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \& \quad T_i T_j = T_j T_i \, , \ |i - j| \neq 1 \, , \tag{3}$$

and the *Hecke relations*

$$(T_i{-}q_1)(T_i - q_2) = 0 \, . \tag{4}$$

One can represent faithfully $\mathcal{H}_n$ by operators on $\mathfrak{Pol}(\mathbf{x}_n) = \mathbb{C}[[q_1, q_2]](\mathbf{x}_n)$, writing $\mathbf{x}_n = \{x_1, \ldots, x_n\}$, such that each T_i acts on x_i, x_{i+1} only, and commutes with symmetric functions in x_i, x_{i+1}. Thus T_i is represented by an

operator, still denoted T_i, which is determined by its action on the basis $\{1, x_{i+1}\}$ of $\mathfrak{Pol}(x_i, x_{i+1})$ as a free $\mathfrak{Sym}(x_i, x_{i+1})$-module.

Writing operators on the right, one puts

$$1\,T_i = q_1 \quad \& \quad x_{i+1}\,T_i = -q_2 x_i\,.$$

In more details, the operators T_i are deformations of Newton's divided differences [15, 16]. Write $f \to f^{s_i}$ for the exchange of x_i, x_{i+1} in a polynomial, ∂_i for the *divided difference* $f \to f\partial_i = (f - f^{s_i})(x_i - x_{i+1})^{-1}$ and $\pi_i = x_i\partial_i$ for the *isobaric divided difference* $f \to f\pi_i = (x_i f - x_{i+1}f^{s_i})(x_i - x_{i+1})^{-1}$. Then

$$T_i = \pi_i(q_1 + q_2) - s_i q_2\,,$$

as one can check by using that $1\pi_i = 1$, $x_{i+1}\pi_i = 0$.

Taking products of simple operators corresponding to reduced decompositions, one obtains operators indexed by permutations [19]. In particular, let $\omega = [n, \ldots, 1]$ be the permutation of maximal length in $\mathfrak{S}_n$, and $\rho = [n-1, \ldots, 0]$. Then

$$\partial_\omega = (\partial_1 \ldots \partial_{n-1})(\partial_1 \ldots \partial_{n-2}) \cdots (\partial_1)\,.$$

This operator has also a global expression given by a summation over the symmetric group [19]:

$$f \to f\partial_\omega = \left(\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\ell(\sigma)} f^\sigma\right) \prod_{1 \le i < j \le n} (x_i - x_j)^{-1}\,.$$

From this expression, one sees that when $v \in \mathbb{N}^n$ is such that $v_1 + \cdots + v_n = \binom{n}{2}$, then one has

$$x^v \partial_\omega = \begin{cases} (-1)^{\ell(\sigma)} & \text{if } \exists \sigma : v = \rho\sigma \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

With products of simple transpositions s_i, one produces only permutations. It is more fruitful to use the Yang-Baxter relations instead of mere braid relations. For example, if one replaces $s_1 s_2 s_1$ by $(1 + s_1)(2^{-1} + s_2)(1 + s_1)$, one obtains by expansion the sum of the 6 permutations of $\mathfrak{S}_3$ instead of the permutation of maximal length.

More generally, for α, β such that $\alpha, \beta, \alpha\beta \ne 1$, and $i = 1, \ldots n-1$, the equalities

$$\left(T_i + \frac{q_1+q_2}{\alpha-1}\right)\left(T_{i+1} + \frac{q_1+q_2}{\alpha\beta-1}\right)\left(T_i + \frac{q_1+q_2}{\beta-1}\right)$$
$$= \left(T_{i+1} + \frac{q_1+q_2}{\beta-1}\right)\left(T_i + \frac{q_1+q_2}{\alpha\beta-1}\right)\left(T_{i+1} + \frac{q_1+q_2}{\alpha-1}\right) \tag{6}$$

3

are called *Yang-Baxter relations* for the Hecke algebra.

These relations can be displayed graphically as follows, writing on the right the corresponding ones for the symmetric group, and taking $i = 1$:
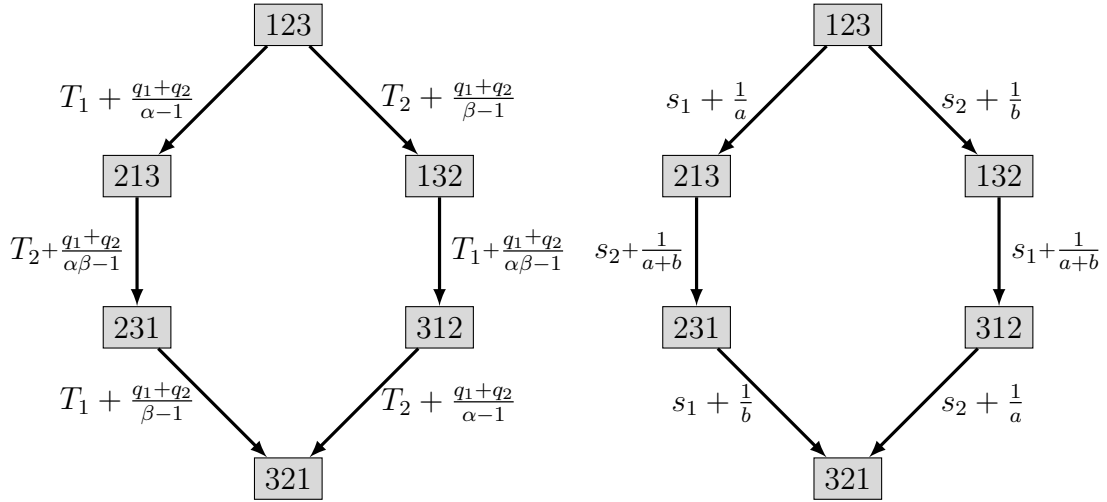
These pictures encode the fact that the two paths from top to bottom evaluate to the same element, when taking the product of the labels in the Hecke algebra or the group algebra respectively.

3   Yang-Baxter graphs and representations

Young's *semi normal representations* of the symmetric group can also be described in terms of a graph. Instead of labeling the vertices by standard Young tableaux in $\mathfrak{Tab}(\lambda)$, the set of standard tableaux of column-shape λ, let us directly take content vectors.

Given a partition $\lambda = [\lambda_1, \ldots, \lambda_\ell]$ of n, let

$$v^\lambda = [0, -1, \ldots, (-\lambda_1+1), 1, 0, \ldots, (-\lambda_2+2), \ldots, (\lambda_\ell-1), \ldots, (-\lambda_\ell+\ell)]$$

One generates a directed labelled graph $\Gamma_\lambda^{\mathfrak{S}}$, starting from v^λ, by the rule: at a vertex v, for each i such that $v_i < v_{i+1}-1$, there is an ensuing edge of label $s_i + (v_i-v_{i+1})^{-1}$, the new vertex being labelled vs_i, i.e. being obtained by exchanging the i-th and $(i+1)$-th component of v. The vectors such obtained are the *content vectors* for the representation of index λ (corresponding to a Ferrers' diagram given by the lengths of its columns). These vectors are in bijection with the tableaux in $\mathfrak{Tab}(\lambda)$, being obtained by reading the contents of the boxes of the Ferrers' diagram in the order indicated by the tableau.

Let us denote their set $\mathcal{C}(\lambda)$. Okounkov and Vershik [23] show that content vectors are forced once one wants to build Gelfand-Tsetlin bases of representations of the symmetric group.

For example, for $\lambda = [3, 2]$, one has $v^\lambda = [0, -1, -2, 1, 0]$ and the graph $\Gamma^{\mathfrak{S}}_{32}$ has five vertices (we use $\bar{a}$ instead of $-a$; we give a copy of the graph with tableaux as labels)
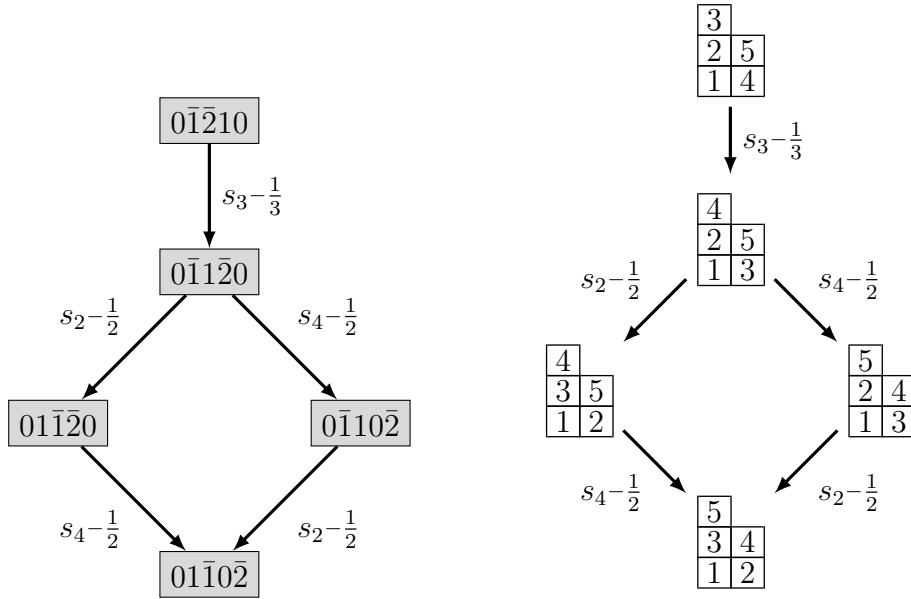
The graph $\Gamma^{\mathfrak{S}}_\lambda$ encodes all the matrices representing $s_1, \ldots, s_{n-1}$. Indeed, given i, one erases all the edges with labels $s_j + c$, with $j \neq i$. The graph decomposes into a disjoint union of two-vertices graphs, and singletons. Taking an order on vertices such that vertices connected by an edge are consecutive, one writes a block-diagonal matrix such that each single vertex v with $v_i = v_{i+1}+1$ gives a block of size 1 filled with -1, each vertex with $v_i = v_{i+1}-1$ gives a block filled with 1, a pair $[\ldots, a, b, \ldots] \xrightarrow{s_i + 1/(a-b)} [\ldots, b, a, \ldots]$ is interpreted as above as a matrix $\begin{bmatrix} g^{-1} & 1 - g^{-2} \\ 1 & -g^{-1} \end{bmatrix}$ with $g = b-a$. These matrices are exactly Young's matrices for semi-normal representations [32, p.451], [25, p.43], [10, p.126]. The fact that the matrices associated to the graph $\Gamma^{\mathfrak{S}}_\lambda$ satisfy the braid relations can be reproved directly, and essentially relies on the Yang-Baxter relations. Cherednik explains how to obtain Young idempotents from the Yang-Baxter relations in [4].

It is easy to extend the preceding considerations to representations of the Hecke algebra. Let us now interpret a graph with two vertices u, v and a

labelled edge

$$u \quad \xrightarrow{\quad T_1 + \dfrac{q_1 + q_2}{\gamma - 1} \quad} \quad v$$

as a two dimensional representation of $\mathcal{H}_2$, with basis $\{u, v\}$ such that v is the image of u under $T_1 + \frac{q_1+q_2}{\gamma-1}$. In other words, the matrix representing T_1 is precisely (2). More generally, the graph $\Gamma^{\mathcal{H}}_\lambda$ is defined by changing the labels of $\Gamma^{\mathfrak{S}}_\lambda$, keeping the set of vertices $\mathcal{C}(\lambda)$. An edge $v \to vs_i$ is now labelled

$$T_i(v_i - v_{i+1}) = T_i + \frac{q_1 + q_2}{(-q_1/q_2)^{v_i-v_{i+1}} - 1} \ .$$

For example the graph $\Gamma^{\mathcal{H}}_{32}$ is equal to
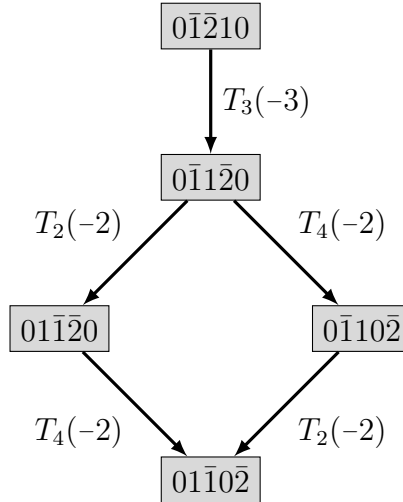
The matrices representing the generators $T_1, \ldots, T_{n-1}$ are read from the graph, after adding the rule that a vertex v with $v_i = v_{i+1}+1$ (resp. $v_i = v_{i+1}-1$ gives a diagonal entry q_2 (resp. q_1). These matrices are given by Murphy [22], up to a different normalization. Once more, the fact that the graph encodes matrices representing $T_1, \ldots, T_{n-1}$ is a consequence of the Yang-Baxter relations [18].

4 Jucys-Murphy elements

The sum θ_n of all transpositions in $\mathfrak{S}_n$ is a central element in the group algebra (because transpositions constitute a conjugacy class). More generally, identifying $\mathfrak{S}_k$ as the subgroup $\mathfrak{S}_k \times \mathfrak{S}_1$ of $\mathfrak{S}_{k+1}$, one sees that the elements $\theta_1, \theta_2, \ldots, \theta_n$ generate a commutative sub-algebra of $\mathbb{C}[\mathfrak{S}_n]$. The elements

$0, \theta_2, \theta_3 - \theta_2, \ldots, \theta_n - \theta_{n-1}$ are called *Jucys-Murphy elements*. Young's orthogonal idempotents [28, 25] for the symmetric group are eigenfunctions of the Jucys-Murphy elements [11, 22, 3], and this property gives the simplest way to characterize them. Notice nevertheless that multiplying a generic element of the group algebra of the symmetric group by sums of transpositions is a costly operation.

The corresponding Jucys-Murphy elements for $\mathcal{H}_n$ can be defined recursively by $\xi_1 = 1$,

$$\xi_i = \frac{-1}{q_1 q_2} T_{i-1} \xi_{i-1} T_{i-1} = 1 + (q_1 + q_2) \sum_{j<i} (-q_1 q_2)^{j-i} T_{(j,i)} \,,$$

and are still called *Jucys-Murphy elements* (compared to the preceding case, one adds a constant term to a sum of transpositions (j, i)). They were in fact first considered by Bernstein. These elements satisfy the following properties which are easy to check [22]. Write $\widehat{T}_i = T_i - q_1 - q_2$.

Lemma 1 *The ξ_i generate a commutative algebra and satisfy the relations*

$$\xi_i T_i = \widehat{T}_i \, \xi_{i+1} \quad \& \quad (\xi_i + \xi_{i+1}) T_i = T_i (\xi_i + \xi_{i+1}) \,, 1 \le i \le n - 1 \,, \quad (7)$$
$$\xi_j T_i = T_i \xi_j \,, j \ne i, i + 1 \quad (8)$$

The following property allows to generate eigenfunctions of Jucys-Murphy elements, starting from one such element.

Lemma 2 *Let f be an eigenvector for the right multiplication by ξ_i and ξ_{i+1}, with eigenvalues α, β. Then $g = f\left(T_i + \frac{q_1 + q_2}{\alpha/\beta - 1}\right)$ is an eigenvector of ξ_i and ξ_{i+1} with eigenvalues β, α.*

Proof.

$$f\left(T_i + \frac{q_1 + q_2}{\alpha/\beta - 1}\right) \xi_{i+1} = f\left(\widehat{T}_i + (q_1 + q_2)\left(1 + \frac{1}{\alpha/\beta - 1}\right)\right) \xi_{i+1}$$
$$= f\left(\xi_i T_i + \xi_{i+1}(q_1 + q_2)\frac{\alpha/\beta}{\alpha/\beta - 1}\right) = f\left(T_i \alpha + (q_1 + q_2)\frac{\alpha}{\alpha/\beta - 1}\right) = g\alpha \,.$$

Since moreover $\xi_i + \xi_{i+1}$ commutes with T_i, the two eigenvalues have been exchanged. QED

In the algebra generated by the Jucys-Murphy elements, Murphy [22] constructs a set of idempotents $\{e_t\}$ indexed by standard Young tableaux, which are simultaneous eigenfunctions of $\xi_1, \ldots, \xi_n$. In particular, if t_λ is the

tableau of column-shape a partition λ filled by consecutive numbers in each column, then the eigenvalues are

$$(-q_1/q_2)^{v_1^\lambda}, \ldots, (-q_1/q_2)^{v_n^\lambda},$$

where v^λ is the content vector defined above. These idempotents generalize Young's orthogonal idempotents.

The space $e_{t_\lambda} \mathcal{H}_n$ is an irreducible representation of the Hecke algebra, the element e_{t_λ} being characterized in this space by the set of equations $e_{t_\lambda} T_i = e_{t_\lambda} q_2$, for all i such that s_i belongs to $\mathfrak{S}_\lambda$. For example, for $\lambda = [3, 2]$, one has $v^\lambda = [0, -1, -2, 1, 0]$, $t_{32} = \begin{array}{|c|c|} \hline 3 \\ \hline 2 & 5 \\ \hline 1 & 4 \\ \hline \end{array}$, the eigenvalues are $1, -q_2/q_1, q_2^2/q_1^2, -q_1/q_2, 1$, and the idempotent $e_{t_{32}}$ satisfies $e_{t_{32}} T_i = e_{t_{32}} q_2$ for $i = 1, 2, 4$.

Notice that one could have used another set of Jucys-Murphy elements, starting with $\widehat{\xi}_1 = 1$, and defining

$$\widehat{\xi}_i = \frac{-1}{q_1 q_2} \widehat{T}_{i-1} \widehat{\xi}_{i-1} \widehat{T}_{i-1}.$$

This amounts using the involution $T_i \to \widehat{T}_i$, $q_1 \to -q_2$, $q_2 \to -q_1$. The eigenvalues of $\widehat{\xi}_1, \ldots, \widehat{\xi}_n$ acting on e_{t_λ} are the inverses of the eigenvalues of $\xi_1, \ldots, \xi_n$.

Idempotents, one expanded in the basis of permutations, are voluminous expressions. I give a way to encode them formally as factorizing polynomials in [17].

5 Polynomial representations

The symmetric group, as well as the Hecke algebra, act on polynomials. Hence representations can be realized as spaces of polynomials.

In the case of the symmetric group, let $\lambda = [\lambda_1, \ldots, \lambda_\ell]$ be a partition n, $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \ldots \mathfrak{S}_{\lambda_r}$ be a Young subgroup. Denote $\mathfrak{Pol}^d$ the subspace of $\mathfrak{Pol}(x_1, \ldots, x_n)$ of polynomials of degree $d = \binom{\lambda_1}{2} + \cdots + \binom{\lambda_\ell}{2}$. The subspace of $\mathfrak{Pol}^d$ of polynomials which are alternating under $\mathfrak{S}_\lambda$ is 1-dimensional and generated by the following product of Vandermonde functions

$$\Delta_\lambda = \Delta([1, \ldots, \lambda_1]) \Delta([\lambda_1+1, \ldots, \lambda_1+\lambda_2]) \ldots \Delta([\lambda_1+ \ldots +\lambda_{\ell-1}+1, \ldots, n])$$

with $\Delta([a, \ldots, b]) = \prod_{a \le i < j \le b} (x_j - x_i)$. The space $\Delta_\lambda \mathbb{C}[\mathfrak{S}_n]$ is an irreducible representation of the symmetric group, called *Specht representation* [27, 8], [10, Ch. 7].

Similarly, the subspace of polynomials f in $\mathfrak{Pol}^d$ which are such that $fT_i = fq_2$ for all i such that s_i belongs to $\mathfrak{S}_\lambda$, is 1-dimensional and generated by

$$\Delta^q_\lambda = \Delta^q([1,\ldots,\lambda_1])\Delta^q([\lambda_1+1,\ldots,\lambda_1+\lambda_2])\ldots\Delta^q([\lambda_1+\ldots+\lambda_{\ell-1}+1,\ldots,n])$$

with $\Delta^q([a,\ldots,b]) = \prod_{a\leq i<j\leq b}(q_2x_i+q_1x_j)$. Moreover, the space $\Delta^q_\lambda\mathcal{H}_n$ is an irreducible representation of the Hecke algebra.

Though starting points are analogous, the Vandermonde function being replaced by its q-generalization, the polynomials obtained under the action of $\mathfrak{S}_n$ or $\mathcal{H}_n$ look very different. For example, the space $\Delta_{22}\mathbb{C}[\mathfrak{S}_4]$ has basis Δ_{22}, $(\Delta_{22})^{s_2} = (x_3-x_1)(x_4-x_2)$, but the polynomial $\Delta^q_{22}T_2$ is not factorizing, being equal to

$$-q_2{}^3x_1x_2 + q_1{}^3x_3x_4 + \left(q_1{}^2q_2 + q_1{}^3\right)x_2x_4 + q_1{}^2q_2x_1x_4 + q_1{}^2q_2x_2x_3\,.$$

We shall nevertheless see that the Yang-Baxter graph still provides a basis which, though not factorizing, satisfies enough vanishing conditions to characterize it.

Let us first go back to a 2-dimensional representation of $\mathcal{H}_2$. Instead of a formal pair u,v, let us take a pair of polynomials $f,g \in \mathfrak{Pol}(x_1,x_2)$ such that $g = f(T_1 + (q_1+q_2)(\gamma-1)^{-1})$. Then the polynomials $g(x_1,x_2)$, $g(x_2,x_1)$ are given by the following formula, in terms of $A = f(x_1,x_2)$, $B = f(x_2,x_1)$:

$$g(x_1,x_2) = A\frac{(q_1+q_2)(\gamma-x_2/x_1)}{(\gamma-1)(1-x_2/x_1)} + B\frac{q_2x_1+q_1x_2}{x_2-x_1} \tag{9}$$

$$g(x_2,x_1) = A\frac{q_1x_1+q_2x_2}{x_1-x_2} + B\frac{(q_1+q_2)(\gamma-x_1/x_2)}{(\gamma-1)(1-x_1/x_2)}\,. \tag{10}$$

Evidently, if $A = 0 = B$, then $g(x_1,x_2) = 0 = g(x_2,x_1)$. For $A \neq 0$ and $B = 0$, one has the following property.

Lemma 3 *Let $f \in \mathfrak{Pol}(x_1,x_2)$, $\alpha,\beta \in \mathbb{C}$ be such that $\alpha \neq \beta$, $q_1\alpha+q_2\beta \neq 0$ and $f(\alpha,\beta) \neq 0$, $f(\beta,\alpha) = 0$. Let moreover $g = f\left(T_1 + (q_1+q_2)(\beta/\alpha-1)^{-1}\right)$. Then*

$$g(\alpha,\beta) = 0 \quad \& \quad g(\beta,\alpha) = \frac{q_1\alpha+q_2\beta}{\alpha-\beta}\,f(\alpha,\beta)\,. \tag{11}$$

In the case of the representation of $\mathcal{H}_2$ generated by the action of T_2 on Δ^q_{22}, one sees that Δ^q_{22} vanishes in the point $p_2 = [1,-q_2/q_1,-q_1/q_2,1]$, but not in $p_1 = [1,-q_1/q_2,-q_2/q_1,1]$. The lemma implies that $f = \Delta^q_{22}\left(T_2 + \right.$

$(q_1+q_2)((-q_2/q_1)^2-1)^{-1}\Big)$ vanishes in the point p_1 and not in p_2. The explicit value of f is

$$f = \frac{q_1^2 q_2^2}{q_1-q_2} \left(\frac{q_2\,(q_1-q_2)\,x_1 x_2}{q_1{}^2} - \frac{q_1\,(q_1-q_2)\,x_3 x_4}{q_2{}^2} + (x_1+x_2)(x_3+x_4) \right) ,$$

an expression which looks rather complicated for just studying a 2-dimensional space. We shall see in the next section that the graph Γ_λ provides polynomials having the required vanishing conditions. In fact, Sahi [26] and Knop [14] use a similar construction to define non-symmetric non-homogeneous Macdonald polynomials, we have rephrased it in terms of Yang-Baxter graphs in [18].

6   Young's basis

Let us define q-integers $[r]$ by $[0] = 0$ and, for $r \in \mathbb{N}\backslash 0$,

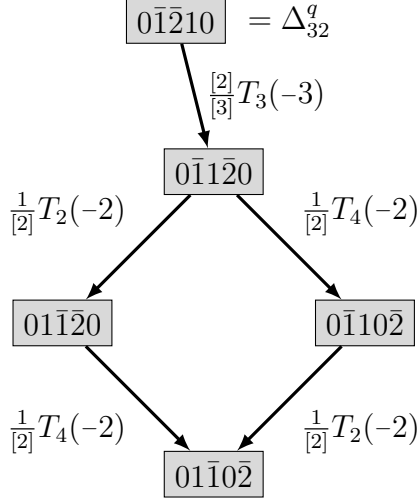$$[r] = q_1^{r-1} - q_1^{r-2} q_2 + \cdots + (-q_2)^{r-1} .$$

To $v \in \mathbb{N}^n$, one associates the vector $\langle v \rangle = [(-q_2/q_1)^{v_1}, \ldots, (-q_2/q_1)^{v_n}]$.

To define Young's basis for the Hecke algebra, we keep the vertices $\mathcal{C}(\lambda)$ of the graph $\Gamma_\lambda^{\mathcal{H}}$, but change the labeling of edges by normalization factors. With $r = v_{i+1}-v_i$, an edge $v \to vs_i$ is labelled

$$\frac{[r-1]}{[r]}\, T_i(-r) = \frac{[r-1]}{[r]} \left(T_i + \frac{q_1+q_2}{(-q_1/q_2)^r - 1} \right) .$$

Finally, we interpret the vertices as indexing polynomials, starting from Δ_λ^q, using the edges as operators, to generate polynomials Y_v from top to bottom. The Yang-Baxter relations are still preserved, so that two paths with the same end points define the same polynomial. We call this set of polynomials *Young's basis*, because these polynomials (we shall see that they are linearly independent) are either preserved or annihilated by the the idempotents generalizing Young's orthogonal idempotents. To the set of vertices $\{v\}$ we associate a second set $Sp(\lambda) = \{\langle v \rangle\}$ that we call the set of *spectral vectors* (they are the eigenvalues with respect to $\widehat{\xi}_1, \ldots, \widehat{\xi}_n$).

For example, for $\lambda = [3,2]$, the graph $\Gamma_{32}^{\mathcal{H}}$ is

$$\boxed{0\bar{1}\bar{2}10} \;\; = \Delta_{32}^q$$

$$\downarrow \tfrac{[2]}{[3]}T_3(-3)$$

$$\boxed{0\bar{1}1\bar{2}0}$$

$$\tfrac{1}{[2]}T_2(-2) \swarrow \qquad \searrow \tfrac{1}{[2]}T_4(-2)$$

$$\boxed{01\bar{1}\bar{2}0} \qquad\qquad \boxed{0\bar{1}10\bar{2}}$$

$$\tfrac{1}{[2]}T_4(-2) \searrow \qquad \swarrow \tfrac{1}{[2]}T_2(-2)$$

$$\boxed{01\bar{1}0\bar{2}}$$

and the spectral vectors are $[1, -\frac{q_1}{q_2}, \frac{q_1{}^2}{q_2{}^2}, -\frac{q_2}{q_1}, 1]$, $[1, -\frac{q_1}{q_2}, -\frac{q_2}{q_1}, \frac{q_1{}^2}{q_2{}^2}, 1]$, $[1, -\frac{q_1}{q_2}, -\frac{q_2}{q_1}, 1, \frac{q_1{}^2}{q_2{}^2}]$, $[1, -\frac{q_2}{q_1}, -\frac{q_1}{q_2}, \frac{q_1{}^2}{q_2{}^2}, 1]$, $[1, -\frac{q_2}{q_1}, -\frac{q_1}{q_2}, 1, \frac{q_1{}^2}{q_2{}^2}]$.

We are finally in position to conclude.

Theorem 4 *Let λ be a partition, and $\{Y_v : v \in \mathcal{C}(\lambda)\}$ be the set of polynomials obtained from the graph $\Gamma_\lambda^{\mathcal{H}}$ as explained above. Let*

$$c_\lambda = q_1^{\binom{n}{3}} q_2^{-n(n-1)(2n-1)/6} [2]^{n-1}[3]^{n-2} \ldots [n] \,.$$

Then, the polynomials $\{Y_v\}$ constitute a basis of an irreducible representation of $\mathcal{H}_n$ and, for any content vector $u \in \mathcal{C}(\lambda)$, one has

$$Y_v(\langle u \rangle) = \delta_{v,u}\, c_\lambda \,. \tag{12}$$

Each Y_v is a simultaneous eigenfunction of the Jucys-Murphy elements,

$$Y_v \xi_i = Y_v(-q_2/q_1)^{v_i^\lambda} \quad \& \quad Y_v \widehat{\xi}_i = Y_v(-q_1/q_2)^{v_i^\lambda} \tag{13}$$

Proof. The q-Vandermonde $\prod_{i<j}(q_2 x_i + q_1 x_j)$ vanishes if some x_j is specialized to $(-q_2/q_1)x_i$. Accordingly, Δ_λ vanish on a spectral vector $\langle v \rangle$ if, cutting v into blocks of successive lengths $\lambda_1, \lambda_2, \ldots$ there are two components of v inside a block such that $v_j - v_i = 1$, $j > i$. But this is the case of all content vectors, except the first one. The evaluation of Δ_λ in the first spectral vector is equal to c_λ. The lemma 3 propagates the fact that each polynomial Y_v vanishes in every spectral vector, except $\langle v \rangle$. The normalization factors have

been chosen in such a way that the non-zero specializations are all equal. This proves (12).

The space $\mathfrak{Pol}^d e_{t_\lambda} \mathcal{H}_n$ coincides with the space $\Delta_\lambda^q \mathcal{H}_n$. Since both e_{t_λ} and Δ_λ^q are annihilated by the $T_i - q_2$ for all i such that s_i belongs to $\mathfrak{S}_\lambda$, the polynomial Δ_λ^q must belong to the space $\mathfrak{Pol}^d e_{t_\lambda}$. This implies that Δ_λ^q is an eigenfunction of the Jucys-Murphy elements, the eigenvalues of $\widehat{\xi}_1, \ldots, \widehat{\xi}_n$ being the components of the spectral vector $\langle v^\lambda \rangle$. Lemma 2 implies that the other polynomials Y_v are still eigenvalues of the Jucys-Murphy elements. QED

We have used normalization factors $[k{-}1][k]^{-1}$ to obtain a constant scalar product in (12). In fact, Young defines a "tableau function" $f(t)$ in [32, p.458] , such that the normalization factor specializes, for $q_1 = 1, q_2 = -1$, to the quotient $f(t)/f(t')$ for a pair of tableaux differing by a simple transposition (see also [10, p.312], and [25, p.47] for a similar function).

There are other families of polynomials which are generated using the Hecke algebra, in particular the non-symmetric Macdonald polynomials. Let us refer to [6] for the fact that some Macdonald polynomials specialize, up to normalization, to the Young basis for $\lambda = [n, n]$ (the extra parameter q occurring in Macdonald's theory is in that case specialized to $-(q_2/q_1)^3$).

One can also use the Kazhdan-Lusztig basis of the Hecke algebra to generate polynomials. For example, among the images of Δ_{32}^q under Kazhdan-Lusztig elements, thus in the span of the Young basis for $\lambda = [3, 2]$, one finds the two factorizing polynomials

$$q_1 \left(q_1 x_4 + q_2 x_3 \right) \left(q_1 x_4 + q_2 x_2 \right) \left(q_2 x_2 + q_1 x_3 \right) \left(q_1{}^2 x_5 - q_2{}^2 x_1 \right),$$

$$q_1{}^2 \left(q_2 x_4 + q_1 x_5 \right) \left(q_1 x_5 + q_2 x_3 \right) \left(q_1 x_4 + q_2 x_3 \right) \left(q_2 x_1 + q_1 x_2 \right),$$

which are known to physicists [12, 5]. In terms of the Young basis, by computing five specializations , one obtains that the first one is equal to

$$(q_1^2 + q_2^2) \left(\frac{q_1{}^2 q_2{}^2}{\left(q_1{}^2 - q_1 q_2 + q_2{}^2 \right) \left(q_1{}^2 + q_2{}^2 \right)} Y_{0\bar{1}\bar{2}10} + \frac{q_1 q_2}{q_1{}^2 - q_1 q_2 + q_2{}^2} Y_{0\bar{1}1\bar{2}0} + Y_{01\bar{1}\bar{2}0} \right),$$

the second one having a similar expression. Computing the action of Jucys-Murphy elements on these polynomials would be of no use to obtain their decomposition in the Young basis.

Young bases for $\lambda = [n, n]$ and $\lambda = [2, \ldots, 2]$ can be used to compute Pfaffians [20].

7 Adjoint basis

One can easily build copies of the irreducible representation $\Delta_\lambda^q \mathcal{H}_n$ in higher degree by starting with a polynomial $f\Delta_\lambda^q$, with f invariant under the Young subgroup $\mathfrak{S}_\lambda$. Taking the same Yang-Baxter graph Γ_λ, one generates, as in the case $f = 1$, a set of polynomials $\{Y_v^f : v \in \mathcal{C}(\lambda)\}$ which satisfies the property

$$Y_v^f(\langle u \rangle) = \delta_{v,u}\, c_\lambda^f \,, \ u \in \mathcal{C}(\lambda)\,, \tag{14}$$

with $c_\lambda^f = f(\langle v^\lambda \rangle)\, c_\lambda$. Therefore, any function f invariant under $\mathfrak{S}_\lambda$ which does not vanish in $\langle v^\lambda \rangle$ provides a Young basis satisfying the same properties as in Theorem 4.

Notice also that reversing the directed graph Γ_λ, keeping the same labels on edges and starting with the polynomial Δ_μ^q, with $\mu = \lambda^\sim$, the conjugate partition, labeling the vertices by the elements of $\mathcal{C}(\mu)$, one obtains a basis of the representation $\Delta_\mu^q \mathcal{H}_n$.
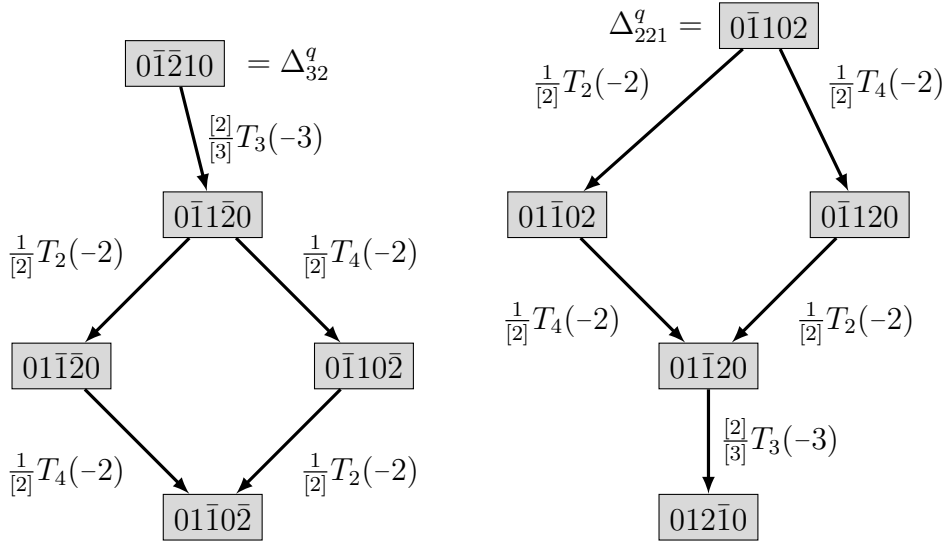
The labels on the vertices are exchanged by $v \to -v$. In particular, the bottom element of the left graph has label $-v^\mu = -[0, -1, 1, 0, 2] = [0, 1, -1, 0, -2]$.

One can build other bases by having recourse to the work of Kazhdan and Lusztig [13]. Kazhdan and Lusztig defined a linear basis $\{C_w : w \in \mathfrak{S}_n\}$ of $\mathcal{H}_n$. This basis is such that, when $w = \omega_\lambda$ is the permutation of maximal length of $\mathfrak{S}_\lambda$, then C_{ω_λ} can be factorized, as an operator on polynomials, as the product of the divided difference $\partial_{\omega_\lambda}$ by Δ_λ^q. In particular, the elements corresponding to simple transpositions s_i are such that

$$C_{s_i} = \partial_i(q_2 x_i + q_1 x_{i+1}) = T_i(-1)\,.$$

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Putting $\rho_\lambda = [\lambda_1{-}1, \ldots, 0, \lambda_2{-}1, \ldots, 0, \ldots, \lambda_\ell{-}1, \ldots, 0]$, one has $x^{\rho_\lambda} \partial_{\omega_\lambda} = 1$, and therefore the module $x^{\rho_\lambda} C_{\omega_\lambda} \mathcal{H}_n$ coincides with the representation that we have studied in the preceding section.

For any standard tableau of (column) shape λ, denote $\sigma(t)$ the inverse of the permutation obtained by reading the tableau column-wise, from left to right. It results from the work of Kazhdan and Lusztig that the set

$$\{L_t = x^{\rho_\lambda} C_{\sigma(t)} : t \in \mathfrak{Tab}(\lambda)\}$$

constitutes a linear basis of the module $x^{\rho_\lambda} C_{\omega_\lambda} \mathcal{H}_n$ (instead of a polynomial representation, Kazhdan and Lusztig [13] take a quotient of $C_{\omega_\lambda} \mathcal{H}_n$). By construction, if $w \in \mathfrak{S}_n$ is such that $w_i > w_{i+1}$, then $C_w C_{s_i} = q_2 C_w$. Accordingly, if $i{+}1$ occurs left of i in the tableau t, then

$$q_2 L_t = L_t \, \partial_i (q_2 x_i + q_1 x_{i+1}) \, . \tag{15}$$

Fill each box of the diagram with row lengths λ with the number of boxes in the rows below it. Let $\wp_\lambda$ be the column reading of this filling, reading the columns from left to right.

$$\lambda = [3,2,2] \, , \qquad \text{filling} \quad \begin{array}{|c|c|c|} \hline 5 & 5 \\ \hline 3 & 3 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \quad , \quad \wp_{322} = [5,3,0,5,3,0,0] \, .$$

Then $x^{\wp_\lambda} \partial_{\omega_\mu}$ is a product of Schur functions in blocks of consecutive variables, and therefore the polynomials $\widetilde{Y}_v$ generated by the Yang-Baxter graph with top element $\widetilde{\Omega}_\lambda = x^{\wp_\lambda} C_{\omega_\mu}$, for v running over $\mathcal{C}(\mu)$, constitute a Young basis.

Let us show that this basis is adjoint to the previous basis $\{Y_v : v \in \mathcal{C}(\lambda)\}$, with respect to a quadratic form, with values in $\mathfrak{Sym}(\mathbf{x}_n)$ (this form occurs in the study of the cohomology ring of the flag variety, [19]). Denote $f \to f^q$ the image of a polynomial under the exchange of q_1, q_2, and

$$(f, g) = f g^q \partial_\omega \in \mathfrak{Sym}(\mathbf{x}_n) \, .$$

In particular, if the product $f g^q$ is homogeneous of degree $\binom{n}{2}$, then according to (5), (f, g) is of degree 0 in $\mathbf{x}_n$ and its value is obtained by extracting in $f g^q$ the monomials which are permuted from x^ρ.

The following lemma describes the compatibility of the action of the Hecke algebra with the quadratic form.

Lemma 5 *Let $f, g \in \mathfrak{Pol}(\mathbf{x}_n)$, $i \in \{1, \ldots, n{-}1\}$, $k \geq 1$. Then*

$$\big(f T_i(-k) \, , \, g\big) \;=\; \big(f \, , \, g T_i(k)\big) \, , \tag{16}$$

$$\Big(f T_i(-k) \frac{[k-1]}{[k]} \, , \, g\Big) \;=\; -\Big(f \, , \, g T_i(k) \frac{[k-1]}{[k]}\Big) \, . \tag{17}$$

Proof. Suppose $k = 1$. Then $T_i(-1) = \partial_i(q_2 x_i + q_1 x_{i+1})$,

$$\left(f T_i(-1), g\right) = f \partial_i(q_2 x_i + q_1 x_{i+1}) g^q \partial_\omega = f \partial_i \left(g(q_1 x_i + q_2 x_{i+1})\right)^q \partial_\omega$$
$$\left(g(q_1 x_i + q_2 x_{i+1})\partial_i\right)^q f \partial_\omega = \left(g \partial_i(q_1 x_{i+1} + q_2 x_i) + q_1 - q_2\right)^q f \partial_\omega$$
$$= \left(f, g(T_i - q_1 + q_1 - q_2)\right) = \left(f, g(T_i(1))\right),$$

using the Leibniz's formula $gh\partial_i = g\partial_i h^{s_i} + h\partial_i g$. Since

$$T_i(-k) = T_i(-1) - q_1 q_2 \frac{[k-1]}{[k]}$$

$$T_i(k) = T_i(1) + q_1 q_2 \frac{[k-1]}{[k]} = T_i(1) - q_1 q_2 \left(\frac{[k-1]}{[k]}\right)^q,$$

the case $k = 1$ implies the case for any k, that is (5). The factor $[k-1][k]^{-1}$ creates a sign in the second equation. QED

Lemma 6 *Let λ be a partition of n, $\mu = \lambda^\sim$. Then $(Y_v, \widetilde{\Omega}_\lambda) = 0$, except $(Y_{-v^\mu}, \widetilde{\Omega}_\lambda) = d_\lambda$ for some non-zero function d_λ of q_1, q_2.*

Proof. The Kazhdan-Lusztig basis is triangular in the Young basis, with respect to any order compatible with the graph $\Gamma_\lambda^{\mathcal{H}}$. If a tableau t and i are such that t contains the subword $i+1, i$, then according to (15), one has $L_t = f(q_2 x_i + q_1 x_{i+1})$, with $f \in \mathfrak{Sym}(x_i, x_{i+1})$. Similarly, if $s_i \in \mathfrak{S}_\mu$, then $\Delta_\mu^q = g(q_2 x_i + q_1 x_{i+1})$, with $g \in \mathfrak{Sym}(x_i, x_{i+1})$. In that case, $(L_t, \Delta_\mu^q) = f g^q (q_2 x_i + q_1 x_{i+1})(q_1 x_i + q_2 x_{i+1}) \partial_\omega$ is null because of the symmetry in x_i, x_{i+1}. Thus one can expect $(L_t, \Delta_\mu^q) \neq 0$ only if t contains the subwords $1 \ldots \mu_1$, $(\mu_1+1) \ldots (\mu_1+\mu_2), \ldots$. In other words t must be the tableau of shape λ with rows filled with consecutive integers, denoted $\aleph$. To show that d_λ is different from 0, one can specialize $q_1 \to 1$, $q_2 \to 0$. The operators $T_i(-k)$ tend to $\partial_i x_{i+1} = \pi_i - 1$, and one sees easily that $L_\aleph$ specializes to $x^{0^{\mu_1} 1^{\mu_2} \cdots}$ and that $\widetilde{\Omega}_\lambda^q$ specializes to $x^{\wp\lambda} \partial_{\omega_\mu} x^{\rho_\mu}$. A little familiarity with divided differences allows us to conclude that $(L_\aleph, \widetilde{\Omega}_\lambda)$ specializes to ± 1, which finishes the proof of the lemma. QED

Combining Lemmas 5 and 6, denoting now the starting element $\widetilde{\Omega}_\lambda$ by $\widetilde{Y}_{v^\mu}$, one otains

Theorem 7 *Let λ be a partition of n, $\mu = \lambda^\sim$. Let $\{Y_v, v \in \mathcal{C}(\lambda)\}$ be the Young basis of the module $\Delta_\lambda^q \mathcal{H}_n$, $\{\widetilde{Y}_u, u \in \mathcal{C}(\mu)\}$ be the Young basis of the module $x^{\wp\lambda} C_{\omega_\mu} \mathcal{H}_n$. Let $\epsilon_u = \pm 1$ according to the parity of the distance of $u \in \mathcal{C}(\mu)$ to v^μ. Then there exists a non-zero function d_λ of q_1, q_2 such that*

$$(Y_v, \widetilde{Y}_u) = \epsilon_u \, \delta_{v,-u} \, d_\lambda. \tag{18}$$

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