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Abstract

We consider continuous state branching processes (CSBP) with additional multiplicative jumps modeling dramatic events in a random environment. These jumps are described by a Lévy process with bounded variation paths. We construct a process of this class as the unique solution of a stochastic differential equation. The quenched branching property of the process allows us to derive quenched and annealed results and to observe new asymptotic behaviors. We characterize the Laplace exponent of the process as the solution of a backward ordinary differential equation and establish the probability of extinction. Restricting our attention to the critical and subcritical cases, we show that four regimes arise for the speed of extinction, as in the case of branching processes in random environment in discrete time and space. The proofs are based on the precise asymptotic behavior of exponential functionals of Lévy processes. Finally, we apply these results to a cell infection model and determine the mean speed of propagation of the infection.

Key words. Continuous State Branching Processes, Lévy processes, Poisson Point Processes, Random Environment, Extinction, Long time behavior

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1 Introduction

Continuous state branching processes (CSBP) are the analogues of Galton-Watson (GW) processes in continuous time and continuous state space. They have been introduced by Jirina [25] and studied by many authors including Bingham [8], Grey [19], Grimvall [20], Lamperti [30, 31], to name but a few.

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A CSBP $Z = (Z_t, t \geq 0)$ is a strong Markov process taking values in $[0, \infty]$, where 0 and ∞ are absorbing states, and satisfying the branching property. We denote by $(\mathbb{P}_x,\, x > 0)$ the law of Z starting from x. Lamperti [31] proved that there is a bijection between CSBP and scaling limits of GW processes. Thus they may model the evolution of renormalized large populations on a large time scale.

The branching property implies that the Laplace transform of Z_t is of the form

$$\mathbb{E}_x\Big[\exp(-\lambda Z_t)\Big] = \exp\{-x u_t(\lambda)\}, \qquad \text{for } \lambda \geq 0,$$

for some non-negative function u_t. According to Silverstein [36], this function is determined by the integral equation

$$\int_{u_t(\lambda)}^{\lambda} \frac{1}{\psi(u)} \mathrm{d}u = t,$$

where ψ is known as the branching mechanism associated to Z. We assume here that Z has finite mean, so that we have the following classical representation

$$\psi(\lambda) = -g\lambda + \sigma^2\lambda^2 + \int_0^\infty \Big(e^{-\lambda z} - 1 + \lambda z\Big)\,\mu(\mathrm{d}z), \tag{1}$$

where $g \in \mathbb{R}$, $\sigma \geq 0$ and μ is a σ-finite measure on $(0, \infty)$ such that $\int_{(0,\infty)} \big(z \wedge z^2\big)\mu(\mathrm{d}z)$ is finite. The CSBP is then characterized by the triplet (g, σ, μ) and can also be defined as the unique non-negative strong solution of a stochastic differential equation. More precisely, from Fu and Li [16] we have

$$Z_t = Z_0 + \int_0^t g Z_s \mathrm{d}s + \int_0^t \sqrt{2\sigma^2 Z_s}\mathrm{d}B_s + \int_0^t \int_0^\infty \int_0^{Z_{s-}} z\widetilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \tag{2}$$

where B is a standard Brownian motion, $N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ is a Poisson random measure with intensity $\mathrm{d}s\mu(\mathrm{d}z)\mathrm{d}u$ independent of B, and $\widetilde{N}_0$ is the compensated measure of N_0.

The stable case with drift, i.e. $\psi(\lambda) = -g\lambda + c\lambda^{1+\beta}$, with β in $(0, 1]$, corresponds to the CSBP that one can obtain by scaling limits of GW processes with a fixed reproduction law. It is of special interest in this paper since the Laplace exponent can be computed explicitly and it can also be used to derive asymptotic results for more general cases.

In this work, we are interested in modeling catastrophes which occur at random and kill each individual with some probability (depending on the catastrophe). In terms of the CSBP representing the scaling limit of the size of a large population, this amounts to letting the process make a negative jump, i.e. multiplying its current value by a random fraction. The process that we obtain is still Markovian whenever the catastrophes follow a time homogeneous Poisson Point Process. Moreover, we show that conditionally on the times and the effects of the catastrophes, the process satisfies the branching property. Thus, it yields a particular class of CSBP in random environment, which can also be obtained as the scaling limit of GW processes in random environment (see [4]). Such processes are motivated in particular by a cell division model; see for instance [5] and Section 5.

We also consider positive jumps that may represent immigration events proportional to the size of the current population. Our motivation comes from the aggregation behavior of some species. We refer to Chapter 12 in [12] for adaptive explanations of these aggregation behaviors, or [35] which shows that aggregation behaviors may result from manipulation by parasites to increase their transmission. For convenience, we still call these dramatic events catastrophes.

The process Y that we consider in this paper is then called *a CSBP with catastrophes*. Roughly speaking, it can be defined as follows: The process Y follows the SDE (2) between catastrophes, which are then given in terms of the jumps of a Lévy process with bounded variation paths. Thus the set of times at which catastrophes occur may have accumulation points, but the mean effect of the catastrophes has a finite first moment. When a catastrophe with effect m_t occurs at time t, we have

$$Y_t = m_t Y_{t-}.$$

We defer the formal definitions to Section 2. We also note that Brockwell has considered birth and death branching processes with another kind of catastrophes, see e.g. [10].

First we verify that CSBP with catastrophes are well defined as solutions of a certain stochastic differential equation, which we give as (5). We characterize their Laplace exponents via an ordinary differential equation (see Theorem 1), which allows us to describe their long time behavior. In particular, we prove an extinction criterion for the CSBP with catastrophes which is given in terms of the sign of $\mathbb{E}[g + \sum_{s\leq 1} \log m_s]$. We also establish a central limit theorem conditionally on survival and under some moment assumptions (Corollary 3).

We then focus on the case when the branching mechanism associated to the CSBP with catastrophes Y has the form $\psi(\lambda) = -g\lambda + c\lambda^{1+\beta}$, for $\beta \in (0, 1]$, i.e. the stable case. In this scenario, the extinction and absorption events coincide, which means that $\{\lim_{t\to\infty} Y_t = 0\} = \{\exists t \geq 0, Y_t = 0\}$. We prove that the speed of extinction is directly related to the asymptotic behavior of exponential functionals of Lévy processes (see Proposition 4). More precisely, we show that the extinction probability of a stable CSBP with catastrophes can be expressed as follows:

$$\mathbb{P}(Y_t > 0) = \mathbb{E}\left[F\left(\int_0^t e^{-\beta K_s} \mathrm{d}s\right)\right],$$

where F is a function with a particular asymptotic behavior and $K_t := gt + \sum_{s\leq t} \log m_s$ is a Lévy process of bounded variation that does not drift to $+\infty$ and satisfies an exponential positive moment condition. We establish the asymptotic behavior of the survival probability (see Theorem 7) and find four different regimes when this probability is equal to zero. Actually, such asymptotic behaviors have previously been found for branching processes in random environments in discrete time and space (see e.g. [21, 18, 1]). Here, the regimes depend on the shape of the Laplace exponent of K, i.e. on the drift g of the CSBP and the law of the catastrophes. The asymptotic behavior of exponential functionals of Lévy processes drifting to $+\infty$ has been deeply studied by many authors, see for instance Bertoin and Yor [7] and references therein. To our knowledge, the remaining cases have been studied only by Carmona et al. (see Lemma 4.7 in [11]) but their result focuses only on one regime. Our result is closely related to the discrete framework via the asymptotic behaviors of functionals of random walks. More precisely, we use in our arguments local limit theorems for semi direct products [34, 21] and some analytical results on random walks [26, 22], see Section 4.

From the speed of extinction in the stable case, we can deduce the speed of extinction of a larger class of CSBP with catastrophes satisfying the condition that extinction and absorption coincide (see Corollary 6). General results for the case of Lévy processes of unbounded variation do not seem easy to obtain since the existence of the process Y and our approximation methods are not so easy to deduce. The particular case when $\mu = 0$ and the environment K is given by a Brownian motion has been studied in [9]. The authors in [9] also obtained similar asymptotics regimes using the explicit law of $\int_0^t \exp(-\beta K_s)ds$.

Finally, we apply our results to a cell infection model introduced in [5] (see Section 5). In this model, the infection in a cell line is given by a Feller diffusion with catastrophes. We derive here the different possible speeds of the infection propagation. More generally, these results can be related to some ecological problems concerning the role of environmental and demographical stochasticities. Such topics are fundamental in conservation biology, as discussed for instance in Chapter 1 in [33]. Indeed, the survival of the population may be either due to the randomness of the individual reproduction, which is specified in our model by the parameters σ and μ of the CSBP, or to the randomness (rate, size) of the catastrophes due to the environment. For a study of relative effects of environmental and demographical stochasticities, the reader is referred to [32] and references therein.

The remainder of the paper is structured as follows. In Section 2, we define and study the CSBP with catastrophes. Section 3 is devoted to the study of the extinction probabilities where special attention is given to the stable case. In Section 4, we analyse the asymptotic behavior of exponential functionals of Lévy processes of bounded variation. This result is the key to deducing the different extinction regimes. In Section 5, we apply our results to a cell infection model. Finally, Section 6 contains some technical results used in the proofs and deferred for the convenience of the reader.

2 CSBP with catastrophes

We consider a CSBP $Z = (Z_t, t \geq 0)$ defined by (2) and characterized by the triplet (g, σ, μ), where we recall that μ satisfies

$$\int_0^\infty (z \wedge z^2)\mu(\mathrm{d}z) < \infty. \tag{3}$$

The catastrophes are independent of the process Z and are given by a Poisson random measure $N_1 = \sum_{i\in I} \delta_{t_i, m_{t_i}}$ on $[0,\infty) \times [0,\infty)$ with intensity $\mathrm{d}t\nu(\mathrm{d}m)$ such that

$$\nu(\{0\}) = 0 \qquad \text{and} \qquad 0 < \int_{(0,\infty)} (1 \wedge |m-1|)\nu(\mathrm{d}m) < \infty. \tag{4}$$

The jump process

$$\Delta_t = \int_0^t \int_{(0,\infty)} \log(m) N_1(\mathrm{d}s, \mathrm{d}m) = \sum_{s\leq t} \log(m_s),$$

is thus a Lévy process with paths of bounded variation, which is non identically zero.

The CSBP (g, σ, μ) with catastrophes ν is defined as the solution of the following stochastic differential equation:

$$
\begin{aligned}
Y_t = Y_0 + \int_0^t gY_s \mathrm{d}s + \int_0^t \sqrt{2\sigma^2 Y_s}\mathrm{d}B_s \quad &+ \quad \int_0^t \int_{[0,\infty)} \int_0^{Y_{s-}} z\widetilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\
&+ \quad \int_0^t \int_{[0,\infty)} \Big(m-1\Big) Y_{s-} N_1(\mathrm{d}s, \mathrm{d}m), \quad (5)
\end{aligned}
$$

where $Y_0 > 0$ a.s.

Let $\mathcal{BV}(\mathbb{R}_+)$ be the set of càdlàg functions on $\mathbb{R}_+ := [0,\infty)$ of bounded variation and C_b^2 the set of all functions that are twice differentiable and are bounded together with their derivatives, then the following result of existence and unicity holds:

Theorem 1. *The stochastic differential equation (5) has a unique non-negative strong solution Y for any $g \in \mathbb{R}, \sigma \geq 0$, μ and ν satisfying conditions (3) and (4), respectively. Then, the process $Y = (Y_t, t \geq 0)$ is a càdlàg Markov process satisfying the branching property conditionally on $\Delta = (\Delta_t, t \geq 0)$ and its infinitesimal generator $\mathcal{A}$ satisfies for every $f \in C_b^2$*

$$
\begin{aligned}
\mathcal{A}f(x) = gxf'(x) + \sigma^2 x f''(x) + &\int_0^\infty \Big(f(mx) - f(x)\Big)\nu(dm) \\
&+ \int_0^\infty \Big(f(x+z) - f(x) - zf'(x)\Big) x\mu(dz).
\end{aligned} \tag{6}
$$

Moreover, for every $t \geq 0$,

$$
\mathbb{E}_y \left[\exp\Big\{ -\lambda \exp\big\{ -gt - \Delta_t \big\} Y_t \Big\} \Big| \, \Delta \right] = \exp\Big\{ -y v_t(0, \lambda, \Delta) \Big\} \qquad a.s.,
$$

where for every $(\lambda, \delta) \in (\mathbb{R}_+, \mathcal{BV}(\mathbb{R}_+))$, $v_t : s \in [0,t] \mapsto v_t(s, \lambda, \delta)$ is the unique solution of the following backward differential equation :

$$
\frac{\partial}{\partial s} v_t(s, \lambda, \delta) = e^{gs+\delta_s} \psi_0\big(e^{-gs-\delta_s} v_t(s, \lambda, \delta)\big), \qquad v_t(t, \lambda, \delta) = \lambda, \tag{7}
$$

and

$$
\psi_0(\lambda) = \psi(\lambda) - \lambda\psi'(0) = \sigma^2\lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z)\mu(dz). \tag{8}
$$

Proof. Under Lipschitz conditions, the existence and uniqueness of strong solutions for stochastic differential equations are classical results (see [24]). In our case, the result follows from Proposition 2.2 and Theorems 3.2 and 5.1 in [16]. By Itô's formula (see for instance [24] Th.5.1), the solution of the SDE (5), $(Y_t, t \geq 0)$ solves the following martingale problem. For every $f \in C_b^2$,

$$
\begin{aligned}
f(Y_t) = f(Y_0) + \text{ loc. mart. } &+ g\int_0^t f'(Y_s) Y_s \mathrm{d}s \\
+ \sigma^2 \int_0^t f''(Y_s) Y_s \mathrm{d}s &+ \int_0^t \int_0^\infty Y_s \Big(f(Y_s + z) - f(Y_s) - f'(Y_s)z\Big)\mu(\mathrm{d}z)\mathrm{d}s \\
+ \int_0^t \int_0^\infty &\Big(f(mY_s) - f(Y_s)\Big)\nu(\mathrm{d}m)\mathrm{d}s,
\end{aligned}
$$

where the local martingale is given by

$$\int_0^t f'(Y_s)\sqrt{2\sigma^2 Y_s}\mathrm{d}B_s + \int_0^t\int_0^\infty \Big(f(mY_{s-}) - f(Y_{s-})\Big)\widetilde{N}_1(\mathrm{d}s,\mathrm{d}m) \qquad (9)$$
$$+ \int_0^t\int_0^\infty\int_0^{Y_{s-}} \Big(f(Y_{s-}+z) - f(Y_{s-})\Big)\widetilde{N}_0(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u),$$

and $\widetilde{N}_1$ is the compensated measure of N_1. Even though the process in (9) is a local martingale, we can define a localized version of the corresponding martingale problem as in Chapter 4.6 of Ethier and Kurtz [15]. We leave the details to the reader. From pathwise uniqueness, we deduce that the solution of (5) is a strong Markov process whose generator is given by (6).

The branching property of Y, conditionally on Δ, is inherited from the branching property of the CSBP and the fact that the additional jumps are multiplicative.

To prove the second part of the theorem, let us now work conditionally on Δ. Applying Itô's formula to the process $\widetilde{Z}_t = Y_t\exp\{-gt-\Delta_t\}$, we obtain

$$\widetilde{Z}_t = Y_0 + \int_0^t e^{-gs-\Delta_s}\sqrt{2\sigma^2 Y_s}\mathrm{d}B_s + \int_0^t\int_0^\infty\int_0^{Y_{s-}} e^{-gs-\Delta_{s-}}z\widetilde{N}_0(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u),$$

and then $\widetilde{Z}$ is a local martingale conditionally on Δ. A new application of Itô's formula ensures that for every $F\in C_b^{1,2}$, $F(t,\widetilde{Z}_t)$ is also a local martingale if and only if for every $t\geq 0$,

$$\int_0^t \frac{\partial^2}{\partial x^2}F(s,\widetilde{Z}_s)\sigma^2\widetilde{Z}_s e^{-gs-\Delta_s}\mathrm{d}s + \int_0^t \frac{\partial}{\partial s}F(s,\widetilde{Z}_s)\mathrm{d}s \qquad (10)$$
$$+\int_0^t\int_0^\infty \widetilde{Z}_s\Big(\Big[F(s,\widetilde{Z}_s+ze^{-gs-\Delta_s}) - F(s,\widetilde{Z}_s)\Big]e^{gs+\Delta_s} - \frac{\partial}{\partial x}F(s,\widetilde{Z}_s)z\Big)\mu(\mathrm{d}z)\mathrm{d}s = 0.$$

In the vein of [24, 5], we choose $F(s,x) := \exp\{-xv_t(s,\lambda,\Delta)\}$, where $v_t(s,\lambda,\Delta)$ is differentiable with respect to the variable s, non-negative and such that $v_t(t,\lambda,\Delta)=\lambda$, for $\lambda\geq 0$. The function F is bounded, so that $(F(s,\tilde{Z}_s), 0\leq s\leq t)$ will be a martingale if and only if for every $s\in[0,t]$

$$\frac{\partial}{\partial s}v_t(s,\lambda,\Delta) = e^{gs+\Delta_s}\psi_0\left(e^{-gs-\Delta_s}v_t(s,\lambda,\Delta)\right), \quad \text{a.s.},$$

where ψ_0 is defined in (8).

Proposition 17 in Section 6 ensures that a.s. the solution of this backward differential equation exists and is unique, which essentially comes from the Lipschitz property of ψ_0 (Lemma 18) and the fact that Δ possesses bounded variation paths. Then the process $(\exp\{-\tilde{Z}_s v_t(s,\lambda,\Delta)\}, 0\leq s\leq t)$ is a martingale conditionally on Δ and

$$\mathbb{E}_y\left[\exp\Big\{-\tilde{Z}_t v_t(t,\lambda,\Delta)\Big\}\Big|\,\Delta\right] = \mathbb{E}_y\left[\exp\Big\{-\tilde{Z}_0 v_t(0,\lambda,\Delta)\Big\}\Big|\Delta\right] \quad \text{a.s.},$$

which yields

$$\mathbb{E}_y\left[\exp\Big\{-\lambda\tilde{Z}_t\Big\}\Big|\,\Delta\right] = \exp\Big\{-yv_t(0,\lambda,\Delta)\Big\} \quad \text{a.s.} \qquad (11)$$

This implies our result. □

Referring to Theorem 7.2 in [27], we recall that a Lévy process has three possible asymptotic behaviors: either it drifts to ∞, $-\infty$, or oscillates a.s. In particular, if the Lévy process has a finite first moment, the sign of its expectation yields the regimes of above. We extend this classification to CSBP with catastrophes.

Corollary 2. *We have the following three regimes.*

i) If $(\Delta_t + gt)_{t\geq 0}$ drifts to $-\infty$, then $\mathbb{P}(Y_t \to 0 \mid \Delta) = 1$ a.s.

ii) If $(\Delta_t + gt)_{t\geq 0}$ oscillates, then $\mathbb{P}(\liminf_{t\to\infty} Y_t = 0 \mid \Delta) = 1$ a.s.

iii) If $(\Delta_t + gt)_{t\geq 0}$ drifts to $+\infty$ and there exists $\varepsilon > 0$, such that

$$\int_0^\infty z \log^{1+\epsilon}(1+z)\mu(dz) < \infty, \tag{12}$$

then $\mathbb{P}(\liminf_{t\to\infty} Y_t > 0 \mid \Delta) > 0$ a.s. and there exists a non-negative finite r.v. W such that

$$e^{-gt-\Delta_t} Y_t \xrightarrow[t\to\infty]{} W \quad a.s., \qquad \{W = 0\} = \Big\{ \lim_{t\to\infty} Y_t = 0 \Big\}.$$

Remark 1. In the regime (*ii*), Y may be absorbed in finite time a.s. (see the next section). But Y_t may also a.s. do not tend to zero. For example, if $\mu = 0$ and $\sigma = 0$, then $Y_t = \exp(gt + \Delta_t)$ and $\limsup_{t\to\infty} Y_t = \infty$.

Assumption (*iii*) of the corollary does not imply that $\{\lim_{t\to\infty} Y_t = 0\} = \{\exists t : Y_t = 0\}$. Indeed, the case $\mu(dx) = x^{-2}\mathbf{1}_{[0,1]}(x)dx$ inspired by Neveu's CSBP yields $\psi(u) \sim u \log u$ as $u \to \infty$. Then, according to Remark 2.2 in [29], $\mathbb{P}(\exists t : Y_t = 0) = 0$ and $0 < \mathbb{P}(\lim_{t\to\infty} Y_t = 0) < 1$.

Proof. We use (10) with $F(s,x) = x$ to get that $\tilde{Z} = (Y_t \exp(-gt - \Delta_t) : t \geq 0)$ is a non-negative local martingale. Thus it is a non-negative supermartingale and it converges a.s. to a non-negative finite random variable W. This implies the proofs of (i-ii).

In the case when $(gt + \Delta_t, t \geq 0)$ goes to $+\infty$, we prove that $\mathbb{P}(W > 0 \mid \Delta) > 0$ a.s. According to Lemma 19 in Section 6, the assumptions of (iii) ensure the existence of a non-negative increasing function k on $\mathbb{R}^+$ such that for all $\lambda > 0$,

$$\psi_0(\lambda) \leq \lambda k(\lambda) \quad \text{and} \quad c(\Delta) := \int_0^\infty k\Big(e^{-(gt+\Delta_t)}\Big) dt < \infty \quad \text{a.s.}$$

For every $(t, \lambda) \in (\mathbb{R}_+^*)^2$, the solution v_t of (7) is non-decreasing on $[0, t]$. Thus for all $s \in [0, t]$, $v_t(s, 1, \Delta) \leq 1$, and

$$\begin{aligned} \psi_0(e^{-gs-\Delta_s} v_t(s,1,\Delta)) &\leq e^{-gs-\Delta_s} v_t(s,1,\Delta) k(e^{-gs-\Delta_s} v_t(s,1,\Delta)) \\ &\leq e^{-gs-\Delta_s} v_t(s,1,\Delta) k(e^{-gs-\Delta_s}) \qquad \text{a.s.} \end{aligned}$$

Then (7) gives

$$\frac{\partial}{\partial s} v_t(s,1,\Delta) \leq v_t(s,1,\Delta) k(e^{-gs-\Delta_s}),$$

implying

$$-\ln(v_t(0,1,\Delta)) \leq \int_0^t k(e^{-gs-\Delta_s}) ds \leq c(\Delta) < \infty \qquad \text{a.s.}$$

Hence, for every $t \geq 0$, $v_t(0,1,\Delta) \geq \exp(-c(\Delta)) > 0$ and conditionally on Δ there exists a positive lower bound for $v_t(0,1,\Delta)$. Finally from (11),

$$\mathbb{E}_y[\exp\{-W\} \,|\Delta] = \exp\Big\{ -y \lim_{t\to\infty} v_t(0,1,\Delta)\Big\} < 1$$

and $\mathbb{P}(W > 0 \mid \Delta) > 0$ a.s.
Moreover, since Y satisfies the branching property conditionally on Δ, we can show (see Lemma 20 in Section 6) that

$$\{W = 0\} = \Big\{ \lim_{t\to\infty} Y_t = 0\Big\} \qquad \text{a.s.},$$

which completes the proof. □

We now derive a central limit theorem in the supercritical regime:

Corollary 3. *Assume that $(gt + \Delta_t, t \geq 0)$ drifts to $+\infty$ and (12) is satisfied. Then, under the additional assumption*

$$\int_{(0,e^{-1}]\cup[e,\infty)} (\log m)^2 \nu(\mathrm{d}m) < \infty, \tag{13}$$

conditionally on $\{W > 0\}$,

$$\frac{\log(Y_t) - \boldsymbol{m}t}{\rho\sqrt{t}} \xrightarrow[t\to\infty]{d} \mathcal{N}(0,1),$$

where $\xrightarrow{d}$ means convergence in distribution,

$$\boldsymbol{m} := g + \int_{\{|\log x|\geq 1\}} \log m \,\nu(\mathrm{d}m) < \infty, \qquad \rho^2 := \int_0^\infty (\log m)^2 \nu(\mathrm{d}m) < \infty,$$

and $\mathcal{N}(0,1)$ denotes a centered Gaussian random variable with variance equals 1.

Proof. We use the central limit theorem for the Lévy process $(gt + \Delta_t, t \geq 0)$ under assumption (13) of Doney and Maller [13], see Theorem 3.5. For simplicity, the details are deferred to Section 6.4. We then get

$$\frac{gt + \Delta_t - \mathbf{m}t}{\rho\sqrt{t}} \xrightarrow[t\to\infty]{d} \mathcal{N}(0,1). \tag{14}$$

From Corollary 2 part *iii*), under the event $\{W > 0\}$, we get

$$\log Y_t - (gt + \Delta_t) \xrightarrow[t\to\infty]{a.s.} \log W \in (-\infty, \infty),$$

and we conclude using (14). □

3 Speed of extinction of CSBP with catastrophes

In this section, we first study the particular case of the stable CSBP with growth $g \in \mathbb{R}$. Then, we derive a similar result for another class of CSBP's.

3.1 The stable case

We assume in this section that

$$\psi(\lambda) = -g\lambda + c_+\lambda^{\beta+1}, \tag{15}$$

for some $\beta \in (0,1]$, $c_+ > 0$ and g in $\mathbb{R}$.

If $\beta = 1$ (i.e. the Feller diffusion), we necessarily have $\mu = 0$ and the CSBP Z follows the continuous diffusion

$$Z_t = Z_0 + \int_0^t gZ_s \mathrm{d}s + \int_0^t \sqrt{2\sigma^2 Z_s}\mathrm{d}B_s, \quad t \geq 0.$$

In the case when $\beta \in (0,1)$, we necessarily have $\sigma = 0$ and the measure μ takes the form $\mu(\mathrm{d}x) = c_+(\beta+1)x^{-(2+\beta)}\mathrm{d}x/\Gamma(1-\beta)$. In other words, the process possesses positive jumps with infinite intensity [28]. Moreover,

$$Z_t = Z_0 + \int_0^t gZ_s \mathrm{d}s + \int_0^t Z_{s^-}^{1/(\beta+1)}\mathrm{d}X_s, \quad t \geq 0,$$

where X is a $(\beta+1)$-stable spectrally positive Lévy process.

For the stable CSBP with catastrophes, the backward differential equation (7) can be solved and in particular, we get

Proposition 4. *For all $x_0 > 0$ and $t \geq 0$:*

$$\mathbb{P}_{x_0}(Y_t > 0 \mid \Delta) = 1 - \exp\left\{ -x_0 \left(c_+\beta \int_0^t e^{-\beta(gs+\Delta_s)}\mathrm{d}s \right)^{-1/\beta} \right\} \qquad a.s. \tag{16}$$

Moreover,

$$\mathbb{P}_{x_0}(\textit{there exists } t > 0,\ Y_t = 0 \mid \Delta) = 1 \qquad a.s.,$$

if and only if the process $(gt + \Delta_t, t \geq 0)$ does not drift to $+\infty$.

Proof. Since $\psi_0(\lambda) = c_+\lambda^{\beta+1}$, a direct integration gives us

$$v_t(u, \lambda, \Delta) = \left[c_+\beta \int_u^t e^{-\beta(gs+\Delta_s)}\mathrm{d}s + \lambda^{-\beta} \right]^{-1/\beta},$$

which implies

$$\mathbb{E}_{x_0}\left[e^{-\lambda \tilde{Z}_t} \middle| \Delta \right] = \exp\left\{ -x_0 \left(c_+\beta \int_0^t e^{-\beta(gs+\Delta_s)}\mathrm{d}s + \lambda^{-\beta} \right)^{-1/\beta} \right\} \quad \text{a.s.} \tag{17}$$

Hence, the absorption probability follows by letting λ tend to ∞ in (17). In other words,

$$\mathbb{P}_{x_0}(Y_t = 0 \mid \Delta) = \exp\left\{ -x_0 \left(c_+\beta \int_0^t e^{-\beta(gs+\Delta_s)}\mathrm{d}s \right)^{-1/\beta} \right\} \quad \text{a.s.}$$

Since $\mathbb{P}_{x_0}(\text{there exists } t \geq 0 : Y_t = 0 \mid \Delta) = \lim_{t\to\infty} \mathbb{P}_{x_0}(Y_t = 0 \mid \Delta)$ a.s., we deduce

$$\mathbb{P}_{x_0}(\text{there exists } t \geq 0 : Y_t = 0 \mid \Delta) = \exp\left\{ -x_0 \left(c_+\beta \int_0^\infty e^{-\beta(gs+\Delta_s)}\mathrm{d}s \right)^{-1/\beta} \right\} \quad \text{a.s.}$$

Finally, according to Theorem 1 in [7], $\int_0^\infty \exp\{-\beta(gs+\Delta_s)\}\mathrm{d}s = \infty$ a.s. if and only if the process $(gt + \Delta_t, t \geq 0)$ does not drift to $+\infty$. This completes the proof. □

In what follows, we assume that the Lévy process Δ admits some positive exponential moments, i.e. there exists $\lambda > 0$ such that $\phi(\lambda) < \infty$. We can then define $\theta_{max} = \sup\{\lambda > 0,\ \phi(\lambda) < \infty\} \in (0, \infty]$ and we have

$$\phi(\lambda) := \log \mathbb{E}[e^{\lambda \Delta_1}] = \int_0^\infty (m^\lambda - 1)\nu(dm) < \infty \qquad \text{for } \lambda \in [0, \theta_{max}). \tag{18}$$

We note that ϕ can be differentiated on the right in 0 and also in 1 if $\theta_{max} > 1$:

$$\phi'(0) := \phi'(0+) = \int_0^\infty \log(m)\nu(dm) \in (-\infty, \infty), \qquad \phi'(1) = \int_0^\infty \log(m) m\nu(dm).$$

Recall that Δ_t/t converges to $\phi'(0)$ a.s. and that $g + \phi'(0)$ is negative in the subcritical case. Proposition 4 then yields the asymptotic behavior of the quenched survival probability :

$$e^{-gt-\Delta_t}\mathbb{P}_{x_0}(Y_t > 0|\ \Delta) \sim x_0\Big(c_+\beta \int_0^t e^{\beta(gt+\Delta_t-gs-\Delta_s)}ds\Big)^{-1/\beta} \qquad (t \to \infty),$$

which converges in distribution to a positive finite limit proportional to x_0. Then,

$$\frac{1}{t}\log \mathbb{P}_{x_0}(Y_t > 0|\ \Delta) \to g + \phi'(0) \qquad (t \to \infty)$$

in probability.

Additional work is required to get the asymptotic behavior of the annealed survival probability, for which four different regimes appear when the process a.s. goes to zero:

Proposition 5. *We assume that ν satisfies (4) and (13), and that ψ and ϕ satisfy (15) and (18) respectively.*

a/ If $\phi'(0) + g < 0$ (subcritical case) *and $\theta_{max} > 1$, then*

- *(i) If $\phi'(1) + g < 0$* (strongly subcritical regime), *then there exists $c_1 > 0$ such that for every $x_0 > 0$,*

$$\mathbb{P}_{x_0}(Y_t > 0) \sim c_1 x_0 e^{t(\phi(1)+g)}, \qquad \textit{as} \quad t \to \infty.$$

- *(ii) If $\phi'(1)+g = 0$* (intermediate subcritical regime), *then there exists $c_2 > 0$ such that for every $x_0 > 0$,*

$$\mathbb{P}_{x_0}(Y_t > 0) \sim c_2 x_0 t^{-1/2} e^{t(\phi(1)+g)}, \qquad \textit{as} \quad t \to \infty.$$

- *(iii) If $\phi'(1) + g > 0$* (weakly subcritical regime) *and $\theta_{max} > \beta + 1$, then for every $x_0 > 0$, there exists $c_3(x_0) > 0$ such that*

$$\mathbb{P}_{x_0}(Y_t > 0) \sim c_3(x_0) t^{-3/2} e^{t(\phi(\tau)+g\tau)}, \qquad \textit{as} \quad t \to \infty,$$

where τ is the root of $\phi' + g$ on $]0, 1[$: $\phi(\tau) + g\tau = \min_{0<s<1}\{\phi(s) + gs\}$.

b/ If $\phi'(0) + g = 0$ (critical case) *and $\theta_{max} > \beta$, then for every $x_0 > 0$, there exists $c_4(x_0) > 0$ such that*

$$\mathbb{P}_{x_0}(Y_t > 0) \sim c_4(x_0) t^{-1/2}, \qquad \textit{as} \quad t \to \infty.$$

Proof. From Proposition 4 we know that

$$\mathbb{P}_{x_0}(Y_t > 0) = 1 - \mathbb{E}\left[\exp\left\{-x_0\left(c_+\beta\int_0^t e^{-\beta(gs+\Delta_s)}\mathrm{d}s\right)^{-1/\beta}\right\}\right] = \mathbb{E}\left[F\left(\int_0^t e^{-\beta K_s}\mathrm{d}s\right)\right],$$

where $F(x) = 1 - \exp\{-x_0(c_+\beta x)^{-1/\beta}\}$ and $K_s = \Delta_s + gs$. The function F satisfies assumption (23) which is required in Theorem 7 (which is stated and proved in the next section). Hence Proposition 5 follows from a direct application of this Theorem. □

In the case of CSBP's without catastrophes ($\nu = 0$), the subcritical regime is reduced to (i), and the critical case differs from b/, since the asymptotic behavior is given by $1/t$. In the strongly and intermediate subcritical cases (i) and (ii), $\mathbb{E}[Y_t]$ provides the exponential decay factor of the survival probability which is given by $\phi(1) + g$. Moreover the probability of non-extinction is proportional to the initial state x_0 of the population. We refer to the proof of Lemma 11 and Section 4.4 for more details.
In the weakly subcritical case (iii), the survival probability decays exponentially with rate $\phi(\tau) + g\tau$, which is strictly smaller than $\phi(1) + g$. In fact, as it appears in the proof of Theorem 7, the quantity which determines the asymptotic behavior in all cases is $\mathbb{E}[\exp\{\inf_{s\in[0,t]}(\Delta_s + gs)\}]$. We also note that c_3 and c_4 may not be proportional to x_0. We refer to [3] for a result in this vein for discrete branching processes in random environment.

More generally, the results stated above can be compared to the results which appear in the literature of discrete (time and space) branching processes in random environment (BPRE), see e.g. [21, 18, 1]. A BPRE $(X_n, n \in \mathbb{N})$ is an integer valued branching process, specified by a sequence of generating functions $(f_n, n \in \mathbb{N})$. Conditionally on the environment, individuals reproduce independently of each other and the offsprings of an individual at generation n has generating function f_n. We present briefly the results of Theorem 1.1 in [17] and Theorems 1.1, 1.2 and 1.3 in [18]. To lighten the presentation, we do not specify here the moment conditions.
In the *subcritical case*, i.e. when $\mathbb{E}[\log f_0'(1)] < 0$, we have the following three asymptotic regimes as n increases,

$$\mathbb{P}(X_n > 0) \sim ca_n, \qquad \text{as} \qquad n \to \infty,$$

where c is a positive constant and a_n is given by

$$a_n = \mathbb{E}\Big[f_0'(1)\Big]^n, \quad a_n = n^{-1/2}\mathbb{E}\Big[f_0'(1)\Big]^n \quad \text{or} \quad a_n = n^{-3/2}\left(\min_{0<s<1}\mathbb{E}\Big[(f_0'(1))^s\Big]\right)^n,$$

when $\mathbb{E}[f_0'(1)\log f_0'(1)]$ is negative, zero or positive, respectively.
In the *critical case*, i.e. $\mathbb{E}[\log f_0'(1)] = 0$, we have

$$\mathbb{P}(X_n > 0) \sim cn^{-1/2}, \qquad \text{as} \qquad n \to \infty,$$

for some positive constant c. In the particular case when $\beta = 1$, these results on BPRE and the approximation techniques implemented in Section 4 can be used to get Proposition 5. We refer to Remarks 2 and 3 for more details.

Finally, in the continuous framework, such results have been established for the Feller diffusion case, i.e. $\beta = 1$, whose drift varies following a Brownian motion (see [9]). In other words the process K is given by a Brownian motion plus a drift. The techniques used by the authors rely on an explicit formula for the Laplace transform of exponential functionals of Brownian motion which we cannot find in the literature for the case of Lévy processes. These results have been completed in the surpercritical regime in [23].

3.2 Beyond the stable case.

In this section, we prove a similar result to Proposition 5 for CSBP's with catastrophes in the case when the branching mechanism ψ_0 is not stable. For technical reasons, we assume that the Brownian coefficient is positive and the associated Lévy measure μ satisfies a second moment condition.

Corollary 6. *Assume that (18) holds and*

$$\int_{(0,\infty)} z^2\mu(dz) < \infty, \qquad \sigma^2 > 0, \qquad \int_{(0,\infty)} (\log m)^2 \nu(dm) < \infty.$$

a/ If $\phi'(0) + g < 0$ and $\theta_{max} > 1$, then

(i) If $\phi'(1) + g < 0$, there exist $0 < c_1 \leq c_1' < \infty$ such that for every x_0,

$$c_1 x_0 e^{t(\phi(1)+g)} \leq \mathbb{P}_{x_0}(Y_t > 0) \leq c_1' x_0 e^{t(\phi(1)+g)} \qquad \text{for sufficiently large } t.$$

(ii) If $\phi'(1) + g = 0$, there exist $0 < c_2 \leq c_2' < \infty$ such that for every x_0,

$$c_2 x_0 t^{-1/2} e^{t(\phi(1)+g)} \leq \mathbb{P}_{x_0}(Y_t > 0) \leq c_2' x_0 t^{-1/2} e^{t(\phi(1)+g)} \quad \text{for sufficiently large } t.$$

(iii) If $\phi'(1) + g > 0$ and $\theta_{max} > \beta + 1$, for every x_0, there exist $0 < c_3(x_0) \leq c_3'(x_0) < \infty$ such that

$$c_3(x_0) t^{-3/2} e^{t(\phi(\tau)+g\tau)} \leq \mathbb{P}_{x_0}(Y_t > 0) \leq c_3'(x_0) t^{-3/2} e^{t(\phi(\tau)+g\tau)} \quad (t > 0),$$

where τ is the root of $\phi' + g$ on $]0,1[$.

b/ If $\phi'(0)+g = 0$ and $\theta_{max} > \beta$, then for every x_0, there exist $0 < c_4(x_0) < c_4'(x_0) < \infty$ such that

$$c_4(x_0) t^{-1/2} \leq \mathbb{P}_{x_0}(Y_t > 0) \leq c_4'(x_0) t^{-1/2} \quad (t > 0).$$

Note that the assumption $\sigma^2 > 0$ is only required for the upper bounds.

Proof. We recall that the branching mechanism associated with the CSBP Z satisfies (1) for every $\lambda \geq 0$. So for every $\lambda \geq 0$,

$$2\sigma^2 \leq \psi''(\lambda) = 2\sigma^2 + \int_{(0,\infty)} z^2 e^{-\lambda z}\mu(dz).$$

Since $c := \int_0^\infty z^2\mu(dz) < \infty$, ψ'' is continuous on $[0,\infty)$. By Taylor-Lagrange's Theorem, we get for every $\lambda \geq 0$, $\psi_-(\lambda) \leq \psi(\lambda) \leq \psi_+(\lambda)$, where

$$\psi_-(\lambda) = \lambda\psi'(0) + \sigma^2\lambda^2 \quad \text{and} \quad \psi_+(\lambda) = \lambda\psi'(0) + (\sigma^2 + c/2)\lambda^2.$$

We first consider the case $\nu(0,\infty) < \infty$, so that Δ has a finite number of jumps on each compact interval a.s., and we also introduce the CSBP's with catastrophes Y^- and Y^+ which have the same catastrophes Δ as Y, but with the characteristics $(g, \sigma^2, 0)$ and $(g, \sigma^2 + c/2, 0)$, respectively. We denote $u_{-,t}$ and $u_{+,t}$ for their respective Laplace exponent, in other words for all $(\lambda, t) \in \mathbb{R}_+^2$,

$$\mathbb{E}\Big[\exp\{-\lambda Y_t^-\}\Big] = \exp\{-u_{-,t}(\lambda)\}, \qquad \mathbb{E}\Big[\exp\{-\lambda Y_t^+\}\Big] = \exp\{-u_{+,t}(\lambda)\}.$$

Thus conditionally on Δ, for every time t such that $\Delta_t = \Delta_{t-}$, we deduce, thanks to Theorem 1, the following identities

$$u'_{-,t}(\lambda) = -\psi_-(u_{-,t}), \qquad u'_{+,t}(\lambda) = -\psi_+(u_{+,t}), \qquad u'_t(\lambda) = -\psi(u_t).$$

Moreover for every t such that $\theta_t = \exp\{\Delta_t - \Delta_{t-}\} \neq 1$,

$$\frac{u_{-,t}(\lambda)}{u_{-,t-}(\lambda)} = \frac{u_t(\lambda)}{u_{t-}(\lambda)} = \frac{u_{+,t}(\lambda)}{u_{+,t-}(\lambda)} = \theta_t,$$

and $u_{-,0}(\lambda) = u_0(\lambda) = u_{+,0}(\lambda) = \lambda$. So for all t, λ, we have

$$u_{+,t}(\lambda) \leq u(t, \lambda) \leq u_{-,t}(\lambda).$$

The extension of the above inequality to the case $\nu(0,\infty) \in [0,\infty]$ can be achieved by successive approximations. We defer the technical details to Section 6.6.
Having into account that the above inequality holds in general, we deduce, taking $\lambda \to \infty$, that

$$\mathbb{P}(Y_t^+ > 0) \leq \mathbb{P}(Y_t > 0) \leq \mathbb{P}(Y_t^- > 0).$$

The result then follows from the asymptotic behavior of $\mathbb{P}(Y_t^- > 0)$ and $\mathbb{P}(Y_t^+ > 0)$, which are inherited from Proposition 5. □

4 Local limit theorem for some functionals of Lévy processes

We proved in Proposition 4 that the probability that a stable CSBP with catastrophes becomes extinct at time t equals the expectation of a functional of a Lévy process. We now prove the key result of the paper. It deals with the asymptotic behavior of the mean of some Lévy functionals.
More precisely, we are interested in the asymptotic behavior at infinity of

$$a_F(t) := \mathbb{E}\left[F\left(\int_0^t \exp\{-\beta K_s\} \mathrm{d}s\right)\right],$$

where K is a Lévy process with bounded variation paths and F belongs to a particular class of functions on $\mathbb{R}_+$. We will focus on functions which decrease polynomially at infinity (with exponent $-1/\beta$). The motivations come from the previous section. In particular, the Proposition 5 is a direct application of Theorem 7.

Thus, we consider a Lévy process $K = (K_t, t \geq 0)$ of the form

$$K_t = \gamma t + \sigma_t^{(+)} - \sigma_t^{(-)}, \qquad t \geq 0, \tag{19}$$

where γ is a real constant, $\sigma^{(+)}$ and $\sigma^{(-)}$ are two independent pure jump subordinators. We denote by Π, $\Pi^{(+)}$ and $\Pi^{(-)}$ the associated Lévy measures of K, $\sigma^{(+)}$ and $\sigma^{(-)}$, respectively. We also define the Laplace exponents of K, $\sigma^{(+)}$ and $\sigma^{(-)}$ by

$$\phi_K(\lambda) = \log \mathbb{E}\Big[e^{\lambda K_1}\Big], \quad \phi_K^+(\lambda) = \log \mathbb{E}\Big[e^{\lambda \sigma_1^{(+)}}\Big] \quad \text{and} \quad \phi_K^-(\lambda) = \log \mathbb{E}\Big[e^{-\lambda \sigma_1^{(-)}}\Big], \tag{20}$$

and assume that

$$\theta_{max} = \sup\left\{\lambda \in \mathbb{R}^+, \int_{[1,\infty)} e^{\lambda x} \Pi^{(+)}(\mathrm{d}x) < \infty\right\} > 0. \tag{21}$$

From the Lévy-Khintchine formula, we deduce

$$\phi_K(\lambda) = \gamma\lambda + \int_{(0,\infty)} \Big(e^{\lambda x} - 1\Big)\Pi^{(+)}(\mathrm{d}x) + \int_{(0,\infty)} \Big(e^{-\lambda x} - 1\Big)\Pi^{(-)}(\mathrm{d}x).$$

Finally, we assume that $\mathbb{E}[K_1^2] < \infty$, which is equivalent to

$$\int_{(-\infty,\infty)} x^2 \Pi(dx) < \infty. \tag{22}$$

Theorem 7. *Assume that (19), (21) and (22) hold. Let $\beta \in (0,1]$ and F be a positive non increasing function such that for $x \geq 0$*

$$F(x) = C_F (x+1)^{-1/\beta}\Big[1 + (1+x)^{-\varsigma} h(x)\Big], \tag{23}$$

where $\varsigma \geq 1$, C_F is a positive constant, and h is a Lipschitz function which is bounded.

a/ If $\phi_K'(0) < 0$

(i) If $\theta_{max} > 1$ and $\phi_K'(1) < 0$, there exists a positive constant c_1 such that

$$a_F(t) \sim c_1 e^{t\phi_K(1)}, \qquad as \qquad t \to \infty.$$

(ii) If $\theta_{max} > 1$ and $\phi_K'(1) = 0$, there exists a positive constant c_2 such that

$$a_F(t) \sim c_2 t^{-1/2} e^{t\phi_K(1)}, \qquad as \qquad t \to \infty.$$

(iii) If $\theta_{max} > \beta + 1$ and $\phi_K'(1) > 0$, there exists a positive constant c_3 such that

$$a_F(t) \sim c_3 t^{-3/2} e^{t\phi_K(\tau)}, \qquad as \qquad t \to \infty,$$

where τ is the root of ϕ_K' on $]0,1[$.

b/ If $\theta_{max} > \beta$ and $\phi_K'(0) = 0$, there exists a positive constant c_4 such that

$$a_F(t) \sim c_4 t^{-1/2}, \qquad as \qquad t \to \infty.$$

This result generalizes Lemma 4.7 in Carmona et al. [11] in the case when the process K has bounded variation paths. More precisely, the authors in [11] only provide a precise asymptotic behavior in the case when $\phi_K'(1) < 0$.

The assumption on the behavior of F as $x \to \infty$ is finely used to get the asymptotic behavior of $a_F(t)$. Lemma 10 gives the properties of F which are required in the proof.

The strongly subcritical case (case (i)) is proved using a continuous time change of measure (see Section 4.4). For the remaining cases, we divide the proof in three steps. The first one (see Lemma 8) consists in discretizing the exponential functional $\int_0^t \exp(-\beta K_s)\mathrm{d}s$ using the random variables

$$A_{p,q} = \sum_{i=0}^{p} \exp\{-\beta K_{i/q}\} = \sum_{i=0}^{p} \prod_{j=0}^{i-1} \exp\Big\{-\beta\big(K_{(j+1)/q} - K_{j/q}\big)\Big\} \quad ((p,q) \in \mathbb{N}\times\mathbb{N}^*). \tag{24}$$

Secondly (see Lemmas 11, 12 and 13), we study the asymptotic behavior of the discretized expectation

$$F_{p,q} := \mathbb{E}\Big[F\Big(A_{p,q}/q\Big)\Big] \quad (q \in \mathbb{N}^*), \tag{25}$$

when p goes to infinity. This step relies on Theorem 2.1 in [21], which is a limit theorem for random walks on an affine group and generalizes theorems A and B in [34].
Finally (see Sections 4.3 and 4.4), we prove that the limit of $F_{\lfloor qt\rfloor,q}$, when $q \to \infty$, and $a_F(t)$ both have the same asymptotic behavior when t goes to infinity.

4.1 Discretization of the Lévy process

The following result, which follows from the property of independent and stationary increments of the process K, allows us to concentrate on $A_{p,q}$, which has been defined in (24).

Lemma 8. *Let $t \geq 1$ and $q \in \mathbb{N}^*$. Then*

$$\frac{1}{q} e^{-\beta(|\gamma|/q+\sigma^{(+)}_{1/q})} A^{(1)}_{\lfloor qt \rfloor -1,q} \leq \int_0^t e^{-\beta K_s} \mathrm{d}s \leq \frac{1}{q} e^{\beta(|\gamma|/q+\sigma^{(-)}_{1/q})} A^{(2)}_{\lfloor qt \rfloor,q},$$

where for every $(p,q) \in \mathbb{N} \times \mathbb{N}^*$, $\sigma^{(+)}_{1/q}$ (resp $\sigma^{(-)}_{1/q}$) is independent of $A^{(1)}_{p,q}$ (resp $A^{(2)}_{p,q}$) and

$$A_{p,q} \overset{(d)}{=} A^{(1)}_{p,q} \overset{(d)}{=} A^{(2)}_{p,q}.$$

Proof. Let (p,q) be in $\mathbb{N} \times \mathbb{N}^*$ and $s \in [p/q,(p+1)/q]$. Then

$$K_s \leq K_{p/q} + |\gamma|/q + [\sigma^{(+)}_{(p+1)/q} - \sigma^{(+)}_{p/q}] \quad \text{and} \quad K_s \geq K_{p/q} - |\gamma|/q - [\sigma^{(-)}_{(p+1)/q} - \sigma^{(-)}_{p/q}]. \tag{26}$$

Now introduce

$$K^{(1)}_{p/q} = K_{p/q} + [\sigma^{(+)}_{(p+1)/q} - \sigma^{(+)}_{p/q}] - \sigma^{(+)}_{1/q} = \gamma p/q + [\sigma^{(+)}_{(p+1)/q} - \sigma^{(+)}_{1/q}] - \sigma^{(-)}_{p/q},$$

and

$$K^{(2)}_{p/q} = K_{p/q} - [\sigma^{(-)}_{(p+1)/q} - \sigma^{(-)}_{p/q}] + \sigma^{(-)}_{1/q} = \gamma p/q + \sigma^{(+)}_{p/q} - [\sigma^{(-)}_{(p+1)/q} - \sigma^{(-)}_{1/q}].$$

Then, we have for all $(p,q) \in \mathbb{N} \times \mathbb{N}^*$

$$(K_0, K_{1/q}, ..., K_{p/q}) \overset{(d)}{=} (K^{(1)}_0, K^{(1)}_{1/q}, ..., K^{(1)}_{p/q}) \overset{(d)}{=} (K^{(2)}_0, K^{(2)}_{1/q}, ..., K^{(2)}_{p/q}).$$

Moreover, the random vector $(K^{(1)}_0, K^{(1)}_{1/q}, ..., K^{(1)}_{p/q})$ is independent of $\sigma^{(+)}_{1/q}$ and $(K^{(2)}_0, K^{(2)}_{1/q}, ..., K^{(2)}_{p/q})$ is independent of $\sigma^{(-)}_{1/q}$. Finally, the definition of

$$A^{(i)}_{p,q} = \sum_{i=0}^{p} \exp\{-\beta K^{(i)}_{i/q}\}$$

for $i \in \{1,2\}$ and the inequalities in (26) complete the proof. □

4.2 Asymptotical behavior of the discretized process

First, we recall Theorem 2.1 of [21] in the case where the test functions do not vanish. This is the key result to obtain the asymptotic behavior of the discretized process.

Theorem 9 (Giuvarc'h, Liu 01). *Let $(a_n, b_n)_{n\geq 0}$ be a $(\mathbb{R}^*_+)^2$-valued sequence of iid random variables such that $\mathbb{E}[\log(a_0)] = 0$. Assume that $b_0/(1-a_0)$ is not constant a.s. and define $A_0 = 1$, $A_n = \prod_{k=0}^{n-1} a_k$ and $B_n = \sum_{k=0}^{n-1} A_k b_k$, for $n \geq 1$. Let η, κ, ξ be three positive numbers such that $\kappa < \xi$, and $\tilde{\phi}$ and $\tilde{\psi}$ be two positive continuous functions on $\mathbb{R}_+$ such that they do not vanish and for a constant $C > 0$ and for every $a > 0$, $b \geq 0$, $b' \geq 0$, we have*

$$\tilde{\phi}(a) \leq Ca^{\kappa}, \quad \tilde{\psi}(b) \leq \frac{C}{(1+b)^{\xi}}, \quad \text{and} \quad |\tilde{\psi}(b) - \tilde{\psi}(b')| \leq C|b-b'|^{\eta}.$$

Moreover, assume that

$$\mathbb{E}\big[a_0^{\kappa}\big] < \infty, \quad \mathbb{E}\big[a_0^{-\eta}\big] < \infty, \quad \mathbb{E}\big[b_0^{\eta}\big] < \infty \quad \text{and} \quad \mathbb{E}\big[a_0^{-\eta} b_0^{-\eta}\big] < \infty.$$

Then there exist two positive constants $c(\tilde{\phi}, \tilde{\psi})$ and $c(\tilde{\psi})$ such that

$$\lim_{n\to\infty} n^{3/2}\mathbb{E}\left[\tilde{\phi}(A_n)\tilde{\psi}(B_n)\right] = c(\tilde{\phi}, \tilde{\psi}) \qquad \text{and} \qquad \lim_{n\to\infty} n^{1/2}\mathbb{E}\left[\tilde{\psi}(B_n)\right] = c(\tilde{\psi}).$$

Let us now state a technical lemma on the tail of function F, useful to get the asymptotical behaviour of the disretized process. Its proof is deferred to Section 6.5 for the convenience of the reader.

Lemma 10. *Assume that F satisfies (23). Then there exist two positive finite constants η and M such that for all (x, y) in $\mathbb{R}_+^2$ and ε in $[0, \eta]$,*

$$\Big|F(x) - C_F x^{-1/\beta}\Big| \leq M x^{-(1+\varepsilon)/\beta}, \tag{27}$$

$$\Big|F(x) - F(y)\Big| \leq M\Big|x^{-1/\beta} - y^{-1/\beta}\Big|. \tag{28}$$

Recall the definitions of $A_{p,q}$ and $F_{p,q}$ in (24) and (25), respectively. The three following lemmas study the asymptotic behavior of $F_{p,q}$ and the mean value of $(A_{p,q}/q)^{-1/\beta}$ in the regimes of (ii), (iii) and b/.

Lemma 11. *Assume that $|\phi_K'(0+)| < \infty$, $\theta_{max} > 1$ and $\phi_K'(1) = 0$. Then there exists a positive and finite constant $c_2(q)$ such that,*

$$F_{p,q} \sim C_F c_2(q)(p/q)^{-1/2} e^{(p/q)\phi_K(1)}, \quad \text{as} \quad p \to \infty, \tag{29}$$

and

$$\mathbb{E}\left[(A_{p,q}/q)^{-1/\beta}\right] \sim c_2(q)(p/q)^{-1/2} e^{(p/q)\phi_K(1)}, \quad \text{as} \quad p \to \infty. \tag{30}$$

Proof. Let us introduce the exponential change of measure known as the Escheer transform

$$\frac{\mathrm{d}\mathbb{P}^{(\lambda)}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{\lambda K_t - \phi_K(\lambda)t} \qquad \text{for } \lambda \in [0, \theta_{max}), \tag{31}$$

where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration generated by K which is naturally completed.

The following equality in law

$$A_{p,q} = e^{-\beta K_{p/q}}\Big(\sum_{i=0}^{p} e^{\beta(K_{p/q} - K_{i/q})}\Big) \overset{(d)}{=} e^{-\beta K_{p/q}}\Big(\sum_{i=0}^{p} e^{\beta K_{i/q}}\Big),$$

leads to $e^{-(p/q)\phi_K(1)}\mathbb{E}\left[A_{p,q}^{-1/\beta}\right] = \mathbb{E}^{(1)}\left[\tilde{A}_{p,q}^{-1/\beta}\right]$, where $\tilde{A}_{p,q} = \sum_{i=0}^{p} e^{\beta K_{i/q}}$. Let $\varepsilon > 0$ be such that (27) holds and observe that $\tilde{A}_{p,q} \geq 1$ a.s. for every (p, q) in $\mathbb{N} \times \mathbb{N}^*$. Thus,

$$\mathbb{E}^{(1)}\left[\tilde{A}_{p,q}^{-(1+\varepsilon)/\beta}\right] \leq \mathbb{E}^{(1)}\left[\tilde{A}_{p,q}^{-1/\beta}\right] \leq \mathbb{E}^{(1)}\left[\inf_{i\in[0,p]\cap\mathbb{N}} e^{-K_{i/q}}\right].$$

Since $\phi_K'(1) = 0$ and $\mathbb{E}[K_{1/q}^2] < \infty$, Theorem A in [26] implies

$$\mathbb{E}^{(1)}\left[\inf_{i\in[0,p]\cap\mathbb{N}} e^{-K_{i/q}}\right] \sim \hat{C}_q (p/q)^{-1/2}, \quad \text{as} \quad p \to \infty,$$

where $\hat{C}_q$ is a finite positive constant. We define for $z \geq 1$,

$$D_q(z,p) = (p/q)^{1/2}\mathbb{E}^{(1)}\left[\tilde{A}_{p,q}^{-z/\beta}\right].$$

Moreover, we note that there exists $p_0 \in \mathbb{N}$ such that for $p \geq p_0$, $D_q(1,p) \leq 2\hat{C}_q$.

Our aim is to prove that $D_q(1,p)$ converges, as p increases, to a finite positive constant $d_2(q)$. Then, we introduce an arbitrary $x \in (0, (C_F/M)^{1/\varepsilon} q^{-1/\beta})$ and apply Theorem 9 with

$$\tilde{\psi}(z) = F(z), \quad \tilde{\phi}(z) = z^{1/(2\beta)}, \qquad (\eta, \kappa, \xi) = (1, 1/(2\beta), 1/\beta).$$

Observe that F is a Lipschitz function and that under the probability measure $\mathbb{P}^{(1)}$, $(a_n, b_n)_{n\geq 0} = (\exp(\beta(K_{(n+1)/q} - K_{n/q})), x^{-\beta}q^{-1})_{n\geq 0}$ is an i.i.d. sequence of random variables with $\mathbb{E}^{(1)}[\log(a_0)] = 0$, since $\phi_K'(1) = 0$. Moreover, a simple computation gives

$$\mathbb{E}^{(1)}[a_0^{-1}] = e^{(\phi_K(1-\beta) - \phi_K(1))/q} < \infty,$$

so that the moment conditions of Theorem 9 are satisfied. We apply the result with

$$B_n = q^{-1}x^{-\beta}\sum_{i=0}^{n-1} e^{\beta K_{i/q}}, \quad n \in \mathbb{N}^*$$

and we get the existence of a positive finite real number $b(q,x)$ such that

$$(p/q)^{1/2}\mathbb{E}^{(1)}\left[F\Big(x^{-\beta}\tilde{A}_{p,q}/q\Big)\right] \to b(q,x), \quad \text{as} \quad p \to \infty.$$

Taking expectation in (27) yields

$$\left|(p/q)^{1/2}\mathbb{E}^{(1)}\left[F\Big(x^{-\beta}\tilde{A}_{p,q}/q\Big)\right] - C_F x q^{1/\beta} D_q(1,p)\right| \leq M x^{1+\varepsilon} q^{(1+\varepsilon)/\beta} D_q(1+\varepsilon, p). \tag{32}$$

Defining $\underline{D}_q := \liminf_{p\to\infty} D_q(1,p)$ and $\overline{D}_q := \limsup_{p\to\infty} D_q(1,p)$, we combine the two last dispalys to get

$$C_F x q^{1/\beta}\overline{D}_q \leq b(q,x) + M x^{1+\varepsilon} q^{(1+\varepsilon)/\beta} \limsup_{p\to\infty} D_q(1+\varepsilon, p),$$

and

$$C_F x q^{1/\beta}\underline{D}_q \geq b(q,x) - M x^{1+\varepsilon} q^{(1+\varepsilon)/\beta} \limsup_{p\to\infty} D_q(1+\varepsilon, p).$$

Adding that $D_q(z,p)$ is non-increasing with respect to z, $D_q(1+\varepsilon,p) \leq D_q(1,p) \leq 2\hat{C}_q$ for every $p \geq p_0$ and

$$\overline{D}_q - \underline{D}_q \leq \frac{4M\hat{C}_q x^{\varepsilon} q^{\varepsilon/\beta}}{C_F}.$$

Finally, letting $x \to 0$, we get that $D_q(1,p)$ converges to a finite constant $d_2(q)$. Moreover, from (32), we get for every integer p:

$$(C_F x q^{1/\beta} + M x^{1+\varepsilon} q^{(1+\varepsilon)/\beta}) D_q(1,p) \geq (p/q)^{1/2}\mathbb{E}^{(1)}\left[F\Big(x^{-\beta}\tilde{A}_{p,q}/q\Big)\right].$$

Letting $p \to \infty$, we get that $d_2(q)$ is positive, which gives (30).

Now, using (27), we get

$$\mathbb{E}\Big|F_{p,q} - C_F\,(A_{p,q}/q)^{-1/\beta}\Big| \le \mathbb{E}\left[(A_{p,q}/q)^{-(1+\varepsilon)/\beta}\right],$$

so the asymptotic behavior in (29) will be proved as soon as we show that

$$\mathbb{E}\Big[A_{p,q}^{-(1+\varepsilon)/\beta}\Big] = o\Big(\mathbb{E}\left[A_{p,q}^{-1/\beta}\right]\Big), \quad \text{as} \quad p\to\infty.$$

From the Escheer transform (31), with $\lambda = 1+\varepsilon$, and the independence of the increments of K, we have

$$\begin{aligned}
\mathbb{E}\Big[A_{p,q}^{-(1+\varepsilon)/\beta}\Big] &= e^{(p/q)\phi_K(1)}\mathbb{E}^{(1)}\Big[\Big(\sum_{i=0}^{p} e^{-\beta K_{i/q}}\Big)^{-\varepsilon/\beta}\Big(\sum_{i=0}^{p} e^{\beta(K_{p/q}-K_{i/q})}\Big)^{-1/\beta}\Big]\\
&\le e^{(p/q)\phi_K(1)}\mathbb{E}^{(1)}\Big[\inf_{0\le i\le\lfloor p/3\rfloor} e^{\varepsilon K_{i/q}} \inf_{\lfloor 2p/3\rfloor\le j\le p} e^{-(K_{p/q}-K_{j/q})}\Big]\\
&= e^{(p/q)\phi_K(1)}\mathbb{E}^{(1)}\Big[\inf_{0\le i\le\lfloor p/3\rfloor} e^{\varepsilon K_{i/q}}\Big]\mathbb{E}^{(1)}\Big[\inf_{0\le j\le\lfloor p/3\rfloor} e^{-K_{j/q}}\Big].
\end{aligned}$$

Using (22), we observe that $\mathbb{E}^{(1)}[K_{1/q}] = 0$ and $\mathbb{E}^{(1)}[K_{1/q}^2] < \infty$. We can then apply Theorem A in [26] to the random walks $(-K_{i/q})_{i\ge1}$ and $(\varepsilon K_{i/q})_{i\ge1}$. Therefore, there exists $C(q)>0$ such that

$$\mathbb{E}\Big[A_{p,q}^{-(1+\varepsilon)/\beta}\Big] \le (C(q)/p)e^{(p/q)\phi_K(1)} = o\Big(\mathbb{E}\left[A_{p,q}^{-1/\beta}\right]\Big), \quad \text{as} \quad p\to\infty.$$

Taking $c_2(q) = d_2(q)q^{1/\beta}$ leads to the result. □

Remark 2. In the particular case when $\beta=1$, it is enough to apply Theorem 1.2 in [18] to a geometric BPRE $(X_n, n\ge 0)$ whose p.g.f's satisfy

$$f_n(s) = \sum_{k=0}^{\infty} p_n q_n^k s^k = \frac{p_n}{1-q_n s},$$

with $1/p_n = 1+\exp\left\{\beta\left(K_{(n+1)/q} - K_{n/q}\right)\right\}$, and $q_n = 1-p_n$. Using $\mathbb{E}[A_{p,q}^{-1}] = \mathbb{P}(X_p>0)$ and $\log f_0'(1) = K_{1/q}$, allows to get the asymptotic behavior of $\mathbb{E}[A_{p,q}^{-1}]$ from the speed of extinction of BPRE in the case of geometric reproduction law (with the extra assumption $\phi_K(2)<\infty$).

Recall that τ is the root of ϕ_K' on $]0,1[$, i.e. $\phi_K(\tau) = \min_{0<s<1}\phi_K(s)$.

Lemma 12. *Assume that $\phi_K'(0)<0$, $\phi_K'(1)>0$ and $\theta_{max} > \beta+1$. Then there exist two positive constants $d(q)$ and $c_3(q)$ such that*

$$F_{p,q} \;\sim\; c_3(q)(p/q)^{-3/2}e^{(p/q)\phi_K(\tau)}, \quad \text{as} \quad p\to\infty, \tag{33}$$

and

$$\mathbb{E}\left[(A_{p,q}/q)^{-1/\beta}\right] \;\sim\; d(q)(p/q)^{-3/2}e^{(p/q)\phi_K(\tau)}, \quad \text{as} \quad p\to\infty. \tag{34}$$

Proof. First we apply Theorem 9 where, for $z \geq 0$,

$$\tilde{\psi}(z) = F(z), \qquad \tilde{\phi}(z) = z^{\tau/\beta}, \qquad (\eta, \kappa, \xi) = (1, \tau/\beta, 1/\beta).$$

Again F is a Lipschitz function, and under the probability measure $\mathbb{P}^{(\tau)}$, $(a_n, b_n)_{n\geq 0} = (\exp(-\beta(K_{(n+1)/q} - K_{n/q})), q^{-1})_{n\geq 0}$, is an i.i.d. sequence of random variables such that $\mathbb{E}^{(\tau)}[\log(a_0)] = 0$, since $\phi_K'(\tau) = 0$. The moment conditions

$$\mathbb{E}^{(\tau)}\left[a_0^{\tau/\beta}\right] = e^{-\phi_K(\tau)/q} < \infty \qquad \text{and} \qquad \mathbb{E}^{(\tau)}\left[a_0^{-1}\right] = e^{(\phi_K(\beta+\tau)-\phi_K(\tau))/q} < \infty,$$

enable us to apply Theorem 9. In this case,

$$B_n = q^{-1}\sum_{i=0}^{n-1} e^{-\beta K_{i/q}}, \quad n \in \mathbb{N}^*.$$

Then there exists $c_3(q) > 0$ such that

$$\mathbb{E}\left[F(A_{p,q}/q)\right] e^{-(p/q)\phi_K(\tau)} = \mathbb{E}^{(\tau)}\left[F(A_{p,q}/q)e^{-\tau K_{p/q}}\right] \sim \;\; c_3(q)(p/q)^{-3/2},$$

as $p \to \infty$. This gives (33).

To prove

$$\mathbb{E}\left[(A_{p,q}/q)^{-1/\beta}\right] \sim d(q)(p/q)^{-3/2}e^{\frac{p}{q}\phi_K(\tau)}, \quad \text{as} \quad p \to \infty$$

for $d(q) > 0$, we follow the same arguments as those used in the proof of Lemma 11. In other words, we define for $z \geq 1$,

$$D_q(z, p) = (p/q)^{3/2}e^{-(p/q)\phi_K(\tau)}\mathbb{E}\left[A_{p,q}^{-z/\beta}\right],$$

which is non-increasing with respect to z. We obtain the same type of inequalities as in Lemma 11, for the random variable A instead of $\tilde{A}$.

Again we take $\varepsilon > 0$ such that (27) holds. Then Lemma 7 in [22] yields the existence of $C_q > 0$ such that for p large enough,

$$\mathbb{E}\left[A_{p,q}^{-(1+\varepsilon)/\beta}\right] \leq \mathbb{E}\left[A_{p,q}^{-1/\beta}\right] \leq \mathbb{E}\left[\inf_{i\in[0,p]\cap\mathbb{N}} e^{-K_{i/q}}\right] \sim C_q(p/q)^{-3/2}e^{(p/q)\phi_K(\tau)}.$$

Finally, we use Theorem 9 to get $0 < \liminf_{n\to\infty} D_q(1, n) = \limsup_{n\to\infty} D_q(1, n) < \infty$, which completes the proof. □

Lemma 13. *Assume that $\phi_K'(0) = 0$ and $\theta_{max} > \beta$. Then there exist two positive constants $b(q)$ and $c_4(q)$ such that*

$$F_{p,q} \sim c_4(q)(p/q)^{-1/2}, \quad \textit{as} \quad p \to \infty, \tag{35}$$

and

$$\mathbb{E}\left[(A_{p,q}/q)^{-1/\beta}\right] \sim b(q)(p/q)^{-1/2}, \quad \textit{as} \quad p \to \infty. \tag{36}$$

Proof. The proof is almost the same as the proof of Lemma 12. We first apply Theorem 9 to the same function $\tilde{\psi}$ and sequence $(a_n, b_n)_{n\geq 0}$ defined in Lemma 12 but under the probability measure $\mathbb{P}$ instead of $\mathbb{P}^{(\tau)}$. Then, we get

$$\mathbb{E}\Big[F(A_{p,q}/q)\Big] \sim c_4(q)(p/q)^{-1/2}, \quad \text{as} \quad p \to \infty.$$

Now, we define for $z \geq 1$,

$$D_q(z,p) = (p/q)^{1/2}\mathbb{E}\Big[A_{p,q}^{-z/\beta}\Big],$$

and from Theorem A in [26] and Theorem 9, we obtain that $D_q(1,p)$ has a positive finite limit when p goes to infinity. □

4.3 From the discretized process to the continuous process

Up to now, the asymptotic behavior of the processes was depending on the step size $1/q$. By letting q tend to infinity, we obtain our results in continuous time. To do this we shall use several times a technical Lemma on limits of sequences.

Lemma 14. *Assume that the non-negative sequences $(a_{n,q})_{(n,q)\in\mathbb{N}^2}$, $(a'_{n,q})_{(n,q)\in\mathbb{N}^2}$ and $(b_n)_{n\in\mathbb{N}}$ satisfy for every $(n,q)\in\mathbb{N}^2$:*

$$a_{n,q} \leq b_n \leq a'_{n,q},$$

and that there exist three sequences $(a(q))_{q\in\mathbb{N}}$, $(c^-(q))_{q\in\mathbb{N}}$ and $(c^+(q)_{q\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} a_{n,q} = c^-(q)a(q), \quad \lim_{n\to\infty} a'_{n,q} = c^+(q)a(q), \quad \textit{and} \quad \lim_{q\to\infty} c^-(q) = \lim_{q\to\infty} c^+(q) = 1.$$

Then there exists a non-negative constant a such that

$$\lim_{q\to\infty} a(q) = \lim_{n\to\infty} b_n = a.$$

Proof. From our assumptions, it is clear that for every $q\in\mathbb{N}$

$$\limsup_{n\to\infty} b_n \leq c^+(q)a(q) \quad \text{and} \quad c^-(q)a(q) \leq \liminf_{n\to\infty} b_n.$$

Then letting q go to infinity, we obtain

$$\limsup_{n\to\infty} b_n \leq \liminf_{q\to\infty} a(q) \quad \text{and} \quad \limsup_{q\to\infty} a(q) \leq \liminf_{n\to\infty} b_n,$$

which ends the proof. □

Recalling the notations (29) to (36), we prove the following limits :

Lemma 15. *There exist five finite positive constants b, d, c_2, c_3 and c_4 such that*

$$(b(q), d(q), c_2(q), c_3(q), c_4(q)) \longrightarrow (b, d, c_2, c_3, c_4), \quad \textit{as} \quad q\to\infty. \tag{37}$$

Proof. First we prove the convergence of $d(q)$. From Lemma 8, we know that for every $n \in \mathbb{N}^*$

$$e^{\frac{\phi_K^-(1)-|\gamma|}{q}} \mathbb{E}\Big[\Big(A_{nq,q}/q\Big)^{-1/\beta}\Big] \le \mathbb{E}\Big[\Big(\int_0^n e^{-\beta K_u} \mathrm{d}u\Big)^{-1/\beta}\Big] \le e^{\frac{\phi_K^+(1)+|\gamma|}{q}} \mathbb{E}\Big[\Big(A_{nq-1,q}/q\Big)^{-1/\beta}\Big]. \tag{38}$$

A direct application of Lemma 14 with

$$a(q) = d(q), \quad c^-(q) = e^{(\phi_K^-(1)-|\gamma|)/q}, \quad \text{and} \quad c^+(q) = e^{(\phi_K^+(1)+|\gamma|)/q},$$

yields that $d(q)$ converges as $q \to \infty$ to a finite non-negative constant d. Let us now prove that d is positive. Let (q_1, q_2) be in $\mathbb{N}^2$. According to (34) and (38) there exists $n \in \mathbb{N}$ such that

$$0 < e^{\frac{\phi_K^-(1)-|\gamma|}{q_1}} d(q_1)/2 \le n^{3/2} e^{-n\phi_K(\tau)} \mathbb{E}\Big[\Big(\int_0^n e^{-\beta K_u} \mathrm{d}u\Big)^{-1/\beta}\Big] \le 2e^{\frac{\phi_K^+(1)+|\gamma|-\phi_K(\tau)}{q_2}} d(q_2).$$

Letting q_2 go to infinity, we conclude that $\liminf_{q\to\infty} d(q) > 0$. Similar arguments imply the convergence of $b(q)$ to a positive constant.

Now, we prove the convergence of $c_2(q)$, $c_3(q)$ and $c_4(q)$. Again the proofs of the three cases are very similar, so we only prove the second one. From Lemmas 8 and 12, we know that for every $(n, q) \in \mathbb{N}^2$,

$$\mathbb{E}\left[F\Big(e^{\beta(|\gamma|/q+\sigma_{1/q}^{(-)})} A_{nq,q}/q\Big)\right] \le a_F(n) \le \mathbb{E}\left[F\Big(e^{-\beta(|\gamma|/q+\sigma_{1/q}^{(+)})} A_{nq-1,q}/q\Big)\right].$$

Using (28), we obtain

$$\begin{aligned} F_{nq,q} + M\mathbb{E}\left[e^{-|\gamma|/q-\sigma_{1/q}^{(-)}} - 1\right] \mathbb{E}\left[\Big(\frac{A_{nq,q}}{q}\Big)^{-\frac{1}{\beta}}\right] & \\ \le a_F(n) \le & \\ F_{nq-1,q} + M\mathbb{E}\left[e^{|\gamma|/q+\sigma_{1/q}^{(+)}} - 1\right] \mathbb{E}\left[\Big(\frac{A_{nq-1,q}}{q}\Big)^{-\frac{1}{\beta}}\right]. \end{aligned}$$

Thus, dividing by $n^{-3/2}\exp(n\phi_K(\tau))$ in the above inequality, we get the convergence using Lemmas 12, 14 and Equation (28) with

$$a(q) = c_3(q), \quad c^-(q) = 1 - \frac{Md(q)(e^{(\phi_K^-(1)-|\gamma|)/q} - 1)}{c_3(q)} \quad c^+(q) = 1 + \frac{Md(q)(e^{(\phi_K^+(1)+|\gamma|)/q} - 1)}{c_3(q)}.$$

We then prove that $\lim_{q\to\infty} c_3(q)$ is positive using similar arguments as previously. □

4.4 Proof of Theorem 7

Proof of Theorem 7 a/ (i). Recall from Lemma II.2 in [6] that the process $(K_t - K_{(t-s)^-}, 0 \le s \le t)$ has the same law as $(K_s, 0 \le s \le t)$. Then

$$\int_0^t e^{-\beta K_s} ds = \int_0^t e^{-\beta K_{(t-s)}} ds = e^{-\beta K_t} \int_0^t e^{\beta K_t - \beta K_{(t-s)}} ds \overset{(d)}{=} e^{-\beta K_t} \int_0^t e^{\beta K_s} ds.$$

We first note that for every $q \in \mathbb{N}^*$ and $t \geq 2/q$, Lemma 8 leads to

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t e^{-\beta K_s} ds\right)^{-1/\beta}\right] &\leq \mathbb{E}\left[\left(\int_0^{2/q} e^{-\beta K_s} ds\right)^{-1/\beta}\right] \\
&\leq q^{1/\beta} e^{|\gamma|/q} \mathbb{E}\Big(e^{\sigma^{(+)}_{1/q}} (A^{(1)}_{1,q})^{-1/\beta}\Big) \\
&\leq q^{1/\beta} \exp\Big(\frac{\phi_K(1) + |\gamma| + \phi_K^+(1)}{q}\Big) < \infty,
\end{aligned}$$

where ϕ_K^+ was defined in (20). Hence using (31), with $\lambda = 1$, we have

$$\mathbb{E}\left[\left(\int_0^t e^{-\beta K_s} ds\right)^{-1/\beta}\right] = \mathbb{E}\left[e^{K_t}\left(\int_0^t e^{\beta K_s} ds\right)^{-1/\beta}\right] = e^{t\phi_K(1)} \mathbb{E}^{(1)}\left[\left(\int_0^t e^{\beta K_s} ds\right)^{-1/\beta}\right].$$

The above identity implies that the decreasing function $t \mapsto \mathbb{E}^{(1)}[(\int_0^t e^{\beta K_s} ds)^{-1/\beta}]$ is finite for all $t > 0$. So it converges to a non-negative and finite limit c_1, as t increases. This limit is positive, since under the probability $\mathbb{P}^{(1)}$, K is still a Lévy process with negative mean $\mathbb{E}^{(1)}(K_1) = \phi'_K(1)$ and according to Theorem 1 in [7], we have

$$\int_0^\infty e^{\beta K_s} ds < \infty, \qquad \mathbb{P}^{(1)}\text{-a.s.}$$

Hence, we only need to prove

$$a_F(t) \quad \sim \quad C_F \mathbb{E}\Big[\Big(\int_0^t e^{-\beta K_s} ds\Big)^{-1/\beta}\Big], \quad \text{as} \quad t \to \infty. \tag{39}$$

Recall that $\theta_{max} > 1$ and $\phi'_K(1) < 0$. So we can choose $\varepsilon > 0$ such that (27) holds, $1 + \varepsilon < \theta_{max}$, $\phi_K(1+\varepsilon) < \phi_K(1)$ and $\phi'_K(1+\varepsilon) < 0$. Therefore

$$\left| F\left(\int_0^t e^{-\beta K_s} ds\right) - C_F \left(\int_0^t e^{-\beta K_s} ds\right)^{-1/\beta}\right| \leq M \left(\int_0^t e^{-\beta K_s} ds\right)^{-(1+\varepsilon)/\beta}.$$

In other words, it is enough to show

$$\mathbb{E}\left[\left(\int_0^t e^{-\beta K_s} ds\right)^{-(1+\varepsilon)/\beta}\right] = o(e^{t\phi_K(1)}), \quad \text{as} \quad t \to \infty.$$

From the Escheer transform (31), with $\lambda = 1 + \varepsilon$, we deduce

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t e^{-\beta K_s} ds\right)^{-(1+\varepsilon)/\beta}\right] &= \mathbb{E}\left[e^{(1+\varepsilon)K_t}\left(\int_0^t e^{\beta K_s} ds\right)^{-(1+\varepsilon)/\beta}\right] \\
&= e^{t\phi_K(1+\varepsilon)} \mathbb{E}^{(1+\varepsilon)}\left[\left(\int_0^t e^{\beta K_s} ds\right)^{-(1+\varepsilon)/\beta}\right].
\end{aligned}$$

Again from Lemma 8, we obtain for $t \geq q/2$,

$$\mathbb{E}\left[\left(\int_0^t e^{-\beta K_s} ds\right)^{-\frac{1+\varepsilon}{\beta}}\right] \leq q^{(1+\varepsilon)/\beta} \exp\Big(\frac{\phi_K(1+\varepsilon) + |\gamma|(1+\varepsilon) + \phi_K^+(1+\varepsilon)}{q}\Big) < \infty,$$

implying that the decreasing function $t \mapsto \mathbb{E}^{(1+\varepsilon)}[(\int_0^t \exp(\beta K_s) ds)^{-(1+\varepsilon)/\beta}]$ is finite for all $t > 0$. This completes the proof. □

Remark 3. In the particular case when $\beta = 1$, it is enough to apply Theorem 1.1 in [18] to the geometric BPRE $(X_n, n \geq 0)$ defined in Remark 2 to get the result.

Proof of Theorem 7 a/ (ii), (iii), and b/. The proofs are very similar for the three regimes, for this reason we only focus on the proof of the regime in a/(iii).

Let $\varepsilon > 0$. Thanks to Lemma 15, we can choose $q \in \mathbb{N}^*$ such that $q \geq 1/\varepsilon$ and $(1-\varepsilon)c_3 \leq c_3(q) \leq (1+\varepsilon)c_3$. Then for every $t \geq 1$, the monotonicty of F yields

$$\mathbb{E}\Big[F(C_{\lfloor qt\rfloor,q}e^{\beta|\gamma|/q}/q)\Big] \leq a_F(t) \leq \mathbb{E}\Big[F(D_{\lfloor qt\rfloor-1,q}e^{-\beta|\gamma|/q}/q)\Big].$$

Applying (28), we obtain :

$$\Big|\mathbb{E}\Big[F(C_{\lfloor qt\rfloor,q}e^{\beta|\gamma|/q}/q)\Big] - F_{\lfloor qt\rfloor,q}\Big| \leq (1-e^{-\varepsilon(|\gamma|-\phi_K^-(1))})M\mathbb{E}\Big[(A_{\lfloor qt\rfloor,q}/q)^{-1/\beta}\Big],$$
$$\Big|\mathbb{E}\Big[F(D_{\lfloor qt\rfloor-1,q}e^{-\beta|\gamma|/q}/q)\Big] - F_{\lfloor qt-1\rfloor,q}\Big| \leq (e^{\varepsilon(|\gamma|+\phi_K^+(1))}-1)M\mathbb{E}\Big[(A_{\lfloor qt\rfloor-1,q}/q)^{-1/\beta}\Big].$$

Taking t to infinity, it is clear from Lemma 12 that both terms are bounded by

$$l(\varepsilon)t^{-3/2}e^{t\phi_K(\tau)} = \Big[2Md(e^{\varepsilon(|\gamma|+\phi_K^+(1))} - e^{-\varepsilon(|\gamma|-\phi_K^-(1))})e^{-\varepsilon\phi_K(\tau)}\Big]t^{-3/2}e^{t\phi_K(\tau)} \tag{40}$$

where ϕ_K^- and ϕ_K^+ are defined in (20), and $l(\varepsilon)$ goes to 0 when ε decreases. On the other hand, for t large enough

$$(1-2\varepsilon)c_3t^{-3/2}e^{t\phi_K(\tau)} \leq F_{\lfloor qt\rfloor,q} \leq a_F(t) \leq F_{\lfloor qt\rfloor-1,q} \leq (1+2\varepsilon)c_3t^{-3/2}e^{t\phi_K(\tau)},$$

which completes the proof of Theorem 7. □

5 Application to a cell division model

When the reproduction law has a finite second moment, the scaling limit of the GW process is a Feller diffusion with growth g and diffusion part σ^2. That is to say, the stable case with $\beta = 1$ and additional drift term g. Such a process is also the scaling limit of birth and death processes. It gives a natural model for populations which die and multiply fast, randomly, without interaction. Such a model is considered in [5] for parasites growing in dividing cells. The cell divides at constant rate r and a random fraction $\Theta \in (0,1)$ of parasites enters the first daughter cell, whereas the remainder enters the second daughter cell. Following the infection in a cell line, the parasites grow as a Feller diffusion process and undergo a catastrophe when the cell divides. We denote by N_t and N_t^* the numbers of cells and infected cells at time t, respectively. We say that the cell population recovers when the asymptotic proportion of contaminated cells vanishes. If there is one infected cell at time 0, $\mathbb{E}[N_t] = e^{rt}$ and $\mathbb{E}[N_t^*] = e^{rt}\mathbb{P}(Y_t > 0)$, where

$$Y_t = 1 + \int_0^t gY_s ds + \int_0^t \sqrt{2\sigma^2 Y_s} dB_s + \int_0^t \int_0^1 (\theta - 1)Y_{s_-}\rho(ds, d\theta). \tag{41}$$

Here B is a Brownian motion and $\rho(ds, d\theta)$ a Poisson random measure with intensity $2rds\mathbb{P}(\Theta \in d\theta)$. Note that the intensity of ρ is twice the cell division rate. This bias follows from the fact that if we pick an individual at random at time t, we are more likely to choose a lineage in which many division events have occurred. Hence the ancestral lineages from typical individuals at time t have a division rate $2r$.

Corollary 2 and Proposition 5 with $\beta = 1$, $\psi(\lambda) = -g\lambda + \sigma^2\lambda$ and $\nu(dx) = 2r\mathbb{P}(\Theta \in dx)$ imply the following result.

Corollary 16. *a/ We assume that $g < 2r\mathbb{E}[\log(1/\Theta)]$. Then there exist positive constants c_1, c_2, c_3 such that*

(i) If $g < 2r\mathbb{E}[\Theta\log(1/\Theta)]$, then

$$\mathbb{E}[N_t^*] \sim c_1 e^{gt}, \quad as \quad t \to \infty.$$

(ii) If $g = 2r\mathbb{E}[\Theta\log(1/\Theta)]$, then

$$\mathbb{E}[N_t^*] \sim c_2 t^{-1/2} e^{gt}, \quad as \quad t \to \infty.$$

(iii) If $g > 2r\mathbb{E}[\Theta\log(1/\Theta)]$, then

$$\mathbb{E}[N_t^*] \sim c_3 t^{-3/2} e^{\alpha t}, \quad as \quad t \to \infty.$$

where $\alpha = \min_{\lambda\in[0,1]}\{g\lambda + 2r(\mathbb{E}[\Theta^\lambda] - 1/2)\} < g$.

b/ We now assume $g = 2r\mathbb{E}[\log(1/\Theta)]$, then there exists $c_4 > 0$ such that,

$$\mathbb{E}[N_t^*] \sim c_4 t^{-1/2} e^{rt}, \quad as \quad t \to \infty.$$

c/ Finally, if $g > 2r\mathbb{E}[\log(1/\Theta)]$, then there exists $0 < c_5 < 1$ such that,

$$\mathbb{E}[N_t^*] \sim c_5 e^{rt}, \quad as \quad t \to \infty.$$

Hence if $g > 2r\mathbb{E}[\log(1/\Theta)]$ (supercritical case c/), the mean number of infected cells is equivalent to the mean number of cells. In the critical case (b/), there are somewhat fewer infected cells, owing to the additional square root term. In the strongly subcritical regime (a/ (i)), the mean number of infected cells is of the same order as the number of parasites. This suggests that parasites do not accumulate in some infected cells. The asymptotic behavior in the two remaining cases is more complex.

We stress the fact that fixing the growth rate g of parasites and the cell division rate r, but making the law of the repartition Θ vary, it changes the asymptotic behavior of the number of infected cells. For example, if we focus on random variables Θ satisfying $\mathbb{P}(\Theta = \theta) = \mathbb{P}(\Theta = 1 - \theta) = 1/2$ for a given $\theta \in]0, 1/2[$, the different regimes can be described easily (see Figure 1).

If $g/r > \log 2$, the cell population either recovers or not, depending on the asymmetry of the parasite sharing. If $g/r \le \log 2/2$, the cell population recovers but the speed of recovery increases with respect to the asymmetry of the parasite sharing, as soon as the weakly subcritical regime is reached. Such phenomena were known in the discrete time, discrete space framework (see [2]), but the boundaries between the regimes are not the same, due to the bias in division rate in the continuous setting. Moreover, we note that if $g/r \in (\log 2/2, \log 2)$, then parasites are in the weakly subcritical regime whatever the distribution of Θ on $]0, 1[$. This phenomenon also only occurs in the continuous setting.

6 Auxiliary results

This section is devoted to the technical results which are necessary for the previous proofs.

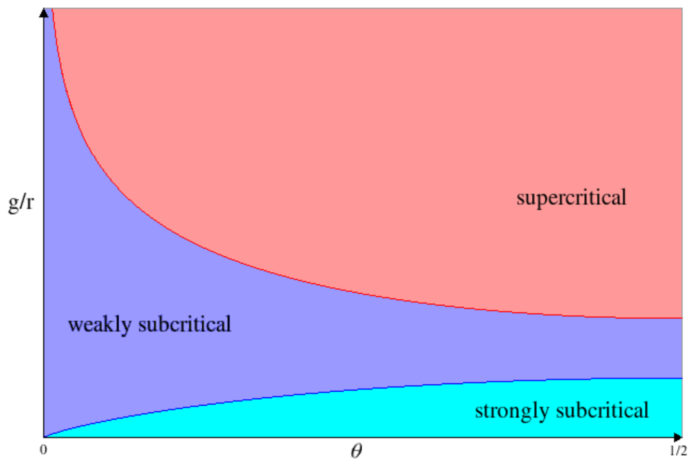

Figure 1: Extinction regimes in the case $\mathbb{P}(\Theta = \theta) = \mathbb{P}(\Theta = 1 - \theta) = 1/2$. Boundaries between the different regimes are given by $g/r = -\log(\theta(1-\theta))$ (supercritical and subcritical) and $g/r = -\theta \log \theta - (1-\theta)\log(1-\theta)$ (strongly and weakly subcritical).

6.1 Existence and uniqueness of the backward ordinary differential equation

The Laplace exponent of $\widetilde{Z}$ in Theorem 1 is the solution of a backward ODE. The existence and uniqueness of this latter are stated and proved below.

Proposition 17. *Let δ be in $\mathcal{BV}(\mathbb{R}^+)$. Then the backward ordinary differential equation (7) admits a unique solution.*

The proof relies on a classical approximation of the solution of (7) and the Cauchy-Lipschitz Theorem. When there is no accumulation of jumps, the latter provides the existence and uniqueness of the solution between two successive jump times of δ. The problem remains on the times where accumulation of jumps occurs. Let us define the family of functions δ^n by deleting the small jumps of δ,

$$\delta_t^n = \delta_t - \sum_{s \leq t} \Big(\delta_s - \delta_{s-}\Big) \mathbf{1}_{\{|\delta_s - \delta_{s-}| < 1/n\}}.$$

We note that ψ_0 is continuous, and $s \mapsto e^{gs+\delta_s^n}$ is piecewise C^1 on $\mathbb{R}^+$ with a finite number of discontinuities. From the Cauchy-Lipschitz Theorem, for every $n \in \mathbb{N}^*$ we can define a solution $v_t^n(., \lambda, \delta)$ continuous with càdlàg first derivative of the backward differential equation:

$$\frac{\partial}{\partial s} v_t^n(s, \lambda, \delta) = e^{gs+\delta_s^n} \psi_0\big(e^{-gs-\delta_s^n} v_t^n(s, \lambda, \delta)\big), \quad 0 \leq s \leq t, \qquad v_t^n(t, \lambda, \delta) = \lambda.$$

We want to show that the sequence $(v_t^n(., \lambda, \delta))_{n \geq 1}$ converges to a function $v_t(., \lambda, \delta)$ which is solution of (7). This follows from the next result. We fix $t > 0$ and define

$$S := \sup_{s \in [0,t], n \in \mathbb{N}^*} \Big\{ e^{gs+\delta_s^n}, e^{-gs-\delta_s^n} \Big\}. \tag{42}$$

Lemma 18. *For every $\lambda > 0$, there exists a positive finite constant C such that for all $0 \leq \eta \leq \kappa \leq \lambda S$,*

$$0 \leq \psi_0(\kappa) - \psi_0(\eta) \leq C(\kappa - \eta). \tag{43}$$

Proof. First, we observe that S is finite and that for all $0 \leq \eta < \kappa \leq \lambda S$, we have $0 \leq e^{-\kappa x} - e^{-\eta x} + (\kappa - \eta)x \leq (\kappa - \eta)x$ for $x \geq 0$ since $x \mapsto e^{-x} + x$ is increasing and $e^{-\kappa x} \leq e^{-\eta x}$. Moreover

$$0 \leq e^{-x} - 1 + x \leq x \wedge x^2, \tag{44}$$

and combining these inequalities yields

$$\begin{aligned}
&\psi_0(\kappa) - \psi_0(\eta) \\
&\quad = \sigma^2(\kappa^2 - \eta^2) + \int_1^\infty \Big(e^{-\kappa x} - e^{-\eta x} + (\kappa - \eta)x\Big)\mu(dx) \\
&\qquad + (\kappa - \eta)\int_0^1 x(1 - e^{-\eta x})\mu(dx) + \int_0^1 \Big(e^{-(\kappa-\eta)x} - 1 + (\kappa - \eta)x\Big)e^{-\eta x}\mu(dx) \\
&\quad \leq \sigma^2(\kappa^2 - \eta^2) + (\kappa - \eta)\int_1^\infty x\mu(dx) + (\kappa - \eta)\eta\int_0^1 x^2\mu(dx) + (\kappa - \eta)^2\int_0^1 x^2\mu(dx) \\
&\quad \leq \Big[2\lambda S\sigma^2 + \int_1^\infty x\mu(dx) + \lambda S\int_0^1 x^2\mu(dx)\Big](\kappa - \eta),
\end{aligned}$$

which proves Lemma 18. □

Next, we prove the existence and uniqueness result.

Proof of Proposition 17. We now prove that $(v_t^n(s, \lambda, \delta), s \in [0, t])_{n \geq 0}$ is a Cauchy sequence. For simplicity, we denote $v^n(s) = v_t^n(s, \lambda, \delta)$, and for all $v \geq 0$:

$$\psi^n(s, v) = e^{gs+\delta_s^n}\psi_0\big(e^{-gs-\delta_s^n}v\big) \quad \text{and} \quad \psi^\infty(s, v) = e^{gs+\delta_s}\psi_0\big(e^{-gs-\delta_s}v\big).$$

We have for any $0 \leq s \leq t$ and $m, n \geq 1$:

$$\begin{aligned}
|v^n(s) - v^m(s)| &= \Big|\int_s^t \psi^n(u, v^n(u))du - \int_s^t \psi^m(u, v^m(u))du\Big| \\
&\leq \int_s^t (R^n(u) + R^m(u))du + \int_s^t \Big|\psi^\infty(u, v^n(u)) - \psi^\infty(u, v^m(u))\Big|du,
\end{aligned} \tag{45}$$

where for any $u \in [0, t]$,

$$\begin{aligned}
R^n(u) &:= \Big|\psi^n(u, v^n(u)) - \psi^\infty(u, v^n(u))\Big| \\
&\leq e^{gu+\delta_u^n}\Big|\psi_0\big(e^{-gu-\delta_u^n}v^n(u)\big) - \psi_0\big(e^{-gu-\delta_u}v^n(u)\big)\Big| + e^{gu}\psi_0\big(e^{-gu-\delta_u}v^n(u)\big)\Big|e^{\delta_u^n} - e^{\delta_u}\Big|.
\end{aligned}$$

Moreover, from (42) to (43), we obtain

$$\begin{aligned}
R^n(u) &\leq SC\lambda\Big|e^{-\delta_u^n} - e^{-\delta_u}\Big| + e^{|g|t}\psi_0(\lambda S)\Big|e^{\delta_u^n} - e^{\delta_u}\Big| \\
&\leq \Big(SC\lambda + e^{|g|t}\psi_0(\lambda S)\Big)\sup_{u \in [0,t]}\Big\{\Big|e^{-\delta_u^n} - e^{-\delta_u}\Big|, \Big|e^{\delta_u^n} - e^{\delta_u}\Big|\Big\} := s_n.
\end{aligned}$$

Using similar arguments as above, we get from (43),

$$\Big|\psi^\infty(u, v^n(u)) - \psi^\infty(u, v^m(u))\Big| \leq CS^2\Big|v^n(u) - v^m(u)\Big|.$$

From (45), we use Gronwall's Lemma (see e.g. Lemma 3.2 in [14]) with

$$R_{m,n}(s) = \int_s^t R^n(u)du + \int_s^t R^m(u)du,$$

to deduce that for all $0 \leq s \leq t$,

$$|v^n(s) - v^m(s)| \leq R_{m,n}(s) + CS^2 e^{CS^2(t-s)} \int_s^t R_{m,n}(u)du.$$

Recalling that $R_n(u) \leq s_n$ and $\int_s^t R^n(u)du \leq ts_n$ for $u \leq t$, we get for every $n_0 \in \mathbb{N}^*$,

$$\sup_{m,n\geq n_0, s\in[0,t]} |v^n(s) - v^m(s)| \leq t\Big[1 + CS^2 e^{CS^2 t} t\Big] \sup_{m,n\geq n_0} (s_n + s_m).$$

Adding that $s_n \to 0$ ensures that $(v^n(s), s \in [0,t])_{n\geq 0}$ is a Cauchy sequence under the uniform norm. Then there exists a continuous function v on $[0,t]$ such that $v^n \to v$, as n goes to ∞.

Next, we prove that v is solution of the Equation (7). As δ satisfies (42), we have for any $s \in [0,t]$ and $n \in \mathbb{N}^*$:

$$\begin{aligned}
&\Big|v(s) - \int_s^t \psi^\infty(s, v(s))ds - \lambda\Big| \\
&\quad \leq \Big|v(s) - v^n(s)\Big| + \int_s^t \Big|\psi^\infty(s, v(s)) - \psi^n(s, v(s))\Big|ds + \int_s^t \Big|\psi^n(s, v(s)) - \psi^n(s, v^n(s))\Big|ds \\
&\quad \leq ts_n + (1 + CS^2) \sup\Big\{\Big|v(s) - v^n(s)\Big|, s \in [0,t]\Big\},
\end{aligned}$$

so that letting $n \to \infty$ yields $\Big|v(s) - \int_s^t \psi^\infty(s, v(s))ds - \lambda\Big| = 0$. It proves that v is solution of (7). The uniqueness follows from Gronwall's lemma. □

6.2 An upper bound for ψ_0

The study of the Laplace exponent of $\widetilde{Z}$ in Corollary 2 requires a fine control of the branching mechanism ψ_0.

Lemma 19. *Assume that the process $(gt + \Delta_t, t \geq 0)$ goes to $+\infty$ a.s. There exists a non-negative increasing function k on $\mathbb{R}^+$ such that for every $\lambda \geq 0$*

$$\psi_0(\lambda) \leq \lambda k(\lambda) \qquad \text{and} \qquad \int_0^\infty k\Big(e^{-(gt+\Delta_t)}\Big)dt < \infty.$$

Proof. The inequality (44) implies that for every $\lambda \geq 0$,

$$\begin{aligned}
\psi_0(\lambda) &\leq \sigma^2\lambda^2 + \int_0^\infty \left(\lambda^2 z^2 \mathbf{1}_{\{\lambda z \leq 1\}} + \lambda z \mathbf{1}_{\{\lambda z > 1\}}\right)\mu(dz) \\
&\leq \left(\sigma^2 + \int_0^1 z^2\mu(dz)\right)\lambda^2 + \lambda^2 \mathbf{1}_{\{\lambda<1\}} \int_1^{1/\lambda} z^2\mu(dz) + \lambda \int_{1/\lambda}^\infty z\mu(dz).
\end{aligned}$$

Now, using condition (12) we obtain the existence of a positive constant c such that

$$\lambda \int_{1/\lambda}^\infty z\mu(dz) \leq \lambda \log^{-(1+\varepsilon)}(1 + 1/\lambda) \int_{1/\lambda}^\infty z \log^{1+\varepsilon}(1+z)\mu(dz) \leq c\lambda \log^{-(1+\varepsilon)}(1 + 1/\lambda).$$

Next, let us introduce the function f, given by

$$f(z) = z^{-1} \log^{1+\varepsilon}(1+z), \quad \text{for } z \in [1, \infty).$$

If we derivate the function f, we deduce that there exists a positive real number $A > 1$ such that f is decreasing on $[A, \infty)$. Therefore, for every $\lambda < 1/A$,

$$\begin{aligned} \int_A^{1/\lambda} \lambda^2 z^2 \mu(dz) &= \lambda \log^{-(1+\varepsilon)}(1+1/\lambda) f(1/\lambda) \int_A^{1/\lambda} \frac{z \log^{1+\varepsilon}(1+z)}{f(z)} \mu(dz) \\ &\leq \lambda \log^{-(1+\varepsilon)}(1+1/\lambda) \int_A^{1/\lambda} z \log^{1+\varepsilon}(1+z) \mu(dz). \end{aligned}$$

Adding that $\lambda^2 \int_1^A z^2 \mu(dz) \leq \lambda^2 A \int_1^\infty z\mu(dz)$ and using again condition (12), we deduce that there exists a positive constant c' such that for every $\lambda \geq 0$,

$$\psi_0(\lambda) \leq c' \Big(\lambda^2 + \lambda \log^{-(1+\varepsilon)}(1+1/\lambda) \Big).$$

Since λ^2 is negligible with respect to $\lambda \log^{-(1+\varepsilon)}(1+1/\lambda)$ when λ is close enough to 0 or infinity, we conclude that there exists a positive constant c'' such that

$$\psi_0(\lambda) \leq c'' \lambda \log^{-(1+\varepsilon)}(1+1/\lambda).$$

Defining the function $k(z) = c'' \log^{-(1+\varepsilon)}(1+1/z)$, for $z > 0$, we get that:

$$k\Big(e^{-(gt+\Delta_t)}\Big) \sim c'' \log^{-(1+\varepsilon)}(2), \quad (t \to 0),$$

thus the integral of $k(\exp(-gt - \Delta_t))$ is finite in a neighborhood of zero, and

$$0 \leq \int_1^\infty k\Big(e^{-(gt+\Delta_t)}\Big) dt \leq c'' \int_1^\infty e^{-(gt+\Delta_t)} (gt+\Delta_t)^{-(1+\varepsilon)} dt,$$

which is finite since the process $(gt + \Delta_t, t \geq 0)$ drifts $+\infty$ and has finite first moment. This completes the proof. □

6.3 Extinction versus explosion

We now verify that the process $(Y_t)_{t \geq 0}$ can be properly renormalized as $t \to \infty$ on the non-extinction event. We use a classical branching argument.

Lemma 20. *Let Y be a non-negative Markov process satisfying the branching property. We also assume that there exists a positive function a_t such that for every $x_0 > 0$, there exists a non-negative finite random variable W such that*

$$a_t Y_t \xrightarrow[t \to \infty]{} W \quad a.s, \qquad \mathbb{P}_{x_0}(W > 0) > 0, \qquad a_t \overset{t\to\infty}{\longrightarrow} 0.$$

Then

$$\{W = 0\} = \Big\{ Y_t \xrightarrow[t \to \infty]{} 0 \Big\} \qquad \mathbb{P}_{x_0} \quad a.s.$$

Proof. First, we prove that

$$\mathbb{P}_{x_0}(\limsup_{t\to\infty} Y_t = \infty \mid \limsup_{t\to\infty} Y_t > 0) = 1. \tag{46}$$

Let $0 < x \leq x_0 \leq A$ be fixed. Since $a_t \to 0$ and $\mathbb{P}_x(W > 0) > 0$, there exists $t_0 > 0$ such that $\alpha := \mathbb{P}_x(Y_{t_0} \geq A) > 0$. By the branching property, the process is stochastically monotone as a function of its initial value. Thus, for every $y \geq x$ (including $y = x_0$),

$$\mathbb{P}_y(Y_{t_0} \geq A) \geq \alpha > 0.$$

We define recursively the stopping times

$$T_0 := 0, \qquad T_{i+1} = \inf\{t \geq T_i + t_0 : Y_t \geq x\} \qquad (i \geq 0).$$

For any $i \in \mathbb{N}^*$, the strong Markov property implies

$$\mathbb{P}_{x_0}(Y_{T_i+t_0} \geq A \mid (Y_t : t \leq T_i),\ T_i < \infty) \geq \alpha.$$

Conditionally on $\{\limsup_{t\to\infty} Y_t > x\}$, the stopping times T_i are finite a.s. and for all $0 < x \leq x_0 \leq A$,

$$\mathbb{P}_{x_0}(\forall i \geq 0 : Y_{T_i+t_0} < A,\ \limsup_{t\to\infty} Y_t > x) = 0.$$

Then, $\mathbb{P}_{x_0}(\limsup_{t\to\infty} Y_t < \infty,\ \limsup_{t\to\infty} Y_t > x) = 0$. Now since $\{\limsup_{t\to\infty} Y_t > 0\} = \cup_{x\in(0,x_0]}\{\limsup_{t\to\infty} Y_t > x\}$, we get (46).

Next, we consider the stopping times $T_n = \inf\{t \geq 0 : Y_t \geq n\}$. The strong Markov property and branching property imply

$$\mathbb{P}_{x_0}(W = 0; T_n < \infty) = \mathbb{E}_{x_0}\Big(\mathbf{1}_{T_n<\infty}\mathbb{P}_{Y_{T_n}}(W = 0)\Big) \leq \mathbb{P}_n(a_t Y_t \underset{t\to\infty}{\longrightarrow} 0) = \mathbb{P}_1(a_t Y_t \underset{t\to\infty}{\longrightarrow} 0)^n,$$

which goes to zero as $n \to \infty$, since $\mathbb{P}_1(a_t Y_t \overset{t\to\infty}{\longrightarrow} 0) = \mathbb{P}_1(W = 0) < 1$. Then,

$$0 = \mathbb{P}_{x_0}(W = 0; \forall n : T_n < \infty) = \mathbb{P}_{x_0}(W = 0, \limsup_{t\to\infty} Y_t = \infty) = \mathbb{P}_{x_0}(W = 0, \limsup_{t\to\infty} Y_t > 0),$$

where the last identity comes from (46). This completes the proof. □

6.4 A Central limit theorem

We need the following central limit theorem for Lévy processes in Corollary 3.

Lemma 21. *Under the assumption (13) we have*

$$\frac{gt + \Delta_t - \boldsymbol{m}t}{\rho\sqrt{t}} \xrightarrow[t\to\infty]{d} \mathcal{N}(0,1).$$

Proof. For simplicity, let η be the image measure of ν under the mapping $x \mapsto e^x$. Hence, assumption (13) is equivalent to $\int_{|x|\geq 1} x^2\eta(\mathrm{d}x) < \infty$, or $\mathbb{E}[\Delta_1^2] < \infty$.

We define $T(x) = \eta\big((-\infty,-x)\big) + \eta\big((x,\infty)\big)$ and $U(x) = 2\int_0^x yT(y)\mathrm{d}y$, and assume that $T(x) > 0$ for all $x > 0$. According to Theorem 3.5 in Doney and Maller [13] there exist two functions $a(t), b(t) > 0$ such that

$$\frac{gt + \Delta_t - a(t)}{b(t)} \xrightarrow[t\to\infty]{d} \mathcal{N}(0,1), \quad \text{if and only if} \quad \frac{U(x)}{x^2 T(x)} \xrightarrow[x\to\infty]{} \infty.$$

If the above condition is satisfied, then b is regularly varying with index $1/2$ and it may be chosen to be strictly increasing to ∞ as $t \to \infty$. Moreover $b^2(t) = tU(b(t))$ and $a(t) = tA(b(t))$, where

$$A(x) = g + \int_{\{|z|<1\}} z\eta(\mathrm{d}z) + \eta\big((1,\infty)\big) - \eta\big((-\infty,-1)\big) + \int_1^x \Big(\eta\big((y,\infty)\big) - \eta\big((-\infty,-y)\big)\Big)\mathrm{d}y.$$

Note that under our assumption $x^2T(x) \to 0$, as $x \to \infty$. Moreover, note

$$U(x) = x^2T(x) + \int_{(-x,0)} z^2\eta(\mathrm{d}x) + \int_{(0,x)} z^2\eta(\mathrm{d}x),$$

and

$$A(x) = g + \int_{\{|z|<x\}} z\eta(\mathrm{d}z) + x\Big(\eta\big((x,\infty)\big) - \eta\big((-\infty,-x)\big)\Big).$$

Hence assumption (13) implies that

$$U(x) \xrightarrow[x\to\infty]{} \int_{(-\infty,\infty)} z^2\eta(\mathrm{d}z) = \rho^2, \quad A(x) \xrightarrow[x\to\infty]{} g + \int_{\mathbb{R}} z\eta(\mathrm{d}z) = \mathbf{m},$$

Therefore, we deduce $U(x)/(x^2T(x)) \to \infty$ as $x \to \infty$, $b(t) \sim \rho\sqrt{t}$ and $a(t) \sim \mathbf{m}t$, as $t \to \infty$.

Now assume that $T(x) = 0$, for x large enough. Define

$$\Psi(\lambda,t) = -\log \mathbb{E}\left[\exp\left\{i\lambda\left(\frac{gt + \Delta_t - a(t)}{b(t)}\right)\right\}\right],$$

where the functions $a(t)$ and $b(t)$ are defined as above. Hence since the process $(\Delta_t, t \geq 0)$ is of bounded variation, from the definition of $a(t)$ and the Lévy-Khintchine formula we deduce

$$\begin{aligned}
\Psi(\lambda,t) &= -i\lambda\left(\frac{gt}{b(t)} - \frac{a(t)}{b(t)}\right) + t\int_{\mathbb{R}}\Big(1 - e^{\frac{i\lambda}{b(t)}x}\Big)\eta(\mathrm{d}x) \\
&= t\int_{\{|x|<b(t)\}}\Big(1 - e^{\frac{i\lambda}{b(t)}x} + \frac{i\lambda}{b(t)}x + \frac{(i\lambda)^2}{2b^2(t)}x^2\Big)\eta(\mathrm{d}x) - \frac{t(i\lambda)^2}{2b^2(t)}\int_{\{|x|<b(t)\}} x^2\eta(\mathrm{d}x) \\
&\qquad + t\int_{\{|x|\geq b(t)\}}\Big(1 - e^{\frac{i\lambda}{b(t)}x}\Big)\eta(\mathrm{d}x) + i\lambda t\Big(\eta(b(t),\infty) - \eta(-\infty,-b(t))\Big).
\end{aligned}$$

Since $T(x) = 0$ for all x large, $b(t) \to \infty$ and $t^{-1}b^2(t) \to \rho^2$, as $t \to \infty$, therefore

$$\Psi(\lambda,t) \xrightarrow[t\to\infty]{} \frac{\lambda^2}{2},$$

which implies the result thanks to Lévy's Theorem. □

6.5 A technical Lemma

We now prove a technical lemma that is needed in the proofs of Section 4.

Proof of Lemma 10. To obtain (27), it is enough to choose $\varepsilon \leq 1$ as we assume in (23) that $\varsigma \geq 1$.

In order to prove (28), we first define the function $\tilde{h} : x \in \mathbb{R}^+ \mapsto (1+x)^{1-\varsigma} h(x)$ and let $0 \leq x \leq y$. Then,

$$\frac{F(x) - F(y)}{C_F} \leq \Big((x+1)^{-1/\beta} - (y+1)^{-1/\beta}\Big) + (1+y)^{-1/\beta-1}\Big|\tilde{h}(x) - \tilde{h}(y)\Big| + \Big|\tilde{h}(x)\Big|\Big((1+x)^{-1/\beta-1} - (1+y)^{-1/\beta-1}\Big). \quad (47)$$

We deal with the second term of the right hand side. Denoting by k the Lipschitz constant of $\tilde{h}$ and applying the Mean Value Theorem to $z \in \mathbb{R}_+ \mapsto (z+1)^{-1/\beta}$ on $[x, y]$, we get

$$(1+y)^{-1/\beta-1}\Big|\tilde{h}(x) - \tilde{h}(y)\Big| \leq k(y+1)^{-1/\beta-1}(y-x) \leq k\beta\left((x+1)^{-1/\beta} - (y+1)^{-1/\beta}\right).$$

Moreover, as $\beta \in (0, 1]$, we have the following inequalities :

$$\left(\frac{1+y}{1+x}\right)^{1+1/\beta} - 1 \leq \left(\left(\frac{1+y}{1+x}\right)^{1/\beta} - 1\right)\left(\frac{1+y}{1+x} - 1\right) \leq \left(\left(\frac{y}{x}\right)^{1/\beta} - 1\right)2\frac{1+y}{1+x}$$

Dividing by $(1+y)^{1/\beta+1}$ and using $(1+y)/[(1+x)(1+y)^{1/\beta+1}] \leq y^{-1/\beta}$ yield

$$(1+x)^{-1/\beta-1} - (1+y)^{-1/\beta-1} \leq 2\Big(x^{-1/\beta} - y^{-1/\beta}\Big).$$

Similarly $(1+x)^{-1/\beta} - (1+y)^{-1/\beta} \leq x^{-1/\beta} - y^{-1/\beta}$ and equation (47) give us

$$0 \leq F(x) - F(y) \leq C_F(1 + 2[\|h\|_\infty + k\beta])\Big(x^{-1/\beta} - y^{-1/\beta}\Big).$$

This completes the proof. □

6.6 Approximations of the survival probability for $\nu(0, \infty) = \infty$

Finally, we prove Corollary 6 in the case when $\nu(0, \infty) = \infty$.

End of the proof of Corollary 6. We let $A^{\varepsilon_1,\varepsilon_2} = (0, 1-\varepsilon_1) \cup (1+\varepsilon_2, \infty)$, where $0 < 1-\varepsilon_1 < 1 < 1+\varepsilon_2$ and define the Poisson random measure $N_1^{\varepsilon_1,\varepsilon_2}$ as the restriction of N_1 to $\mathbb{R}^+ \times A^{\varepsilon_1,\varepsilon_2}$. We denote by $\mathrm{d}t\nu^{\varepsilon_1,\varepsilon_2}(\mathrm{d}m)$ for its intensity measure, where $\nu^{\varepsilon_1,\varepsilon_2}(\mathrm{d}m) = \mathbf{1}_{\{m \in A^{\varepsilon_1,\varepsilon_2}\}}\nu(\mathrm{d}m)$, and the corresponding Lévy process $\Delta^{\varepsilon_1,\varepsilon_2}$ is defined by

$$\Delta_t^{\varepsilon_1,\varepsilon_2} = \int_0^t \int_{(0,\infty)} \log m \; N_1^{\varepsilon_1,\varepsilon_2}(\mathrm{d}s, \mathrm{d}m).$$

We also consider the CSBP's $Y^{\varepsilon_1,\varepsilon_2}$ (resp $Y^{\varepsilon_1,\varepsilon_2,-}$ and $Y^{\varepsilon_1,\varepsilon_2,+}$) with branching mechanism ψ (resp. ψ_- and ψ_+) and the same catastrophes $\Delta^{\varepsilon_1,\varepsilon_2}$ via (5). Since $\nu^{\varepsilon_1,\varepsilon_2}(0,\infty) < \infty$, from the first step we have $u_{+,t}^{\varepsilon_1,\varepsilon_2}(\lambda) \leq u^{\varepsilon_1,\varepsilon_2}(t, \lambda) \leq u_{-,t}^{\varepsilon_1,\varepsilon_2}(\lambda)$, where as expected $\mathbb{E}[\exp\{-\lambda Y_t^{\varepsilon_1,\varepsilon_2,*}\}] = \exp\{-u_{*,t}^{\varepsilon_1,\varepsilon_2}(\lambda)\}$ for each $* \in \{+, \emptyset, -\}$.

Similarly, let $A^{\varepsilon_1} = (0, 1-\varepsilon_1) \cup (1, \infty)$ and define the Poisson random measure $N_1^{\varepsilon_1}$ as the restriction of N_1 to $\mathbb{R}^+ \times A^{\varepsilon_1}$ with intensity measure $\mathrm{d}t\nu^{\varepsilon_1}(\mathrm{d}m)$, where $\nu^{\varepsilon_1}(\mathrm{d}m) = \mathbf{1}_{\{m \in A^{\varepsilon_1}\}}\nu(\mathrm{d}m)$. Let us fix t in $\mathbb{R}_+^*$, and define Y^{ε_1} as the unique strong solution of

$$Y_t^{\varepsilon_1} = Y_0 + \int_0^t gY_s^{\varepsilon_1}\mathrm{d}s + \int_0^t \sqrt{2\sigma^2 Y_s^{\varepsilon_1}}\mathrm{d}B_s + \int_0^t \int_{[0,\infty)} \int_0^{Y_{s-}^{\varepsilon_1}} z\widetilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[0,\infty)} \Big(m-1\Big)Y_{s-}^{\varepsilon_1} N_1^{\varepsilon_1}(\mathrm{d}s, \mathrm{d}m). \quad (48)$$

We already know from Theorem 1 that Equation (48) has a unique non-negative strong solution. Moreover, from Theorem 5.5 in [16] and the fact that $N_1^{\varepsilon_1}$ has the same jumps as $N_1^{\varepsilon_1,\varepsilon_2}$ plus additional jumps greater than one, we conclude

$$Y_t^{\varepsilon_1,\varepsilon_2} \leq Y_t^{\varepsilon_1}, \qquad \text{a.s.}$$

Using assumption (4), we can apply Gronwall's Lemma to the non-negative function $t \mapsto \mathbb{E}[Y_t^{\varepsilon_1} - Y_t^{\varepsilon_1,\varepsilon_2}]$ and obtain

$$\mathbb{E}\Big[\big|Y_t^{\varepsilon_1,\varepsilon_2} - Y_t^{\varepsilon_1}\big|\Big] \xrightarrow[\varepsilon_2 \to 0]{} 0.$$

Now, since $Y^{\varepsilon_1,\varepsilon_2}$ is decreasing with ε_2, we finally get, $Y_t^{\varepsilon_1,\varepsilon_2} \xrightarrow{a.s.} Y_t^{\varepsilon_1}$, as $\varepsilon_2 \to 0$. Using similar arguments as above for $Y^{\varepsilon_1,\varepsilon_2,+}$ and $Y^{\varepsilon_1,\varepsilon_2,-}$, we deduce

$$u_{+,t}^{\varepsilon_1}(\lambda) \leq u^{\varepsilon_1}(t,\lambda) \leq u_{-,t}^{\varepsilon_1}(\lambda).$$

In order to complete the proof, we let ε_1 tend to 0. □

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