DYNAMICS OF IRREDUCIBLE ENDOMORPHISMS OF F_N

BY

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DISSERTATION

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Abstract

We consider the class non-surjective irreducible endomorphisms of the free group F_n. We show that such an endomorphism ϕ is topologically represented by a simplicial immersion $f : G \to G$ of a marked graph G; along the way we classify the dynamics of $\partial\phi$ acting on ∂F_n: there are at most $2n$ fixed points, all of which are attracting. After imposing a necessary additional hypothesis on ϕ, we consider the action of ϕ on the closure $\overline{CV}_n$ of the Culler-Vogtmann Outer space. We show that ϕ acts on $\overline{CV}_n$ with "sink" dynamics: there is a unique fixed point $[T_\phi]$, which is attracting; for any compact neighborhood N of $[T_\phi]$, there is $K = K(N)$, such that $\overline{CV}_n\phi^{K(N)} \subseteq N$. The proof uses certian projections of trees coming from invariant length measures. These ideas are extended to show how to decompose a tree T in the boundary of Outer space by considering the space of invariant length measures on T; this gives a decomposition that generalizes the decomposition of geometric trees coming from Imanishi's theorem.

The proof of our main dynamics result uses a result of independent interest regarding certain actions in the boundary of Outer space. Let T be an $\mathbb{R}$-tree, equipped with a very small action of the rank n free group F_n, and let $H \leq F_n$ be finitely generated. We consider the case where the action $F_n \curvearrowright T$ is indecomposable–this is a strong mixing property introduced by Guirardel. In this case, we show that the action of H on its minimal invariant subtree T_H has dense orbits if and only if H is finite index in F_n. There is an interesting application to dual algebraic laminations; we show that for T free and indecomposable and for $H \leq F_n$ finitely generated, H carries a leaf of the dual lamination of T if and only if H is finite index in F_n. This generalizes a result of Bestvina-Feighn-Handel regarding stable trees of fully irreducible automorphisms.

In fond memory of Dr. George H. Connor Jr.

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Thanks also go to Vincent Guirardel for indicating that finite covers of indecomposable trees are also indecomposable and for mentioning an idea that eventaully lead to a proof of this fact. I am also grateful to Vincent Guirardel for explaining some ideas from his new work with G. Levitt [26]; this lead to an improved exposition in Sections 5 and 5.3 coming from the use of the iterative procedure explained in Remark 5.11. More importantly, Guirardel pointed out a gap in an earlier approach: instead of Proposition 5.16, I was using a result of Plante [39, Theorem 3.1], which Guirardel noticed to be incorrect. Guirardel gave an enlightening counterexample and pointed out that [39, Theorem 3.2] can be proved under a stronger hypothesis (mixing). After appropriately weaking the hypothesis of Guirardel to suit my needs, I arrived at the proof of Proposition 5.16, which generalizes a result of [26] and is essential to the approach presented here.

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Chapter 1

Introduction

1.1 Overview of Outer Space

In [17], Culler and Vogtmann defined a contractible, finite dimensional topological space on which the outer automorphism group $Out(F_n)$ of the free group acts properly; this space, which we denote by CV_n, has come to be known as Outer space. Culler and Vogtmann showed that there is an $Out(F_n)$-equivariant deformation retraction of CV_n onto its spine $K_n \subseteq CV_n$, on which the $Out(F_n)$-action is cocompact; this allowed them to prove that $Out(F_n)$ is of type VFL and that $Out(F_n)$ has finite virtual cohomological dimension. The idea of Culler-Vogtmann was to introduce an $Out(F_n)$-analogue of Teichmüller space, so as to allow for an adaptation of the Thurston program for studying (mapping classes of) surface diffeomorphisms to the study of elements of $Out(F_n)$.

We now give an intuitive explanation of Outer space; see also Section 2.3. Let $\mathscr{B}'$ denote the set of bases of F_n; there is a transitive action of $Aut(F_n)$ on $\mathscr{B}'$, and the stabilizer of a point $B \in \mathscr{B}'$ is just the group of permutations of the elements of B. Taking the point of view that two bases $B_1, B_2 \in \mathscr{B}$ are not so different if they are conjugate, *i.e.* if there is $g \in F_n$ such that $B_2 = \{gbg^{-1} | b \in B_1\}$, one is left with the set $\mathscr{B}$ of conjugacy classes of bases of F_n, and the group of inner automorphisms acts trivially on $\mathscr{B}$. Hence the action of $Aut(F_n)$ on $\mathscr{B}$ factors through to an action of $Out(F_n)$ on $\mathscr{B}$, and this action is transitive and has finite point stabilizers. One wants to embed $\mathscr{B}$ into some space in which it is possible to continuously deform a particular (conjugacy class of a) basis into a different basis.

Fix a basis $\{x_1, ..., x_n\}$ for F_n, and let Γ denote the associated Cayley graph, which is a regular $2n$-valent tree, and carries a free action of F_n; the quotient Γ/F_n is a wedge of n circles, which one may think of as being labeled by $x_1, ..., x_n$. Now consider the automorphism $\alpha \in Aut(F_n)$, $\alpha(x_1) = x_1x_2$, and $\alpha(x_i) = x_i$ for $i \neq 1$. One gets a "twisted" action F_n on Γ, where for each

$x \in \Gamma$, and each $g \in F_n$, one defines $g.x := \alpha(g)x$. Again, the quotient Γ/F_n is a wedge of n circles; however, the labeling has changed: the edge formerly labeled by x_1 is now labeled by $x_1 x_2^{-1}$, representing the fact that in the twisted action x_1 acts as $x_1 x_2$ on the original tree Γ. There is a clear way to "deform" the original labeled graph to this new labeled graph: drag the endpoint of the x_1-edge across the x_2 edge in the "negative direction." To make sense of this, one needs to orient the edges of Γ/F_n; further, metrizing Γ/F_n gives a way of parametrizing the process described.

A finite graph G has *rank n* if $rank(\pi_1(G)) = n$. Let X'_n denote the set of rank n graphs G such that G is not homotopy equivalent to a proper subgraph, such that no vertex of G is of valence-2, such that every edge of G has been identified with closed interval $\mathbb{R}$ of finite, non-zero length, and such that the sum of the lengths of these intervals is 1. A *marking* on a (oriented, metric) graph $G \in X'_n$ is an identification $\rho : \pi_1(G) \cong F_n$; one calls (G, ρ) a *marked metric graph.* Let X_n denote the set of marked metric graphs. For any $(G, \rho) \in X_n$, one has that the universal cover $\tilde{G}$ is a *simplicial* $\mathbb{R}$-tree (see Chapter 2), and ρ gives an isometric action of F_n on $\tilde{G}$. According to [16], the set X_n maps injectively into the space of *projective length functions* on F_n, and this gives a way of topologizing X_n (see Chapter 2); the resulting space is CV_n, the Culler-Vogtmann Outer space. In a precise sense, CV_n is a "deformation space of hyperbolic structures on F_n," in strong analogy with Teichmüller space. Further, by [16], CV_n has compact closure in the space of length functions, so CV_n has a "Thurston compactification" $\overline{CV}_n$, whose elements are *homothety classes* of (very small) actions of F_n on $\mathbb{R}$-trees [17, 9, 2].

1.2 Irreducible Endomorphisms

In what follows, F_n denotes the rank n free group; $\overline{CV}_n$ denotes the "Thurston compactification" of the Culler-Vogtmann Outer space; and $\overline{cv}_n$ denotes the space of very small actions of F_n on $\mathbb{R}$-trees, so $\overline{CV}_n$ consists of projective classes $[T]$ of trees $T \in \overline{cv}_n$; see Section 2.3 for definitions.

In [6] Bestvina and Handel introduce the notion of an *irreducible* outer automorphism of the free group F_n (see Section 3); this class of (outer) automorphisms serves as an analog of the class of (mapping classes of) pseudo-Anosov diffeomorphisms of a hyperbolic surface. Bestvina and Handel introduce *train track represetatives* for irreducible elements of $Out(F_n)$; these are topological representatives that allow for very close control over rates of growth of conjugacy classes. It is shown in [6] that any irreducible outer automorphism of F_n has a train track representative; see Section 3.1.2.

The terminiology of train tracks in [6] seems to anticipate the work in [3], where to each irreducible automorphism of F_n there is associated a pair of abstract laminations. These abstract laminations on F_n are formalized by Coulbois, Hilion, and Lustig in [12] via the notion of an *algebraic lamination* (see Subsection 2.4); this formalism, as well as its applications in [13, 14], does well to compliment the theory of train tracks for studying free group (outer) automorphisms. The algebraic laminations associated to a free group automorphism are analogous to the geodesic surface laminations associated to a pseudo-Anosov surface diffeomorphism: they are a sort of asymptotic invariant encoding limits of iterates of the automorphism (and its inverse) on primitive elements–the free group analogs of essential simple closed curves on a surface. Generalizations of the tools of [3] and [6] were used by Bestvina-Feighn-Handel to prove the Tits Alternative for $Out(F_n)$ [4, 5].

Building on the techniques of [3], Levitt and Lustig show in [34] that any irreducible outer automorphism of F_n acts on $\overline{CV}_n$ with north-south dynamics. This is analogous to the well-known result that a pseudo-Anosov mapping class acts on $\overline{\mathscr{T}(S)}$ with north-south dynamics, though, as appears to be typical, the result for $Out(F_n)$ is much more difficult to prove.

Inspired by the applicability of these dynamical techniques for understanding elements of $Out(F_n)$, we study non-surjective irreducible (outer) endomorphisms of F_n from a dynamical viewpoint. An endomorphism $\phi : F_n \to F_n$ is *irreducible* if no power of ϕ maps a non-trivial, proper free factor of F_n into a conjugate of itself, and if this condition holds for any power of ϕ as well; see Section 3.

Suppose that $\phi : F_n \to F_n$ is irreducible; it follows from work of Bestvina-Feighn-Handel [3] that one may associate to ϕ an algebraic lamination Λ_ϕ and a (projective) *stable tree* $[T_\phi] \in \overline{CV_n}$; see Subsections 3.4.1 and 3.1.2. There is a natural right action of ϕ on the set of $\mathbb{R}$-trees, equipped with minimal, isometric actions of F_n, and $[T_\phi]$ has the property that $[T_\phi \phi] = [T_\phi]$. Using the techniques of [3], Coulbois-Hilion have shown that for irreducible $\alpha \in Out(F_n)$, the stable tree T_α has a strong mixing property–it is *indecomposable* [10]; see Definition 4.16. In constrast with the case of outer automorphisms, we obtain (Proposition 3.21):

Proposition 1.1. *Let $\phi : F_n \to F_n$ an irreducible endomorphism that is non-surjective. There is a free simplicial F_n-tree T_ϕ such that $[T_\phi]\phi = [T_\phi]$.*

This immediately gives (Corollary 3.23):

Corollary 1.2. *Let $\phi : F_n \to F_n$ an irreducible endomorphism that is non-surjective. Then (the outer class of) ϕ is topologically-represented by a train track map with no illegal turns.*

Building on the techniques of [3], Levitt and Lustig show in [34] that any irreducible outer automorphism of F_n acts on $\overline{CV}_n$ with north-south dynamics: there are exactly two fixed points, one attracting and one repelling, such that convergence to the attractor is uniform on compact subsets avoiding the repeller.

Unlike in the case of $Out(F_n)$ one needs to impose an additional assumption on a non-surjective irreducible endomorphism ϕ to ensure that the action of ϕ on the set of F_n-trees induces an action on $\overline{CV_n}$; we call such ϕ *admissible* (see Section 3.5). In this case we consider the dynamics of the action of ϕ on $\overline{CV_n}$; we show (Theorem 5.21):

Theorem 1.3. *Let $\phi : F_n \to F_n$ be an admissible irreducible endomorphism that is non-surjective. Then ϕ has a unique fixed point $[T_\phi] \in \overline{CV_n}$, which is free and simplicial; for any $[T] \neq [T_\phi]$ one has that $[T]\phi^k \to [T_\phi]$; and for any compact nieghborhood N of $[T_\phi]$, there is $k = k(N)$ such that $\overline{CV}_n\phi^k \subseteq N$.*

It should be noted that Theorem 1.9 is novel in the sense that ϕ is not assumed to be invertible. This result turns out to be much more difficult to prove than North-South dynamics for irreducible automorphisms of F_n [34], which is in turn much more difficult to prove than North-South dynamics for pseudo-Anosov surface automorphisms. The latter two results use “backwards iteration” in an essential way, and it is reasonable to say that many of the complications in the present work stem from the lack of an inverse.

Theorem 1.3 implies that for k sufficiently large, the subgroups $\phi^k(F_n)$ have a strong rigidity property (Corollary 5.22), which seems interesting to us:

Corollary 1.4. *For any $C > 1$, there is a finitely generated, non-abelian subgroup $H \leq F_n$, such* *that for any non-trivial* $h, h' \in H$ *and any trees* $T, T' \in \overline{cv}_n$*, one has* $l_T(h) > 0$ *and*

$$\frac{1}{C} \leq \frac{l_T(h)/l_T(h')}{l_{T'}(h)/l_{T'}(h')} \leq C$$

1.3 Indecomposable Trees in the Boundary of Outer Space

The proof of Theorem 5.21 makes use of a result that, for certain actions $T \in \overline{cv}_n$, strongly restricts the way that a finitely generated subgroup $H \leq F_n$ can act on its minimal invariant subtree.

Let G be a finitely generated group, and suppose that $G \curvearrowright T$ is an action by isometries of G on an $\mathbb{R}$-tree T.

Definition 1.5. Following [25], we say that the action $G \curvearrowright T$ is *indecomposable* if for any non-degenerate arcs $I, J \subseteq T$, there are elements $g_1, ..., g_r \in G$ such that $J \subseteq g_1 I \cup ... \cup g_r I$ and such that $g_i I \cap g_{i+1} I$ is non-degenerate for $i \leq r - 1$.

It is important to note that the intersections $g_i I \cap g_{i+1} I$ need not be contained in J, or even interect J non-degenerately; see [25] for further discussion. Indecomposability of the action $G \curvearrowright T$ is a strong mixing property; it prohibits the existence of a *transverse family* for the action $G \curvearrowright T$ (see Definition 3.13). In particular, if the action $G \curvearrowright T$ is indecomposable, then $G \curvearrowright T$ cannot be written as a non-trivial *graph of actions* (see [32, 25]). If $H \leq G$ is a finitely generated subgroup containing a hyperbolic isometry of T, then there is a canonical minimal subtree T_H for the action $H \curvearrowright T$; notice that if the action $G \curvearrowright T$ has dense orbits, and if $H \leq G$ is a finitely generated, finite index subgroup, then the action $H \curvearrowright T$ has dense orbits as well. The main result of this paper says that, in some sense, certain indecomposable actions cannot contain any interesting subactions other than the obvious ones.

Let $\overline{cv}_n$ denote the unprojectivised closed Outer space, *i.e.* the space of very small actions of F_n on $\mathbb{R}$-trees (see Definition 2.4); we show (Theorem 4.11):

Theorem 1.6. *Suppose that* $T \in \overline{cv}_n$ *is indecomposable, and let* $H \leq F_n$ *be finitely generated. The action* $H \curvearrowright T_H$ *has dense orbits if and only if* H *has finite index in* F_n*.*

There is a nice application of Theorem 4.11 to *algebraic laminations*: associated to any action $F_n \curvearrowright T$ of F_n on a tree is a *dual lamination* $L^2(T) \subseteq \partial^2(F_n)$, which is an algebraic analog of a surface lamination; here $\partial^2(F_n) := \partial F_n \times \partial F_n - \Delta$ (see section 2.3 for an brief introduction, and [12, 13] for details). We say that a finitely generated subgroup $H \leq F_n$ *carries* a leaf $l \in L^2(T)$ if $l \in \partial^2(H) \subseteq \partial^2(F_n)$; the following result appears as Corollary 4.15 below.

Corollary 1.7. *Suppose that $T \in \overline{cv}_n$ is indecomposable and free with dual lamination $L^2(T)$, and let $H \leq F_n$ be finitely generated. Then H carries a leaf of $L^2(T)$ if and only if H is finite index in F_n.*

The reason for the assumption that the action be free comes from the definition of the dual lamination of an action $F_n \curvearrowright T$; namely, if $K \leq F_n$ has a fixed point in T, then $\partial^2(K) \subseteq L^2(T)$. Further, since the action $F_n \curvearrowright T$ is minimal, it is the case that K is infinite index in F_n.

The results of this paper can be thought of as a dynamical-algebraic analogy between indecomposable trees in the boundary of Outer space and ending laminamtions on surfaces. A lamination L on a compact surface S (possibly with boundary) is *minimal* if every half leaf of L is dense in L, and L is *filling* if all complimentary regions are ideal polygons or crowns. If L is minimal and filling, then L is called an *ending lamination* (see [8] for background on suface laminations). If $S_1 \to S$ is a finite cover, and if L_1 is the lift of L to S_1, then L_1 is an ending lamination. Indeed, a finite cover of an ideal polygon (resp. crown) is an ideal polygon (resp. crown), and it is an exercise to check that L_1 is minimal.

In [41] Scott proves that suface groups are subgroup separable (or LERF); his proof is geometric: he finds, for any finitely generated subgroup $H \leq \pi_1(S)$, a finite cover $S_1 \to S$, a compact surface S', along with a π_1-injective embedding $\iota : S' \to S_1$ such that $\pi_1(\iota)(\pi_1(S')) = H$. This geometric description of subgroups of $\pi_1(S)$ gives a clear picture of which subgroups of $\pi_1(S)$ are able to "encode" leaves of the lamination on S. Say that a finitely generated subgroup $H \leq \pi_1(S)$ *carries* a leaf l of L if there are S_1, ι, and S' as above, such that a lift of l in S_1 is contained in $\iota(S')$. If S is equipped with an ending lamination L, it is evident that the lifted lamination L_1 on S_1 intersects $\iota(S')$ in finite arcs, unless $\iota(S') = S_1$, *i.e.* unless H is finite index in $\pi_1(S)$. It follows that no finitely generated subgroup of infinite index carries a leaf of L. Now suppose that $\mathscr{L} = (L, \mu)$ is a measured lamination with L an ending lamination, and let $\mathscr{L}_1 = (L_1, \mu_1)$ the lift of $\mathscr{L}$ to S_1. Let $T_{\mathscr{L}}$ denote the $\mathbb{R}$-tree dual to $\mathscr{L}$, and let $T_H \subseteq T$ the minimal invariant subtree for the action of H on T. Evidently, T_H is "dual" to $\mathscr{L}_1' := (L_1 \cap \iota(S'), \mu_1|_{\iota(S')})$, so the action $H \curvearrowright T_H$ is discrete, again unless H is finite index in $\pi_1(S)$. Hence, it follows that H carries a leaf of L if and only if H is finite index in $\pi_1(S)$ if and only if the action $H \curvearrowright T_H$ is indiscrete.

It is an exercise to check that if an action $\pi_1(S) \curvearrowright T_{\mathscr{L}}$ is dual to a measured ending lamination $\mathscr{L}$ on S, then the action is indecomposable; see [25, Proposition 1.25]. On the other hand, it follows from Skora's duality theorem [43] and the Rips theory ([1], [21]) that any indecomposable, relatively elliptic action $\pi_1(S) \curvearrowright T$ is dual to an ending lamination on S; here *relatively elliptic* means that the (maximal) elliptic subgroups of the action $\pi_1(S) \curvearrowright T$ are precisely the peripheral subgroups of $\pi_1(S)$. There are other natural examples of indecomposable trees. The first come from the Rips theory: any geometric tree dual to a minimal band complex is indecomposable (see [1] for explanation of terminology and [25] for a proof); this includes the "surface trees" mentioned above as well as the so-called thin (or exotic, or Levitt) trees (see [1, 21, 30, 25] for details). Finally, stable trees of fully irreducible (iwip) automorphisms are indecomposable; this can be shown using the machinery of [6] and [3]. There are examples of such "iwip trees" that are not geometric [1].

As mentioned above, the main results of this paper are known for surface trees. Using train track machinery, Bestvina-Feighn-Handel establish these results in the special case of stable trees of fully irreducible automorphisms ([3, Theorem 5.4] and [3, Proposition 2.4]. We remark that it follows from the North-South dynamics [34] that any stable tree of a fully irreducible automorphism is uniquely ergodic (see Section 3.0 below); on the other hand, [35] establishes the existence of non-uniquely ergodic thin band complexes, so the theorem is saying something new even in the case of geometric trees.

The inspiration for the proof of the main result is precisely the discussion presented above regarding the dynamical-algebraic properties of ending laminations and their dual trees; in fact, the skeleton of the current proof is essentially identical to that surface theory argument. The first ingredient is Lemma 4.8, which says that any "finite cover" of an indecomposable action $G \curvearrowright T$ is also indecomposable. We then establish a certain measure-theoretic approximation of actions $F_n \curvearrowright T \in \overline{cv}_n$ with dense orbits: we show that any such action is "supported almost everywhere" on a finite forest of arbitrarily small measure, and this allows us to construct from the action $F_n \curvearrowright T$ a finitely generated pseudogroup (see Definition 2.14) with well-controlled dynamics (see Lemma 4.2). All this is combined with an inequality of Gaboriau-Levitt-Paulin to greatly restrict the "shape" of families $\{gT_H\}_{g\in F_n}$ for e-algebraically closed subgroups $H \leq F_n$ (see Definition 4.4). Finally, the strong subgroup separability of F_n is used to conclude.

1.4 Structure of Trees in the Boundary of Outer Space

Along the way to proving Theorem 1.3, we introduce the rudiments of a decomposition theory of individual trees $T \in \overline{cv}_n$. At the heart of this approach is a study of the space $M_0(T)$ of invariant, non-atomic *length measures* on T (Section 5); these objects, introduced by Paulin, generalize measured laminations on surfaces. In [23] Guirardel uses length measures as part of an approach to study the dynamics of $Out(F_n)$ acting on the boundary of $\overline{CV}_n$; there he shows that for $T \in \overline{cv}_n$ with dense orbits, the projectivization of $M_0(T)$ embeds in the boundary of $\overline{CV}_n$. This shows, in particular, that the space $M_0(T)$ is finite dimensional.

We now briefly recall the structure of trees dual to measured laminations on surfaces, in order to contrast with the trees in $\overline{cv}_n$. Let $\mathscr{L} = (L, \mu)$ be a measured lamination on a surface S, and let $T = T_{\mathscr{L}}$ denote the dual tree; see, for example, [38] or [30]. If L is not minimal, then T can be decomposed in a way that parallels the decompositon of L into minimal components–T is a *graph of actions*; see Section 3.2. A feature of (minimal) *arational measured laminations* is that every half-leaf is dense; this implies that a tree dual to an arational measured laminiation is indecomposable. It follows that, if L has no compact leaf, then either T is indecomposable, or T splits as a graph of indecomposable actions.

The structure of some trees in $\overline{cv}_n$ is divergent from this picture: there are trees $T \in \overline{cv}_n$ such that T is neither indecomposable nor a graph of actions; see Example 5.9. There is a holonomy pseudogroup associated to T, which is completely analogous to the holonomy pseudogroup associated to a lamination, and this psuedogroup contains an *exceptional* set, in contrast with the surface case; see Section 5. To understand the dynamical structure of T, it is useful to consider certain *projections* of T; for this approach it is critical that the dynamics of the action $F_n \curvearrowright T$ are "visible" to length measures. This is accomplished via the following (Proposition 5.16):

Proposition 1.8. *Let $T \in \overline{cv}_n$ have dense orbits. Suppose that T does not split as a graph of actions and that T contains an exceptional set X. Then there is $\mu \in M_0(T)$ supported on X.*

A non-empty F_n-invariant subset $X \subseteq T$ is called *exceptional* if for any finite subtree $K \subseteq T$, $X \cap K$ is empty, finite, or a Cantor set with finitely many points added. According to Proposition 1.8, given a tree $T \in \overline{cv}_n$ with dense orbits, such that T is not a graph of actions and such that T contains an exceptional set X, we can find an invariant measure μ supported on X. We may then pass to a *projection* of T: the measure μ gives rise to a pseudometric d_μ on T, where $d_\mu(x, y) := \mu([x, y])$. Making this pseudometric Hausdorff gives a tree T_μ, equipped with an isometric action of F_n (see [23, 11]); in a precise sense, the action $F_n \curvearrowright T_\mu$ isolates the dynamics of $F_n \curvearrowright X$.

Note that an F_n-tree T can be non-uniquely ergodic, even if it has strong mixing properties: examples already come from non-uniquely ergodic, arational laminations on surfaces [31]. Define a partial order $\leq$ on $M_0(T)$ via $\mu \leq \mu'$ if and only if $Supp(\mu) \subseteq Supp(\mu')$; this gives an equivalence with classes $[[\mu]] = \{\nu \in M_0(T) | Supp(\nu) = Supp(\mu)\}$, which serve as candidates for the "components" of T. Indeed, in the case that T is dual to a measured lamination (λ, μ_0) on a surface with boundary, the set of $[[.]]$-classes of invariant (non-atomic) length measures on T bijectively corresponds to the set of sublaminations of λ.

Having understood that some dynamical structure of T is clarified by considering projections of T, we complete our analogy with measured laminations by associating to every $[[.]]$-class of ergodic measures in $M_0(T)$ a canonical *mixing* action (Definition 5.27, Proposition 5.29, and Corollary 5.32). Below we give a simplified form of our decomposition result (Theorem 5.34).

Theorem 1.9. *Let $T \in \overline{cv}_n$ have dense orbits, and let $\{\nu_1, ..., \nu_r\}$ be a maximal set of mutually-singular ergodic measures.*

(i) For each ν_i with non-degenerate support, there is associated to $[[\nu_i]]$ a mixing action $H([[\nu_i]]) \curvearrowright T([[\nu_i]])$, such that $T([[\nu_i]])$ is unique up to translation in T,

(ii) For each ν_j with degenerate support, there is a projection $T \to T_{[[\nu_j]]}$, such that:

(a) $dim(M_0(T_{[[\nu_j]]})) < dim(M_0(T))$,

(b) for all $\nu_j' \in [[\nu_j]]$, $M_0(T_{\nu_j'})$ is naturally identified with $M_0(T_{[[\nu_j]]})$,

(c) for all $\nu_j' \in [[\nu_j]]$, $L^2(T_{\nu_j'}) = L^2(T_{[[\nu_j]]})$

In the statement, $M_0(T)$ denotes the space of non-atomic invariant length measures on T. The subtree $T([[\nu_i]]) \subseteq T$ is "equal to" the support set $Supp(\nu_i)$, and $H([[\nu_i]])$ is the setwise stabilizer of $T([[\nu_i]])$. In short, part (i) of Theorem 2 is analogous to the usual dynamical decomposition of measured surface laminations, and, more generally, the decomposition of *geometric trees* coming from Imanishi's Theorem; see [21]. So, part (ii) illustrates only *non-geometric* phenomena.

Theorem 1.9 is related to ongoing work (in preparation) of Guirardel and Levitt about actions of finitely presented groups on R-trees [26]; see the Acknowledgements, as well as Sections 5 and 5.3 for further discussion.

1.5 Organization

In Chapter 2 we collect basic background material about $\mathbb{R}$-trees, Outer space, algebraic laminations, the observers' topology on trees, length measures, and pseudogroups; this section is expository, except for Lemma 2.12.

Chapter 3 introduces topological representatives, train tracks and stable trees; there we introduce *expansive* (Definition 3.8) endomorphisms and show (Proposition 3.11) that an irreducible endomorphism is either expansive or an automorphism; this gives, via Corollary 3.10, that stable trees of non-surjective irreducible endomorphisms are free.

Section 3.2 is devoted to defining and giving basic properties of *graphs of actions* (Definition 3.14), which are used in Section 3.3 to show (Proposition 3.21) that the stable tree of a non-surjective irreduicble endomorphism is free and simplicial. This immediately implies that such an endomorphism is topologically represented by a simplicial immersion (Corollary 3.23). The proof of Proposition 3.21 shows (Corollary 3.22) that for $\phi : F_n \to F_n$ irreducible and non-surjective, $\partial\phi$ acts on ∂F_n with finitely-many fixed points, all of which are attracting.

We then turn to the question of the dynamics of ϕ acting on Outer space, denoted CV_n, and its closure, denoted $\overline{CV}_n$. Section 3.4 introduces the stable lamination, denoted Λ_ϕ, associated to ϕ. We then state the convergenge criterion that will be used for the sequel: Proposition 3.24, which is due to Bestvina-Feighn-Handel. This immediately gives that ϕ acts on CV_n with precisely one (attracting) fixed point–the stable tree of ϕ.

Before proceeding to consider the dynamics of ϕ acting on $\overline{CV}_n$, we must impose a condition to ensure that ϕ acts on $\overline{CV}_n$; we explain what can go wrong in Section 3.5, and then give the definition for *admissible* endomorphisms (Definition 3.27). So that we may apply our convergence criterion, we show that being admissible is equivalent to a condition on Λ_ϕ, namely that no leaf of Λ_ϕ is carried by a vertex group of a *very small splitting* of F_n (Proposition 3.32).

Using Proposition 3.32 along with Proposition 3.24, in Section 3.6 we get convergence for trees in the boundary of Outer space that split as a non-trivial graph of actions.

In Chapter 4, we prove the results needed to handle convergence for indecomposable actions (Definition 4.16); the main result is Theorem 4.11. In Section 4.3 we handle convergence for indecomposable actions.

Chapter 5 collects the relevant measure theory on trees. We use the key result of Guirardel (Proposition 2.15), that for T with dense orbits, the space $M_0(T)$ of invariant non-atomic measures on T is finite dimensional. Next, we define *exceptional sets* (Definition 5.8) and provide Example 5.9 to show that such things actually occur. We present an iterative procedure (see Remark 5.11), due to Guirardel-Levitt [26], for building *transverse families* (Definition 3.13) of subtrees. In Subsection 5.1.4, we prove the critical result (Proposition 5.16) for the rest of the paper: if T is a tree in the boundary of Outer space with dense orbits, and if T does not split as a graph of actions, then any exceptional set in T supports an invaraint measure.

In Section 5.2 we combine Proposition 5.16 with Lemma 2.12 to get convergence for the remainder of actions in $\overline{CV}_n$. The dynamics of ϕ acting on $\overline{CV}_n$ (Theorem 5.21) then easily follows.

Section 5.3 elaborates upon the measure-theoretic techniques introduced in Section 5 to present an approach to decomposing trees in the boundary of Outer space. For the remainder of the paper, we consider a tree T with dense orbits. In Subsection 5.3.1, we define a transverse family $\mathscr{F}$ that gives a coarse decomposition of T (Lemma 5.23). We then bring Proposition 5.29 and Corollary 5.32 to show how to associate to every invariant measure on T a canonical mixing action; these actions are “building blocks” of T. We collect the results of Section 5.3 to give our decomposition result, Theorem 5.34.

Chapter 2

Background

In what follows F_n denotes the free group of rank n; for $g \in F_n$ let $[g]$ denote the conjugacy class of g.

2.1 Basics About $\mathbb{R}$-Trees

A metric space (T, d) is called an $\mathbb{R}$-*tree* (or just a *tree*) if for any two points $x, y \in T$, there is a unique topological arc $p_{x,y} : [0, 1] \to T$ connecting x to y, and the image of $p_{x,y}$ is isometric to the segment $[0, d(x, y)] \subseteq \mathbb{R}$. We let $[x, y]$ stand for $\mathrm{Im}(p_{x,y})$, and we call $[x, y]$ the *segment* (also called an *arc*) in T from x to y. A segment is called *non-degenerate* if it contains more than one point. Hence, the first examples of $\mathbb{R}$-trees are just connected subsets of $\mathbb{R}$. We let $\overline{T}$ stand for the metric completion of T. Unless otherwise stated, we regard T as a topological space with the metric topology.

2.1.1 Simplicial $\mathbb{R}$-Trees

A *simplicial tree* is (the geometric realization of) a 1-dimensional CW-complex containing no embedded copy of S^1. If T is a simplicial tree, then T can be given the structure of an $\mathbb{R}$-tree as follows: choose for each edge e of T a number $l(e) \in \mathbb{R}_{>0}$, and identify e with the segment $[0, l(e)] \subseteq \mathbb{R}$. It is important to note that the metric topology on T will be weaker than the cellular topology in the case that T is not locally finite.

We will say that an $\mathbb{R}$-tree T is *simplicial* if T can be obtained from a simplicial tree via the above procedure.

Example 2.1. Let $||.||$ denote the Euclidean norm on $\mathbb{R}^n$, and consider a metric ρ given by $\rho(x, y) := ||x - y||$, if there is $r \in \mathbb{R}$ such that $y = rx$, and $\rho(x, y) := ||x|| + ||y||$ otherwise. Then

$(\mathbb{R}^n, \rho)$ is a simplicial $\mathbb{R}$-tree.

Intuitively, the metric ρ regrads $\mathbb{R}^n$ as the union of rays eminating from the origin.

2.1.2 non-Simplicial $\mathbb{R}$-Trees and Branch Points

In the definition of a simplicial $\mathbb{R}$-tree, there is a well-defined notion of "vertex," coming from the underlying combinatorial structure; "branching" can occur at only such points, and though the set of these vertices is not necessarily discrete, it is, in some sense, quite small.

More precisely, if T is an $\mathbb{R}$-tree, and $x \in T$, then x is called a *branch point* if the cardinality of $\pi_0(T - \{x\})$ is greater than two. For $x \in T$, the elements of $\pi_0(T - \{x\})$ are called *directions* at x. We see that there is single branch point in the tree of Example 2.1.

Example 2.2. Consider a metric ρ' on $\mathbb{R}^2$ given by $\rho'((x_1, y_1), (x_2, y_2)) := |y_1 - y_2|$ if $x_1 = x_2$, and $\rho'((x_1, y_1), (x_2, y_2)) := |y_1| + |y_2| + |x_1 - x_2|$ otherwise. Then $(\mathbb{R}^2, \rho')$ is an $\mathbb{R}$-tree that is not simplicial, as every point on the "x-axis" is a branch point.

2.2 Group Actions on Trees

2.2.1 Isometries of $\mathbb{R}$-trees

We now recall some standard facts about isometries of $\mathbb{R}$-trees; see, for example, [16]. Let T be an $\mathbb{R}$-tree, and let $h \in Isom(T)$. If $h \in Isom(T)$ fixes a point of T, then h is called *elliptic*, and the collection $A(h) := \{y \in T | h(y) = y\}$ is a closed, convex subset of T. If h is not elliptic, *i.e.* for each $x \in T$, $h(x) \neq x$, then h leaves invariant a unique isometric copy $A(h)$ of $\mathbb{R}$ contained in T; $A(h)$ is called the *axis* of h. In this case, one says that h is *hyperbolic*, and for each $x \in A(h)$, one has $d(x, h(x)) = \inf_{y \in T} d(y, h(y)) =: l_T(h)$, the *translation length* of h. Hence, h acts on $A(h)$ as a translation.

2.2.2 Group Actions on $\mathbb{R}$-trees

Let T be an $\mathbb{R}$-tree. An isometric (left) action of finitely generated group G on T is a group morphism $\rho : G \to Isom(T)$; as usual, we always supress the morphism ρ and identify $g \in G$ with $\rho(g)$. A tree T equipped with an isometric action will be called an *G-tree*, and we denote this situation by $G \curvearrowright T$. An action $G \curvearrowright T$ induces an action of G on the set of directions at branch points of T. We identify two G-trees T, T' if there is an G-equivariant isometry between them.

Given a G-tree T, we have the so-called *hyperbolic length function* $l_T : G \to \mathbb{R}$, where

$$l_T(g) := \inf\{d(x, gx) | x \in T\}$$

The number $l_T(g)$ is called the *translation length* of g, and for any $g \in G$, the infimum is always realized on $A(g)$, so that g acts on $A(g)$ as a translation of length $l_T(g)$. The fuction l_T is constant on conjugacy classes in G. If $H \leq G$ is a finitely generated subgroup containing a hyperbolic isometry, then H leaves invariant the set

$$T_H := \cup_{l_T(h)>0} A(h)$$

which is a subtree of T, and is minimal in the set of H-invariant subtrees of T; T_H is called the *minimal invariant subtree for* H. An action $G \curvearrowright T$ is called *minimal* if $T = T_G$; a minimal action $G \curvearrowright T$ is *non-trivial* if T contains more than one point.

For an action $G \curvearrowright T$, and for $x \in T$, let $Gx := \{gx | g \in G\}$ denote the *orbit* of x. An action $G \curvearrowright T$ has *dense orbits* if for some $x \in T$, we have $\overline{Gx} = T$. Note that if some orbit is dense, then every orbit is dense.

2.2.3 Topology on the Set of G-Trees

Fix a group G. Recall that for each action $G \curvearrowright T$ of G on a tree, one has the associated length function $l_T : G \to \mathbb{R}_{\geq 0}$. Note that $l_T = l_{T_G}$, *i.e.* length fuctions encode information about minimal actions. Hence, an action $G \curvearrowright T$ is trivial if and only if $l_T = 0$.

One says that an action $G \curvearrowright T$ is *abelian* if $l_T(ghg^{-1}h^{-1}) = 0$ for all $g, h \in G$.

Lemma 2.3. *[16] Let $G \curvearrowright T_1$, $G \curvearrowright T_2$ be non-abelian actions on trees. If $l_{T_1} = l_{T_2}$, then there is a G-equivariant isometry $T_1 \to T_2$.*

Hence, setting $X(G) :=$ {minimal, non-trivial, non-abelian G-trees}, we get an injective map $X(G) \to \mathbb{R}^G$, and we take the induced topology on $X(G)$, *i.e.* a sequence (T_i) of non-ablelian G-trees converges to a G-tree T if and only if $l_{T_i}(g) \to l_T(g)$ for all $g \in G$.

The group $\mathbb{R}_{>0}$ acts on $X(G)$ by scaling the metric on a tree, and we let $\mathbb{P}X(G)$ denote the quotient space.

2.2.4 $Out(G)$ acts on $X(G)$

There is a natural action of $Aut(G)$ on $X(G)$: for $\alpha \in Aut(G)$, $l_{T\alpha}(g) := l_T(\alpha(g))$. As noted before, length functions are constant on conjugacy classes, so $Inn(G)$ lies in the kernel of this action; hence, we have an induced action of $Out(G) = Aut(G)/Inn(G)$ on $X(G)$. This action descends to an action of $Out(G)$ on $\mathbb{P}X(G)$.

2.3 Outer Space and its Closure

Recall that F_n denotes the rank-n free group. An action $F_n \curvearrowright T$ is *free* if for any $1 \neq g \in F_n$ one has $l_T(g) > 0$. If $X \subseteq T$, then the *stabilizer* of X is $Stab(X) := \{g \in F_n | gX = X\}$–the setwise stabilizer of X. We say that an action $F_n \curvearrowright T$ is *very small* if:

(i) $F_n \curvearrowright T$ is minimal,

(ii) for any non-degenerate arc $I \subseteq T$, $Stab(I) = \{1\}$ or $Stab(I)$ is a maximal cyclic subgroup of F_n,

(iii) stabilizers of tripods are trivial.

An action $F_n \curvearrowright T$ is called *discrete* (or *simplicial*) if the F_n-orbit of any point of T is a discrete subset of T; in this case T is obtained by equivariantly assigning a metric to the edges of a (genuine) simplicial tree; see subsection 2.1.1.

Let T, T' be trees; a map $f : T \to T'$ is called a *homothety* if f is F_n-equivariant and bijective, and if there is some positive real number λ such that for any $x, y \in T$, we have $d_{T'}(f(x), f(y)) = \lambda d_T(x, y)$; in this case T, T' are called *projectively equivalent* or *homothetic.*

Definition 2.4.

1. *The* unprojectivised Outer space *of rank n, denoted cv_n, is the topological space whose underlying set consists free, minimal, discrete, isometric actions of F_n on $\mathbb{R}$-trees; it is equipped with the* length function topology.

2. *[17] The* Culler-Vogtmann Outer space *of rank n, denoted CV_n, is the topological space whose underlying set consists of homothety classes of free, minimal, discrete, isometric actions of F_n on $\mathbb{R}$-trees; it is equipped with the* projective length function topology*.*

3. *The* unprojectivised closed Outer space *of rank n, denoted $\overline{cv}_n$, is the topological space whose underlying set consists of very small isometric actions of F_n on $\mathbb{R}$-trees; it is equipped with the* length function topology*.*

4. *The* closed Outer space *of rank n, denoted $\overline{CV}_n$, is the topological space whose underlying set consists of homothety classes of very small isometric actions of F_n on $\mathbb{R}$-trees; it is equipped with the* projective length function topology*.*

As in Subsection 2.2.3 points in CV_n can be thought of as projective classes of such length functions, *i.e.* $CV_n \subseteq \mathbb{PR}^{F_n}$; and CV_n is topologized via the quotient of the weak topology on length functions. It is the case that the closure $\overline{CV}_n$ of CV_n is compact and consists precisely of homothety classes of very small F_N-actions on $\mathbb{R}$-trees [9, 2]. For more background on CV_n and its closure, see [46] and the references therein.

2.4 Algebraic Laminations

Here, we present a brief and restricted view of dual laminations of F_n-trees; see [12] and [13] for a careful development of the general theory. Let ∂F_n denote the Gromov boundary of F_n–*i.e.* the Gromov boundary of any Cayley graph of F_n; let $\partial^2(F_n) := \partial F_n \times \partial F_n - \Delta$, where Δ is the diagonal. The left action of F_n on a Cayley graph induces actions by homeomorphisms of F_n on ∂F_n and $\partial^2 F_n$. Let $i : \partial^2 F_n \to \partial^2 F_n$ denote the involution that exchanges the factors. An *algebraic lamination* is a non-empty, closed, F_n-invariant, i-invariant subset $\mathscr{L} \subseteq \partial^2 F_n$.

Fix an action $F_n \curvearrowright T$ with dense orbits; following [34] (see also [13]), we associate an algebraic lamination $L^2(T)$ to the action $F_n \curvearrowright T$. Let $T_0 \in cv_n$ (*i.e.* the action $F_n \curvearrowright T_0$ is free and discrete), and let $f : T_0 \to T$ be an F_n-equivariant map, isometric when restricted to edges of T_0. Say that f has *bounded backtracking* if there is $C > 0$ such that $f([x, y]) \subseteq N_C([f(x), f(y)])$, where N_C denotes the C-neighborhood. For $T_0 \in cv_n$, denote by $vol(T_0) := vol(T_0/F_n)$ the sum of lengths of edges of the finite metric graph T_0/F_n.

Proposition 2.5. *[34, Lemma 2.1] Let $T \in \overline{cv}_n$; let $T_0 \in cv_n$; and let $f : T_0 \to T$ be equivariant and isometric on edges. Then f has bounded backtracking with $C = vol(T_0)$.*

For $T_0 \in cv_n$, we have an identification $\partial T_0 \cong \partial F_n$. If ρ is a ray in T_0 representing $X \in \partial F_n$, we say that X is T-*bounded* if $f \circ \rho$ has bounded image in T; this does not depend on the choice of T_0 (see [3]).

Proposition 2.6. *[34, Proposition 3.1] Let $T \in \overline{cv}_n$ have dense orbits, and suppose that $X \in \partial F_n$ is T-bounded. There there is a unique point $Q(X) \in \overline{T}$ such that for any $f : T_0 \to T$, equivariant and isometric on edges, and any ray ρ in T_0 representing X, the point $Q(X)$ belongs to the closure of the image of $f \circ \rho$ in $\overline{T}$. Further the image of $f \circ \rho$ is a bounded subset of $\overline{T}$.*

The (partially-defined) map Q given above is clearly F_n-equivariant; in fact, it extends to an equivariant map $Q : \partial F_n \to \overline{T} \cup \partial T$, which is surjective (see [34]). The crucial property for us is that Q can be used to associate to T an algebraic lamination.

Proposition 2.7. *[13] Let $T \in \overline{cv}_n$ have dense orbits. The set $L^2_Q(T) := \{(X,Y) \in \partial^2(F_n) | Q(X) = Q(Y)\}$ is an algebraic lamination.*

Following [13], we mention that there is different, perhaps more intuitive, procedure for defining $L^2(T)$. Let $T \in \overline{cv}_n$ (not necessarily with dense orbits, but not free and discrete), and let $\Omega_\epsilon(T) := \{g \in F_n | l_T(g) < \epsilon\}$, where l_T is the hyperbolic length function for the action $F_n \curvearrowright T$. The set $\Omega_\epsilon(T)$ generates an algebraic lamination $L^2_\epsilon(T)$, which is the smallest algebraic lamination containing $(g^{-\infty}, g^{\infty}) = (...g^{-1}g^{-1}, gg...) \in \partial^2(F_n)$ for every $g \in \Omega_\epsilon$. One then defines $L^2_\Omega(T) := \cap_{\epsilon>0} L^2_\epsilon(T)$. In [13] it is shown that for an action $F_n \curvearrowright T \in \overline{cv}_n$ with dense orbits, $L^2_\Omega(T) = L^2_Q(T)$, as defined above.

Definition 2.8. *Let $F_n \curvearrowright T \in \overline{cv}_n$ be an action with dense orbits. The* dual lamination of $F_n \curvearrowright T$ *is $L^2(T) := L^2_Q(T) = L^2_\Omega(T)$.*

2.4.1 The Observers Topology

In [11], a weaker topology on $\mathbb{R}$-trees is considered. Let T be a tree with Gromov boundary ∂T and metric completion $\overline{T}$; put $\hat{T} := \overline{T} \cup \partial T$. The metric topology on T canonically extends to $\overline{T}$, and we may extend this topology to $\hat{T}$ as follows: for a ray ρ in T representing $[\rho] \in \partial T$, a neighborhood basis at $[\rho]$ is taken to be the set of components of $T \setminus \{pt.\}$ meeting ρ in a non-compact set. For $p, q \in \hat{T}$, the *direction* of q at p the component $d_p(q)$ of $\hat{T} \setminus \{p\}$ containing q. The *observers topology* $\hat{T}$ is the topology with subbbasis the collection of directions in $\hat{T}$.

With this topology the map $Q : \partial F_n \to \overline{T} \cup \partial T$ is continuous [11, Proposition 2.3]. Further, when restricted to finite subtrees of T, the observers topology agrees with the metric topology; in particular, we have the following:

Lemma 2.9. *[11, 13] Let $T \in \overline{cv}_n$ have dense orbits. For any $x \in T$, the set $Q^{-1}(x) \subseteq \partial F_n$ is compact.*

The aim of [11] is to investigate to what extent $L^2(T)$ determines T for trees $T \in \overline{cv}_n$ with dense orbits. For the following equip trees $T \in \overline{cv}_n$ with the metric topology, and equip $\hat{T}$ with the observers topology.

Proposition 2.10. *[11, Theorem I] Let $T_1, T_2 \in \overline{cv}_n$ have dense orbits. Then $L^2(T_1) = L^2(T_2)$ if and only if $\hat{T}_1$ is homeomorphic to $\hat{T}_2$.*

Let $T \in \overline{cv}_n$ have dense orbits; fix $q \in T$; and let (p_k) be a sequence in T. Put $I_m := \cap_{k \geq m}[q, p_k]$, so $I_m = [q, r_m]$, and we have $I_m \subseteq I_{m+1}$. The *inferior limit* of (p_k) from q is the limit $\underline{\lim}_q\, p_k := \lim r_m$. The following gives a characterization of convergence in $\hat{T}$:

Lemma 2.11. *[11, Lemma 1.12] If a sequence (p_k) in $\hat{T}$ converges to p, then for any $q \in \hat{T}$, $p = \underline{\lim}_q\, p_k$.*

A map $f : T \to T'$ between trees T, T' is *continuous on segments* if for any finite segment $I \subseteq T$, the restriction $f|_I : I \to T'$ is continuous. The following result, along with the approach of Sections 5 and 5.3 provide a sort of converse of the work in [15].

Lemma 2.12. *Let $T, T' \in \overline{cv}_n$ have dense orbits, and suppose that there is an equivariant bijection $f : T \to T'$ that is continuous on segments. Then f extends to a unique homeomorphism $\hat{f} : \hat{T} \to \hat{T}'$; in particular, $L^2(T) = L^2(T')$.*

Proof. Let T, T', and f as in the statement, and let T_{obs}, T'_{obs} denote T, T' regarded as subspaces of $\hat{T}, \hat{T}'$. We first show that f induces a homeomorphism $T_{obs} \to T'_{obs}$. Let $p, q \in T$, and notice that $d_p(q) = \cup_{p \notin [q,q']}[q, q']$. As f is continuous on segments and bijective, we have that $f(d_p(q)) = \cup_{f(p) \notin [f(q), f(q')]}[f(q), f(q')] = d_{f(p)}(f(q))$, hence f is open. Applying essentially the same argument to f^{-1} gives that f is continuous, hence f is a homeomorphism $T_{obs} \to T'_{obs}$.

Let $p_k \in \hat{T}$ with $p_k \to p \in \hat{T} \setminus T$. By the discussion following Proposition 2.10, we have for any $q \in \hat{T}$, $p = \lim_q p_k$. Set $I_m = \cap_{k \geq m}[q, p_k] = [q, r_m]$, so that $\lim r_m = p$. Since f is continuous on segments and bijective, we have that $f(I_m) = [f(q), f(r_m)]$, hence the sequence $f(r_m)$ has a well-defined limit $r' \in \hat{T}'$. If $r' \in T' \subseteq \hat{T}'$, then there is $r'' \in T$ such that $f(r'') = r'$; in this case $f([q, r'']) = [f(q), r']$. Further $[q, r'']$ evidently contains each I_m, hence $[q, r'']$ contains $\overline{\cup_m I_m} \ni \lim r_m = p$, a contradiction. Hence, $r' \in \hat{T}' \setminus T'$, and we define $\hat{f}(p) = r'$.

Now note that for any $p' \in \hat{T}' \setminus T'$ and any $q' \in T'$ there is a sequence r'_m in T' such that $[q, r'_m] \subseteq [q, r'_{m+1}]$ with $p' = \lim r'_m$. We find $q, r_m \in T$ such that $f(q) = q'$ and $f(r_m) = r'_m$, and it follows from the preceding arguments that there is a unique $p \in \hat{T} \setminus T$ with $\hat{f}(p) = p'$; hence $\hat{f}$ is bijective. Futher, it is easy to check as above that $\hat{f}$ is continuous and open, so $\hat{f}$ is a homeomorphism. The fact that $L^2(T) = L^2(T')$ then follows from Proposition 2.10. $\square$

2.5 Subgroups of Free Groups

In this section, we briefly recall Stallings' approach to finitely generated subgroups of free groups; see [44, 29] for details. Put $R_n := \bigvee_{i=1}^n S^1$. Fix a basis $A = \{a_1, ..., a_n\}$ for F_n; this gives an identification $F_n \cong \pi_1(R_n)$ by labeling the i^{th} copy of S^1 with a_i.

The universal cover $T := \tilde{R_n}$ is a simplicial tree, which we may regard as an $\mathbb{R}$-tree by identifying each edge with $[0, 1] \subseteq \mathbb{R}$. Lift the labeling of R_n to T. Let $H \leq F_n$ be finitely generated; then H acts (freely) on T; as before, let T_H denote the minimal invariant subtree for the action $H \curvearrowright T$. We have an embedding $T_H/H =: \Gamma_H \to T/H$, and the restriction f_H of the covering $f : T/H \to R_n$ to Γ_H is an immersion. The immersion $f_H : \Gamma_H \to R_n$ is called the *core Stallings graph* for H with respect to the basis A. The immersion $f_H : \Gamma_H \to R_n$ differs from the usual Stallings graph in that there is no "base point" involved; hence Γ_H provides information about the conjugacy class of H, rather than H.

Let Γ be a directed graph, labeled by elements of A. One says that Γ is *folded* if for every vertex $v \in V(\Gamma)$ and every $a_i \in A$, there is at most one each of incoming and outgoing edges labeled by a_i at v. Choose a base vertex $v_0 \in \Gamma$, and let $*$ denote the vertex of R_n. There is a map $f = f(\Gamma, v_0) : (\Gamma, v_0) \to (R_n, *)$, which is an immersion as long as no vertex other than v_0 has valence-1. Any finitely generated subgroup $H \leq F_n$ is represented by such an immersion: choose a basis $\{h_1, ..., h_k\}$ for H, and let w_i be the unique reduced work in A representing h_i, and let $|.|$ denote word length. Subdivide the i^{th} edge of R_k into $|w_i|$ segments and label these segments by w_i to get a pointed, labeled, directed graph $(\Gamma_0, *)$. Then Γ_0 is homotopy equivalent to a folded graph, in which no vertex other than the image of $*$ has valence-1.

One says that a folded, labeled graph Γ is *regular* if for every vertex $v \in V(\Gamma)$ and every $a_i \in A$, there are incoming and outgoing edges at v with label a_i. In this case, the immersion $f : (\Gamma, v_0) \to (R_n, *)$ is a covering.

2.6 Length Measures

Let T be an $\mathbb{R}$-tree.

Definition 2.13. *[23] A* length measure *(or just* measure*) μ on T is a collection $\mu = \{\mu_I\}_{I \subseteq T}$ of finite positive Borel measures on the finite arcs $I \subseteq T$; it is required that for $J \subseteq I$, $\mu_J = (\mu_I)|_J$.*

As these measures are defined locally on finite arcs, all the usual measure-theoretic definitions are similarly defined: a set $X \subseteq T$ is *μ-measurable* if $X \cap I$ is μ_I-measurable for each $I \subseteq T$; X has *μ-measure zero* if $X \cap I$ is μ_I-measure zero for each I; and so on. The *Lebesgue length measure*, denoted μ_L, on T is the collection of Lebesgue measures on the finite arcs of T. If T is equipped with an action of a group G, then we say that a measure μ is *G-invariant* if $\mu_I(X \cap I) = \mu_{gI}(gX \cap gI)$ holds for each μ-measurable set X and each $g \in G$. Note that if the action $G \curvearrowright T$ is by isometries, then the Lebesgue measure is invariant. We let $M(T) = M(G \curvearrowright T)$ stand for the set of invariant measures on T.

Suppose that $G \curvearrowright T$ is an action by isometries, with G a countable group. Say that the action is *finitely supported* if there is a finite subtree $K \subseteq T$ such that any finite arc $I \subseteq T$ may be covered by finitely many translates of K by elements of G; in this case, we say that the action $G \curvearrowright T$ is *supported on* K. Note that, if G is finitely generated, then any minimal action $G \curvearrowright T$ is finitely supported.

Let K be a compact topological space.

Definition 2.14. *A collection Γ of homeomorphisms between subsets of K is called a* pseudogroup *if the following are satisfied:*

(1) the identity mapping is an element of Γ,

(2) if $\gamma \in \Gamma$, then $\gamma^{-1} \in \Gamma$, where $Dom(\gamma^{-1}) = Ran(\gamma)$,

(3) if $\gamma_1, \gamma_2 \in \Gamma$, then $\gamma_1 \circ \gamma_2 \in \Gamma$

(4) if $\gamma_1, \gamma_2 \in \Gamma$, and if $\gamma_1(x) = \gamma_2(x)$ for all $x \in Dom(\gamma_1) \cap Dom(\gamma_2)$, then if $\gamma_1 \cup \gamma_2$ is a homeomorphism, then $\gamma_1 \cup \gamma_2 \in \Gamma$, and

(5) if $\gamma_1 \in \Gamma$, then the restriction of γ_1 to any Borel subset of $Dom(\gamma_1)$ is in Γ.

We say that $\{\gamma_1, ..., \gamma_k, ...\}$ *generate* Γ if any $\gamma \in \Gamma$ can be obtained from the γ_i via the operations in the definition of a pseudogroup. A measure μ on K is said to be Γ-*invariant* if for any measurable $X \subseteq K$, we have $\mu(X \cap dom(\gamma)) = \mu(\gamma(X \cap dom(\gamma)))$ for each $\gamma \in \Gamma$. We let $M(K) = M(\Gamma, K)$ stand for the set of Γ-invariant measures on K.

Let $G \curvearrowright T$ be an isometric action supported on the finite subtree $K \subseteq T$. We consider the (countably generated) pseudogroup $\Gamma := \{g|_{K'} : g \in G, K' \subseteq K, gK' \subseteq K\}$ of restrictions of the isometries G to Borel subsets of K. Since the action is supported on K, there is a bijective correspondence between $M(T)$ and $M(K)$.

A non-trivial measure $\mu \in M(T)$ is called *ergodic* if any G-invariant subset is either full measure or zero measure. A G-tree T is called *uniquely ergodic* if there is a unique, up to scaling, G-invariant measure μ on T; in this case μ must be ergodic. Let $M_0(T)$ denote the set of non-atomic, G-invariant measures on T, and let $M_1(T) := \{\nu \in M_0(T) | \nu \leq \mu_L\}$. Note that both $M_0(T)$ and $M_1(T)$ are convex.

Proposition 2.15. *[23, Corollary 5.4] Let $T \in \overline{cv}_n$ be with dense orbits. Then $M_0(T)$ is a finite dimensional convex set, which is projectively compact. Moreover, T has at most $3n - 4$ non-atomic ergodic measures (up to homothety), and every measure in $M_0(T)$ is a sum of these ergodic measures. Further $M_1(T)$ is compact.*

2.6.1 Finite Systems of Isometries

A (closed) *finite tree* is a tree that is the convex hull of a finite set; a *finite forest* is a finite union of finite trees. A *finite pseudogroup* is a finitely generated pseudogroup $S = (F, A)$, where F is a finite forest. Let $S = (F, A)$ be a finite pseudogroup generated by $A = \{a_1, ..., a_n\}$; we require that $dom(a_i)$ be a closed finite tree. For $a_i \in A$, let $B_i := dom(a_i) \times I$; regard B_i as foliated by leaves of the form $\{pt.\} \times I$. Form the *suspension* $\Sigma(S)$ of S from the disjoint union $K \sqcup B_1 \sqcup ... \sqcup B_n$ by identifying $dom(a_i) \times \{0\}$ with $dom(a_i)$ and $y = (x, 1) \in dom(a_i) \times \{1\}$ with $a_i(x)$. Put a relation R_l on points of $\Sigma(S)$, where $x, y \in R_l$ if and only if x, y are contained in a leaf of some B_i; let $\overline{R_l}$ be the smallest equivalence relation containing R_l; and regard $\Sigma(S)$ as foliated by leaves that are the classes of $\overline{R_l}$. Note that for $x \in K$, the leaf $l(x)$ containing x intersects K precisely in the orbit $S.x$.

Let B denote the set of branch points of K, and let E denote the set containing all endpoints of all $dom(a_i)$; put $C := B \cup E$. A leaf l of $\Sigma(S)$ is called *singular* if $l \cap C \neq \emptyset$; any leaf that is not singular is called *regular*. Suppose that $\Sigma(S)$ contains a finite regular leaf $l = l(x)$, then for $y \in K$ close to x, $l(y)$ is finite and regular. It follows that there are $y_1, y_2 \in K$ with $x \in [y_1, y_2]$ and $d(y_1, y_2)$ maximal, such that for $z \in (y_1, y_2)$, $l(z)$ is finite and regular. Hence, $F_x := \cup_{z \in (y_1, y_2)} l(z)$ is a (y_1, y_2)-bundle over some leaf $l(z) \in F_x$. The set F_x is called a *maximal family of finite orbits*, and the *transverse measure* of F_x is $d(y_1, y_2)$. Evidently, $l(y_i)$ are singular, so there are finitely many maximal families of finite orbits in $\Sigma(S)$. This gives a coarse decomposition of $\Sigma(S)$, which is the starting point for a refined decomposition of $\Sigma(S)$, see [21] for the statement as well as for details regarding the above discussion.

Suppose that $S = (F, A)$ is a finite pseudogroup; define the following:

1. $m :=$ total measure of F
2. $d :=$ the sum of measures of domains of generators
3. $e :=$ the sum of transverse measures of maximal families of finite orbits.

We regard m, d, and e as functions $\{\text{finite pseudogroups}\} \to \mathbb{R}$. Say that S has *independent generators* if no reduced word in the generators A and their inverses defines a partial isometry of F that fixes a non-degenerate arc.

Proposition 2.16. *[21, Proposition 6.1] Let S, F, A as above, and suppose that S has independent generators, then $e(S) + d(S) = m(S)$.*

Chapter 3

Irreducible Endomorphisms

Let $\phi : F_n \to F_n$ be an endomorphism. The *outer class of* ϕ is the set $\Phi := \{\iota_g \circ \phi | g \in F_n\}$, where ι_g is the inner automorphism $\iota_g(f) = gfg^{-1}$; we call Φ an *outer endomorphism.* We will frequently be discussing both outer classes of endomorphisms and particular endomorphisms in a class; we will always use capital letters to denote outer classes and lower case letters to denote particular endomorphisms, surpressing futher comment when confusion is unlikely to arise. Further, we will frequently take liberties in replacing ϕ by a power with little or no comment, as throughout we are studying asymptotic behavior.

3.1 Train Tracks

3.1.1 Topological Representatives

This subsection closely follows [6], to which the reader should refer for details. The (n-petal) *rose* is $R_n := \vee_{i=1}^n S^1$, the wedge of n copies of S^1; once and for all, we make the identification $F_n = \pi_1(R_n)$. A *marked graph* is a finite graph G of rank n, along with a homotopy equivalence $\tau : R_n \to G$; this gives an action of F_n on $\tilde{G}$ by deck transformations, hence an identification of $\pi_1(G)$ with F_n. This action is well-defined up to conjugation, *i.e.* up to choosing a preferred lift of a base point in G. Denote by $V = V(G) = \{v_1, ..., v_l\}$ and $E = E(G) = \{e_1, ..., e_k\}$ the sets of vertices and edges of G, respectively.

Let Φ be an outer endomorphism of F_n, and let G be a marked graph. A map $f : G \to G$ is a *topological representative* for Φ if:

- $f(V) \subseteq V$,
- for any $e \in E$, $f|_e$ is either locally injective, or $f(e)$ is a vertex, and

- f induces Φ.

Topological representatives always exist; one can take the obvious map with $G = R_n$. Fix a topological representative $f : G \to G$. The *transition matrix* of f is the $k \times k$ matrix $M(f)$ whose (i,j)-entry is the number of times the f-image of e_j crosses e_i (in either direction). Any transition matrix is a non-negative integral matrix, and it is evident that $M(f)^r = M(f^r)$. We say that a non-negative integral matrix M is (fully) *irreducible* if:

- for any (i,j), there is $N(i,j)$ such that $(M^{N(i,j)})_{i,j} > 0$, and
- the prior condition holds for any power of M.

A subgraph $G_0 \subseteq G$ is called *invariant* if $f(G_0) \subseteq G_0$. The topological representative f is *admissible* if there is no invariant non-degenerate forest.

Definition 3.1. *[6] An endomorphism $\phi : F_n \to F_n$ is* irreduicble *if any admissible topological representative for Φ has irreducible transition matrix.*

A *free factor system $\mathscr{F}$* for F_n is a decomposition $F_n = F_{n_1} * ... * F_{n_r} * F'$, where each F_{n_i} is a non-trivial proper subgroup. An endomorphism $\psi : F_n \to F_n$ *preserves $\mathscr{F}$* if $\psi(F_{n_i}) \leq F_{ni}^{g_i}$ for some elements $g_i \in F_n$.

Lemma 3.2. *[6] If $\phi : F_n \to F_n$ be irreducible, then ϕ does not preserve any free factor system for F_n.*

Corollary 3.3. *Let $\phi : F_n \to F_n$ be an irreducible endomorphism. Then ϕ is injective.*

Proof. An easy argument using Nielsen moves shows that $ker(\phi)$ contains a non-trivial free factor of F_n; the corollary then follows from Lemma 3.2. □

Let $f : G \to G$ be a topological representative with $M(f)$ irreducible. A *turn* in G is a set $T = \{e_i, e_j\}$ of directed edges of G with a common initial vertex; a turn is *degenerate* if $e_i = e_j$. The topological representative f induces a map Tf on the set of turns in G by sending an edge e to the first edge in the path $f(e)$. A non-degenerate turn T is called *illegal* if $(Tf)^r(T)$ is degenerate for some k. A turn is called *legal* if it is not illegal, and a path is called legal if it crosses only legal turns. For any path α in G, denote by $[f(\alpha)]$ the immersed path homotopic to $f(\alpha)$ (rel endpoints).

Definition 3.4. *[6] An admissible topological representative $f : G \to G$ for Φ is a* train track map *for Φ if $[f^r(e)] = f^r(e)$ for every $e \in E$.*

3.1.2 Train Tracks and the Stable Tree

The following result is proved in [6] for irreducible outer automorphisms of F_n; however, with no modification their proof works for all irreducible endomorphisms. This result is established by different means by Dicks-Ventura in [18].

Proposition 3.5. *[6, 18] Let $\phi : F_n \to F_n$ be irreducible, then Φ has a topological representative that is a train track map.*

The Perron-Frobenius theory gives for any any irreducible matrix M a unique positive normalized eigenvector $\mathbf{v}$ with associated eigenvalue $\lambda > 1$ (see [42]). Let $\phi : F_n \to F_n$ be irreducible, and let $f : G \to G$ be a train track representative for ϕ. We equip G with the *Perron-Frobenius metric*: identify edge e_i with the segment of length $\mathbf{v}_i$. With this metric, the map f expands lengths of legal paths by the factor λ. For $g \in F_n$ we let α_g stand for the immersed loop representing $[g]$.

Lemma 3.6. *[3] Let $\phi : F_n \to F_n$ be irreducible, and let $f : G \to G$ be a train track representative for Φ with Perron-Frobenius eigenvalue λ. For $g \in F_n$ put*

$$l_{T_\Phi}(g) := \lim_k \lambda^{-k} L([f^k(\alpha_g)])$$

Then the following hold:

(i) l_{T_Φ} is the length function for an $\mathbb{R}$-tree $T_\Phi \in \overline{cv}_n$,

(ii) T_Φ is independent of the choice of train track representative, and

(iii) $l_{T_\Phi}(\phi(g)) = \lambda_\Phi l_{T_\Phi}(g)$,

Let $\phi : F_n \to F_n$ be irreducible, and let $f : G \to G$ be a train track representative for Φ with associated Perron-Frobenius eigenvalue λ. Note that it follows from that above lemma that λ is determined by Φ. Equip G with the Perron-Frobenius metric, and put $T_0 := \tilde{G}$ with the lifted metric. Let $\tilde{f} : T_0 \to T_0$ be a lift of f; the choice of $\tilde{f}$ amounts to choosing some representative

$\psi \in \Phi$, and we prefer to take $\tilde{f}$ corresponding to ϕ when convenient. Note that for any $g \in F_n$, one has $\tilde{f}(gx) = \phi(g)\tilde{f}(x)$. Let T'_k denote the minimal F_n-invariant subtree of T_0 with the action twisted by ϕ^k; so $T'_k = \tilde{f}^k(T_0)$. Define T_k to be T'_k with the metric rescaled by λ^{-k}. The sequence of F_n-trees (T_k) converges in the Gromov-Hausdorff topology to the tree T_Φ. The map $\tilde{f} : T_0 \to T_0$ gives maps $f_k : T_k \to T_{k+1}$, which give rise to a map $f_\phi : T_\Phi \to T_\Phi$ satisfying:

- $\text{Length}(f_\phi([x, y])) = \lambda_\Phi \text{Length}([x, y])$,
- $f_\phi(gx) = \phi(g) f_\phi(x)$

Definition 3.7. *The tree T_Φ is called the* stable tree of Φ.

Any endomorphism $\psi : F_n \to F_n$ acts (on the right) on the set of F_n-trees via

$$l_{T\psi}(g) = l_T(\psi(g))$$

If one restricts attention to a space X of nontrivial trees such that ϕ acts on X, then the action of ϕ on X gives an action on the set of projective classes of trees coming from X. If $[T]\psi = [T]$, then $T\psi$ is F_n-equivariantly isometric to T with the metric rescaled by some number c. This data is witnessed by a function $H : T \to T$ satisfying:

- $\text{Length}(H([x, y])) = c\text{Length}([x, y])$,
- $H(gx) = \psi(g)H(x)$

Call such a map H a *ψ-compatible c-homothety*, or just a *homothety* if ψ and c are clear from context (see [19]). Conversely, if Y is an F_n tree, $\eta : F_n \to F_n$ some endomorphism, then the existence of a η-compatible homothety $H : Y \to Y$ implies that $[Y]\eta = [Y]$. The map $f_\phi : T_\Phi \to T_\Phi$ is a ϕ-compatible λ_Φ-homothety, so $[T_\Phi]\phi = [T_\Phi]$.

3.1.3 Expansive Endomorphisms

Definition 3.8. *Fix a basis B for F_n. An endomorphism $\phi : F_n \to F_n$ is* expansive *with respect to B if for any real number L there is a number K such that for any $1 \neq g \in F_n$, one has $||\phi^k(g)||_B \geq L$ whenever $k \geq K$.*

The definition of expansive involves a particular basis for F_n; however, it is clear that if an endomorphism is expansive with respect to some basis then it is expansive with respect to any basis. The following lemma is an easy application of the definition of a train track map.

Lemma 3.9. *[3] Let $f : G \to G$ be a train track map with associated eigenvalue λ, and let p be a path in G. Then the sequence $L([f^k(p)])$ either is uniformly bounded or grows like $Const.\lambda^k$.*

Corollary 3.10. *Let $\phi : F_n \to F_n$ be irreducible, and suppose that ϕ is expansive. Then T_Φ is free.*

Proof. Considering Lemma 3.9 and the construction of T_Φ we see that an elements $g \in F_n$ is elliptic in T_Φ if and only if the conjugacy class of g is represented by a loop α_g in G such that the length of $[f^k(\alpha_g)]$ is uniformly bounded. If ϕ is expansive, this is only possible for $g = 1$. □

We now establish a dichotomy for irreducible endomorphisms of the free group. The result follows easily from a theorem of M. Takahasi [45](see also [29]), but we include a proof, as the techniques are relevant to the sequel.

Proposition 3.11. *Let $\phi : F_n \to F_n$ be irreducible, then either $\phi \in Aut(F_n)$ or ϕ is expansive.*

Proof. Suppose that $\phi : F_n \to F_n$ is irreducible and not expansive. Let $S_k = S(\phi^k(F_n))$ denote the Stallings subgroup graph of $\phi^k(F_n)$, and let $x_k \in S_k$ be the base point (see [29] for background). Denote by i_k the injectivity radius of S_k (with the simplicial metric). Since ϕ is not expansive, it follows that the sequence $(i_k)_{k\in\mathbb{N}}$ is bounded. Hence, by replacing ϕ by a suitable power, we can find a labeled graph S' with basepoint $x' \in S'$ along with embeddings $f_k : S' \to S_k$ sending x' to the projection of x_k onto the image of f_k.

Let $f : G \to G$ be a train track representative for Φ. Since the action $F_n \curvearrowright \tilde{R}_n$ is quasi-isometric to the action $F_n \curvearrowright \tilde{G}$, after replacing ϕ by a power if necessary, we get that the collection of subgraphs of S_k that are unions of short loops gives rise to a collection of subgroups that invariant under ϕ up to conjugacy. Each of these subgroups is a free factor of F_n as its conjugacy class corresponds to a subgraph of each S_k for $k >> 0$, which implies that there is a free factorization of F_n mapping onto this collection. Since ϕ is injective and irreducible, we get that S' surjects onto each S_k so that ϕ is an automorphism. □

Remark 3.12. *The above proof shows that for any endomorphism $\psi : F_n \to F_n$ that is not expansive, after possibly replacing ψ by a power, we can find a collection of free factors $F_{n_1}, ..., F_{n_r}$ of F_n that are preserved by ψ up to conjugacy such that the restriction of ψ to each F_{n_i} is an automorphism. In this case, there are inner automorphisms $\iota_{g_{n_i}}$ such that $\cap_k(\iota_{g_{n_i}} \circ \psi)^k(F_n) = F_{n_i}$.*

3.2 Graphs of Actions

To begin in earnest our study of the structure of stable trees of irreducible endomorphisms, we recall the following notion of decomposability for group actions on trees. Let G be finitely generated group acting on an $\mathbb{R}$-tree T.

Definition 3.13. *A G-invariant family $\mathscr{Y} = \{Y_v\}_{v \in V}$ of non-degenerate subtrees $Y_v \subseteq T$ is called a* transverse family *for the action $G \curvearrowright T$ if for $Y_v \neq Y_{v'}$, one has that $Y_v \cap Y_{v'}$ contains at most one point.*

Note that if $\mathscr{Y}$ is a transverse family for the action $G \curvearrowright T$, we may replace each Y_v by its closure in T; the resulting collection also will be a transverse family. Let $\{Y_v\}_{v \in V}$ is a transverse family of closed subtrees of T. If, in addition, for any finite arc $I \subseteq T$, one has that I is contained in a finite union $Y_{v_1} \cup ... \cup Y_{v_r}$, then the collection $\mathscr{Y}$ is called a *transverse covering* of T [25].

Definition 3.14. *[32, 25] A* graph of actions *$\mathscr{G} = (S, \{Y_v\}_{v \in V(S)}, \{p_e\}_{e \in E(S)}$ consists of:*

(i) a simplicial tree S, called the skeleton*, equipped with an action (without inversions) of G,*

(ii) for each vertex $v \in V(S)$ of S an $\mathbb{R}$-tree Y_v, called a vertex tree*, and*

(iii) for each oriented edge $e \in E(S)$ with terminal vertex $v \in V(S)$ a point $p_e \in Y_v$, called an attaching point*.*

It is required that the projection sending $Y_v \to p_e$ is equivariant and that for $g \in G$, one has $gp_e = p_{ge}$. Associated to a graph of actions $\mathscr{G}$ is a canonical action of G on an $\mathbb{R}$-tree $T_{\mathscr{G}}$: define a pseudo-metric d on $\coprod_{v \in V(S)} Y_v$: if $x \in Y_u$, $y \in Y_v$, let $e_1...e_k$ be the reduced edge-path from u to v in S, *i.e.* $\iota(e_1) = u$, $\tau(e_k) = v$, and $\tau(e_i) = \iota(e_{i+1})$, then

$$d(x,y) = d_{Y_u}(x, p_{\overline{e_1}}) + d_{Y_{\tau(e_1)}}(p_{e_1}, p_{\overline{e_2}}) + ... + d_{Y_v}(p_{e_r}, y) \tag{3.1}$$

Making this pseudo-metric Hausdorff gives an $\mathbb{R}$-tree, called the *dual* of $\mathscr{G}$, which we denote by $T_{\mathscr{G}}$. If T is an $\mathbb{R}$-tree equipped with an action of G by isometries, and if there is an equivariant isometry $T \to T_{\mathscr{G}}$ to the dual of a graph of actions, then we say that T *splits* as a graph of actions. See [24, 32] for details.

The following result shows that graphs of actions and transverse coverings are equivalent ideas.

Lemma 3.15. *[25, Lemma 1.5] Assume that T splits as a graph of actions with vertex trees $\{Y_v\}_{v\in V(S)}$, then $\{Y_v\}_{v\in V(S)}$ is a transverse covering for T. Conversely, if T has a transverse covering $\{Y_v\}_{v\in V}$, then T splits as a graph of actions whose non-degenerate vertex trees are $\{Y_v\}_{v\in V}$.*

We now recall a sketch of the proof of the second statement of Lemma 3.15. Suppose that the action $G \curvearrowright T$ has transverse covering $\mathscr{Y} = \{Y_v\}_{v\in V}$; we find a graph of actions structure for $G \curvearrowright T$. First we define the skeleton S; $V(S) = V_0 \cup V_1$, where the elements of V_0 are in one-to-one correspondence with the elements of $\mathscr{Y}$, and the elements of V_1 are in one-to-one correspondence with intersection points of distinct elements of $\mathscr{Y}$. There is an edge from $v_1 \in V_1$ to $v_0 \in V_0$ if and only if the point corresponding to v_1 is contained in the tree corresponding to v_0. One checks that there is an induced action of G on S without inversions and that association given above of trees to the elements of $V(S)$ defines a graph of actions structure on $G \curvearrowright T$ (see [25]).

The following is a simple application of the preceding discussion.

Lemma 3.16. *Let $G \curvearrowright T$ be an action of a finitely generated group on an $\mathbb{R}$-tree, and suppose that $\mathscr{T} = \{T_v\}_{v\in V}$ is a transverse covering for $G \curvearrowright T$. If the action $G \curvearrowright T$ is free, then each $Stab(T_v)$ is a free factor of G.*

Proof. Let $\mathscr{G} = (S, \{T_v\}_{v\in V(S)}, \{p_e\}_{e\in E(S)})$ be the graph of actions structure on $G \curvearrowright T$ defined above. Note that edge stabilizers in the action $G \curvearrowright S$ arise from stabilizers of attaching points. Since $G \curvearrowright T$ is free, edge stabilizers in $G \curvearrowright S$ are trivial. Since vertex stabilizers in $G \curvearrowright S$ correspond to the stabilizers of the trees Y_v, we see from the Bass-Serre theory that each such stabilizer is a free factor of G. $\square$

To conclude this section, we state the following result of Levitt.

Proposition 3.17. *[32, Theorem 5] Let $G \curvearrowright T$ be an action of a finitely generated group on an $\mathbb{R}$-tree; suppose that the action is not simplicial and not with dense orbits. Then $G \curvearrowright T$ splits as a graph of actions $\mathscr{G} = (S, \{T_v\}_{v \in V(S)}, \{p_e\}_{e \in E(S)}$, where each T_v is either a finite segment or $Stab(T_v) \curvearrowright T_v$ is with dense orbits.*

If $T \in \overline{cv}_n$ does not have dense orbits, then there is some discrete orbit; according to the above result the union of discrete orbits in T is a forest F with a positive lower bound on the diameter of each component. The set of components of $T \setminus F$ consists of finitely many orbits of subtrees of T, such that the stabilizer of each component acts on it with dense orbits. The union of components of $T \setminus F$ with closures of components of F is a transverse covering of the action $F_n \curvearrowright T$. We call the set of closures of components of F the *simplicial part of T*.

Remark 3.18. *Graphs of actions are ubiquitous in the sequel, so it seems appropriate to give a bit of motivation; for this we reach to the source of the idea. The definition of a graph of actions generalizes the decomposition of a tree dual to surface lamination that comes from the decomposition of the lamination into its minimal components. Indeed, if a surface S is equipped with measured lamination (L, μ), and if T is the $\mathbb{R}$-tree dual to (L, μ), then there is, for each sublamination L' of L, a transverse family $\mathscr{T}_{L'}$ of subtrees in T that are dual to the various lifts of L' to $\tilde{S}$. It is easy to see that if $L_1, ..., L_k$ are the minimal sublaminations of L, then there is a transverse covering, namely $\mathscr{T} = \mathscr{T}_{L_1} \cup ... \cup \mathscr{T}_{L_k}$, of T, containg k orbits of trees; this corresponds to the decomposition of L into minimal components.*

3.3 Structure of the Stable Tree

In this section we investigate the structure of the stable tree of an irreducible non-surjective endomorphism of F_n; the first step is to show that if some orbit is discrete, then every orbit must be discrete. To that end we recall the following result of Levitt-Lustig:

Lemma 3.19. *[34, Corollary 2.5] Let $T \in \overline{cv}_n$ have dense orbits. Given $p \in T$ and $\epsilon > 0$, there is a basis $\{a_1, ..., a_n\}$ of F_n such that $\Sigma_{i=1}^n d(p, a_i p) < \epsilon$.*

The next lemma is our first step in characterizing the structure of the stable tree of an irreducible, non-surjective endomorphism of F_n.

Lemma 3.20. *Let $\phi : F_n \to F_n$ be an irreducible endomorphism, and let T_Φ be its stable tree. Then either the action $F_n \curvearrowright T_\Phi$ has dense orbits or $F_n \curvearrowright T_\Phi$ is free and simplicial.*

Proof. In the case that $\phi \in Aut(F_n)$ this result follows from [3]. Hence, in light of Proposition 3.11, we may assume that ϕ is expansive, and by Corollary 3.10 we have that T_Φ is free. Toward a contradiction we assume that the action $F_n \curvearrowright T_\Phi$ is not discrete but does not have dense orbits. In this case Corollary 3.17 gives that $F_n \curvearrowright T_\Phi$ splits as a graph of actions with vertex trees simplicial edges or trees with dense orbits.

Put $T = T_\Phi$, and let $f : T \to T$ be a homothety witnessing $[T]\phi = [T]$. Immediately one has that for $\epsilon > 0$, ϕ takes elements of ϵ-short translation length to elements of $\lambda\epsilon$-short translation length. Recall that the action $F_n \curvearrowright T\phi$ is precisely the action $\phi(F_n) \curvearrowright T_{\phi(F_n)}$ of $\phi(F_n)$ on its minimal invariant subtree $T_{\phi(F_n)}$. There are finitely many orbits of vertices in the skeleton of the graph of actions structure on T; each vertex group either acts with dense orbits on the corresponding vertex tree or is trivial, in the case that the corresponding vertex tree is contained in the simplicial part of T.

Let $\mathscr{G} = (S, \{T_v\}_{v\in V(S)}, \{p_e\}_{e\in E(S)})$ be the graph of actions structure on T guaranteed by Proposition 3.17 and described above. As the action $F_n \curvearrowright T$ is free, the action $F_n \curvearrowright S$ is a free decomposition of F_n. Choose representatives $V_1, ..., V_r$ of conjugacy classes of vertex groups with $V_i = Stab(T_{v_i})$ such that the action $V_i \curvearrowright T_{v_i}$ has dense orbits. According to Lemma 3.19, for any $\epsilon > 0$ and points $p_i \in T_{v_i}$, we can find bases B_i for V_i such that $\Sigma_{b\in B_i} d(p_i, bp_i) < \epsilon$. Taking ϵ small with respect to the minimal length of a simplicial edge in T and recalling Formula 3.1, we see that each B_i is mapped under ϕ into a single vertex group of the graph of actions structure on $F_n \curvearrowright T$. Since there finitely many conjugacy classes of these vertex groups, it follows that there is some V_j such that $\phi(V_j) \leq V_j^g$ for some $g \in F_n$; by Lemma 3.2, we arrive at a contradiction to irreducibility of ϕ. □

Let T be a tree with base point $x \in T$. To each $x \neq y \in T$, there is associated a (one-sided) *cylinder* $C_x(y)$ that consists of rays ρ in T based at x that contain the segment $[x, y]$. The cylinder $C_x(y)$ is regarded as a subset of ∂T.

To complete this section, we bring the following result, ruling out the possiblity that the stable tree T of a non-surjective irreducible endomorphism ϕ could have dense orbits. The proof of this result contains a characterization of the dynamics of $\partial\phi$ acting on ∂F_n (Corollary 3.22), which we see to be incompatible with the existence of a map Q for T (refer to Subsection 2.4).

Proposition 3.21. *Let $\phi : F_n \to F_n$ be an irreducible endomorphism, and assume that $\phi \notin Aut(F_n)$. Then T_Φ is free and simplicial.*

Proof. Put $T = T_\Phi$, $f = f_\phi : T \to T$, $\lambda = \lambda_\Phi$, and note that since ϕ is not an automorphism, we have $\lambda > 1$ and that ϕ is expansive. Further, by Corollary 3.10, T_Φ is free, and by Lemma 3.20 T_Φ either has dense orbits, or T_Φ is simplicial.

Toward a contradiction suppose that the action $F_n \curvearrowright T$ is free with dense orbits. By [20] there are finitely many F_n-orbits of branch points in T and finitely many orbits of directions at branch points in T. By the equation $f(gx) = \phi(g)f(x)$ we have that f induces a well-defined map on the set of orbits of branch points in T. By replacing f with some power, we get a branch point $x \in T$ such that $f(x) = gx$, for some $g \in F_n$. Replace f by $g^{-1}f$, which is easily seen to be a homothety representing $\iota_{g^{-1}} \circ \phi$. This gives $f(x) = x$. As the map f is a homothety, it is injective; since there are finitely many directions at x, we may replace f by a power to ensure that f fixes each direction at x.

Let d be some direction at x, and let ρ be a ray in T based at x in direction d. It follows that there is $y \in d$ such that $[x, y] \subseteq f(\rho) \cap \rho$. Since f is a λ-homothety and since $\lambda > 1$, we can find a sequence $y_k \in d$ such that $f^k([x, y_k]) = [x, y]$. It follows that $[x, y] \subseteq \cap_k f^k(T)$.

Let $Q = Q_T : \partial F_n \to T$ be the map defined in Proposition 2.6. Recall that Q is F_n-equivariant and surjective, so for any $z \in [x, y]$ the set $Q^{-1}(z) \subseteq \partial F_n$ is non-empty, and by Lemma 2.9 $Q^{-1}(z)$ is compact. The commutativity of the below diagram follows easily from the definition of the map Q; see Subsection 2.4.

$$\begin{tikzcd} \partial F_n \arrow[r, "\partial\phi"] \arrow[d, "Q"'] & \partial F_n \arrow[d, "Q"] \\ T \arrow[r, "h"] & T \end{tikzcd}$$

By definition $(\partial\phi)^k(\partial F_n) = \partial\phi^k(F_n)$. As $[x, y] \subseteq \cap_k f^k(T)$, for each $z \in [x, y]$, we have that the sets $Z_k := Q^{-1}(z) \cap (\partial\phi)^k(\partial F_n)$ form a nested sequence of non-empty compact subsets of ∂F_n, so $Z := \cap_k Z_k$ is non-empty. Hence, $\partial\phi(\cup_{z\in[x,y]} Z) = \cup_{z\in[x,y]} Z \subseteq \cap_k \partial(\phi^k(F_n))$; in particular, $\cap_k \partial(\phi^k(F_n))$ is infinite. We show that this is impossible.

Fix a Cayley tree T' for F_n. Let $S'_k := S(\phi^k(F_n))$ be the Stallings subgroup graph for $\phi^k(F_n)$, and let $S_k := Core(S'_k)$ be the core of S'_k (see [29]). A fundamental domain for the action $\phi^k(F_n) \curvearrowright T'_{\phi^k(F_n)}$ can be got by "unfolding" S_k in $T_{\phi^k(F_n)}$, and such a fundamental domain is the union of exactly $2n$ (possibly overlapping) segments eminating from $1 \in T'$. It follows that $\partial\phi^k(F_n)$ is contained in the union of $2n$ cylinders, say $C_{1,k}, ..., C_{2n,k}$. Let $g_{i,k} \in F_n$ be chosen to define $C_{i,k}$. Notice that since ϕ is expansive we have for any N some $k(N)$ such that $l_{T'}(g_{i,k(N)}) \geq N$ for each i. It follows that $\cap_k(\cup_i C_{i,k})$ is a finite set; on the other hand $\cap_k \partial(\phi^k(F_n)) \subseteq \cap_k(\cup_i C_{i,k})$, a contradiction. □

As the dynamics of $\partial\phi$ acting on ∂F_n is of independent interest, we include the following corollary, which follows immediately from the above proof.

Corollary 3.22. *Let $\phi : F_n \to F_n$ be an irreducible endomorphism, and assume that ϕ is not an automorphism. The induced map $\partial\phi : \partial F_n \to F_n$ has finitely many fixed points $X_1, ..., X_r$ such that $r \leq 2n$, and each X_i is attracting. If $N \subseteq \partial F_n$ is some compact neighborhood of $\{X_1, ..., X_r\}$ then there is K such that $(\partial\phi)^k(\partial F_n) \subseteq N$ for any $k \geq K$.*

The following corollary is a restatement of Proposition 3.21 in the language of train tracks.

Corollary 3.23. *Let $\phi : F_n \to F_n$ be an irreducible endomorphism, and assume that ϕ is not an automorphism. Then Φ is topologically represented by a train track map with no illegal turns.*

Proof. From Proposition 3.21 we have that the action $F_n \curvearrowright T_\Phi$ is free and simplicial. Let $f : T_\Phi \to T_\Phi$ be a homothety witnessing the fact that $[T_\Phi]\phi = [T_\Phi]$; then f descends to a map $\overline{f} : T_\Phi/F_n \to T_\Phi/F_n$ that is easily seen to be a simplicial immersion inducing Φ, *i.e.* a train track representative with no illegal turns. □

3.4 Dynamics on CV_n

In this section we classify the dynamics of an irreducible non-surjective endomorphism ϕ acting on CV_n. Recall that, in this case, by Proposition 3.11, we have that ϕ is expansive; and by Proposition 3.21 there is a fixed point for the action, namely $[T_\Phi]$.

3.4.1 The Stable Lamination

Let $\phi : F_n \to F_n$ be irreducible. Following [3] we associate to Φ an algebraic lamination. Let $f : G \to G$ be a train track representative for Φ with transition matrix $M = M(f)$, and equip G with the Perron-Frobenius metric (see Subsection 3.1.2). By Corollary 3.23, we can assume that f is an immersion.

Let $e_i \in E(G)$; by irreducibility of M, there is a natural number k such that the (i,i)-entry of M^k is non-zero. Since $M^k = M(f^k)$, this gives that the f-image of e_i crosses e_i. This gives a fixed point x of f^k in the interior of e_i. Let $N(x)$ be a small ϵ-neighborhood of x in the interior of e_i. There is a unique orientation-preserving isometry $l_0 : (-\epsilon, \epsilon) \to N(x) : 0 \mapsto x$. Each $f^r|_{e_i}$ is an immersion; hence, there are unique orientation-preserving isometric immersions $l_n : (-\lambda^{nk}\epsilon, \lambda^{nk}\epsilon) \to G : 0 \mapsto x$ satisfying $l_n(y) = f^k(l_{n-1}(\lambda^{-k}y))$. The sequence (l_n) gives an isometric immersion $l : \mathbb{R} \to G$ that is f^k-invariant in the sense that $f^k \circ l : \mathbb{R} \to G$ is a reparametrization of l.

Let L_Φ stand for the set of isometric immersions $l' : \mathbb{R} \to G$ obtained via the above procedure; this set is essentially the lamination defined in [3]. The marking $\tau : R_n \to G$ gives a free action of $F_n = \pi_1(R_n)$ on $\tilde{G}$, which gives an identification $\partial F_n \cong \partial G$. This gives a homeomorphism from the space of immersed lines in $\tilde{G}$ (with the weak topology) to $\partial^2 F_n$. For any $l \in L_\Phi$ there are various lifts of l to $\tilde{G}$, and the collection of lifts to $\tilde{G}$ of lines $l \in L_\Phi$ evidently gives an F_n-invaraiant subset $\mathscr{L}_\Phi \subseteq \partial^2 F_n$. The *stable lamination of* Φ, denoted Λ_Φ, is defined to be the smallest algebraic lamination containing $\mathscr{L}_\Phi$.

3.4.2 The Convergence Criterion

In this subsection we state a result of Bestvina-Feighn-Handel from [3] that gives a sufficient condition on a tree $T \in \overline{cv}_n$ to ensure that $[T]\phi^k$ converges to $[T_\Phi]$; this will immediately give a dynamics statement for an irreducible, non-surjective endomorphism acting on Outer space.

Let $T_0 \in cv_n$ and $T \in \overline{cv}_n$; an equivariant map $f : T_0 \to T$ has *bounded backtracking* if there is a constant C such that the f-image of a segment $[p, q]$ is contained in the C-neighborhood of the segment $[f(p), f(q)]$. The smallest such C is called the *backtracking constant of* f, denoted $BBT(f)$. It is a fact that for T_0, T, and f as above, it is always the case that f has bounded backtracking (see [34] and the references therein).

Proposition 3.24. *[3][34, Proposition 6.1] Let $T \in \overline{cv}_n$. Suppose that there is a tree $T_0 \in cv_n$, an equivariant map $f : T_0 \to T$, and a bi-infinite geodesic $\gamma_0 \subseteq T_0$ representing a leaf of Λ_Φ such that $f(\gamma_0)$ has diameter greater than $2BBT(f)$. Then $f(\gamma_0)$ has infinite diameter and there exists a neighborhood V of $[T]$ in $\overline{CV}_n$ such that $\phi^p|_V$ converges uniformly to $[T_\Phi]$.*

We cite the result of [34], as it is completely clear that their proof works in our context; Proposition 6.1 is proved for laminations associated to irreducible outer automorphisms of F_n, but the proof goes through without modification for the case of non-surjective irreducible endomorphisms. Actually, the proof could be simplified by considering only the case of an irreducible expansive endomorphism, as one has in this case the luxury of a train track representative with no illegal turn.

Proposition 3.24 can be restated in terms of dual laminations. Let $T \in \overline{cv}_n$ have dense orbits. For any $\epsilon > 0$, Proposition 2.2 of [34] ensures the existence of a simplicial tree $T_0 \in cv_n$ and an equivariant map $f : T_0 \to T$ with $BBT(f) < \epsilon$. If $Z = (X, Y) \in \partial^2 F_n$ is some point such that for all f with small backtracking, a line representing Z in T_0 is sent under f to a small diameter subset of T, then $Z \in L^2(T)$. Hence, we may apply Proposition 3.24 to get convergence for a tree $T \in \overline{cv}_n$ as long as some leaf of Λ_Φ is not contained in $L^2(T)$; note that by irreducibility of ϕ, if some leaf of Λ_Φ is contained in $L^2(T)$, then every leaf of Λ_Φ is contained in $L^2(T)$.

Corollary 3.25. *Let $\phi : F_n \to F_n$ be irreducible and non-surjective. For any $[T] \in CV_n$, we have $[T]\phi^k \to [T_\Phi]$.*

Proof. For any $T \in cv_n$ and any $Z \in \partial^2 F_n$, Z is represented by an infinite line in T; the result follows by applying Proposition 3.24. □

The convergence of Corollary 3.25 is uniform on compact subsets of CV_n; the goal of the next several sections is to show that the convergence is actually uniform over all of CV_n. The next section deals with obvious obstructions.

3.5 Endomorphisms Acting on $\overline{CV}_n$

It is evident that ϕ acts on CV_n as long as ϕ is injective. However, for a tree $T \in \overline{cv}_n$, it could be the case that $T\phi$ is trivial even if ϕ is injective, and in this case ϕ would not act on $\overline{CV}_n$. The aim of this section is to first illustrate exactly how an endomorphism can fail to act on $\overline{CV}_n$ and to give a sufficient condition for an irreducible endomorphism to act on $\overline{CV}_n$.

3.5.1 Admissible Endomorphisms

Example 3.26. Let $F_3 = F(a, b, c)$, and define $\phi : F_3 \to F_3$ by:

$$a \mapsto a$$

$$b \mapsto bab^{-1}$$

$$c \mapsto b^2ab^{-2}$$

Suppose that $T \in \overline{cv}_3$ is the Bass-Serre tree of the splitting $F(a, b, c) = \langle a, b\rangle * \langle c\rangle$. The endomorphism ϕ is injective, but the tree $T\phi$ is trivial, since $\phi(F_3)$ fixes the vertex of T corresponding to $\langle a, b\rangle$.

The endomorphism ϕ does not act on $\overline{CV}_3$, hence we must restrict attention to a proper subsemigroup of the semigroup of injective endomorphisms of F_n.

Definition 3.27. *An endomorphism $\phi : F_n \to F_n$ is called* admissible *if for all $T \in \overline{cv}_n$, one has that $T\phi$ is non-trivial.*

It follows from [22] that any very small action $F_n \curvearrowright T$ with trivial arc stabilizers can be approximated by a simplicial very small action $F_n \curvearrowright T'$ such that a subgroup $V \leq F_n$ fixes a point $x \in T$ if and only if V fixes some vertex $x' \in T'$; hence we get the following characterization of admissibility.

Lemma 3.28. *An endomorphism $\phi : F_n \to F_n$ is admissible if and only if for any simplicial tree $T \in \overline{cv}_n$, one has that $T\phi$ is non-trivial.*

Lemma 3.28 shows that Example 3.26 is quite generic and immediately emphasizes the importance of vertex stabilizers in simplicial trees in $\overline{cv}_n$ in the present context. Hence, we bring the following:

Definition 3.29. *A splitting of F_n is called* very small *if it corresponds to a simplicial tree in $\overline{cv}_n$.*

The following is a translation of [3, Definition 2.2] into the formalism of algebraic laminations.

Definition 3.30. *Let $H \leq F_n$ be finitely generated; say that (the conjugacy class of) H* carries *a point $Z \in \partial^2 F_n$ if for any $T \in cv_n$, there is $g \in F_n$ such that $Z \in \partial^2 T_{H^g}$.*

The stable lamination Λ_Φ and its relationship to the dual lamination $L^2(T)$ of a tree $T \in \overline{cv}_n$ is of primary importance to us if we wish to apply the convergence criterion given by Proposition 3.24; hence, we now begin working to develop a characterization of admissibility involving only Λ_Φ.

Lemma 3.31. *Let $T \in cv_n$, and let $H \leq F_n$ be finitely generated. Then $T_H = T$ if and only if H is finite index in F_n.*

Proof. Let T and H be as in the statement, and suppose that $T_H = T$. First suppose that $T = \tilde{R_n}$ is the "standard" Cayley tree for $F_n = F(A)$, and regard T as a labeled directed tree. Then T/H is a labeled directed finite graph, which is A-regular (see [29]), and choosing a basepoint in T/H gives an immersion representing a subgroup $H' \leq F_n$ that is conjugate to H. As T/H is A-regular, this immersion is a covering map, and it follows that H' is finite index in F_n; hence, H is finite index in F_n.

Now let $T \in cv_n$ be arbitrary, and choose a spanning tree $G_0 \subseteq T/F_n$; collapsing the lifts of G_0 in T to points gives a map $f : T \to T_0$ onto a Cayley tree. By replacing H by its image under some $\alpha \in Aut(F_n)$, we may suppose that T_0 is the Cayley tree $\tilde{R_n}$. It is easy to see that $(T_0)_H = f(T_H)$, and so by above, we get that H is finite index in F_n.

Conversely, suppose that $H \leq F_n$ is finite index, so there is k such that for all $f \in F_n$, one has $f^k \in H$. Since $A(f^k) = A(f)$, it follows that $T_H = \cup_{1 \neq h \in H} A(h) = T$. $\square$

3.5.2 The Admissibility Criterion

We now establish a characterization of admissibility in the case of irreducible endomorphisms; the following lemma allows us to use the convergence criterion of Proposition 3.24 to understand the action of an admissible irreducible endomorphism on the simplicial trees in $\overline{CV}_n$.

Proposition 3.32. *Let $\phi : F_n \to F_n$ be irreducible. Then ϕ is admissible if and only if no leaf of Λ_Φ is carried by a vertex group of a very small splitting of F_n.*

Proof. If ϕ is not admissible, then, by Lemma 3.28, there is some simplicial tree $Y \in \overline{cv}_n$, such that $Y\phi$ is trivial. In this case some vertex group of the splitting corresponding to Y carries every leaf of Λ_Φ.

So, assume that ϕ is admissible. Put $T := T_\Phi$, and let $f : T \to T$ be the ϕ-compatible $\lambda := \lambda_\Phi$ homothety of T. As in the proof of Proposition 3.21, after possibly passing to a power of f, we may find a branch point $x \in T$ that is fixed by f and such that every direction at x is fixed by f as well. It is easy to see that there is, in each diretion at x, an infinite ray based at x that is fixed setwise by f. Denote this infinite multipod by X; it follows that $\partial^2 X \subseteq \Lambda_\Phi$.

Toward a contradiction, suppose that some leaf of $\Lambda = \Lambda_\Phi$ is carried by a vertex group V of a very small splitting of F_n; without loss, we can assume that V is a vertex group of a one-edge splitting. As ϕ is irreducible, we have that every leaf of Λ is carred by V (see [3]). The following is easily verified:

Claim 3.33. Let $H \leq F_n$ be finitely generated, and let $Y \in cv_n$. There is a constant $C = C(H, Y)$ such that if $Y_H \cap Y_{H^g}$ has diameter greater than C, then $Y_H \cap Y_{H^g}$ has infinite diameter, and $H \cap H^g$ is non-trivial.

Note that if V is a vertex group of a very small 1-edge splitting and if $V \neq V^g$ with $V \cap V^g$ nontrivial, then $V \cap V^g$ is cyclic and is conjugate to the edge group of the splitting.

Claim 3.34. No leaf of Λ is periodic: for any tree $T \in cv_n$ and any non-trivial $g \in F_n$, there is a constant $K = K(T, g)$ such that if l is a line in T representing a leaf of Λ, then the diameter of $l \cap A(g)$ is bounded above by K.

Proof. The existence of a periodic leaf would give an element $f \in F_n$ such that $\phi(f) = hf^r h^{-1}$ for some r and some $h \in F_n$. If $|r| = 1$, expansivity of ϕ is contradicted. If $|r| > 1$, it follows from [2, Lemma 4.1] that f is primitive, contradicting Lemma 3.2. □

According to the above claims, after possibly replacing V with a conjugate, we have that $X \subseteq T_V$. As ϕ is irreducible, any line in X crosses every $\phi(F_n)$-orbit of branch points in $T_{\phi(F_n)}$ infinitely often. Let $C = C(V, T)$ be the constant guaranteed by Claim 3.33. By replacing ϕ by a power if necessary, we can assume that branch points in $T_{\phi(F_n)} = f(T)$ are separated by distance at least C. Suppose that $c \in F_n$ is a generator for some (non-trivial) cyclic intersection $V \cap V^f$, and put $K = K(T, c)$ as in Claim 3.34; increase C if necessary to ensure that $C \geq K$.

Let $y \in T$ be a branch point of $T_{\phi(F_n)}$ such that $[x, y]$ contains no other branch points of $T_{\phi(F_n)}$. By the above discussion, there is $g \in F_n$ such that gT_V contains an infinite multipod centered at y that also contains the point x. Hence the diameter of $T_v \cap gT_v$ is infinite and $V \cap V^g$ is nontrivial. If it were the case that $V \neq V^g$, then $V \cap V^g$ is cyclic; however, this is impossible by choice of C. It follows that $V = V^g$, and by iterating this argument, we get that $T_{\phi(F_n)} \subseteq T_V$.

Claim 3.35. Let $Stab(T_V)$ denote the setwise stabilizer, then $Stab(T_V) = V$.

Proof. If $Stab(T_V) \neq V$, there is a finitely generated V' containing V such that $T_{V'} = T_V$. It follows from Lemma 3.31 that V is finite index in V'. Let Y be a Bass-Serre tree for a very small splitting of F_n with vertex group V. It is easy to see that V' fixes the vertex of Y corresponding to V, hence $V' = V$. □

It follows from the above claim that for any finitely generated $K \leq F_n$, if $T_K \leq T_V$, then $K \leq V$. Therefore, we conclude that for some k, $\phi^k(F_n) \leq V$, contradicting admissibility of ϕ. □

3.6 Convergence for Simplicial Actions and Graphs of Actions

In this section we apply Proposition 3.32 along with Proposition 3.24 to understand the action of a non-surjective admissible irreducible endomorphism on tree that splits as a non-trivial graph of actions.

3.6.1 Simplicial Actions in $\overline{CV}_n$

Proposition 3.36. *Let $\phi : F_n \to F_n$ be irreducible and non-surjective, and suppose that ϕ is admissible. For any simplicial $T \in \overline{cv}_n$, one has $[T]\phi^k \to [T_\Phi]$.*

Proof. By Proposition 3.32 we have that no leaf of Λ_Φ is carried by a vertex group of a very small splitting of F_n. By [7, Proposition 1.3] ∂F_n is naturally identified with the disjoint union of ∂T with the union of boundaries of the vertex stabilizers. It follows that for any leaf $Z \in \Lambda_\Phi$ and any equivariant map $f : T_0 \to T$ from a simplicial tree T_0 to T, if l is a line in T_0 representing Z, then $f(l)$ has infinite diameter in T. Convergence follows from Proposition 3.24. □

3.6.2 Graphs of Actions in $\overline{CV}_n$

Lemma 3.37. *Let $\phi : F_n \to F_n$ be irreducible, non-surjective, and admissible; let $T \in \overline{cv}_n$. There is k such that $T\phi^k$ is free.*

Proof. Toward a contradiction suppose that there is $T \in \overline{cv}_n$ such that for all i, $T\phi^i$ is not free. By [22] point stabilizers in T are vertex groups of a very small splitting of F_n; there are finitely many orbits of points in T with nontrivial stabilizer, so finitely many conjugacy classes of such vertex groups appear. Hence, we may find a sequences $g_k \in \phi^{i_k}(F_n)$ and $h_k \in F_n$, and a vertex group V of a very small splitting of F_n such that $g_k^{h_k} \in V$.

Again by [22], there is a simplicial tree $T' \in \overline{cv}_n$ such that the set of point stabilizers in T is equal to the set of vertex stabilizers in T'. As $g_k^{h_k}$ fixes a point in $T\phi^{i_k}$, we have that $g_K^{h_k}$ fixes a point in $T'\phi^{i_k}$. Since, T_Φ is free, we arrive at a contradiction to Proposition 3.36. □

We now consider trees in $\overline{cv}_n$ that split as graphs of actions. Let $T \in \overline{cv}_n$, and suppose that T splits as a grpah of actions $T = T_{\mathscr{G}}$, for $\mathscr{G} = (S, \{T_v\}_{v \in V(S)}, \{p_e\}_{e \in E(S)})$ (refer to Section 3.2 for definitions).

Lemma 3.38. *Let $\phi : F_n \to F_n$ be irreducible, non-surjective, and admissible; and let $T \in \overline{cv}_n$. If T splits as a graph of actions, then $[T]\phi^k \to [T_\Phi]$.*

Proof. Suppose that T splits as a graph of action $\mathscr{G} = (S, \{T_v\}_{v \in V(S)}, \{p_e\}_{e \in E(S)})$. Let ϕ as in the statement. According to Lemma 3.37, there is k such that the action $T\phi^k$ is free. Keep in mind that the action $T\phi^k$ is equivariantly isometric to the action $\phi^k(F_n) \curvearrowright T_{\phi^k(F_n)}$. Put $H_l := \phi^l(F_n)$, and put $T_l := T\phi^l$; we regard the action T_l as a subaction of the action $F_n \curvearrowright T$; namely, the action $F_n \curvearrowright T_l$ is precisely the action $H_l \curvearrowright T_{H_l}$.

The subgroups H_l act on S, and it is evident that the union of vertex trees in T corresponding to vertices of S_{H_l} give a transverse covering of T_l, whence T_l inherits a graph of actions structure from $\mathscr{G}$, with skeleton $S_l := S_{H_l}$. As T_k is free, the action $H_l \curvearrowright S_l$ is with trivial arc stabilizers; this is because arc stabilizers in the action $F_n \curvearrowright S$ correspond to stabilizers of attaching points. Thus it follows from Proposition 3.36 that there is M such that $H_l \curvearrowright S_l$ is free for $l \geq M$ and such that for any non-trivial $h \in H_l$, the translation length of h is at least two in the simplicial metric on S_l. Hence, for l big enough, S_l is locally finite, and it follows from the distance formula for graphs of actions (Formula 3.1) that there is a positive lower bound for translation lengths of non-trivial elements for the action $H_l \curvearrowright T_l$. Hence, for l big enough, T_l is free and simplicial, and by Proposition 3.36 we have that $[T]\phi^k \to [T_\Phi]$. $\square$

Chapter 4

Indecomposable Trees

4.1 Approximations of Trees

Fix a basis A for F_n and an action $F_n \curvearrowright T \in \overline{cv}_n$ with dense orbits, and let $\mu \in M(T)$.

Definition 4.1. *Say that the action $F_n \curvearrowright T$ is* supported μ-a.e. *on a μ-measurable set $X \subseteq T$ if for any arc $I \subseteq T$ and any $\delta > 0$, there are $g_1, ..., g_r \in F_n$ such that $\mu(I - (g_1 X \cup ... \cup g_r X)) < \delta$.*

For a finite forest $F \subseteq T$, we write $S = (F, A)$ to denote the pseudogroup generated by restrictions of elements of A to F. Recall that μ_L denotes Lebesgue measure on T.

Lemma 4.2. *Let $T \in \overline{cv}_n$ be with dense orbits. For any $\epsilon > 0$ and any finite forest $K \subseteq T$, there are finite forests F_ϵ and F such that:*

(i) $\mu_L(F_\epsilon) < \epsilon$,

(ii) $F_n \curvearrowright T$ *is supported* μ_L*-a.e. on* F_ϵ

(iii) $\mu_L(F \cap K) > \mu_L(K) - \epsilon$,

(iv) $S = (F, A)$ *satisfies* $m(S) - d(S) < \epsilon$

Proof. Let T, ϵ, and K as in the statement. By Proposition 2.15 we have that $\mu_L = \sum_{i=1}^{p} \nu_i$, with each ν_i ergodic. Take $J_i \subseteq T$ finite arcs such that $\nu_i(J_i) > 0$ and $\sum_i \mu_L(J_i) < \epsilon$; put $F_\epsilon := \cup_i J_i$. By ergodicity of the measures ν_i, we get that $\bigcup_{g \in F_n} gF_\epsilon$ is a full measure subset of T, so the action $F_n \curvearrowright T$ is supported μ_L-a.e. on F_ϵ. Hence, there are $g_1, ..., g_r \in F_n$ such that $\mu_L(K \cap (\cup_i g_i F_\epsilon)) > \mu_L(K) - \epsilon$. Let $S_0 := (F_\epsilon, A) = (F_0, A)$ the finite pseudogroup of restrictions of elements of A to F_0, and note that $m(S_0) - d(S_0) \leq \mu_L(F_\epsilon) < \epsilon$. Define $F_i := F_{i-1} \cup \bigcup_{a \in A^\pm} aF_{i-1}$, and let $S_i = (F_i, A)$. Immediately, one has $m(S_i) - d(S_i) \leq m(S_{i-1}) - d(S_{i-1})$. The claim follows by observing that for $i \geq max\{|g_i|_A\}$ we have $\cup_i g_i F_0 \subseteq F_i$. ∎

Remark 4.3. *If an action $F_n \curvearrowright T \in \overline{cv}_n$ is indiscrete, but not with dense orbits, then T splits as a graph of actions with vertex trees either finite arcs of T (the* simplicial part *of T) or subtrees T_v such that the action $Stab(T_v) \curvearrowright T_v$ is with dense orbits (see [32, 25]). In this case, it follows from the above argument that for any $\epsilon > 0$, the action $F_n \curvearrowright T$ is supported μ_L-a.e. on a finite forest F'_ϵ with $\mu_L(F'_\epsilon) < vol(T/F_n) + \epsilon$, where $vol(T/F_n) = \inf \mu_L(S)$ with the infimum taken over all measurable $S \subseteq T$ projecting onto T/F_n under the natural map.*

Definition 4.4. *[37] A finitely generated subgroup $H \leq F_n$ is* e-algebraically closed *if for any $g \in F_n - H$, one has $\langle H, g\rangle \cong H * \langle g\rangle$.*

Equivalently, H is e-algebraically closed if there is no non-trivial equation $w(\overline{h}, x)$ over H with a solution $w(\overline{h}, g)$ for $g \in F_n - H$. Any free factor of F_n is necessarily e-algebraically closed; further, if $H \leq F_n$ has rank r and is maximal in the poset of rank r subgroups of F_n, then H is e-algebraically closed (see [37] for details). For the sequel, our interest in Definition 4.4 is that free factors are e-algebraically closed; we work in the more general framework because the present techniques potentially admit generalization; see the Remark 4.10.

Recall that for T an $\mathbb{R}$-tree, equipped with an action of a group G, a G-invariant collection $\{T_v\}_{v\in V}$ of non-degenerate proper subtrees of T is called a *transverse family* if whenever $T_v \neq T_{v'}$, $T_v \cap T_{v'}$ contains at most one point.

Lemma 4.5. *Let $T \in \overline{cv}_n$ be with dense orbits; let $H \leq F_n$ a finitely generated subgroup with minimal invariant tree $T_H \subseteq T$. Suppose that the action $H \curvearrowright T_H$ has dense orbits and that H is e-algebraically closed. The family of translates $\{gT_H\}_{g\in F}$ is a transverse family.*

Proof. Let T and H as in the statement of the lemma. Note that since H is e-algebraically closed, if H is a proper subgroup of F_n, then H is infinite index in F_n. If $H = F_n$, then the statement is trivial, so we suppose that H has infinite index in F_n. Choose a basis B for H. Let $F \subseteq T_H$ be a finite forest; since the action $F_n \curvearrowright T$ has dense orbits and is very small, arc stabilizers are trivial [34], so $H \curvearrowright T_H$ has trivial arc stabilizers as well. Hence, the pseudogroup $S = (F, B)$ generated by restrictions of the elements of B to F has independent generators; this follows from the fact that any non-trivial word in $B^{\pm}$ represents a non-trivial element of H. Further, since H is e-algebraically closed, it is the case that for any $f \in F_n - H$, we have $\langle H, f\rangle = H * \langle f\rangle$, so the restrictions of $B \cup \{f\}$ to F give a finite pseudogroup with independent generators.

Toward a contradiction, suppose that there is $f \in F - H$ such that $fT_H \cap T_H$ contains more than one point. Since the intersection of two trees is convex, we have that $fT_H \cap T_H$ contains a non-degenerate arc I. Choose $\epsilon > 0$ small with respect to $\mu_L(I)$. Set $K := I \cup f^{-1}I$; by Lemma 4.2, we may find a finite forest $F \subseteq T_H$ such that $\mu_L(F \cap K) > \mu_L(K) - \epsilon$ and such that $S = (K, B)$ satisfies $m(S) - d(S) < \epsilon$.

Now, consider $S' := (K, B \cup \{f\})$; as noted above, S' has independent generators. On the other hand, it is clear from the construction that $m(S') - d(S') < 0$, a contradiction to Proposition 2.16. It follows that $fT_H \cap T_H$ contains at most one point for each $f \in F_n - H$, so $\{gT_H\}_{g \in F_n}$ is a transverse family. $\square$

Remark 4.6. *The proof actually shows something stronger: if the action $H \curvearrowright T_H$ is indiscrete and if H is e-algebraically closed, then for $g \in F_n$ with $gT_H \neq T_H$, we have that any non-degenerate intersection $gT_H \cap T_H$ is contained in the simplicial part of T_H (See Remark 4.3).*

Further, the proof shows that if H is e-algebraically closed and if H is a proper subgroup of F_n, then T_H is a proper subtree of T; later (see Remark 4.12), we will see that for H finitely generated, $T_H = T$ if and only if H is finite index in F_n.

4.2 Indecomposable Trees

Recall that a G-tree T is called *indecomposable* if for any non-degenerate arcs $I, J \subseteq T$, there are elements $g_1, ..., g_r$ such that $J \subseteq g_1 I \cup ... \cup g_r I$, and $g_i I \cap g_{i+1} I$ is non-degenerate for $i \leq r - 1$.

Lemma 4.7. *If an action $G \curvearrowright T$ is indecomposable, then there is no transverse family for the action $G \curvearrowright T$.*

Proof. Suppose that the action $G \curvearrowright T$ is indecomposable; and, toward contradiction, suppose that $\{T_v\}_{v \in V}$ is a transverse family for the action $G \curvearrowright T$. Recall that each T_v is a proper, non-degenerate subtree of T and that the collection $\{T_v\}_{v \in V}$ is G-invariant. Hence, we may find distinct T_v, $T_{v'}$ along with an arc $I \subseteq T$ such that $I \cap T_v$ and $I \cap T_{v'}$ are non-degenerate. Define $I_0 := I \cap T_v$; by indecomposability of the action $G \curvearrowright T$, there are $g_1, ..., g_r \in G$ such that $I \subseteq g_1 I_0 \cup ... \cup g_r I_0$ with $g_i I_0 \cap g_{i+1} I_0$ non-degenerate. Without loss we may assume that $g_1 I_0 \cap T_v$ and $g_r I_0 \cap T_{v'}$ are non-degenerate. Since $\{T_v\}_{v \in V}$ is a transverse family and since $g_i I_0 \cap g_{i+1} I_0$ are non-degenerate, it follows that $g_i I_0 \subseteq T_v$ for each i. On the other hand, again since $\{T_v\}_{v\in V}$ is a transverse family and since $g_i I_0 \cap g_{i+1} \cap I_0$ are non-degenerate, it follows that $g_i I_0 \subseteq T_{v'}$. This implies $T_v = T_{v'}$, a contradiction. □

4.2.1 Lifting Indecomposability

We need the following technical result; an idea helpful for constructing the proof was communicated to us by Vincent Guirardel.

Lemma 4.8. *Suppose that the action $G \curvearrowright T$ is indecomposable and that $H \leq G$ is finitely generated and finite index. Then the action $H \curvearrowright T$ is indecomposable.*

Proof. We remark that since $H \leq G$ is finite index, $T_H = T$; without loss, we may assume that H is normal. For an arc $I \subseteq T$, define a subtree $Y_I \subseteq T$ as follows. Put $Y_0 := I$, and define $Y_{i+1} := Y_i \cup \bigcup_h hI$, where the union is taken over elements $h \in H$ such that $hI \cap Y_i$ is non-degenerate. Finally set $Y_I := \cup_i Y_i$. Toward a contradiction assume that the action $H \curvearrowright T$ is not indecomposable; it follows that we may find a non-degenerate arc $I \subseteq T$ such that $Y_I \subsetneq T$. By construction, the collection $\{hY_I\}_{h\in H}$ is a transverse family for the action $H \curvearrowright T$. The idea is to use $\{hY_i\}_{h\in H}$ to produce a transverse family for the action $G \curvearrowright T$, which will contradict Lemma 4.7.

Let $\{1 = g_1, ..., g_l\}$ be a left transversal to H in G, and let $[g_i]$ denote the coset corresponding to g_i. Note that by indecomposibility of the action $G \curvearrowright T$, there is $g \in G - H$ such that $gY_I \cap Y_I$ is non-degenerate and $gY_I \neq Y_I$; say $g \in [g_i]$. Consider the collection of non-degenerate intersections $gY_I \cap hY_I$ for $g \in [g_i]$ and $h \in H$. This collection is a transverse family for the action $H \curvearrowright T$; indeed, normality of H ensures invariance, so suppose that $h(g_i h_1 Y_I \cap h_2 Y_I) \cap (g_i h_3 Y_I \cap h_4 Y_I)$ is non-degenerate. We have:

$$\begin{aligned} h(g_i h_1 Y_I \cap h_2 Y_I) \cap (g_i h_3 Y_I \cap h_4 Y_I) &= g_i h' h_1 Y_I \cap h h_2 Y_I \cap g_i h_3 Y_I \cap h_4 Y_I \\ &= g_i (h' h_1 Y_I \cap h_3 Y_I) \cap (h h_2 Y_I \cap h_4 Y_I) \end{aligned}$$

As $\{hY_I\}_{h\in H}$ is a transverse family for the action $H \curvearrowright T$, it follows that $h' h_1 Y_I = h_3 Y_I$ and $h h_2 Y_I = h_4 Y_I$. Hence $h(g_i h_1 Y_I \cap h_2 Y_I) = (g_i h_3 Y_I \cap h_4 Y_I)$, as desired.

Suppose that for any $g \in G$ with $g \notin H \cup [g_i]$ every intersection of the form $gY_I \cap g_ih_1Y_I \cap h_2Y_I$ is degenerate. We claim that this implies that the family of G-translates of non-degenerate intersections of the form $g_ih_1Y_I \cap h_2Y_I$ is a transverse family for the action $G \curvearrowright T$. Indeed, if $g \notin H \cup [g_i]$, then $g(g_ih_1Y_I \cap h_2Y_I) \cap (g_ih_3Y_I \cap h_4Y_I) = \emptyset$, and we have already checked that the collection of translates of non-degenerate intersections of the form $g_ih_1Y_I \cap h_2Y_I$ is a transverse family for the action of H. Hence, we are left to consider $g \in [g_i]$; we have:

$$\begin{aligned} g_ih(g_ih_1Y_I \cap h_2Y_I) \cap (g_ih_3Y_I \cap h_4Y_I) &= g_i^2h'h_1Y_I \cap g_ihh_2Y_I \cap g_ih_3Y_I \cap h_4Y_I \\ &= g_i(hh_2Y_I \cap h_3Y_I) \cap g_i^2h'h_1Y_I \cap h_4Y_I \end{aligned}$$

Since $\{hY_I\}_{h\in H}$ is a transverse family for the action $H \curvearrowright T$, if the above intersection intersection is non-degenerate, then $hh_2Y_I = h_3Y_I$, and the intersection $g_i^2h'h_1Y_I \cap g_ih_3Y_I \cap h_4Y_I$ is also non-degenerate. On the other hand, $g_i^2 \notin [g_i]$, so it must be the case that $g_i^2 \in H$. In this case we get

$$\begin{aligned} g_ih(g_ih_1Y_I \cap h_2Y_I) \cap (g_ih_3Y_I \cap h_4Y_I) &= g_ih(g_ih_1Y_I \cap h_2Y_I) \\ &= (g_ih_3Y_I \cap h_4Y_I) \end{aligned}$$

It follows that, under this assumption, the family of G-translates of non-degenerate intersections of the form $g_ih_1Y_I \cap h_2Y_I$ is a transverse family for the action $G \curvearrowright T$, contradicting Lemma 4.7.

Hence, there must be a non-degenerate intersection of the form $gY_I \cap g_jh_1Y_I \cap h_2Y_I$, for $g \notin [g_i]$. We may continue in this way to get a non-degenerate intersection $g_1'Y_I \cap ... \cap g_l'Y_I$, for $g_i' \in [g_i]$. By the same arguments as above, the collection of all such non-degenerate intersections is a transverse family for the action $H \curvearrowright T$; we claim that it is also a transverse family for the action $G \curvearrowright T$.

We have $Y = g_1h_1Y_I \cap ... \cap g_lh_lY_I$ non-degenerate. Let $g \in G$, then by normality of H, we get

$$\begin{aligned} gY \cap Y &= g(g_1h_1Y_I \cap ... \cap g_lh_lY_I) \cap (g_1h_1Y_I \cap ... \cap g_lh_lY_I) \\ &= g_ih(g_1h_1Y_I \cap ... \cap g_lh_lY_I) \cap (g_1h_1Y_I \cap ... \cap g_lh_lY_I) \\ &= (g_ig_1h_1'h_1Y_I \cap ... \cap g_ig_lh_l'h_lY_I) \cap (g_1h_1Y_I \cap ... \cap g_lh_lY_I) \\ &= (g_{i_1}h_1'h_1Y_I \cap ... \cap g_{i_l}h_l'h_lY_I) \cap (g_1h_1Y_I \cap ... \cap g_lh_lY_I) \end{aligned}$$

Here g_{i_j} is the representative for the coset $g_i[g_j]$. Since $\{hY_I\}_{h \in H}$ is a transverse family for $H \curvearrowright T_H$, and since $g_{i_j}(h_ih_i'Y_I \cap h_jY_I)$ is non-degenerate, we get that $h_ih_i'Y_I = h_jY_I$. Hence, $\{gY\}_{g \in G}$ is a transverse family, a contradiction to Lemma 4.7. □

4.2.2 Infinite Index Subgroups Act Discretely

The following result is of central importance to us. It is a partial restatement of Marshall Hall's Theorem; for a particularly beautiful proof, see [44] (see [29] for extensions of the ideas of [44]).

Theorem 4.9. *[27] Let $H \leq F_n$ be finitely generated. There is finitely generated $F' \leq F_n$ of finite index, such that $F' = H * K$.*

Recall that if $F_0 \leq F_n$ is a free factor, then F_0 is e-algebraically closed in F_n. In light of this, the above theorem states that for any finitely generated $H \leq F_n$, we can find a finitely generated, finite index subgroup $F' \leq F$ such that H is e-algebraically closed in F'.

Remark 4.10. *Our reason for proving Lemma 4.5 in the case of e-algebraically closed subgroups of an arbitrary group G and not just in the case of free factors comes from the fact that the proof of our main result (Theorem 4.11 below) goes through for any indecomposable action $G \curvearrowright T$, where G satisfies Theorem 4.9,* i.e. *finitely generated subgroups of G are "virtually e-algebraically closed." It would be interesting to describe the class of groups satisfying this condition.*

Theorem 4.11. *Let $T \in \overline{cv}_n$ be indecomposable, and let $H \leq F_n$ be a finitely generated subgroup. The action $H \curvearrowright T_H$ is indiscrete if and only if H is finite index in F_n.*

Proof. Let T and H as in the statement; toward a contradiction suppose that the action $H \curvearrowright T_H$ is indiscrete. It follows from the discussion in Remark 4.3 that there is finitely generated $H' \leq H$

such that the action $H' \curvearrowright T_{H'}$ is with dense orbits, so we may suppose that the action $H \curvearrowright T_H$ is with dense orbits. By Theorem 4.9, there is a finitely generated $F' \leq F_n$ of finite index, such that $H \leq F'$ is a free factor; hence H is e-algebraically closed in F'. By Lemma 4.8 we have that the action $F' \curvearrowright T$ is indecomposable; on the other hand, by Lemma 4.5 as H is e-algebraically closed in F', the family of F'-translates of T_H is a transverse family for the action $F' \curvearrowright T$. By Lemma 4.7, we arrive at a contradiction to the indecomposability of the action $F' \curvearrowright T$. □

Remark 4.12. *A similar line of reasoning as above shows that for any finitely generated, infinite index $H \leq F_n$, we have that T_H is a proper subtree of T. Indeed, by Theorem 4.9 we may find $F' \leq F_n$, finite index, such that H is e-algebraically closed in F'. It follows from Remark 4.6 that $T_H \subsetneq T_{F'} = T$.*

Corollary 4.13. *Let $F_n \curvearrowright T \in \overline{cv}_n$ be an action with dense orbits, and let $H \leq F_n$ be finitely generated. Then $T_H = T$ if and only if H is finite index in F_n.*

To complete the analogy with the dynamical-algebraic properties of ending laminations, we bring Corollary 4.15 below; as mentioned in the introduction, the hypothesis that the action $F_n \curvearrowright T$ be free is essential–it is a by-product of the definition of the dual lamination of a tree. Corollary 4.15 follows immediately from Theorem 4.11 and the following:

Lemma 4.14. *Let $T \in \overline{cv}_n$ be free with dense orbits, and let $H \leq F_n$ finitely generated. The action $H \curvearrowright T_H$ is indiscrete if and only if H carries a leaf of $L^2(T)$.*

Proof. Suppose that the action $H \curvearrowright T_H$ is indiscrete, then $L^2(T_H) := L^2(H \curvearrowright T_H)$ is non-empty; from the definition of $L^2(T)$, it is evident $L^2(T_H) \subseteq L^2(T)$.

Conversely, suppose that H carries a leaf $l \in L^2(T)$. Toward a contradiction suppose that the action $H \curvearrowright T_H$ is discrete. Let $T^0 \in cv_n$, and choose an F_n-equivariant map $f : T^0 \to T$; then f restricts to an H-equivarant map $f_H : T^0_H \to T_H$, which descends to $\overline{f}_H : T^0_H/H \to T_H/H$, which is a homotopy equivalence, since the action $H \curvearrowright T_H$ is free. It follows that f_H is a quasi-isometry. On the other hand, by Proposition 2.6, if $l \in L^2(T)$ is carried by H, there is a line $l_0 \subseteq T^0_H$ representing l that is mapped via f_H to a bounded subset of T, a contradiction. □

Corollary 4.15. *Suppose that $T \in \overline{cv}_n$ is indecomposable and free with dual lamination $L^2(T)$, and let $H \leq F_n$ be finitely generated. Then H carries a leaf of $L^2(T)$ if and only if H is finite index in F_n.*

4.3 Convergence for Indecomposable Actions in $\overline{CV}_n$

In this section we consider trees with the following strong mixing property introduced by Guirardel in [25]; this definition is crucial for the sequel.

Definition 4.16. *An action $G \curvearrowright T$ of a finitely generated group on an $\mathbb{R}$-tree is called* indecomposable *if for any finite, non-degenerate arcs $I, J \subseteq T$, there are elements $g_1, ..., g_r \in G$ such that $J \subseteq g_1 I \cup ... \cup g_r I$ and such that $g_i I \cap g_{i+1} I$ is non-degenerate for $i \leq r-1$.*

It is important to note that the intersections $g_i I \cap J$ can be degenerate; see [25] for further discussion.

Theorem 4.11 allows us to handle convergence for indecomposable trees in $\overline{cv}_n$; note that Theorem 4.11 implies that for finitely generated, infinite index $H \leq F_n$, it must be the case that $H \curvearrowright T_H$ is simplicial. Indeed, if not, by Proposition 3.17 there would be some finitely generated $K \leq H$ such that $K \curvearrowright T_K$ has dense orbits.

Corollary 4.17. *Let $\phi : F_n \to F_n$ be irreducible, non-surjective, and admissible, and let $T \in \overline{cv}_n$ be indecomposable. Then $[T]\phi^k \to [T_\Phi]$.*

Proof. Let ϕ and T as in the statement. As ϕ is non-surjective, we have that $\phi(F_n)$ has infinite index in F_n; it then follows from Theorem 4.11 and the discussion above that $\phi(F_n) \curvearrowright T_{\phi(F_n)}$ is simplicial. Convergence then follows from Proposition 3.36. □

Chapter 5

Invariant Measures and Projections

In this section we establish some structure theory for trees $T \in \overline{cv}_n$ that do not split as graphs of actions and are not indecomposable; in short, we show how to find $T' \in \overline{cv}_n$ such that $L^2(T) \subseteq L^2(T')$ and such that either T' splits as a graph of actions, or T' is indecomposable. The aim, of course, is to obtain convergence for the remainder of trees in $\overline{cv}_n$. The main technical tool is the notion of a *length measure*; as mentioned in the Introduction, this tool treats a tree $T \in \overline{cv}_n$ as a generalization of a measured lamination on a surface: the length measures are analogs of the transverse measures.

5.1 Length Measures

Let T be an $\mathbb{R}$-tree. The following definition appears in [23], where it is attributed to F. Paulin.

Definition 5.1. *A* length measure *(or just* measure*) μ on T is a collection $\mu = \{\mu_I\}_{I \subseteq T}$ of finite positive Borel measures on the finite arcs $I \subseteq T$; it is required that for $J \subseteq I$ $\mu_J = (\mu_I)|_J$.*

As these measures are defined locally on finite arcs, all the usual measure-theoretic definitions are similarly defined: a set $X \subseteq T$ is *μ-measurable* if $X \cap I$ is μ_I-measurable for each $I \subseteq T$; X is *μ-measure zero* if $X \cap I$ is μ_I-measure zero for each I; and so on. The *Lebesgue length measure* on T, denoted μ_L, is the collection of Lebesgue measures on the finite arcs of T.

If T is equipped with an action of a group G, then we say that a (length) measure μ is *G-invariant* if $\mu_I(X \cap I) = \mu_{g.I}(g.X \cap g.I)$ holds for each $g \in G$. Note that if the action $G \curvearrowright T$ is by isometries, then the Lebesgue measure is invariant. We let $M(T) = M(G \curvearrowright T)$ stand for the set of invariant measures on T. The following lemma shows that the existence of an invariant atomic measure has a simple interpretation.

Lemma 5.2. *Suppose that $G \curvearrowright T$ has an invariant atomic measure, then T splits as a graph of actions*

Proof. Let μ be a G-invariant atomic measure on T; without loss, we suppose that μ is ergodic. Let $x \in T$ with $\mu(x) > 0$. Since the measures μ_I are finite, it follows that $G.x$ meets any finite subtree of T in a finite set. Consider the collection $\{T_v\}_{v \in V}$ of closures of components of $T \setminus G.x$; this family is evidently a transverse covering of T. Hence, $G \curvearrowright T$ splits as a graph of actions by Lemma 3.15. □

Later we will restrict our attention to non-atomic measures; Lemma 5.2 shows that this restriction is vacuous as long as the tree in question does not split as a graph of actions. The following definition from [25] is convenient when dealing with length measures; the discussion following it shows that global properties of length measures can be seen in finite subtrees.

Definition 5.3. *Let G a group and T an $\mathbb{R}$-tree equipped with an action of G be isometries; and let $K \subseteq T$ be a subtree. We say that the action $G \curvearrowright T$ is* supported *on K if for any finite arc $I \subseteq T$, there are $g_1, ..., g_r \in G$ such that $I \subseteq g_1K \cup ... \cup g_rK$.*

Let G and T as above, and suppose that G is finitely generated with generating set X. Then for any $y \in T$ the convex hull of $\{gy\}_{g \in X^{\pm}}$ is a finite supporting subtree for the action $G \curvearrowright T$. For a measure $\mu \in M_0(T)$ and a finite tree $K = I_1 \cup ... \cup I_l \subseteq T$ for finite arcs $I_j \subseteq T$, let $Supp_K(\mu)$ denote the union of support sets $Supp(\mu_{I_1}) \cup ... \cup Supp(\mu_{I_l})$. The set $Supp_K(\mu)$ is called the *K-support* of μ; if K is clear from context, then $Supp_K(\mu) = Supp(\mu)$ is called the *support* of μ. If $X \subseteq T$ is some subset, say that the support of μ is contained in X, if for every finite $K \subseteq T$, one has $Supp_K(\mu) \subseteq K \cap X$; similarly write $Supp(\mu) = X$ if for every finite $K \subseteq T$, one has $Supp_K(\mu) = K \cap X$.

Recall that given an action $G \curvearrowright T$, $M(T)$ denotes the positive convex cone of G-invariant measures on T. A non-trivial measure $\mu \in M(T)$ is called *ergodic* if any G-invariant subset is either full measure or measure zero; the G-tree is called *uniquely ergodic* if there is a unique, up to scaling, G-invariant measure μ on T; in this case μ must be ergodic. Let $M_0(T)$ denote the set of non-atomic, G-invariant measures on T, and let $M_1(T) := \{\nu \in M_0(T) | \nu \leq \mu_L\}$. Note that both $M_0(T)$ and $M_1(T)$ are convex.

We equip $M_0(T)$ and $M_1(T)$ with the *weak topology*: a sequence μ_i converges to μ if for every finite arc I and every continuous functional f on I, we have $\int_I f d(\mu_i)_I \to \int_I f d\mu_I$. If $G \curvearrowright T$ is an action with finite supporting subtree $K \subseteq T$, then $M(T)$ can be identified with the space of (ordinary) maeasures ν on K that are invariant under the (closed) pseudogroup $\Gamma = \Gamma_K$ generated by the restrictions $g|_K : g^{-1}K \cap K \to K \cap gK$ It should be noted that Γ differs from a pseudogroup in the usual sense in that the domains of elements of Γ are closed. It is important to keep in mind the following issue: choose an ennumeration $G = \{g_1, g_2, ...\}$, and suppose that there is a sequence μ_l of probability measures on K with μ_l invariant under the restrictions $\{g_1|_K, ..., g_l|_K\}$. Since the domains of $g_i|_K$ are closed, it does not follow that $\mu = \lim \mu_l$ is invariant under $\{g_1|_K, g_2|_K, ...\}$; see [23] for further discussion.

Let $T \in \overline{cv}_n$, and suppose that T has dense orbits; then Proposition 2.15 enusres that there is a finite set $\{\nu_1, ..., \nu_k\} \subseteq M_0(T)$ of mutually singular ergodic measures spanning $M_0(T)$. The following simple proposition shows that the supports of these ergodic measures are arranged in T in a simple way; the result follows from the definition of ergodicity and the fact that the "topological dyanmics" of $F_n \curvearrowright T$ determine the way F_n-invaraint sets can be arranged in T.

Proposition 5.4. *With notation as above:*

(i) if $I \subseteq Supp(\nu_i)$ is non-degenerate, then $\nu_i(I) > 0$,

(ii) $K = \cup Supp(\nu_i)$,

(iii) if $Supp(\nu_i) \cap Supp(\nu_j)$ contains a set of positive ν_i-measure, then $Supp(\nu_i) \subseteq Supp(\nu_j)$,

(iv) if $Supp(\nu_i) \cap Supp(\nu_j)$ contains a non-degenerate arc, then $Supp(\nu_i) = Supp(\nu_j)$.

Proof. The statement (i) is immediate from the definition. For (ii), we have that $\cup Supp(\nu_i)$ is a μ_L full measure subset of K, hence dense; but it is closed, so $\cup Supp(\nu_i) = K$. For (iii) note that, by ergodicity of ν_i, the union of Γ-translates of $Supp(\nu_i) \cap Supp(\nu_j)$ is a ν_i full measure subset of $Supp(\nu_i)$, hence dense in $Supp(\nu_i)$. On the other hand, the union of Γ-transaltes of $Supp(\nu_i) \cap Supp(\nu_j) \subseteq Supp(\nu_j)$ as $Supp(\nu_j)$ is Γ-invariant. Since $Supp(\nu_j)$ is closed, it follows that $Supp(\nu_i) \subseteq Supp(\nu_j)$. The claim (iv) follows from (i) and (iii). ☐

This immediately gives:

Corollary 5.5. *With notation as above, $K = \cup Supp(\nu_{i_j})$, where ν_{i_j} runs over measures in $M_1(T)$ whose supports contain a non-degenerate arc.*

5.1.1 Pullbacks and Projections

Definition 5.6. *[23] Let T and T' be $\mathbb{R}$-trees. A map $f : T \to T'$ is* alignment-preserving *if for any $x \in T'$, $f^{-1}(x)$ is a convex subset of T.*

Suppose now that T and T' are $\mathbb{R}$-trees, equipped with actions by isometries of a finitely generated group G; and suppose that $f : T \to T'$ is G-equivariant and alignment-preserving. It is observed in [23] that any non-atomic $\mu' \in M(T')$ can be pulled-back to a measure $\mu = f_*(\mu') \in M(T)$. Indeed, we let μ be the unique non-atomic measure such that for all $J \subseteq I \subseteq T$, $\mu_I(J) = \mu'_{f(I)}(f(J))$. As pointed out in [23], the hypothesis on f can be weakened; the construction goes through as long as f is equivariant and any finite arc $I \subseteq T$ can be subdivided into finitely-many subarcs such that the restriction of f to each subarc is alignment-preserving.

Let $G \curvearrowright T$ be an action by isometries. Given a G-invariant non-atomic measure μ on T, one may consider a pseudo-metric d_μ on T defined by $d_\mu(x, y) := \mu([x, y])$. It is easy to see that making this pseudo-metric Hausdorff gives an $\mathbb{R}$-tree T_μ, equipped with an isometric action of G. In the situation above, the natural map $f_\mu : T \to T_\mu$ is alignment-preserving, and if $\mu \leq \mu_L$, then f_μ is 1-Lipschitz.

Definition 5.7. *If there is an equivariant, alignment-preserving map $f : T \to T'$, then we say that T' is a* projection *of T.*

Note that if $G \curvearrowright T$ is indiscrete but not with dense orbits, then the map $T \to T'$ collapsing each component of the simplicial part of T to a point is a projection. Indeed, in this case, by Proposition 3.17 splits as a graph of actions with vertex trees either simplicial or with dense orbtis; it is easy to see that restricting the Lebesgue measure of T to the trees with dense orbits gives and invariant measure $\mu \in M_0(T)$ such that $T' = T_\mu$.

5.1.2 Exceptional Sets

Let $T \in \overline{cv}_n$ have dense orbits.

Definition 5.8. *An invariant subset $X \subseteq T$ is called* exceptional *if for any finite subtree $K \subseteq T$, $X \cap K$ is closed and nowhere dense and if there is a finite subtree $K_0 \subseteq T$ such that $X \cap K_0$ contains a Cantor set.*

A famous theorem of Imanishi [28], rediscovered by Morgan-Shalen [38] and proved in the present context by Levitt [21], states that given a finite 2-complex A, equipped with a codimension-1 singular measured foliation, one is able to cut the 2-complex A along certain subsets of singular leaves to arrange that every leaf is either finite or locally dense (see [21] or [1]); the key property is that no leaf closure is a Cantor set.

An action $G \curvearrowright T$ of a finitely preseted group G on a tree T is called *geometric* if there is a finite 2-complex A, equipped with a codimension-1 singular measured foliation, such that $\pi_1(A) = G$ and such that the action $G \curvearrowright T$ is dual to the natural action of G on the space of leaves in $\tilde{A}$–the metric comes from the transverse measure (see [38, 21, 1]). Given a geometric action $G \curvearrowright T$ with dense orbits, the Imanishi theorem implies that $G \curvearrowright T$ splits as a graph of indecomposable actions (see [25]). In particular, no geometric action can contain an exceptional set.

In [15], Coulbois-Hilion-Lustig show that any $T \in \overline{cv}_n$ with dense orbits is weakly geometric in the following sense: for any basis B of F_n, there is a canonical compact subtree $K_A \subseteq T$ such that the action $F_n \curvearrowright T_{K_A}$ dual to the restrictions of elements of A to K_A contains $F_n \curvearrowright T$ as its unique minimal subaction. Here, the action $F_n \curvearrowright T$ is dual to a compact 2-complex, equipped with a codimension-1 singular measured foliation, and one might hope to generalize the theorem of Imanishi to this context, ruling out the possibility of an exceptional set in T. However, Imanishi's theorem fails in this case, as is evidenced by the following example.

This example was pointed out to us independently by M. Lustig and V. Guirardel. Here, we use the language of relative train tracks; the reader is directed to [6] for background.

Example 5.9. Let $\alpha \in Aut(F_n)$ be represented by a relative train track map $f : G \to G$, and suppose that there are exactly two strata, both exponential, such that the PF eiqenvalue λ_l of the lower stratum is strictly larger than the PF eigenvalue λ_u of the upper stratum. Equip G with a metric that restricts to the PF metrics on the exponential strata. Let $\tilde{f}$ be a lift of f to the universal cover $\tilde{G}$ of G; set $T'_k := \tilde{f}^k(\tilde{G})$; and finally define $T_k := \lambda^{-k} T'_k$. It is easy to check that the sequence $(T_k)_{k \in \mathbb{N}}$ is convergent in $\overline{cv}_n$ to an action $F_n \curvearrowright T$.

Color the lower stratum green and the upper stratum red; it is evident that each action $F_n \curvearrowright T_k$ is color-preserving. By the assumption $\lambda_l > \lambda_u$, in the limit we get an invariant (red) set that intersects a finite supporting subtree in a Cantor set. Hence the action $F_n \curvearrowright T$ contains an exceptional set. One can check that the action $F_n \curvearrowright T$ is neither indecomposable nor a graph of actions.

The following result establishes a useful trichotomy.

Proposition 5.10. *Let $T \in \overline{cv}_n$ have dense orbits, and let $K \subseteq T$ be a finite supporting subtree. Suppose that for each $\mu \in M_1(T)$, one has that $Supp_K(\mu) = K$. Then one of the following holds:*

(i) the action $F_n \curvearrowright T$ is indecomposable,

(ii) the action $F_n \curvearrowright T$ splits as a graph of actions, or

(iii) there is an exceptional subset of T.

The proof of Proposition 5.10 uses a technique of Guirardel-Levitt [26] that is described in the next subsection, where we first present the proof, as it illustrates the effectiveness of the procedure.

5.1.3 The Procedure of Guirardel-Levitt

Let $T \in \overline{cv}_n$. For a nondegenerate arc $I \subseteq T$, let Y_I be the subtree of T that is the union of all segments $J \subseteq T$ such that there are $1 = g_0, g_1, ..., g_r \in F_n$ with $J \subseteq g_0 I \cup g_1 I \cup ... \cup g_r I$ and such that $g_i I \cap g_{i+1} I$ is nondegenerate. By construction, the collection $\mathscr{Y} := \{gY_I\}_{g \in F_n}$ is a transverse family for the action $F_n \curvearrowright T$; and the same holds for the collection $\mathscr{Y}_1 := \{g\overline{Y_I}\}_{g \in F_n}$ of translates of the closure $\overline{Y_I}$ of Y_I.

Proof. (Proposition 5.10) Assume that the action $F_n \curvearrowright T$ is not indecomposable; there is a nondegenerate arc $I \subseteq T$ such that $Y_I \neq T$. If $\mathscr{Y}_1$ is a transverse covering of T, then T splits as a graph of actions by Lemma 3.15. Otherwise, the set $X_1 \subseteq K$ of points of K that are not covered by trees in $\mathscr{Y}_1$ is a non-empty subset that is invariant under the pseudogroup Γ_K generated by restrictions $g|_K : g^{-1}K \cap K \to K \cap gK$ for $g \in F_n$. Notice that X_1 cannot contain a nondegenerate arc; this follows from the assumption that any ergodic $\mu \in M_1(T)$ satisfies $Supp_K(\mu) = K$. In fact X_1 is nowhere dense; indeed, let $\mu \in M_1(T)$ be ergodic. For any nondegenerate arc $J' \subseteq T$ and any $\epsilon > 0$, there are $g_1, ..., g_r \in F_n$ such that $\mu(K \setminus \cup_i g_i J') < \epsilon$. It follows that for any nondegenerate arc $J \subseteq T$, there is some tree $Y \in \mathscr{Y}$ meeting J nondegenerately.

Recursively define $\mathscr{Y}_{i+1}$ to be the collection obtained from $\mathscr{Y}_i$ via the following procedure: take unions of intersecting trees in $\mathscr{Y}_i$ to get $\mathscr{Y}'_{i+1}$, then take closures of the trees in $\mathscr{Y}'_{i+1}$ to get $\mathscr{Y}_{i+1}$. Since $\mathscr{Y}_1$ is a transverse family, so is each $\mathscr{Y}_i$. Put X_i to be the collection of points of K not covered by $\mathscr{Y}_i$; each X_i is a totally disconnected, nowhere dense invariant subset.

Note that each $Y \in \mathscr{Y}'_{i+1}$ carries the structure of a graph of actions, with vertex trees coming from $\mathscr{Y}_i$. It may be the case that for some i, $\mathscr{Y}_i = \{T\}$; in this case T splits as a graph of actions. Alternatively, it could be that for some i, we have for $Y, Y' \in \mathscr{Y}_i$, $Y \cap Y' \neq \emptyset$ if and only if $Y = Y'$. In this case we claim that X_i is an exceptional subset of T. We already know that X_i is totally disconnected and nowhere dense; so we need only see that X_i is perfect. This follows from the fact that X_i could contain no isolated point (the trees $Y \in \mathscr{Y}_i$ are closed). $\square$

Remark 5.11. *The procedure of Guirardel-Levitt has two parts:*

(I) Given an action $G \curvearrowright T$ that is not indecomposable, one is able to find a subtree $Y_I \neq T$ such that the collection $\mathscr{Y}_1 := \{g\overline{Y_I}\}_{g \in F_n}$ is a transverse family for the action $G \curvearrowright T$.

(II) Second, given a transverse family $\mathscr{Y}_i$, one applies the iterative procedure from the above proof to obtain $\mathscr{Y}_{i+1}$:

- *(a) $\mathscr{Y}'_{i+1}$ consists of trees that are (maximal) unions of intersecting trees from $\mathscr{Y}_i$,*
- *(b) $\mathscr{Y}_{i+1}$ consists of closures of trees from $\mathscr{Y}'_{i+1}$.*

If the transverse family $\mathscr{Y}_1$ is "large enough," then the result is either a graph of actions structure for the action $G \curvearrowright T$ or an exceptional subset of T.

Definition 5.12. *Let $T \in \overline{cv}_n$, and let $\mathscr{Y}_1$ be a transverse family. If iteratively applying procedure II to $\mathscr{Y}_1$ eventually gives $\mathscr{Y}_i = \{T\}$, then we say that T is obtained by an* iterated graph of actions *starting from $\mathscr{Y}_1$.*

5.1.4 Invariant Measures on Exceptional Sets

We now begin the analysis of trees that do not split as a graph of actions and contain an exceptional subset; we will show that such a tree T admits a projection onto a tree T', such that either T' splits as a graph of actions, or T' is indecomposable. We begin with a definition from [33], attributed there to M. Bestvina.

Definition 5.13. *Let G a finitely generated group, and let T an $\mathbb{R}$-tree. An action of G on T by homeomorphisms is called* non-nesting *if for any nondegenerate arc $I \subseteq T$ and any $g \in G$, if $gI \subseteq I$, then $gI = I$.*

In [33] Levitt shows that if a finitely presented group G admits a non-trivial, non-nesting action by homeomorphisms on an $\mathbb{R}$-tree T, then G admits a non-trivial isometric action on some $\mathbb{R}$-tree T_0. The key observation is the following:

Proposition 5.14. *[33, Proposition 4] Let $\mathscr{K}$ be a non-nesting closed system of maps on a finite tree K. Assume that $\mathscr{K}$ has an infinite orbit. Then there exists a $\mathscr{K}$-invariant probability measure μ on $\mathscr{K}$ with no atom.*

A closed system of maps on a finite tree K is a variant of a finitely generated pseudo-group of partial homeomorphisms of K, where each partial homeomorphism is required to have closed domain. Levitt observes that if such a $\mathscr{K}$ has no infinite orbit, then the existence of a measure as in the conclusion is easy to see; this is because the system decomposes into a finite union of parallel families of finite orbits, and such a system has many invariant non-atomic measures (see [33], also [21]). Our goal now is to use Proposition 5.14 to construct invariant measures on exceptional subsets of certain trees in $\overline{cv}_n$.

We will need the following result of Guirardel.

Proposition 5.15. *[23, Proposition 5.5] Let T be a minimal non-abelian action with dense orbits of a finitely generated group G, and assume that T is not a line. Assume that we are given actions T_p, T'_p, and T' such that $T_p \to T$ and $T'_p \to T'$; further assume that we have equivariant 1-Lipshitz maps preserving alignment $q_p : T_p \to T'_p$. Then there is an equivariant 1-Lipshitz map preserving alignment $q : T \to T'$.*

As mentioned in the Acknowledgements, the proof of Proposition 5.16 was inspired by a conversation with Vincent Guirardel. Before talking to Guirardel, we were using [39, Theorem 3.1] instead of proving Proposition 5.16. Guirardel pointed out that [39, Theorem 3.1] is not correct and mentioned that it is possible to construct an invariant measure on actions that are non-nesting and *mixing*.

The idea is as follows: given an action $F_n \curvearrowright T$ with dense orbits such that T contains an exceptional set X, we form an action $F_n \curvearrowright T'$ on the space T' obtained from T by collapsing the components of the complement of X to points. One checks that $F_n \curvearrowright T'$ is a non-nesting action on a tree T'. We then find an invariant, non-atomic measure on T', and then this measure pulls back to an invariant measure on T, supported on X.

The idea of Guirardel ensures the existence of an invariant, non-atomic measure as long as the action $F_n \curvearrowright T'$ is mixing. However, we cannot assume that our quotient action is mixing; a resultant problem is that our attempt to build an invariant measure may result in an atomic measure, which may not be invariant. Fortunately, in this case we are able to use Proposition 5.15 to obtain a graph of actions structure on $F_n \curvearrowright T$.

Proposition 5.16. *Let $T \in \overline{cv}_n$ have dense orbits. Suppose that T does not split as a graph of actions and that T contains an exceptional set X. Then there is $\mu \in M_0(T)$ supported on X.*

Proof. Let T and X as in the statement, and fix a basis $B = \{g_1, ..., g_n\}$ for F_n. Let $x \in T$, and set K to be the convex hull of $\{gx | g \in B^{\pm}\}$, so K is a finite supporting subtree.

Define a relation R_0 on T by xR_0y if $[x, y]$ is contained in the closure of some component of $T \setminus X$. Let R be the equivalence relation generated by R_0. We want to see that the classes of R are closed subtrees of T. Let Y be some class of R, and let $y_i \in Y$ be a sequence of points converging to $y \in T$. The segments $I_m := \cap_{k \geq m}[y_1, y_k] = [y_1, p_m]$ are contained in Y, and p_m converges to y. If $y \notin Y$, then $y \in X$. For $k >> 0$, there is $g \in F_n$ such that $g[p_m, y] \subseteq K$; as $X \cap K$ is a Cantor set, by increasing m if necessary, we can assume that $g[p_m, y] \cap X = \{y\}$, hence $y \in Y$.

The set $\mathscr{T} = \{Y_i\}_{i \in I}$ of non-degenerate classes of R is clearly a transverse family; further, it is easy to see that for $Y \neq Y' \in \mathscr{T}$, $Y \cap Y' = \emptyset$. Put T' to be the quotient of T by R; the natural map $f : T \to T'$ is continuous on segments of T and has convex point preimages. As classes of R are closed, T' is a regular Hausdorff space. Further, it is easy to see that T' is uniquely arc connected and locally arc connected, hence by [36] T' is an $\mathbb{R}$-tree. There is by construction an action of F_n on T' such that each $f \in F_n$ acts as a homeomorphism on any finite subtree of T' . It is easy to check that the action is non-nesting and supported on the finite subtree $K' := f(K)$.

Let $K_i \subseteq T$ be the convex hull of $\{gx : ||g||_B \leq i\}$, so $K_1 = K$, and the sequence K_i is an invasion of T by finite subtreees. Let T_i be the geometric action corresponding to the restrictions of elements of B to K_i; see [21]. The set $X \cap K$ gives rise to an exceptional sets $X_i \subseteq T_i$; as above, let $f_i : T_i \to T_i'$ be the quotient map, so that we have non-nesting actions $F_n \curvearrowright T_i'$ that are dual to finite systems of maps as in [33]. By Proposition 5.14, each such action supports an invariant measure with no atoms, so we get actions by isometries $F_n \curvearrowright T_i''$ along with equivariant, alignment preserving maps $f_i'' : T_i \to T_i''$. Passing to a subsequence if necessary and rescaling, we get a sequence of actions $F_n \curvearrowright T_i''$ that is convergent in $\overline{cv_n}$ to some action $F_n \curvearrowright T''$.

Pulling back via f_i'' the Lebesgue measure on T_i'', we get invariant measures μ_i on T_i supported on X_i; let Y_i be the tree with underlying set T_i and Lebesgue measure $\mu_L(Y_i) := \mu_L(T_i) + \mu_i$. We get equivariant 1-Lipschitz, alignment preserving maps $g_i : Y_i \to T_i$ and $h_i : Y_i \to T_i''$; further, as each μ_i is non-atomic, each g_i is a bijection.

For an F_n-tree U and an element $g \in F_n$, let $A_U(g)$ denote the characteristic set of g in U. Let $g \in F_n$ be hyperbolic in T; then for $i >> 0$, g is hyperbolic in T_i. It follows from [23, Lemma 5.1] that $f_i''(A_{T_i}(g)) = A_{T_i''}(g)$; hence it follows that $l_{Y_i}(g) = l_{T_i}(g) + l_{T_i''}(g)$. On the other hand, we have arranged that the sequences $l_{T_i}(g)$ and $l_{T_i''(g)}$ are convergent; hence, the sequence $l_{Y_i}(g)$ is convergent. It follows that the sequence of actions $F_n \curvearrowright Y_i$ converges to an action $F_n \curvearrowright Y \in \overline{cv}_n$; further, Proposition 5.15 gives 1-Lipschitz alignment preserving maps $g : Y \to T$ and $h : Y \to T''$.

If the map $g : Y \to T$ is a bijection, then we are finished; indeed, in this case $\mu_L(Y) - g^*(\mu_L(T))$ is an invariant measure with no atoms supported on X. Hence, we suppose that the sequence μ_i converges to an atomic measure ν on K', and let $y \in K'$ have $\nu(\{y\}) = m > 0$. It is easy to see that in this case there is a germ $\hat{d}$ of a direction d at y in K' such that the set $O(\hat{d}) := \{g \in F_n | gy \in K'$ and $g[y, \hat{d}] \cap K'$ is non-degenerate$\}$ is finite (else, the sequence $\mu_i(K')$ is unbounded). It follows that there is a positive lower bound for the translation lengths of hyperbolic elements $g \in F_n$ such that $A(g)$ meets $F_n y$ and non-degenerately interects $F_n d$. This shows that T'' does not have dense orbits, *i.e.* T'' has non-empty simplicial part.

As the map $h : Y \to T''$ is equivariant, 1-Lipschitz, and alignment-preserving, Y has a non-empty simplicial part as well. Let $\mathscr{C} = \{Y_i\}_{i \in I}$ be the transverse covering of Y by subtrees Y_i, which are either simplicial edges or have dense orbits (see Proposition 3.17); and put $\mathscr{C}_0 \subseteq \mathscr{C}$ to be collection of trees with dense orbits. Evidently, $g(\mathscr{C}_0) := \{g(Y_i) | Y_i \in \mathscr{C}_0\}$ is a transverse covering of T, hence by Lemma 3.15 T splits as a graph of actions, a contradiction.

$\square$

This immediately gives.

Corollary 5.17. *Let $T \in \overline{cv}_n$ have dense orbits, and let $K \subseteq T$ be a finite supporting subtree. Suppose that T does not split as a graph of actions. If for each $\mu \in M_0(T)$, one has that $Supp_K(\mu) = K$, then T is indecomposable.*

Proof. By the hypothesis on $M_0(T)$ and by Proposition 5.16, we have that T contains no exceptional set; the result is then a restatement of Proposition 5.10, $\square$

To finish this section, we provide an example showing that the hypothesis of Proposition 5.16 is necessary and not an artifact of our proof.

Example 5.18. Let $F_n \curvearrowright T$ be the action constructed in Example 5.9; as aforementioned, $F_n \curvearrowright T$ is neither indecomposable nor a graph of actions. By Proposition 5.16 we can find an invariant measure ν supported on the exceptional set $E \subseteq T$; we remark that it is easy to check that E is μ_L-measure zero, where $\mu_L = \mu_L(T)$ is the Lebesgue measure for T. Further, one can show that $dim(M_0(T)) = 2$, *i.e.* μ_L and ν are the only ergodic measures on T up to rescaling; see [10].

Let T' be the tree with underlying set T and Lebesgue measure $\mu_L(T') = \mu_L + \nu$; as ν is non-atomic, the "identity map" $f : T \to T'$ is continuous on segments and bijective. By Lemma 2.12, f extends to a unique homeomorphism $\hat{f} : \hat{T} \to \hat{T}'$; refer to Subsection 2.4.1. On the other hand, since ν is singular with respect to $\mu_L(T)$, the map f is not continuous with respect to the metric topologies on T and T'; we leave this as an exercise to the reader.

As in Subsection 2.4.1, let $q \in T$ be a base point, and let (p_k) be a sequence in T. Put $I_m := \cap_{k \geq m}[q, p_k]$, so $I_m = [q, r_m]$, and we have $I_m \subseteq I_{m+1}$. Recall that the *inferior limit* of (p_k) from q is the limit $\lim_q p_k := \lim r_m$. If (p_k) is convergent in the metric topology to $p \in T$, then $p = \lim_q p_k$. Since f is not continuous with respect to the metric topology, we can find a convergent sequence (p_k) of points in T such that $(f(p_k))$ is not convergent in T'. Since f is continuous on segments and since $\lim p_k = \lim_q p_k = p$, we have that the sets $I'_m := [f(q), f(r_m)]$ satisfy $\overline{\cup_m I'_m} = f([q,p])$; it follows that the distances $d_{T'}(f(p_k), [f(q), f(p)])$ are bounded below by some number $c > 0$. On the other hand, since $\lim p_k = \lim_q p_k = p$, we have that $d_T(p_k, [q,p]) \to 0$.

Replace (p_k) be a subsequence to ensure that $\sum_k d(p_k, [q,p])$ is finite. Let $y_k \in [q,p]$ be defined so that $d(p_k, y_k) = d(p_k, [q,p])$, and set $J_k := [p_k, y_k]$. By the above paragraph, the lengths of $f(J_k)$ are bounded below by $c > 0$. It is an easy exercise using the fact that T has dense orbits to use the intervals J_k to produce a finite length ray ρ in T, such that $f(\rho)$ is unbounded in T'. Since f is continuous on segments, and since $\hat{f}$ is a homeomorphism, we can conclude that ρ converges to a point $w \in \overline{T} \setminus T$, while $f(\rho)$ converges to a point of $\partial T'$.

Let $w' \in T$ be a point with trivial stabilizer; we are going to form an "HNN-extension" of T using w and w'. Let S be the Bass-Serre tree for the splitting $F_{n+1} = F_n *_{\langle 1 \rangle}$. Associate to each vertex of S is a copy of the tree T. Let $\tau \subseteq S$ be a lift of a spanning tree of S/F_{n+1}, *i.e.* τ is an edge e of S; put $p_e := w$ and $p_{\overline{e}} := w'$, and extend this equivariantly to associate to each directed edge of S an attaching point to get a graph of actions $\mathscr{G}$. Let $Y := T_{\mathscr{G}}$ be the tree dual to $\mathscr{G}$.

By construction, the measure ν on T does not give an invariant measure on Y; this is because we have arranged that any finite arc of the form $[z, w]$ would have infinite measure. On the other hand, any invariant measure μ' on Y clearly gives rise to an invariant measure on T. Hence, $dim(M_0(Y)) = 1$, *i.e.* Y is uniquely ergodic. The existence of the exceptional set $E \subseteq T$ gives rise to an exceptional set $E' \subseteq Y$, and it follows that there is no measure supported on E'.

It should be noted that the trick in the example can be applied to any tree T with dense orbits such that there exist two mutually-singular ergodic measures $\mu, \mu' \in M_0(T)$ such that $Supp(\mu) \subseteq Supp(\mu')$; the result will be an HNN-extension of T in which the measure μ has been "hidden."

5.2 Dynamics on $\overline{CV}_n$

We are now in a position to prove our main dynamics result; we are left to handle convergence for trees $T \in \overline{cv}_n$ that are not indecomposable and do not split as a graph of actions. The results of Section 5 shows that in this case, there is a projection T' of T that is either indecomposable or a graph of actions. We are left to show that this projection does not distort the tree T in non-obvious ways: we need to control $L^2(T')$.

Lemma 5.19. *Let $T \in \overline{cv}_n$ have dense orbits, and let $\mu \in M_0(T)$. Then $L^2(T) \subseteq L^2(T_\mu)$.*

Proof. Let T as in the statement, and note that if μ is absolutely continuous with respect to μ_L then the result is obvious. Hence, we may reduce to the case that μ is "generic" in the sense that there is no $\nu' \in M_0(T)$ singular with respect to μ; any generic μ can be rescaled so that $\mu_L \leq \mu$, so we assume that $\mu_L \leq \mu$. Hence, a 1-Lipschitz, alignment-preserving map $f : T_\mu \to T$, and we need to establish that $L^2(T_\mu) = L^2(T)$. It is easy to check that f is a bijection that is continuous on segments; therefore, by Lemma 2.12, f induces a homeomorphism $\hat{f} : \hat{T_\mu} \to \hat{T}$ so that $L^2(T) = L^2(T_\mu)$. □

This immediately gives:

Corollary 5.20. *Let $T \in \overline{cv}_n$ have dense orbits, and suppose that T is not indecomposable and does not split as a graph of actions. There is a projection $f : T \to T'$ such that:*

(i) either T' is indecomposable, or T' splits as a graph of actions,

(ii) $L^2(T) \subseteq L^2(T')$.

Proof. Let T as in the statement; choose $\mu \in M_0(T)$ such that $Supp(\mu)$ is set-theoretically minimal. It follows from Propositions 5.4 and 5.16 that if $T' := T_\mu$ contains an exceptional set, then T' splits as a graph of actions; hence, by Corollary 5.17, either T' is indecomposable or T' splits as a graph of actions. Further, it follows from Lemma 5.19 that $L^2(T) \subseteq L^2(T')$. □

We are now prepared to prove the main dynamics result of the paper.

Theorem 5.21. *Let $\phi : F_n \to F_n$ be an irreducible endomorphism, and suppose that $\phi \notin Aut(F_n)$. Further suppose that for any $T \in \overline{cv}_n$, $T\phi$ is non-trivial. Then ϕ acts on $\overline{CV}_n$; there is a unique fixed point $[T_\Phi] \in CV_n \subseteq \overline{CV}_n$; and for any compact neighborhood N of $[T_\Phi]$, there is $k = k(N)$ such that for any $[T] \in \overline{CV}_n$, $[T]\phi^k \in N$.*

Proof. Let ϕ as in the statement. First note that it follows from Propositions 3.36 and 4.17, Lemma 3.38, and Corollary 5.20 that for any $T \in \overline{cv}_n$, we have that $[T]\phi^k \to [T_\Phi]$, hence $[T_\Phi]$ is the unique fixed point of ϕ acting on $\overline{CV}_n$.

Toward contradiction suppose that there is a compact neighborhood N of $[T_\Phi]$ in $\overline{CV}_n$ and actions $[T_k]$ such that $[T_k]\phi^k \notin N$. Note that ϕ induces a continuous function on $\overline{CV}_n$. It follows that the set of accumulation points $\{[T_k]\}_{k\in\mathbb{N}}$ is ϕ-invariant and does not contain $[T_\Phi]$, hence this set must be empty, a contradiction. $\square$

To finish this section we bring an immediate corollary of Theorem 5.21, showing the existence of a certain type of rigid subgroup of F_n.

Corollary 5.22. *For any $C > 1$, there is a finitely generated, non-abelian subgroup $H \leq F_n$, such that for any non-trivial $h, h' \in H$ and any trees $T, T' \in \overline{cv}_n$, one has $l_T(h) > 0$ and*

$$\frac{1}{C} \leq \frac{l_T(h)/l_T(h')}{l_{T'}(h)/l_{T'}(h')} \leq C$$

Proof. Let $\phi : F_n \to F_n$ be an irreducible, non-surjective, admissible endomorphism, and set $H_k := \phi^k(F_n)$. It is immediate from Theorem 5.21 that for any $C > 1$, there is K such that for all $k \geq K$, H_k satisfies the desired properties. $\square$

5.3 Structure of Actions in $\overline{cv}_n$

In this section we expand the results of Section 5 to give a dynamical decomposition of trees in $\overline{cv}_n$ that generalizes the dynamical decomposition of geometric trees coming from Imanishi's theorem. The results we obtain are similar in spirit to new work of Guirardel-Levitt [26].

5.3.1 The Transverse Family $\mathscr{F}$

Suppose that $T \in \overline{cv}_n$ is not simplicial; if T does not have dense orbits, then by Proposition 3.17 T splits as a graph of actions, with vertex trees either simplicial edges or trees with dense orbits. So, let $T \in \overline{cv}_n$ have dense orbtis. By Proposition 2.15 we have that $M_0(T)$ is $\mathbb{R}_{>0}$-spanned by a finite set $B = \{\nu_1, ..., \nu_r\}$ of mutually-singular ergodic measures, and by Proposition 5.4 we have for $\nu_i, \nu_j \in B$ with $Supp(\nu_i), Supp(\nu_j)$ non-degenerate, if $Supp(\nu_i) \neq Supp(\nu_j)$, then for any finite tree $K \subseteq T$, $Int(Supp_K(\nu_i)) \cap Int(Supp_K(\nu_j)) = \emptyset$. We define a family $\mathscr{F}$ of subtrees of T; a subtree $Y \subseteq T$ is a member of $\mathscr{F}$ if:

(i) Y is non-degenerate,

(ii) there is $\nu_i \in B$ and an invasion $\{F_k\}_{k \in \mathbb{N}}$ of Y by finite subtrees F_k such that $F_k = Supp_{F_k}(\nu_i)$, and

(iii) Y is maximal with respect to (ii).

It follows from Proposition 5.4 that $\mathscr{F}$ is a transverse family for the action $F_n \curvearrowright T$; further, also by Proposition 5.4, there are finitely-many F_n-orbits of trees in $\mathscr{F}$.

Lemma 5.23. *Let $T \in \overline{cv}_n$ have dense orbits, and suppose that there are $\mu, \mu' \in M_0(T)$ such that $Supp(\mu) \neq Supp(\mu')$ are both non-degenerate. Then T splits as a graph of actions.*

Proof. Let $\mathscr{F}$ be the transverse family defined above. Since there are $\mu, \mu' \in M_0(T)$ with $Supp(\mu) \neq Supp(\mu')$ both non-degenerate, the family $\mathscr{F}$ contains more than one tree. We apply the iterative procedure II of Remark 5.11 to get a sequence $\mathscr{F}_k$ of transverse families in T. This procedure might terminate with an exceptional set $E \subseteq T$; in this case, by Proposition 5.16, either T splits as a graph of actions, or there would be ergodic $\nu_i \in B$ supported on E. By Proposition 5.4, $Supp(\nu_i)$ would be contained in the interior of $Supp(\nu_j)$ for some $\nu_j \in B$ with non-degenerate support; but this is impossible by definition of $\mathscr{F}$ and by construction of E. So, for some k_0, we have that $\mathscr{F}_{k_0} = \{T\}$; suppose that k_0 is minimal with respect to this property. Then T splits as a graph of actions corresponding to the transverse covering $\mathscr{F}_{k_0-1}$. □

The proof of Lemma 5.23 also shows that if for any finite arc $I \subseteq T$, I is the union of finitely many subintervals $I_1, ..., I_k$ such that $I_j = Supp_{I_j}(\nu_{i_j})$, then the set orbits of vertex trees in this graph of actions structure for T bijectively corresponds to the set of non-degenerate support sets of the ergodic measures $\nu \in M_0(T)$. In particular, this would be the case if the action $F_n \curvearrowright T$ happened to be geometric.

Lemma 5.24. *Let $T \in \overline{cv}_n$ have dense orbits, and let $\mathscr{F}$ be the transverse family constucted above. Suppose that $\mathscr{F}$ contains at least two orbits of trees. Then for each $Y \in \mathscr{F}$, the set-wise stabilizer $Stab(Y)$ is a vertex group of a very small splitting of F_n.*

Proof. Let T and $\mathscr{F}$ as in the statement. By hypothesis, we can find ergodic $\nu, \nu' \in M_0(T)$ such that $Supp(\nu)$ and $Supp(\nu')$ are non-degenerate and such that $Supp(\nu) \neq Supp(\nu')$. Let $\mu_L = \nu_1 + ... + \nu_l$ be the decomposition of the Lebesgue measure on T as a sum of muntually-singular ergodic measure $\nu_i \in M_0(T)$. Define a measures $\mu \in M_1(T)$ by $\mu := \Sigma_{Supp(\nu_i)=Supp(\nu)}\nu_i$; we get an equivariant 1-Lipschitz alignment-preserving map $f : T \to T_\mu$. We observe that if $Y \in \mathscr{F}$ corresponds to $Supp(\nu')$, then the image of Y under f is a point. It follows that $Stab(Y)$ fixes a point $y \in T_\mu$. By [22] $Stab(Y)$ is contained in a vertex group of a very small splitting of F_n.

On the other hand, it is immediate that $Stab(\{y\}) = Stab(\overline{Y})$. Further, it is clear that if $f \in F_n$ is hyperbolic in T, then $f \in Stab(Y)$ if and only if $f \in Stab(\overline{Y})$. Toward a contradiction suppose that there is $g \in Stab(\overline{Y}) \setminus Stab(Y)$, then g acts elliptically on $\overline{Y}$, and so g must fix a point $p \in \overline{Y} \setminus Y$. There is only one direcction (in $\overline{Y}$) at p, so g must fix a non-degenerate arc $[p', p] \subseteq \overline{Y}$. We show this is impossible.

Let $\eta \in M_0(T)$ ergodic such that $Supp(\eta) = Y$, and let $I_0 \subseteq Y$ be a small arc. By ergodicity, for any $J \subseteq Y$ and $\epsilon > 0$, there are $g_1, ..., g_r \in F_n$ such that $\eta(J \setminus g_1 I_0 \cup ... \cup g_r I_0) < \epsilon$. It easily follows that there are elements $g' \in Stab(Y)$ with arbitrarily short translation length, hence $Stab(Y)$ acts on Y with dense orbits, and the same holds for the action $Stab(\overline{Y}) \curvearrowright \overline{Y}$. This contradicts the above observation that g must fix a non-degenerate arc of $\overline{Y}$ and completes the proof. $\square$

5.3.2 The Case $\mathscr{F} = \{T\}$

We turn to analyzing trees in $\overline{cv}_n$ for which the family $\mathscr{F}$ is as simple as possible.

Proposition 5.25. *Let $T \in \overline{cv}_n$ have dense orbits, and suppose that for each $\nu \in M_0(T)$, if $Supp(\nu)$ is non-degenerate, then $Supp(\nu) = T$. There is a finite set of projections $\{P_i\}_{i\in\{1,...,r\}} = \{f_i : T \to T_i\}_{i\in\{1,...,r\}}$ such that:*

(i) $dim(M_0(T_i)) < dim(M_0(T))$,

(ii) there is a partial order $\leq$ on $\{P_1, ..., P_r\}$ such that:

(a) if $P_i \leq P_j$, then f_i factors through f_j,

(b) if $P_i < P_j$, then $dim(M_0(T_i)) < dim(M_0(T_j))$,

(c) for P_i minimal, if T_i contains an exceptional set, then T (and T_i) splits as a graph of actions.

(iii) for any projection $T \to T'$, there is T_i and is $\mu \in M_0(T_i)$ such that T' is equivariantly isometric to $(T_i)_\mu$ and such that the natural map $T_i \to (T_i)_\mu$ is a bijection.

Proof. Let T as in the statement. We first arrange that μ_L is contained in the interior of $M_0(T)$, *i.e.* there is no $\nu \in M_0(T)$ singular with respect to μ_L. This is accomplished by replacing T with T_μ for some $\mu \in M_0(T)$ with the desired property; the natural map $T \to T_\mu$ is a bijection. By Proposition 2.15 $M_0(T)$ is spanned by a finite set $\{\nu_1, ..., \nu_r\}$ of mutually-singular ergodic measures. In particular, there are finitely many sets $Supp(\nu)$ for ergodic $\nu \in M_0(T)$; let $\mathscr{X} = \{X_1, ..., X_s\}$ denote the collection of these support sets. Then $\mathscr{X}$ carries the obvious partial order $\leq$, with unique maximal element $X_i = T$. By Proposition 5.16, if $X_j = Supp(\nu)$ is minimal with respect to $\leq$, then if T_ν contains an exceptional, then T_μ splits as a graph of actions. Define a set of measures $\mathscr{M} = \{\mu_1, ..., \mu_s\} \subseteq M_0(T)$ by $\mu_i := \mu_L|_{X_i}$. It is then easy to check that the collection $\{f_i : T \to T_{\mu_i} | Supp(\mu) \neq T\}$ satisfy the conclusions of the proposition. □

Let $T \in \overline{cv}_n$ have dense orbits, and assume that no exceptional subset of T has non-zero measure. For any ergodic $\mu \in M_1(T)$, and for any finite arc $I \subseteq T$ with $Supp_I(\mu)$ non-degenerate, the restriction $\mu|_I$ determines μ in the sense that μ is the unique minimal element of $M_1(T)$ restricting to $\mu|_I$ on I. To extend the analogy between length measures and measured laminations on surfaces, we look for invariant subsets of T that encode μ as above. As any two non-degenerate, invariant subsets supporting μ must intersect non-degenerately, we are naturally led to consider transverse families with one orbit of trees. We need the following result, whose proof was sketched to us by V. Guirardel.

Lemma 5.26. *[26] Let $T \in \overline{cv}_n$ have dense orbits. Let $Y \subseteq T$ be a non-degenerate subtree such that for all $g \in F_n$, either $gY = Y$ or $gY \cap Y = \emptyset$. Then $Stab(Y)$ is a free factor of F_n.*

Proof. We prove the result in the case that $\mathscr{F} = \{T\}$, as the general case follows immediately from this case. First note that $Stab(Y)$ is non-trivial. Indeed, let $I \subseteq Y$ be a non-degenerate arc, then there is ergodic $\nu \in M_1(T)$ such that $\nu(I) > 0$. For a small subarc $I_0 \subseteq I$ with $\nu(I_0) > 0$ and any $\epsilon > 0$, we can find elements $g_1, ..., g_r \in F_n$ such that $\nu(I \setminus \cup_i g_i I_0) > \nu(I) - \epsilon$. It follows that there is some $g \in F_n$ such that $A(g) \cap I$ contains a fundamental domain for the action $\langle g \rangle \curvearrowright A(g)$, and since Y is disjoint from its translates, it follows that $g \in Stab(Y)$.

Fix a basis $B = \{x_1, ..., x_n\}$ for F_n and a point $y \in T$; define K_i to be the convex hull of the set $\{gy : ||g||_B \leq i\}$; and let T_i be the geometric action dual to the restrictions of elements of B to K_i. So, T is the strong limit of the sequence T_i, and as T has trivial arc stabilizers, each T_i has trivial arc stabilizers as well.

Replace Y with a translate if necessary to ensure that $Y \cap K_1$ is non-degenerate. Define $Y_i \subseteq T_i$ as follows: $Y_i^1 := Y \cap K_i$; Y_i^{r+1} is the union of Y_i^r and all translates of Y_i^r meeting Y_i^r; $Y_i' := \cup_r Y_i^r$; and $Y_i := \overline{Y_i'}$. Hence Y_i is a closed subtree of T_i disjoint from its translates. Being geometric, T_i splits as a graph of actions, with each vertex tree either simplicial edge or an indecomposable tree. Evidently, if Y_i meets an indecomposable vertex tree V in a non-degenerate arc, then $V \subseteq Y_i$. By the hypothesis $\mathscr{F} = \{T\}$, we have that for any non-degenerate $I, J \subseteq T$, there is $g \in F_n$ such that $gI \cap J$ is non-degenerate. It follows that the splitting of each T_i into a graph of indecomposable trees and simplicial edges can contain at most one orbit of indecomposable trees; further, since T is a strong limit of the T_i's, if V is an indecomposable vertex tree of some T_i, then there is $j \geq i$ such that $V \subseteq Y_j$.

We collapse to a point each tree in the orbit of Y_i tree to get a tree S_i, equipped with a non-trivial action of F_n. From the above discussion, S_i is a simplicial tree with trivial arc stabilizers such that $Stab(Y_i)$ is a vertex stabilizer in S_i. Hence, $Stab(Y_i)$ is a free factor of F_n. To conclude, just note that if $g_1, ..., g_k \in Stab(Y)$ are hyperbolic, then there is i_k such that each g_j is hyperbolic in T_{i_k} and such that $A(g_j) \subseteq Y_{i_k}$; therefore eventually $Stab(Y_i) = Stab(Y)$. $\square$

Definition 5.27. *An action $T \in \overline{cv}_n$ is called* mixing *if for any non-degenerate arcs $I, J \subseteq T$, there are $g_1, ..., g_r \in F_n$ such that $J \subseteq g_1 I \cup ... \cup g_r I$.*

Mixing differs from indecomposable in that we place no requirements on the overlaps $g_i I \cap g_{i+1} I$. Mixing is equivalent to the following condition: for any non-degenerate arcs $I, J \subseteq T$, J can be subdivided into arcs $J_1, ..., J_r$ such that there are $g_1, ..., g_r \in F_n$ with $g_i J_i \subseteq I$.

Lemma 5.28. *Let $T \in \overline{cv}_n$. The following are equivalent*

(i) For any direction d at $x \in T$, and any non-degenerate arc $I \subseteq T$ there is $g \in F_n$ such that $gx \in I$ and $gd \cap I$ is non-degenerate,

(ii) the action $F_n \curvearrowright T$ is mixing.

Proof. To see (i) implies (ii), let $T \in \overline{cv}_n$, and assume that T is not mixing. There are non-degenerate arcs $I, J \subseteq T$ such that $J \setminus \cup_{g \in F_n} gI \neq \emptyset$; let $y \in J \setminus \cup_{g \in F_n} gI$, and let d be a direction at y meeting J non-degenerately. Evidently, there is no $g \in F_n$ such that $gy \in I$ and such that $gd \cap I$ is non-degenerate.

To see (ii) implies (i), assume that T is mixing, and let $I \subseteq T$ be any non-degenerate arc. For any direction d at $x \in T$, we take $y \in d \subseteq T$, so $[x, y]$ is a non-degenerate arc. Since T is mixing $[x, y]$ can be divided into finitely many subarcs $[x, y] = [x = y_0, y_1] \cup [y_1, y_2] \cup ... \cup [y_{r-1}, y_r]$ such that there are $g_1, ..., g_r \in F_n$ with $g_i[y_{i-1}, y_i] \subseteq I$. Hence, $g_o x \in I$, and $g_0 d \cap I$ is non-degenerate. ☐

Proposition 5.29. *Suppose that $T \in \overline{cv}_n$ has dense orbits. There is a transverse family $\mathscr{T} = \{Y_i\}_{i \in I}$, with orbits in $\mathscr{T}$ in bijective correspondence with orbits in $\mathscr{F}$, such that the actions $Stab(Y_i) \curvearrowright Y_i$ are mixing.*

Proof. Let T as in the statement; we suppose that for any $\mu \in M_0(T)$ with non-degenerate support, we have $Supp(\mu) = T$, as the general case follows immediately from this case by considering the trees in $\mathscr{F}$. We proceed by induction on n. It follows from Harrison's theorem that every $T \in \overline{cv}_2$ is geometric, and the result is clear in this case; indeed, it follows from Imanishi's theorem that any geometric tree with dense orbits is a graph of indecomposable actions. So, we suppose the result holds for all $T \in \overline{cv}_m$ for $m < n$.

If $T \in \overline{cv}_n$ contains an exceptional set, then we are done by Lemma 5.26 and induction, so we suppose this is not the case. It follows that for each $\mu \in M_0(T)$, $Supp(\mu) = T$, hence applying Procedure II of Remark 5.11 to any transverse family in T will eventually produce T.

Assume that there is some $y \in T$ and direction d at y such that for any finite arc $J \subseteq T$, the set $\{g \in F_n | gy \in J \text{ and } gd \cap J \text{ is non-degenerate}\}$ is finite. In this case, we may blow-up the orbit of d as follows; split T open at y, gluing directions not in the orbit of d back to y, and for each d'

at y in the orbit of d, glue a simplicial edge of length 1 to y; finally glue each direction in the orbit of d to its corresponding simplicial edge. This gives a tree T', which, by the finiteness assumption on d, carries an isometric action of F_n.

The obvious map $f : T' \to T$ is equivaraint, 1-Lipschitz and alignment-preserving. The graph of actions structure on T' gives a graph of actions structure T. Further, collasping to a point every tree with dense orbits in T' gives an action of F_n on a simplicial tree with trivial arc stabilizers, where vertex stabilizers correspond to stabilizers of the trees in T' with dense orbits. Hence, we get a transverse covering of T by subtrees $\{Y_i\}_{i\in I}$ such that $Stab(Y_i)$ is a free factor of F_n, and the result follows from induction.

We are left to consider the case that no direction d in T satisfies the above finiteness condition.

Claim 5.30. Let $T \in \overline{cv}_n$, and suppose that there is no direction d at $x \in T$ such that for every non-degenerate arc $I \subseteq T$, $\{y | y = gx \text{ and } gd \cap I \text{ is non-degenerate}\}$ is finite. Then T is mixing.

Proof. If it were the case that for any direction d at y in T and any non-degenerate arc $J \subseteq T$, the collection $\{gy | gd \cap J \text{ is non-degenerate}\}$ is dense in J, the T would be mixing by Lemma 5.28. Toward contradiction, we suppose that there is a point $x \in T$, a direction d at x, and a non-degenerate segment $I \subseteq T$ such that there is no $g \in F_n$ with $gd \cap I$ non-degenerate.

Define $X := \cup I$ where I runs over all non-degenerate arcs of T not meeting d non-degenerately; the collection $\mathscr{X}_1$ of path components of X is a transverse family for the action $F_n \curvearrowright T$. Note that since for each $\mu \in M_0(T)$, $Supp(\mu) = T$, we have that $T \backslash X$ contains no non-degenerate arc; further, by applying Procedure II of Remark 5.11 to $\mathscr{X}_1$ will eventually produce T. By construction, the family $\mathscr{X}$ cannot be a transverse covering of T.

As in Remark 5.11, let $\mathscr{X}'_{i+1}, \mathscr{X}_{i+1}$ denote the results of applying Procedure II to $\mathscr{X}_i$. As $\mathscr{X}$ is not a transverse covering of T, some member of $\mathscr{X}_2$ is not closed in T. Let i minimal such that $\mathscr{X}_{i+1} = \{T\}$, then there is $Y \in \mathscr{X}'_i$, which is not closed in T, and such that the F_n-translates of its closure $\overline{Y}$ give a transverse cover of T. Let $x_1 \in \overline{Y} \setminus Y$, and let d_1 denote the unique direction in $\overline{Y}$ at x_1. Note that for any non-degenerate arc $I \subseteq \overline{Y}$, the orbit $F_n x_1$ can meet I only at its endpoints. For any non-degenerate arc $J \subseteq T$, let $g_1, ..., g_r \in F_n$ such that $J \subseteq g_1\overline{Y} \cup ... \cup g_r\overline{Y}$; it follows that $\{y \in J | y = gx_1 \text{ and } gd_1 \cap J \text{ non-degenerate}\}$ is finite (with cardinality bounded by $2r$), a contradiction. $\square$

$\square$

Remark 5.31. *One should note that if an action $F_n \curvearrowright T$ has dense orbits and is free, then T is (uniquely) a graph of indecomposable actions. Indeed, as noted in Lemma 4.7, T is not indecomposable if and only if there is a transverse family for the action $F_n \curvearrowright T$; for simplicity, we assume that for any $\mu \in M_0(T)$, $Supp(\mu) = T$. If T contains a transverse family, then by Procedure II of Remark 5.11, T splits as a graph of actions, say with skeleton S. Since the action $F_n \curvearrowright T$ is free, the action $F_n \curvearrowright S$ has trivial arc stabilizers;* i.e. *S encodes a non-trivial free decomposition of F_n, and the claim follows by induction on rank.*

Corollary 5.32. *Let $T \in \overline{cv}_n$ have dense orbits, and suppose that for all $\mu \in M_0(T)$, $Supp(\mu) = T$. There is a unique conjugacy class $[H]$ of finitely generated subgroups $H \leq F_n$ such that:*

(i) the action $H \curvearrowright T_H$ is mixing,

(ii) $H = Stab(T_H)$, and

(iii) T_H is maximal with respect to (i), (ii).

Proof. Let T as in the statement. As in the proof of Proposition 5.29, there are finitely-many (orbits of) directions $\{d_i\}_{i=1,...,k}$ at points $\{x_i\}_{i=1,...,k}$ such that for any non-degenerate arc $I \subseteq T$, there are finitely many elements $g \in F_n$ taking x_i into I such that $gd_i \cap I$ is non-degenerate. Splitting T apart on the orbits of these directions gives a transverse family $\{gY\}_{g \in F_n}$ such that $Stab(Y) \curvearrowright Y$ is mixing. The following claim follows easily from the defintion of Y.

Claim 5.33. Let Y as defined above, and let $Y_I := \cup J$, where J runs over all non-degenerate arcs contained in T such that there are $g_1, ..., g_r \in F_n$ with $J \subseteq g_1 I \cup ... \cup g_r I$ and such that $\cup_i g_i I$ is connected. For any non-degenerate $I \subseteq Y$, $Y_I = Y$.

Now, let $K \curvearrowright X$ be a subaction satisfying (i)-(iii). Since for all $\mu \in M_0(T)$, one has $Supp(\mu) = T$, it is the case that up to replacing X by a translate and replacing K with a conjugate, we can assume that $Y \cap X$ is non-degenerate. Further, since $K \curvearrowright X$ is mixing, we can assume that no direction d_i at x_i as in the first paragraph of the proof meets K non-degenerately; indeed, it is immediate that if d_i met K non-degenerately, then $x_i \in \overline{X} \setminus X$, but then the orbit Kx_i would not be dense in every non-degenerate arc of K, contradicting mixing. Let $I \subseteq X \cap Y$ be a non-degenerate arc. From the claim above, $Y = Y_I$; from maximality of X, we can assume that $Y \subseteq X$. On the other hand, from the fact that $Y = Y_J$ for any non-degenerate arc $J \subseteq Y$, it follows that Y is the maximal mixing subaction of $F_n \curvearrowright T$ containing I. Hence, $X = Y$, and by (ii) $K = Stab(Y)$.

□

5.3.3 Decomposing Actions in the Boundary of Outer Space

In this subsection we collect the preceding results of Section 5.3 to associate to any action $T \in \overline{cv}_n$ a diagram of actions that encodes the structure of T. Let $\leq$ denote the obvious partial order on the finite set $\{Supp(\mu)\}_{\mu \in M_0(T)}$; also denote by $\leq$ the partial order inherited by $M_0(T)$, *i.e.* $\mu \leq \mu'$ if and only if $Supp(\mu) \leq Supp(\mu')$. Define $[[\mu]] := \{\nu \in M_0(T) | \nu \leq \mu, \mu \leq \nu\}$; for any $\mu \in [[\mu_L]]$, we have that the natural map $T \to T_\mu$ is a bijection, and there is an identification $M_0(T) = M_0(T_\mu)$. Further, by Lemma 5.19, we have that $L^2(T) = L^2(T_\mu)$.

Let $T \in \overline{cv}_n$ have dense orbits, and let $\mathscr{F} = \mathscr{F}(T)$ be the transverse family constructed in Subsection 5.3.1. By Proposition 5.29 and Corollary 5.32 we have associated to each F_n-orbit O of trees in $\mathscr{F}$ a (canonical) mixing action $H_O \curvearrowright T_{H_O}$, defined up to translation in T, *i.e.* up to replacing H_O by a conjugate. Let $\mu \in [[\mu_L]]$, with $f : T \to T_\mu$ the natural map. Then it is easy to see that $f(\mathscr{F}) := \{f(Y) | Y \in \mathscr{F}\} = \mathscr{F}(T_\mu)$ and that $H_O \curvearrowright f(T_{H_O})$ is the mixing action associated to the orbit $f(O)$ in $\mathscr{F}(T_\mu)$.

By Lemma 5.23 if $\mathscr{F} \neq \{T\}$, then T can be recovered by the iterated graph of actions procedure of Remark 5.11 starting from $\mathscr{F}$. The family $\mathscr{F}$ is canonical, and $Y \in \mathscr{F}$ satisfies the hypotheses of Proposition 5.25. Note that the projections of Proposition 5.25 are canonical if we consider them to be defined only up to $[[.]]$. Hence, we obtain:

Theorem 5.34. *Let $T \in \overline{cv}_n$ have dense orbits. With notation as above:*

(i) there is a transverse family $\mathscr{F}$ such that:

(a) the set of orbits of trees in $\mathscr{F}$ bijectively corresponds to the set of classes $[[\mu]]$ of ergodic measures $\mu \in M_0(T)$ with non-degenerate support,

(b) *associated to each orbit O of trees in $\mathscr{F}$ is a subtree $T(O) = T([[\mu]]) \subseteq T$, unique up to translation in T, such that the action $H(O) = Stab(T(O) \curvearrowright T(O)$ is mixing,*

(c) *the action $F_n \curvearrowright T$ can be recovered via an iterated graphs of actions construction (Remark 5.11, procedure II) starting from $\mathscr{F}$,*

(d) *if $\mathscr{F} \neq \{T\}$, then for any $Y \in \mathscr{F}$, $Stab(Y)$ is a vertex group of a very small splitting of F_n.*

(ii) *there is a diagram of projections of T: associated to the class $[[\mu]]$ of an ergodic measure $\mu \in M_0(T)$ with degenerate support is a projection $f_{[[\mu]]} : T \to T_{[[\mu]]}$ such that:*

(a) *$dim(M_0(T_{[[\mu]]})) < dim(M_0(T))$,*

(b) *for ergodic $\mu' \in M_0(T)$ with degenerate support, if $\mu \leq \mu'$, then the projection $f_{[[\mu]]} : T \to T_{[[\mu]]}$ factors through the projection $f_{[[\mu']]} : T \to T_{[[\mu']]}$.*

(c) *there is a unique class $[[\nu]]$ of ergodic measures $\nu' \in M_0(T)$ with non-degenerate support such that $\mu \leq \nu$; the subgroup $H([[\nu]]) = Stab(T([[\nu]])$ is a point stabilizer in $T_{[[\mu]]}$.*

(d) *if $[[\mu]]$ is minimal, and if $T_{[[\mu]]}$ contains an exceptional set, the T (and $T_{[[\mu]]}$) splits as a graph of actions.*

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