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**Antoine Billot
Vassili Vergopoulos**

JEL Codes: D81, D83, D90

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PARIS-JOURDAN SCIENCES ÉCONOMIQUES

48, Bd JOURDAN – E.N.S. – 75014 PARIS
TÉL. : 33(0) 1 43 13 63 00 – FAX : 33 (0) 1 43 13 63 10
www.pse.ens.fr

Dynamic Consistency and Expected Utility with State Ambiguity

Antoine Billot* and Vassili Vergopoulos†

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Abstract

While models of ambiguity are reputed to generate a basic tension between dynamic consistency and the Ellsberg choices, this paper identifies a third implicit ingredient of this tension, namely the *parsimony rule*, which enforces each state of nature to encode a well-defined unique observation. This paper then develops nonparsimonious interpretations of the state space to make the Ellsberg choices compatible with both expected utility and dynamic consistency. The state space may contain nonobservable states: a state is allowed to encode more than one observation, a pattern called *state ambiguity*. The presence of such ambiguous states motivates an explicit distinction between the decision-maker and the theory-maker, the latter designing the state space and eliciting the former's preferences.

Keywords: ambiguity, state of nature, epistemics, dynamic consistency

JEL classification: D81, D83, D90

1 Introduction

The theory of Subjective Expected Utility (SEU) seems to forbid the epistemics of an individual to affect her observable behavior. For instance, climate change mitigation or bailout of financial system can both be seen as costly preventive activities believed to decrease causally the likelihood of some bad outcome such as global warming or bankruptcy. A blind application of SEU would justify the following argument: at any state of nature at which the *good* outcome occurs, there is no more gain to expect from prevention and not engaging in this activity is preferred. At any state at which the *bad* outcome occurs, there is again no reason to choose this activity. Since prevention is dominated at each possible state, it is optimal not to engage in it even when the true state is ignored. As a result, SEU

*LEMMA-Université Paris 2 Panthéon-Assas, and iuF: billot@u-paris2.fr

†Paris School of Economics-Université Paris 1 Panthéon-Sorbonne: vassili.vergopoulos@univ-paris1.fr
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with respect to the set of observations $\{bad, good\}$ forces the individual to behave as if she did not perceive causality between her own decisions and the resolution of uncertainty. In this spirit, climate change mitigation and financial bailout appear to be not worthwhile.

Several recent contributions relate epistemics to observable behavior through more general versions of the SEU framework. Examples include Gilboa and Schmeidler (1994), Mukerji (1997), Lipman (1999), Machina (2011) or Grabiszewski (2013) and involve unforeseen contingencies, failures of logical omniscience or belief in logically impossible states. Such epistemics are introduced through an unusual distinction between the set over which subjective probability judgements are formulated and the set of objective observations. In the example, causal relationships can indeed be taken into account through an enriched state space $\{(good, good), (good, bad), (bad, bad), (bad, good)\}$, where the first (resp. second) component of each state encodes the outcome that would causally result from prevention (resp. no prevention) at that state. The preventive activity is then undertaken whenever the subjective probability of state $(good, bad)$ is high enough.

In this context, to what extent is a theorist observing an individual choosing prevention allowed to infer that she perceives causality? This paper makes a first step towards theoretical foundations of such epistemic rationalizations of behavior. It seems impossible though to determine, on the sole basis of observable behavior, whether an individual actually perceives causality, unforeseen contingencies or logically impossible states. There is indeed nothing in observable behavior that determines the one appropriate epistemic structure in which to represent it. Then, any specific epistemic rationalization can not but ultimately rely upon a certain amount of subjectivity on the part of the theorist. The remainder of this introduction reexamines these stakes from the point of view of the Ellsberg paradox.

The Ellsberg experiment. In Ellsberg (1961), an urn contains 90 balls, 30 of which are known to be red while the remaining ones are blue or green in unknown proportions. A ball is about to be drawn in the urn and some decision-maker (DM) must choose between a bet on red and a bet on blue in situation 1 and between a bet on not-blue and a bet on not-red in situation 2. The corresponding outcomes are given in the tables hereafter.

Situation 1	r	b	g	Situation 2	r	b	g
<u>A bet on r</u>	1	0	0	A bet on b^c	1	0	1
A bet on b	0	1	0	<u>A bet on r^c</u>	0	1	1

Feasible acts are defined as functions mapping a set \mathcal{O} of possible observations $\{r, b, g\}$ onto a certain set of outcomes. In such a context, the DM is offered the possibility to choose an alternative within any possible subset of the set of feasible acts. Observable behavior in the Ellsberg experiment is taken to consist of the choices underlined in the tables, which are referred to as the *Ellsberg choices*. They are meant to capture the way confidence in probability judgements affects behavior: since the average probability of success is the same for each pair of bets, the Ellsberg choices express a preference for bets involving better known probabilities, a pattern known as ambiguity aversion.

Parsimony rule vs. state ambiguity: the static case. Savage's axiomatic treatment of the SEU assumption (Savage, 1954), meant to represent behavior in the face of uncer-

tainty, involves some primitive notions consisting of a set Ω of states of nature, a set X of outcomes and a preference relation \succsim_{Ω} . Each Savage act is defined as a function mapping states onto outcomes and \succsim_{Ω} is defined on the set of Savage acts. As usual in axiomatics, these primitives are left unspecified. Therefore, there are as many ways to interpret the Savage theory as specifications of its primitives and what is here called the *parsimony rule* delivers the most natural specification. It is basically motivated by the argument that nothing can be said in a meaningful way about unobservable entities and thus appears to be quite reminiscent of Samuelson's conception of operationalism in economics (see Bolland, 2008). Actually, parsimony tolerates such entities like a state space Ω only to the extent that Ω is directly observable, and preferences \succsim_{Ω} only to the extent that it is possible to reconstruct them from the DM's observable behavior. More formally, the parsimony rule consists of the two following assumptions:

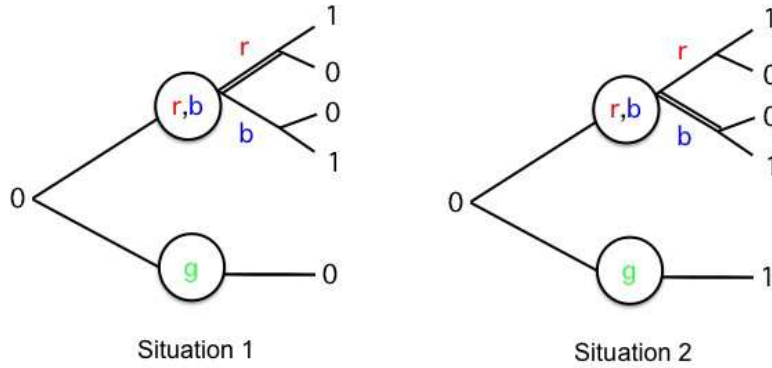
- (a) States of nature are nothing more than possible observations: $\Omega = \mathcal{O}$.
- (b) Savage preferences \succsim_{Ω} represent observable behavior.

Parsimony is however quite restrictive and ends up transforming the Ellsberg choices into a paradox for SEU: if the DM were to conform to the Savage theory in the parsimonious sense, then her subjective probability measure would apply to observations and put more weight on red than on blue and, simultaneously, more weight on not-red than on not-blue. This pattern can not be supported by any standard probability measure, hence the paradox. Ellsberg constructs this thought experiment to argue that SEU forbids confidence in probability judgements to affect observable behavior. However, following the intuitions of the introductory prevention example, it is not clear whether this impossibility is really due to SEU or to its parsimonious interpretation. Put differently, the Ellsberg paradox, which is usually analyzed as a failure of the DM to conform to SEU or a failure of SEU to describe accurately her behavior, might also be thought as a failure of the parsimony rule. In this perspective, the fact that the probabilities of observable events $\{b\}$ and $\{r, g\}$ can not sum up to 1, which motivates the introduction of non-additive probabilities in models of ambiguity as in Schmeidler (1989), may now be rationalized through an *ambiguous state of nature* $\omega \in \Omega$ that encodes simultaneously both observable events $\{b\}$ and $\{r, g\}$.

Parsimony rule vs. state ambiguity: the dynamic case. The dynamic extension of the Ellsberg choices is reputed to generate an inconsistency in the DM's behavior, which remains an important obstacle to certain economic applications of ambiguity (see Machina, 1989; Epstein and Le Breton, 1993; Wakker, 1997; Hanany and Klibanoff, 2007; or Al-Najjar and Weinstein, 2009). Epistemic rationalizations of behavior through state ambiguity might shed, in return, some light on this issue and lead to dynamically consistent reformulations of the Ellsberg choices. Consider the following dynamic version of the Ellsberg decision situations. The DM knows *ex ante* that she is about to be told whether the ball drawn from the urn is green or not. In the former case, there is no real decision to make while, in the latter case, she must choose whether to bet on red or to bet on blue.

Under reduction and consequentialism¹, optimal behavior is fully determined by the

¹Reduction and consequentialism capture respectively the ideas that feasible plans are compared by applying preferences to the acts they induce and that choices only depend upon information and constraints.



static Ellsberg choices and indicated by a double line. The inconsistency lies in that behavior at upper decision nodes $\{r, b\}$ is affected by the yet counterfactual and thus irrelevant outcome associated to $\{g\}$. However, if state ambiguity is allowed, then such inconsistency may precisely be rationalized through an ambiguous state at which $\{g\}$ and $\{r, b\}$ are both possibly observable: if the DM understands at upper decision nodes that $\{r, b\}$ is possible but not certain, then $\{g\}$ remains possible as well and the corresponding outcome is still relevant. In contrast, if no such states were perceived, this outcome would indeed be irrelevant and could not affect behavior.

Theory-making. Models of state ambiguity have the potential to lead to epistemic rationalizations of the Ellsberg choices that conform to SEU and ensure dynamic consistency. However, contrary to parsimony, it relies upon unobservable states and fails to provide operational elicitation of preferences and beliefs. If states of nature differ from observations and if Savage preferences do not represent the DM's observable behavior, then what are these objects exactly and how do they relate to observable behavior? This paper introduces explicitly a theory-maker (TM) supposed to design the various decision problems submitted to the DM and to elicit operationally her preferences. The state space Ω is the set of states used by the TM to represent the uncertainty generated by the DM's observable behavior and Savage preferences represent the TM's own behavior in the face of that uncertainty. Such a framework is enriched with one more object, namely a *hermeneutic mapping*. First, this mapping makes it possible to derive from the TM's Savage preferences the observable preferences he ascribes to the DM. These observable preferences are systematically assumed to be consistent with the DM's observable behavior. Put differently, the TM's model of the DM is assumed to be nonrefuted by observable data. Thus, observable behavior does determine as usual the TM's subjective primitive notions, not in an operationalist way though, but rather upon the more flexible mode of nonrefutation. Second, the hermeneutic mapping specifies the one epistemic structure used by the TM to rationalize the DM's observable behavior. Within the TM's representation, this mapping makes it thus possible to interpret observable behavior in terms of epistemics.

In this setting, the Ellsberg choices are shown to constrain the TM to appeal to non-parsimonious representations of behavior (Section 2). Consistently with the intuitions

developed in this introduction, parsimony is shown to be more than a neutral methodological principle: within the TM's representation, parsimony is proved to characterize the fact that the DM perceives no state ambiguity. The Ellsberg choices then constrain the TM to consider that the DM perceives ambiguous states (Section 3). At last, different representations of the DM's static and dynamic behavior are given. It is shown in particular that observable behavior can always be reformulated within the TM's subjective state space Ω in a lattice-theoretic way that generalizes the classical Savage framework and that systematically delivers dynamic consistency, even when the DM is inconsistent over the set of observations (Section 4).

2 Preferences and parsimony

This section defines a framework and axioms for the TM to represent the behavior of the DM and formalizes the parsimony rule to show how the Ellsberg choices forces the TM to reject it.

2.1 Framework

The TM faces the uncertainty generated by DM's behavior in the same way that the DM faces the uncertainty generated by some urn (or any other random system) and is assumed to follow the Savage theory: he is then characterized by a finite *state space* Ω , a set of *outcomes* X and a *preference relation* \succsim_{Ω} . In addition, the framework is enriched by some more primitive notions: a finite set of *observations* \mathcal{O} and a *hermeneutic mapping* φ . More precisely, the set \mathcal{O} can be seen as the set over which the TM elicits preferences for the DM and the mapping φ is the one tool through which the TM interprets the DM's behavior. All these primitive notions characterize not only the TM but also the DM in the sense that she is herself the object described by these entities.

A subset $E \subseteq \Omega$ is referred to as a *Savage event* and is said *to obtain*, while a subset $A \subseteq \mathcal{O}$ is referred to as an *observable event* and is said *to be realized*. A *Savage act* F is defined as a function mapping states onto outcomes and the set of such acts is denoted \mathcal{A}_{Ω} . A *feasible act* f is defined as a function mapping observations onto outcomes and the set of such acts is denoted $\mathcal{A}_{\mathcal{O}}$. Preferences \succsim_{Ω} are defined over \mathcal{A}_{Ω} . To simplify notations, each outcome $x \in X$ is identified with the constant Savage act $F \in \mathcal{A}_{\Omega}$ defined by $F(\omega) = x$, for all $\omega \in \Omega$, but also with the constant feasible act $f \in \mathcal{A}_{\mathcal{O}}$ defined by $f(o) = x$, for all $o \in \mathcal{O}$. Moreover, for any two Savage acts $F, G \in \mathcal{A}_{\Omega}$ and any Savage event $E \subseteq \Omega$, $F_E G$ stands for the Savage act equal to F over E and to G over E^c . The fact that F and G are equal over E is denoted $F =_E G$. Similar notations are used for feasible acts.

At last, the *hermeneutic mapping* φ maps each feasible act $f \in \mathcal{A}_{\mathcal{O}}$ onto a Savage act $\varphi(f) \in \mathcal{A}_{\Omega}$. The TM interprets each outcome $\varphi(f)(\omega)$ as one of the DM's certainty equivalent of $f \in \mathcal{A}_{\mathcal{O}}$ at state $\omega \in \Omega$. This mapping φ is proved below to imply the existence of a family of preferences $(\succsim_{\omega})_{\omega \in \Omega}$ over feasible acts and thus operates as a particular specification of the TM's state space Ω according to which the TM basically

appears to be uncertain of the DM's true preferences.

2.2 Axioms

The next two axioms further structure the TM's state space. First, the state space is assumed to be rich enough to give rise to well-behaved Savage preferences.

Axiom 1 *The preference relation \succsim_Ω satisfies:*

A1.1 (*completeness*): $\forall F, G \in \mathcal{A}_\Omega$, $F \succsim_\Omega G$ or $G \succsim_\Omega F$.

A1.2 (*transitivity*): $\forall F, G, H \in \mathcal{A}_\Omega$, if $F \succsim_\Omega G$ and $G \succsim_\Omega H$, then $F \succsim_\Omega H$.

A1.3 (*monotonicity*): $\forall F \in \mathcal{A}_\Omega$, $\forall x, y \in X$, $\forall E \in \Omega^*$, $x_E F \succsim_\Omega y_E F \iff x \succsim_\Omega y$.

A1.4 (STP): $\forall F, G, H, K \in \mathcal{A}_\Omega$, $\forall E \subseteq \Omega$, $F_E H \succsim_\Omega G_E H \iff F_E K \succsim_\Omega G_E K$.

In A1.3, Ω^* denotes the set of nonnull events. A Savage event $E \subseteq \Omega$ is said to be null if any two Savage acts equal over E^c are indifferent for \succsim_Ω . These four properties which are all part of the Savage axiomatic system and include in particular the Sure Thing Principle (STP) translate in a minimal way the idea that Savage preferences are of the SEU type. Second, the TM's state space is assumed to be rich enough to give rise to a well-behaved hermeneutic mapping.

Axiom 2 *The hermeneutic mapping φ satisfies:*

A2.1: $\forall f \in \mathcal{A}_\mathcal{O}$, $\forall x \in X$, if $f(o) \sim_\Omega x$, $\forall o \in \mathcal{O}$, then $\varphi(f)(\omega) \sim_\Omega x$, $\forall \omega \in \Omega$.

A2.2: $\forall o \in \mathcal{O}$, $\exists \omega \in \Omega$ such that $\forall f \in \mathcal{A}_\mathcal{O}$, $\varphi(f)(\omega) \sim_\Omega f(o)$.

A2.3: $\forall \omega, \omega' \in \Omega$, if $\forall f \in \mathcal{A}_\mathcal{O}$, $\varphi(f)(\omega) \sim_\Omega \varphi(f)(\omega')$, then $\omega = \omega'$.

A2.4: $\forall \omega \in \Omega$, $\exists f, g \in \mathcal{A}_\mathcal{O}$ such that $\varphi(f)(\omega) \succ_\Omega \varphi(g)(\omega)$.

Under A2.1, the mapping preserves constant acts. In following A2.2, the TM considers that, for each observation, there exists a state of nature such that the DM behaves at that state as if that observation was necessarily realized. Then, A2.3 delivers uniqueness in A2.2. At last, A2.4 stands for a standard nontriviality requirement.

2.3 Preferences

The following definition explains how the TM's preferences reveal, through the hermeneutic mapping φ , the observable preferences he assigns to the DM. The same process, applied statewise, also delivers a family of preferences indexed by states.

Definition 1 *The preference relations $\succsim_\mathcal{O}$ and $(\succsim_\omega)_{\omega \in \Omega}$ are defined by:*

(i) $\forall f, g \in \mathcal{A}_\mathcal{O}$, $f \succsim_\mathcal{O} g \iff \varphi(f) \succsim_\Omega \varphi(g)$.

(ii) $\forall f, g \in \mathcal{A}_\mathcal{O}$, $f \succsim_\omega g \iff \varphi(f)(\omega) \succsim_\Omega \varphi(g)(\omega)$.

The binary relation $\succsim_\mathcal{O}$ is referred to as observable preferences and, strictly speaking, stands for the preferences assigned to the DM by the TM. In addition, these preferences are assumed, throughout the paper, to be consistent with the DM's observable behavior.

From this second point of view, Definition 1(i) is rather a constraint of nonrefutation of the TM's model of the DM's behavior. Besides, Definition 1(ii) explains how the hermeneutic mapping induces a family $(\succsim_\omega)_{\omega \in \Omega}$ of local preferences. Each state thus encodes a specific preference relation over feasible acts and the TM basically appears to be uncertain of the DM's true preferences. The next proposition presents a property of monotonicity of $\succsim_{\mathcal{O}}$ with respect to (wrt) $(\succsim_\omega)_{\omega \in \Omega}$ and shows that each $\varphi(f)(\omega)$ is viewed indeed as a certainty equivalent of feasible act f at state ω .

Proposition 1 *The two assertions hereafter hold:*

(P1.1) $\forall f, g \in \mathcal{A}_{\mathcal{O}}$, if $\forall \omega \in \Omega$, $f \succsim_\omega g$, then $f \succsim_{\mathcal{O}} g$.

(P1.2) $\forall f \in \mathcal{A}_{\mathcal{O}}$, $\forall \omega \in \Omega$, $\varphi(f)(\omega) \sim_\omega f$.

Proof. A???.3 implies $\forall F, G \in \mathcal{A}_\Omega$, $F(\omega) \succsim_\Omega G(\omega) \Rightarrow F \succsim_\Omega G$, which, given Def. ??, delivers P1.1. Moreover, let $x = \varphi(f)(\omega) \in X$ and define $g \in \mathcal{A}_{\mathcal{O}}$ by $g(o) = x$, $\forall o \in \mathcal{O}$. By A???.1, $\varphi(f)(\omega) \sim_\Omega x \sim_\Omega \varphi(g)(\omega)$ and therefore $f \sim_\omega g$ and $f \sim_\omega \varphi(f)(\omega)$. ■

Ellsberg Urn 1 Consider the following version of the Ellsberg decision situation which will be used throughout the paper to illustrate some results. The space of observations is given by $\mathcal{O} = \{r, b, g\}$, while the space of states is given by $\Omega = \{R, B, G, BG\}$. The mapping φ is indirectly specified by a family $(\succsim_\omega)_{\omega \in \Omega}$. Fix a utility function $u : X \rightarrow R$ and, for all $f, g \in \mathcal{A}_{\mathcal{O}}$, let:

$$\left\{ \begin{array}{l} f \succsim_R g \iff u(f(r)) \geq u(g(r)), \\ f \succsim_B g \iff u(f(b)) \geq u(g(b)), \\ f \succsim_G g \iff u(f(g)) \geq u(g(g)), \\ f \succsim_{BG} g \iff E_{\nu_{BG}} u(f) \geq E_{\nu_{BG}} u(g), \end{array} \right.$$

where ν_{BG} is a Choquet capacity over \mathcal{O} defined by $\nu_{BG}(\emptyset) = \nu_{BG}(\{r\}) = 0$, $\nu_{BG}(\{b\}) = \nu_{BG}(\{r, b\}) = p$, $\nu_{BG}(\{g\}) = \nu_{BG}(\{r, g\}) = q$ and $\nu_{BG}(\{b, g\}) = \nu_{BG}(\{r, b, g\}) = 1$ and the expectation refers to the Choquet integral (Schmeidler, 1989).

2.4 Parsimony

Through the following definition, the parsimony rule captures the idea that, for each state, there exists an observation such that the TM considers that the DM behaves at this state as if that observation was necessarily realized.

Definition 2 *The TM is said to conform to the parsimony rule if for each state $\omega \in \Omega$ there exists an observation $o \in \mathcal{O}$ such that: $\forall f \in \mathcal{A}_{\mathcal{O}}$, $\varphi(f)(\omega) \sim_\Omega f(o)$.*

Proposition ?? makes the connection between the formal definition of parsimony and its intuitive content, namely, the identification of states of nature with observations and the consistency of Savage preferences with observable behavior.

Proposition 2 *The conditions hereafter are equivalent:*

(P2.1) *The TM conforms to the parsimony rule.*

(P2.2) *There exists a bijection $\epsilon : \mathcal{O} \rightarrow \Omega$ such that $\forall o \in \mathcal{O}, \forall f \in A_{\mathcal{O}}, \varphi(f)(\epsilon(o)) \sim_{\Omega} f(o)$.*

(P2.3) *φ is an order isomorphism² from $(\mathcal{A}_{\mathcal{O}}, \succsim_{\mathcal{O}})$ onto $(\mathcal{A}_{\Omega}, \succsim_{\Omega})$.*

Proof. Assume first that the TM is parsimonious. The combination of A???.2 and A???.3 delivers the existence of an injective mapping $\epsilon : \mathcal{O} \rightarrow \Omega$ such that for each $o \in \mathcal{O}$: $\forall f \in A_{\mathcal{O}}, \varphi(f)(\epsilon(o)) \sim_{\Omega} f(o)$. In fact, parsimony makes sure that ϵ is also surjective: let $\omega \in \Omega$ and let $o \in \mathcal{O}$ be as in Def. 2 so that $\forall f \in A_{\mathcal{O}}, \varphi(f)(\omega) \sim_{\Omega} f(o)$. Then, $\forall f \in A_{\mathcal{O}}, \varphi(f)(\epsilon(o)) \sim_{\Omega} \varphi(f)(\omega)$ and, by A???.3, $\omega = \epsilon(o)$, hence surjectivity and P2.2. Assume now P2.2 to show P2.3. Fix a Savage act $F \in A_{\Omega}$ and let $f \in A_{\mathcal{O}}$ be defined by $f(o) = F(\epsilon(o))$ for each $o \in \mathcal{O}$. Then, for each $\omega \in \Omega$, $F(\omega) = F(\epsilon(\epsilon^{-1}(\omega))) = f(\epsilon^{-1}(\omega)) \sim_{\Omega} \varphi(f)(\omega)$ and, by A1.3, $\varphi(f) \sim_{\Omega} F$. Finally, since φ also preserves the order, it is indeed an order isomorphism. Finally, assume P2.3 to show P2.1 and consider a state $\omega_0 \in \Omega$. If ω_0 is one of the states implied by A???.2, then there is nothing left to prove. If not, then there must exist, by assumption, a feasible act $f \in A_{\mathcal{O}}$ such that, $\forall \omega \in \Omega, F(\omega) \sim_{\Omega} \varphi(f)(\omega)$, where $F = x_{\{\omega_0\}}y$ with $x \succ_{\Omega} y$. A2.4 ensures the existence of such x, y . Then, $\varphi(f)(\omega) \sim_{\Omega} y$, for all the states implied by A???.2 and, precisely by A???.2, $f(o) \sim_{\Omega} y, \forall o \in \mathcal{O}$. By A???.1, $\varphi(f)(\omega) \sim_{\Omega} y, \forall \omega \in \Omega$. However, $\varphi(f)(\omega_0) \sim_{\Omega} F(\omega_0) = x$, hence a contradiction. ■

The next proposition proves that a parsimonious TM necessarily assigns to the DM observable preferences satisfying STP.

Proposition 3 *If the TM conforms to the parsimony rule, then the preference relation $\succsim_{\mathcal{O}}$ he assigns to the DM satisfies STP.*

Proof. Let ϵ be the bijective mapping constructed in P2.2. Fix $A \subseteq \mathcal{O}$ and let $E = \epsilon(A) \subseteq \Omega$. Then, for any feasible acts $f, h \in A_{\mathcal{O}}$ and any $\omega \in \Omega$, $\varphi(f_A h)(\omega) \sim_{\Omega} f_A h(\epsilon^{-1}(\omega)) \sim_{\Omega} \varphi(f)_E \varphi(h)(\omega)$. By A???.3, this implies $\varphi(f_A h) \sim_{\Omega} \varphi(f)_E \varphi(h)$ and, given STP in A???, delivers STP for $\succsim_{\mathcal{O}}$. ■

Finally, STP for the DM's observable preferences is the price to pay by a parsimonious TM: the parsimony rule forces the TM to consider that the DM does not perceive ambiguity. Equivalently, when facing the Ellsberg choices, the TM is led to give parsimony up.

Ellsberg Urn 2 Under parsimony, if the TM's preferences \succsim_{Ω} are SEU wrt u and \mathbb{P} , then the preferences $\succsim_{\mathcal{O}}$ he assigns to the DM are SEU themselves with respect to u and some probability measure P over \mathcal{O} . For instance, let the state space be given by $\{R, B, G\}$ with corresponding relations as in Ellsberg Urn ??. Then, the TM conforms to the parsimony rule and P is defined by $P(\{r\}) = \mathbb{P}(\{R\})$, $P(\{b\}) = \mathbb{P}(\{B\})$ and $P(\{g\}) = \mathbb{P}(\{G\})$.

²See Appendix 5.1 for a formal definition.

3 State ambiguity and epistemics

This section studies the way the TM translates observable events within the state space and proposes two equivalent characterizations. The first one is semantical-like and involves an orthogonality relation between states (Moore, 1999) while the second one is syntactical-like and involves a knowledge operator (Aumann, 1999). This epistemic structure assigned to the DM by the TM delivers the tools needed for the representational results.

3.1 Intuitions

With parsimony, each state of nature encodes a well-defined and unique observation. However, when facing the Ellsberg choices, the TM gives parsimony up and now a state possibly encodes more than one observation as in Ellsberg Urn 1. Such a state is called *ambiguous*. Actually, *state ambiguity* might capture different types of concrete situations. Consider the following version of the Ellsberg urn:

- (1) each ball is characterized by two numbers (x, y) with $x \in \{0, 1, 2, 3\}$ and $y \in \{0, 1\}$
- (2) these numbers fully work out the colour of the ball according to the following rules:

	$x = 0$	$x = 1$	$x = 2$	$x = 3$
$y = 0$	r	b	g	b
$y = 1$	r	b	g	g

Suppose for instance that the two numbers x and y represent the last two digits of a ball's serial number and that the TM thinks that the DM understands the importance of x but fails to see the relevance of y . He then considers that, from her own point of view, the color of a ball is determined by the following nonparsimonious rule:

$x = 0$	$x = 1$	$x = 2$	$x = 3$
r	b	g	$\{b, g\}$

The TM might deal with this situation by introducing a state space $\Omega = \{R, B, G, BG\}$ as in Ellsberg Urn 1 where each state corresponds to one of the values of x . Each of R , B and G then defines preferences \succsim_R , \succsim_B and \succsim_G of the SEU type with respect to Dirac measures putting all weight on r , b and g respectively. Such states are therefore nonambiguous. But state BG defines a preference relation \succsim_{BG} , possibly SEU or of the Schmeidler type, that describes the local uncertainty with support $\{b, g\}$ that prevails when $x = 3$. Hence the TM considers that the DM perceives state ambiguity at state BG . Finally, state ambiguity might accomodate not only unforeseen contingencies as in the above example (Dekel, Lipman and Rustichini, 2001) but also bounded rationality (Lipman, 1999; Grabiszewski, 2013), causality/moral hazard (Gibbard and Harper, 1978) or even indeterministic uncertainty.

3.2 Evidences and orthogonality

With parsimony, any state of nature encodes a single *necessary* observation. With state ambiguity, a state encodes more than one *possible* observation. For instance, in Ellsberg Urn 1, state BG encodes observations b and g . More generally, let $N_{\mathcal{O}}(\omega)$ be the set of observable events that are necessarily realized at state ω and $I_{\mathcal{O}}(\omega)$ be the set of observable events that are impossible at state ω . Hence $N_{\mathcal{O}}(\omega)$ and $I_{\mathcal{O}}(\omega)$ respectively stand for what the TM sees as the DM's *necessity evidence* and *impossibility evidence* at ω and are formally defined as follows:

Definition 3 For each state $\omega \in \Omega$, let:

- (i) $N_{\mathcal{O}}(\omega) = \{A \subseteq \mathcal{O}, \forall f, g \in \mathcal{A}(\mathcal{O}), f =_A g \implies f \sim_{\omega} g\}$.
- (ii) $I_{\mathcal{O}}(\omega) = \{A \subseteq \mathcal{O}, \forall f, g \in \mathcal{A}(\mathcal{O}), f =_{A^c} g \implies f \sim_{\omega} g\}$.
- (iii) $n_{\mathcal{O}}(\omega) = \bigcap_{A \in N_{\mathcal{O}}(\omega)} A$.
- (iv) $i_{\mathcal{O}}(\omega) = \bigcup_{A \in I_{\mathcal{O}}(\omega)} A$.

The next proposition delivers the appropriate interpretations for $n_{\mathcal{O}}(\omega)$ and $i_{\mathcal{O}}(\omega)$ which will be referred to respectively as *necessity support* and *impossibility support* when ω obtains. They stand respectively for the *smallest* observable event necessarily realized and for the *largest* observable event impossible at ω . The proposition also presents some basic duality relations between necessity and impossibility as well as the way supports generate evidences.

Proposition 4 For each state $\omega \in \Omega$:

- (P4.1) $n_{\mathcal{O}}(\omega) \in N_{\mathcal{O}}(\omega)$.
- (P4.2) $i_{\mathcal{O}}(\omega) \in I_{\mathcal{O}}(\omega)$.
- (P4.3) $\forall A \subseteq \mathcal{O}, A \in N_{\mathcal{O}}(\omega) \iff A^c \in I_{\mathcal{O}}(\omega)$.
- (P4.4) $\forall A \subseteq \mathcal{O}, A \in N_{\mathcal{O}}(\omega) \iff n_{\mathcal{O}}(\omega) \subseteq A$.
- (P4.5) $\forall A \subseteq \mathcal{O}, A \in I_{\mathcal{O}}(\omega) \iff A \subseteq i_{\mathcal{O}}(\omega)$.
- (P4.6) $n_{\mathcal{O}}(\omega) = i_{\mathcal{O}}(\omega)^c$.
- (P4.7) $\forall o \in \mathcal{O}, \exists \omega \in \Omega, n_{\mathcal{O}}(\omega) = i_{\mathcal{O}}(\omega)^c = \{o\}$.

Proof. First, note that each necessity evidence is stable by intersections. This follows from the equality $f_{A \cap B} g = (f_A g)_B$ holding for all feasible acts f and g and observable events A and B . P4.1 follows from this remark, while P4.3 follows from Def. 3(i-ii). Then, P4.2 is a consequence of P4.3 and P4.1. In addition, the remark that $N_{\mathcal{O}}(\omega)$ contains any observable event that contains itself some observable event in $N_{\mathcal{O}}(\omega)$, together with P4.1, implies P4.4. Then, P4.3 implies P4.5-6. At last, for P4.7, fix $o \in \mathcal{O}$ and $\omega \in \Omega$ as in A???.2. Then: $f \succsim_{\omega} g \iff f(o) \succsim_{\Omega} g(o)$. Therefore, $\{o\} \in N_{\mathcal{O}}(\omega)$ and $n_{\mathcal{O}}(\omega) \subseteq \{o\}$. By A2.4, $n_{\mathcal{O}}(\omega)$ cannot be empty and finally $n_{\mathcal{O}}(\omega) = \{o\}$. ■

With parsimony, complementarity within Ω reflects exactly complementarity within \mathcal{O} : any two distinct states encode two distinct observations. With state ambiguity, two distinct states might encode a common observation: for instance, states B and BG in

Ellsberg Urn 1 both encode b . An appropriate notion of complementarity within Ω should then allow such distinct states not to be treated as substantially different, which can be achieved through an *orthogonality relation* (see Moore, 1999): two states are defined to be orthogonal if there exists an observable event A necessarily realized at the one state and impossible at the other one. By extension, each Savage event $E \subseteq \Omega$ is associated with its orthogonal event $E^\perp = \{\omega \in \Omega, \forall \omega' \in E, \omega \perp \omega'\}$. (See Appendix for some properties of orthogonality used throughout the paper.)

Definition 4 *Two states of nature $\omega, \omega' \in \Omega$ are said to be orthogonal, which is denoted $\omega \perp \omega'$, if $N_{\mathcal{O}}(\omega) \cap I_{\mathcal{O}}(\omega') \neq \emptyset$.*

The next proposition presents the basic properties of orthogonality as well as its relationship to supports: two states are orthogonal if and only if their supports are disjoint.

Proposition 5 *The two assertions hereafter hold:*

(P5.1) \perp is symmetric and antireflexive on Ω .

(P5.2) $\forall \omega, \omega' \in \Omega, \omega \perp \omega' \iff n_{\mathcal{O}}(\omega) \cap n_{\mathcal{O}}(\omega') = \emptyset$.

Proof. Symmetry is implied by P4.3. If $\omega \perp \omega'$, then $A \in N_{\mathcal{O}}(\omega) \cap I_{\mathcal{O}}(\omega')$ and, by P4.3, $A^c \in N_{\mathcal{O}}(\omega)$. Since $N_{\mathcal{O}}(\omega)$ is stable by intersections, $\emptyset = A \cap A^c \in N_{\mathcal{O}}(\omega)$, which contradicts A2.4, hence P5.1. Moreover, if $\omega \perp \omega'$, then there exists $A \in N_{\mathcal{O}}(\omega) \cap I_{\mathcal{O}}(\omega')$ and, by P3, $n_{\mathcal{O}}(\omega) \subseteq A$ and $n_{\mathcal{O}}(\omega') \subseteq A^c$, which shows that $n_{\mathcal{O}}(\omega)$ and $n_{\mathcal{O}}(\omega')$ are disjoint. This argument can be reversed and delivers P5.2. ■

Ellsberg Urn 3 Necessity evidences and corresponding supports take the following form in the context of Ellsberg urn ??:

$$\left\{ \begin{array}{ll} N_{\mathcal{O}}(R) = \{A \subseteq \mathcal{O}, r \in A\} & \text{and } n_{\mathcal{O}}(R) = \{r\}, \\ N_{\mathcal{O}}(B) = \{A \subseteq \mathcal{O}, b \in A\} & \text{and } n_{\mathcal{O}}(B) = \{b\}, \\ N_{\mathcal{O}}(G) = \{A \subseteq \mathcal{O}, g \in A\} & \text{and } n_{\mathcal{O}}(G) = \{g\}, \\ N_{\mathcal{O}}(BG) = \{\{b, g\}, \{r, b, g\}\} & \text{and } n_{\mathcal{O}}(BG) = \{b, g\}. \end{array} \right.$$

Orthogonality is defined in the following way: $R \perp B$, $R \perp G$, $B \perp G$ and $R \perp BG$, while $B \not\perp BG$ and $G \not\perp BG$. A Savage event and its orthogonal always encode disjoint observations which, for instance, becomes transparent with $E = \{BG, G\}$ and $E^\perp = \{R\}$.

3.3 Knowledge and belief

The TM uses the *knowledge* and *belief* operators K_Ω and B_Ω defined hereafter to represent observable events within Ω .

Definition 5 *For each $A \subseteq \mathcal{O}$, let:*

(i) $K_\Omega(A) = \{\omega \in \Omega, A \in N_{\mathcal{O}}(\omega)\}$.

(ii) $B_\Omega(A) = \{\omega \in \Omega, A \notin I_{\mathcal{O}}(\omega)\}$.

The subset $K_\Omega(A)$ is the set of states at which A is necessarily realized while $B_\Omega(A)$ is the set of states at which A is possible. For the TM, the DM knows (resp. believes) exactly what she sees as necessary (resp. possible). The next proposition presents some properties for K_Ω and B_Ω .

Proposition 6 *For each $A \subseteq \mathcal{O}$:*

- (P6.1) $K_\Omega(A) = \{\omega \in \Omega, n_{\mathcal{O}}(\omega) \subseteq A\}$.
- (P6.2) $B_\Omega(A) = \{\omega \in \Omega, A \not\subseteq i_{\mathcal{O}}(\omega)\}$.
- (P6.3) $K_\Omega(A^c) = B_\Omega(A)^c$.
- (P6.4) K_Ω preserves intersections.
- (P6.5) B_Ω preserves reunions.
- (P6.6) $A = \cup_{\omega \in K_\Omega(A)} n_{\mathcal{O}}(\omega) = \cap_{\omega \notin B_\Omega(A)} i_{\mathcal{O}}(\omega)$.
- (P6.7) Both K_Ω and B_Ω are injections.
- (P6.8) $K_\Omega(A^c) = K_\Omega(A)^\perp$.
- (P6.9) $B_\Omega(A^c) = B_\Omega(A)^{c\perp c}$.

Proof. P6.1-3 follow respectively from P4.3-5. For P6.4, note that $n_{\mathcal{O}}(\omega) \subseteq A \cap B \iff n_{\mathcal{O}}(\omega) \subseteq A$ and $n_{\mathcal{O}}(\omega) \subseteq B$, for all $\omega \in \Omega$. P6.5 follows from P6.3-4. For P6.6, if $o \in A$, then, by P4.7, there exists $\omega \in \Omega$ such that $n_{\mathcal{O}}(\omega) = \{o\}$. Furthermore, $n_{\mathcal{O}}(\omega) \subseteq A$, which shows that $\omega \in K_\Omega(A)$. Finally: $o \in \cup_{\omega \in K_\Omega(A)} n_{\mathcal{O}}(\omega)$. Conversely, let $o \in n_{\mathcal{O}}(\omega)$ for some $\omega \in K_\Omega(A)$, then $o \in n_{\mathcal{O}}(\omega) \subseteq A$. The second equality can therefore be obtained as a consequence of the first one and P4.6. In turn, P6.7 follows from P6.6 while P5.8 follows from:

$$\begin{aligned}
\omega_0 \in K_\Omega(A)^\perp &\iff \forall \omega \in K_\Omega(A), n_{\mathcal{O}}(\omega_0) \cap n_{\mathcal{O}}(\omega) = \emptyset \text{ by P5.2,} \\
&\iff n_{\mathcal{O}}(\omega_0) \cap (\cup_{\omega \in K_\Omega(A)} n_{\mathcal{O}}(\omega)) = \emptyset, \\
&\iff n_{\mathcal{O}}(\omega_0) \cap A = \emptyset \text{ by P6.6,} \\
&\iff n_{\mathcal{O}}(\omega_0) \subseteq A^c, \\
&\iff \omega_0 \in K_\Omega(A^c) \text{ by P6.1.}
\end{aligned}$$

At last P6.9 can be obtained from P6.3 and P6.8, which ends the proof. ■

The knowledge operator offers a specific way to represent observable events as Savage events. Due to injectivity (P6.7), the observable event A corresponding to a Savage event $E = K_\Omega(A)$ is unique and claiming that this event E obtains is just equivalent for the TM to considering that the DM thinks that A is necessarily realized. Such a Savage event that lies in the range of the knowledge operator is said to be realizable. Another possible approach would involve the belief operator and lead to a dual notion of realizability.

Definition 6 *A Savage event $E \subseteq \Omega$ is said to be realizable if there exists an observable event $A \subseteq \mathcal{O}$ such that $E = K_\Omega(A)$.*

Realizability can be characterized in terms of orthogonality. Let \mathcal{R} stand for the set of Savage events $E \subseteq \Omega$ such that $E^{\perp\perp} = E$. It has actually the structure of an ortholattice with operations (\wedge, \vee, \perp) given by $E \wedge F = E \cap F$ and $E \vee F = (E \cup F)^{\perp\perp}$ and orthocomplementation given by E^\perp , for all $E, F \in \mathcal{R}$. Due to P6.8, any realizable event

belongs to \mathcal{R} . In fact, the next proposition proves that the knowledge operator is an ortholattice isomorphism and therefore that the set of realizable events is equal to \mathcal{R} and has the structure of an ortholattice. (See Appendix for more mathematical details on the notion of an ortholattice.)

Proposition 7 $K_\Omega : 2^{\mathcal{O}} \rightarrow \mathcal{R}$ is an ortholattice isomorphism.

Proof. Previous results have already shown that K_Ω is injective, preserves intersections and complements. P6.8 entails that K_Ω has values respectively in \mathcal{R} . In addition, to show surjectivity, take any $E \subseteq \Omega$ and let $A = \cup_{\omega \in E^{\perp\perp}} n_{\mathcal{O}}(\omega)$. Then:

- (1) $E \subseteq K_\Omega(A) \subseteq E^{\perp\perp}$,
- (2) If $E \in \mathcal{R}$, then $K_\Omega(A) = E$.

Indeed, by P6.1: $\omega_0 \in E \Rightarrow \omega_0 \in E^{\perp\perp} \Rightarrow n_{\mathcal{O}}(\omega_0) \subseteq \cup_{\omega \in E^{\perp\perp}} n_{\mathcal{O}}(\omega) = A \Rightarrow \omega_0 \in K_\Omega(A)$. On the other hand, fix $\omega_0 \in K_\Omega(A)$ and $\omega \in E^\perp$. It is sufficient to show $\omega \perp \omega_0$. Since $\omega \in E^\perp$, for any $\omega_1 \in E^{\perp\perp}$, it follows $\omega \perp \omega_1$ and $n_{\mathcal{O}}(\omega) \cap n_{\mathcal{O}}(\omega_1) = \emptyset$. Then: $n_{\mathcal{O}}(\omega) \cap A = n_{\mathcal{O}}(\omega) \cap (\cup_{\omega_1 \in E^{\perp\perp}} n_{\mathcal{O}}(\omega_1)) = \cup_{\omega_1 \in E^{\perp\perp}} (n_{\mathcal{O}}(\omega) \cap n_{\mathcal{O}}(\omega_1)) = \emptyset$. Therefore, $\omega \in K_\Omega(A^c)$. Since $\omega_0 \in K_\Omega(A)$, one obtains $\omega \perp \omega_0$ which finally implies (1). At last, (2) follows from the definition of \mathcal{R} . There remains to show that K_Ω preserves unions. Let $A, B \subseteq \mathcal{O}$ and $E = K_\Omega(A^c)$ and $F = K_\Omega(B^c)$:

$$\begin{aligned}
K_\Omega(A) \vee K_\Omega(B) &= (K_\Omega(A) \cup K_\Omega(B))^{\perp\perp} \text{ by definition of } \vee, \\
&= [K_\Omega(A)^\perp \cap K_\Omega(B)^\perp]^\perp \text{ (see Appendix),} \\
&= [K_\Omega(A^c) \cap K_\Omega(B^c)]^\perp \text{ by P6.8,} \\
&= K_\Omega(A^c \cap B^c)^\perp \text{ by P6.4,} \\
&= K_\Omega((A \cup B)^c)^\perp, \\
&= K_\Omega(A \cup B) \text{ by P6.8.}
\end{aligned}$$

Finally, K_Ω is an ortholattice isomorphism. ■

With parsimony, each observable partition over \mathcal{O} corresponds to a genuine Savage partition over Ω . With state ambiguity, an observable partition might correspond to more than one Savage partition. For instance, in Ellsberg Urn 1, the partition $(\{r, b\}, \{g\})$ corresponds to both Savage partitions $(\{R, B, BG\}, \{G\})$ and $(\{R, B\}, \{G, BG\})$. However, the knowledge operator offers a specific way to translate observable partitions into *orthopartitions* over Ω . A family $(E_i)_{i=1}^n$ of realizable events is an orthopartition over Ω if for all $i \in [1, n]$, $E_i^\perp = \bigvee_{j \neq i} E_j$, which implies in particular $\bigvee_{i=1}^n E_i = \Omega$ and $E_i \wedge E_j = \emptyset$, for all $i \neq j$, while respecting the orthogonality structure, i.e. any two states in two different cells are orthogonal. As an example, each realizable event $E \in \mathcal{R}$ induces an orthopartition (E, E^\perp) . Then, each observable partition $(A_i)_{i=1}^n$ induces, through the knowledge operator, an orthopartition $(K_\Omega(A_i))_{i=1}^n$. In general, orthopartitions are not standard partitions, as suggested in Ellsberg Urn ?? below.

Ellsberg Urn 4 The mappings K_Ω and B_Ω are defined as follows:

A	$\{r\}$	$\{b\}$	$\{g\}$	$\{r, b\}$	$\{r, g\}$	$\{b, g\}$
$K_\Omega(A)$	$\{R\}$	$\{B\}$	$\{G\}$	$\{R, B\}$	$\{R, G\}$	$\{B, G, BG\}$
$B_\Omega(A)$	$\{R\}$	$\{B, BG\}$	$\{G, BG\}$	$\{R, B, BG\}$	$\{R, G, BG\}$	$\{B, G, BG\}$

Nontrivial orthopartitions are given by:

$$\left\{ \begin{array}{l} (\{R\}, \{B\}, \{G\}), \\ (\{R\}, \{B, G, BG\}), \\ (\{B\}, \{R, G\}), \\ (\{G\}, \{R, B\}). \end{array} \right.$$

Only $(\{R\}, \{B, G, BG\})$ is also a standard partition over Ω .

3.4 Measurability

Each family $(K_\Omega(A), K_\Omega(A^c))$, for $A \subseteq \mathcal{O}$, defines an orthopartition over Ω and can be rewritten as $(K_\Omega(A), B_\Omega(A)^c)$ by P6.3. Hence, it naturally amounts to a standard partition over Ω if $K_\Omega(A) = B_\Omega(A)$. When this happens, A is either necessarily realized or impossible at any state. The event A is therefore not affected by state ambiguity and is referred to as a measurable observable event.

Definition 7 An observable event $A \subseteq \mathcal{O}$ is said to be measurable if $K_\Omega(A) = B_\Omega(A)$. A Savage event $E \subseteq \Omega$ is said to be measurable if there exists a measurable observable event $A \subseteq \mathcal{O}$ such that $E = K_\Omega(A)$.

Let $\mathcal{M}_\mathcal{O}$ and $\mathcal{A}_\mathcal{O}^M$ stand respectively for the sets of *measurable events* and *measurable feasible acts*. Similarly, let \mathcal{M}_Ω and \mathcal{A}_Ω^M stand for the sets of *measurable Savage events* and *measurable Savage acts*.

Proposition 8 The following assertions hold:

- (P8.1) For each $A \in \mathcal{M}$, $(K_\Omega(A), K_\Omega(A^c))$ is a partition of Ω .
- (P8.2) For each $A \subseteq \mathcal{O}$, A is measurable iff $B_\Omega(A)$ is realizable.
- (P8.3) For each $E \subseteq \Omega$, E is measurable iff E is realizable and $E^\perp = E^c$.
- (P8.4) $\mathcal{M}_\mathcal{O}$ and \mathcal{M}_Ω are algebras.
- (P8.5) Each of K_Ω and B_Ω induces an algebra isomorphism from $\mathcal{M}_\mathcal{O}$ onto \mathcal{M}_Ω .

Proof. Consider a measurable event $A \subseteq \mathcal{O}$. Then, $K_\Omega(A^c) = B_\Omega(A)^c = K_\Omega(A)^c$ so that $(K_\Omega(A), K_\Omega(A^c))$ is indeed a genuine partition. In addition, if A is measurable, then $B_\Omega(A) = K_\Omega(A)$ is realizable. If $B_\Omega(A)$ is realizable, then there exists $B \subseteq \mathcal{O}$ such that $B_\Omega(A) = K_\Omega(B)$, which, given P6, can be rewritten as: $\forall \omega \in \Omega, n_\mathcal{O}(\omega) \subseteq B \iff n_\mathcal{O}(\omega) \cap A \neq \emptyset$. Fix $o \in B$ and, by P4.7, let $\omega \in \Omega$ be such that $n_\mathcal{O}(\omega) = \{o\}$. Then, $n_\mathcal{O}(\omega) \subseteq B$ implies $n_\mathcal{O}(\omega) \cap A \neq \emptyset$ and $o \in A$. Conversely, fix $o \in A$ and let $\omega \in \Omega$

such that $n_{\mathcal{O}}(\omega) = \{o\}$. Then, $n_{\mathcal{O}}(\omega) \cap A \neq \emptyset$ implies $o \in n_{\mathcal{O}}(\omega) \subseteq B$. Finally, $A = B$ and $K_{\Omega}(A) = B_{\Omega}(A)$, which shows that A is measurable. For P8.3, if E is measurable, then $E = K_{\Omega}(A)$ with A measurable and E clearly realizable. Furthermore, $E^{\perp} = K_{\Omega}(A)^{\perp} = K_{\Omega}(A^c) = B_{\Omega}(A)^c = K_{\Omega}(A)^c = E^c$. Conversely, if E is realizable with $E^{\perp} = E^c$, then there exists $A \subseteq \mathcal{O}$ such that $E = K_{\Omega}(A)$ and $K_{\Omega}(A)^{\perp} = K_{\Omega}(A)^c$. Therefore, $B_{\Omega}(A)^c = K_{\Omega}(A^c) = K_{\Omega}(A)^c$, which delivers the measurability of A and E . Then, first, $\mathcal{M}_{\mathcal{O}}$ contains both \emptyset and \mathcal{O} ; second, it is stable by complementation: if $A \in \mathcal{M}_{\mathcal{O}}$, then $K_{\Omega}(A^c) = B_{\Omega}(A)^c = K_{\Omega}(A)^c = B_{\Omega}(A^c)$; third, it is stable by finite intersections: for all $A, B \subseteq \mathcal{O}$, one gets $B_{\Omega}(A \cap B) \subseteq B_{\Omega}(A) \cap B_{\Omega}(B)$ by monotonicity and, in particular, if $A, B \in \mathcal{M}_{\mathcal{O}}$, then, by P6.4, $B_{\Omega}(A \cap B) \subseteq K_{\Omega}(A) \cap K_{\Omega}(B) = K_{\Omega}(A \cap B) \subseteq B_{\Omega}(A \cap B)$. As a result, $\mathcal{M}_{\mathcal{O}}$ is an algebra. In the same way, note first \emptyset and Ω are measurable; second, an intersection of measurable Savage events is measurable due to P6.4; third, the complement of a measurable Savage event is also measurable due to P6.8. Then, \mathcal{M}_{Ω} is an algebra as well. For P8.5, note that K_{Ω} and B_{Ω} are equal over $\mathcal{M}_{\mathcal{O}}$. Their common restriction to $\mathcal{M}_{\mathcal{O}}$ preserves intersections, like K_{Ω} , and reunions as well, like B_{Ω} . Moreover, if $A \in \mathcal{M}_{\mathcal{O}}$, then, given P6.3, $K_{\Omega}(A^c) = B_{\Omega}(A)^c = K_{\Omega}(A)^c$ so that complements are also preserved. At last, the common restriction is injective, like K_{Ω} or B_{Ω} , takes values in \mathcal{M}_{Ω} and is surjective since, for any $E \in \mathcal{M}_{\Omega}$, there exists $A \subseteq \mathcal{O}$ such that $E = K_{\Omega}(A)$ and necessarily then, by P8.3, $B_{\Omega}(A)^c = K_{\Omega}(A^c) = K_{\Omega}(A)^{\perp} = K_{\Omega}(A)^c$, which gives the measurability of A . ■

Ellsberg Urn 5 The algebra of measurable events can be read directly on the table of Ellsberg Urn ??: $\mathcal{M}_{\mathcal{O}} = \{\emptyset, \{r\}, \{b, g\}, \mathcal{O}\}$. More generally, the algebra of measurability can sometimes be trivial. Take for instance a two-color urn containing red and black balls: $\mathcal{O} = \{r, b\}$. Assume, in addition, that there are three states R, B and RB such that $n_{\mathcal{O}}(R) = \{r\}$, $n_{\mathcal{O}}(B) = \{b\}$ and $n_{\mathcal{O}}(RB) = \{r, b\}$. Then, the corresponding algebra of measurability only contains \emptyset and \mathcal{O} .

Measurability can now be used to define and characterize more formally the notion of state ambiguity and to relate it to the parsimony rule.

Definition 8 (*The TM is said to consider that*) the DM perceives state ambiguity if there exists an observable event that is nonmeasurable.

There are several equivalent definitions of state ambiguity. For instance, the DM perceives state ambiguity if there exists an orthopartition over Ω that is not a standard partition or, equivalently, if all Savage events are realizable. At last, the next proposition relates state ambiguity and parsimony within the epistemic structure constructed in this section.

Proposition 9 *The following assertions are equivalent:*

(P9.1) *The TM conforms to the parsimony rule.*

(P9.2) *(The TM considers that) the DM does not perceive state ambiguity.*

Proof. Assume parsimony and consider two distinct states $\omega, \omega' \in \Omega$. Let $o, o' \in O$ be the respective observations they encode as in the definition of parsimony. Assume $o = o'$ so that for all feasible acts $f, g \in A_{\mathcal{O}}$, $f \succsim_{\omega} g \iff f(o) \succsim_{\Omega} g(o) \iff f(o') \succsim_{\Omega} g(o') \iff f \succsim_{\omega'} g$. Now let $x = \varphi(f)(\omega) \in X$ and let g be the constant feasible act equal to x in each observation. Then, by A2.1, $\varphi(g)(\omega) \sim_{\Omega} x \sim_{\Omega} \varphi(f)(\omega)$, which is equivalent to $f \sim_{\omega} g$ and, therefore, to $f \sim_{\omega'} g$, finally implies $\varphi(f)(\omega') \sim_{\Omega} \varphi(g)(\omega')$. Using again A2.1, one obtains $\varphi(f)(\omega') \sim_{\Omega} x \sim_{\Omega} \varphi(f)(\omega)$ and since this holds for all f , A2.3 implies $\omega = \omega'$, which is absurd. Then, $o \neq o'$. Now, let $A \subseteq O$ be the observable event defined by $A = \{o\}$. On the one hand, $n_{\mathcal{O}}(\omega) = \{o\} \subseteq A$, which can be rewritten as $A \in N_{\mathcal{O}}(\omega)$. On the other hand, $i_{\mathcal{O}}(\omega') = \{o'\}^c \supseteq A$, which can be rewritten as $A \in I_{\mathcal{O}}(\omega')$. Finally, ω and ω' are indeed orthogonal. As a result, under parsimony, any two distinct states are orthogonal. Then, for any $E \subseteq \Omega$, $E^{\perp} = E^c$ and therefore $E^{\perp\perp} = E^{cc} = E$ so that any Savage event is realizable and even measurable, hence P9.2. Conversely, consider a state $\omega \in \Omega$ and let $o, o' \in n_{\mathcal{O}}(\omega)$. Let ω' correspond to o' as in A2.2 so that $n_{\mathcal{O}}(\omega') = \{o'\}$. Then, $\omega \in B_{\Omega}(n_{\mathcal{O}}(\omega'))$ and, since all observable events are assumed to be measurable, $\omega \in K_{\Omega}(n_{\mathcal{O}}(\omega'))$, which, by P6.1, delivers $n_{\mathcal{O}}(\omega) \subseteq n_{\mathcal{O}}(\omega')$. Finally $o = o'$ and each support $n_{\mathcal{O}}(\omega)$ is a singleton. For each $\omega \in \Omega$, let $o \in O$ be such that $n_{\mathcal{O}}(\omega) = \{o\}$. Fix $f \in A_{\mathcal{O}}$ to show $\varphi(f)(\omega) \sim_{\Omega} f(o)$. Let $x = f(o) \in X$ so that $f = x$ over $\{o\}$. Then, $f \sim_{\omega'} x$, for all $\omega' \in K_{\Omega}[\{o\}]$ and, in particular, $f \sim_{\omega} x$. Using A2, it follows that $\varphi(f)(\omega) \sim_{\Omega} x \sim_{\Omega} f(o)$, which delivers parsimony. ■

This result shows that parsimony is not a neutral methodological principle: it necessarily constrains the TM's representation of the DM's behavior. Especially, it rules state ambiguity out. However, when the TM faces a DM making the Ellsberg choices, then the combination of Propositions 3 and 9 prove that he can not but reject parsimony and is finally led to consider that the DM perceives state ambiguity.

4 Representation results

This section derives, from the epistemic tools and notions, the TM's representations of the static and dynamic DM's behavior.

4.1 Representation and measurability

By its simple definition, the algebra $\mathcal{M}_{\mathcal{O}}$ is a state ambiguity-free domain in the sense that each observable event $A \in \mathcal{M}_{\mathcal{O}}$ is either necessarily realized or impossible at any state. The next proposition shows consequently that $\mathcal{M}_{\mathcal{O}}$ is a natural domain of validity of the parsimony rule. At last, since the restriction of observable preferences to $\mathcal{M}_{\mathcal{O}}$ satisfies STP, then measurability is also an ambiguity-free domain.

Proposition 10 *The following assertions hold:*

(P10.1) φ induces an order isomorphism between $(\mathcal{A}_{\mathcal{O}}^M, \succsim_{\mathcal{O}})$ and $(\mathcal{A}_{\Omega}^M, \succsim_{\Omega})$.

(P10.2) *The restriction to $\mathcal{A}_{\mathcal{O}}^{\mathcal{M}}$ of the preference relation $\succsim_{\mathcal{O}}$, assigned to the DM by the TM, satisfies STP.*

Proof. Fix a Savage act $F \in \mathcal{A}_{\Omega}^{\mathcal{M}}$. It is necessarily of the form $F = \sum_{i=1}^n \mathbb{1}_{E_i} x_i$ with x_i pairwise distinct and all $E_i \in \mathcal{M}_{\Omega}$. Let $f \in \mathcal{A}_{\mathcal{O}}^{\mathcal{M}}$ be defined by $f = \sum_{i=1}^n \mathbb{1}_{A_i} x_i$ with $A_i = K_{\Omega}^{-1}(E_i) \in \mathcal{M}_{\mathcal{O}}$. Since, for any i , $f = x_i$ over A_i , one obtains that $\varphi(f)(\omega) \sim_{\Omega} x_i$, $\forall \omega \in E_i$. Then, $\varphi(f)(\omega) \sim_{\Omega} F(\omega)$, $\forall \omega \in \Omega$, and finally, by A1.3, $\varphi(f) \sim_{\Omega} F$. To obtain P10.1, note that φ preserves the order. For P10.2, consider two measurable feasible acts $f, h \in \mathcal{A}_{\mathcal{O}}^{\mathcal{M}}$ and a measurable observable event $A \in \mathcal{M}_{\mathcal{O}}$ so that $(K_{\Omega}(A), K_{\Omega}(A^c))$ is a genuine partition over Ω . Then, for any $\omega \in K_{\Omega}(A)$, since $f_A h = f$ over A , it follows that $\varphi(f_A h)(\omega) \sim_{\Omega} \varphi(f)(\omega)$ and in the same way, for any $\omega \in K_{\Omega}(A^c)$, $\varphi(f_A h)(\omega) \sim_{\Omega} \varphi(h)(\omega)$. Therefore, for any $\omega \in \Omega$, $\varphi(f_A h)(\omega) \sim_{\Omega} (\varphi(f)_{K_{\Omega}(A)} \varphi(h))(\omega)$ and, by A1.3, $\varphi(f_A h) \sim_{\Omega} \varphi(f)_{K_{\Omega}(A)} \varphi(h)$. Then, STP for the restriction of observable preferences follows from their definition and A1.4. ■

Strengthening A1 up to a SEU representation of Savage preferences, Theorem ?? delivers a SEU representation of the restriction of observable preferences to measurable feasible acts.

Theorem 1 *If the preference relation \succsim_{Ω} is SEU wrt some utility function u and some probability measure \mathbb{P} , the following assertions hold:*

(Th1.1) $\forall A \in \mathcal{M}_{\mathcal{O}}$, $P(A) = \mathbb{P}[K_{\Omega}(A)]$ defines a probability measure over $(\mathcal{O}, \mathcal{M}_{\mathcal{O}})$.

(Th1.2) $\forall f, g \in \mathcal{A}_{\mathcal{O}}^{\mathcal{M}}$, $f \succsim_{\mathcal{O}} g \iff \mathbb{E}_{Pu}(f) \geq \mathbb{E}_{Pu}(g)$.

Proof. Th1.1 is a consequence of P8.5. Consider now a measurable feasible act $f \in \mathcal{A}_{\mathcal{O}}^{\mathcal{M}}$. It is of the form: $f = \sum_{i=1}^n \mathbb{1}_{A_i} x_i$ with all x_i pairwise distinct and all A_i measurable. Since $f = x_i$ over A_i , $u(\varphi(f)(\omega)) = u(x_i)$, for all $\omega \in K_{\Omega}(A_i)$. Moreover, the family $(K_{\Omega}(A_i))_{i=1}^n$ is a partition of Ω . Then, the expected utility induced by $\varphi(f)$ can be written in the following way:

$$\mathbb{E}_{\mathbb{P}u}(\varphi(f)) = \sum_{i=1}^n u(x_i) \mathbb{P}[K_{\Omega}(A_i)] = \mathbb{E}_{Pu}(f).$$

Th1.2 then follows from the definition of observable preferences. ■

4.2 Representation and observability

Theorem 2 gives a functional representation of observable preferences over the full set $\mathcal{A}_{\mathcal{O}}$ of feasible acts under the assumption that Savage preferences are SEU. It also derives more explicit representations through more specific assumptions on the mapping φ that involve Choquet Expected Utility (CEU) preferences (Schmeidler, 1989). These results could be adapted to the case of Maxmin Expected Utility (MEU) preferences (Gilboa and Schmeidler, 1989).

Theorem 2 *Assume that Savage preferences \succsim_{Ω} are SEU wrt some utility function u and some probability measure \mathbb{P} over Ω .*

(Th2.1) $\forall f, g \in \mathcal{A}_{\mathcal{O}}, f \succsim_{\mathcal{O}} g \iff \mathbb{E}_{\mathbb{P}}u(\varphi(f)) \geq \mathbb{E}_{\mathbb{P}}u(\varphi(g))$.

Furthermore, if each \succsim_{ω} is CEU wrt u and some capacity ν_{ω} , then:

(Th2.2) $\succsim_{\mathcal{O}}$ is CEU wrt u and capacity ν_0 defined by $\nu_0(A) = \sum_{\omega \in \Omega} \nu_{\omega}(A) \mathbb{P}[\{\omega\}]$, $\forall A \subseteq \mathcal{O}$,

(Th2.3) $\forall A \subseteq \mathcal{O}, \mathbb{P}[K_{\Omega}(A)] \leq \nu_0(A) \leq \mathbb{P}[B_{\Omega}(A)]$.

Proof. Th2.1 is obtained through Def. 1 and the SEU assumption for Savage preferences. Moreover, Th2.2 follows from the equality $\mathbb{E}_{\mathbb{P}}u(\varphi(f)) = \mathbb{E}_{\nu_0}u(f)$, for all feasible act $f \in \mathcal{A}_{\mathcal{O}}$. At last, if $\omega \in K_{\Omega}(A)$, then $x_A y \sim_{\omega} x$, for all $x, y \in X$ and $\nu_{\omega}(A) = 1$. Similarly, if $\omega \notin B_{\Omega}(A)$, then $x_A y \sim_{\omega} y$, for all $x, y \in X$ and $\nu_{\omega}(A) = 0$, hence Th2.3. ■

First, Th2.2 gives conditions under which observable preferences are consistent with CEU. Then, Th2.3 imposes a consistency requirement between the capacity ν_0 revealed by observable preferences and the epistemic operators of knowledge and belief implied by the hermeneutic mapping: the epistemic operators set bounds to the degree of ambiguity aversion revealed by observable preferences (Ghirardato and Marinacci, 2002). Using Proposition 7 and the properties of orthogonality, the lower bound $\mathbb{P}[K_{\Omega}(\cdot)]$ can be shown to be a *belief function* while the upper bound $\mathbb{P}[B_{\Omega}(\cdot)]$ corresponds to the dual *plausibility function* (Dempster, 1967). Consistently with Theorem 1, these two bounds coincide over the algebra of measurability. In addition, the definition of ν_0 is reminiscent of the logic of Möbius inversion (Dempster, 1967) and its decision-theoretic implementation (Mukerji, 1997). In these contributions, the family $(\nu_{\omega})_{\omega \in \Omega}$ degenerates into a linearly independent family of unanimity games on \mathcal{O} and, therefore, the probability measure \mathbb{P} can be uniquely inferred from the capacity ν_0 . This raises the question of whether it is possible here to enrich A2 with an additional axiom that would ensure the linearly independence of $(\nu_{\omega})_{\omega \in \Omega}$ and thus the possibility for the TM to derive uniquely \mathbb{P} from observable preferences.

4.3 Representation and dynamics

Models of ambiguity (Schmeidler, 1989; Gilboa and Schmeidler, 1989) rationalize the Ellsberg choices interpreted as violations of STP. In these approaches, parsimony is typically maintained while dynamic consistency is typically lost. Here, the Ellsberg choices are interpreted as revealing state ambiguity and lead to an alternative rationalization in which the parsimony rule is violated while dynamic consistency is always preserved.

Theorem 3 *There exists a family of preference relations $(\succsim_E)_{E \subseteq \Omega}$ over feasible acts such that for all $E \subseteq \Omega$:*

(Th3.1) $\forall f, g \in \mathcal{A}_{\mathcal{O}}, f \succsim_E g$ and $f \succsim_{E^c} g \implies f \succsim_{\mathcal{O}} g$.

(Th3.2) $\forall f, g \in \mathcal{A}_{\mathcal{O}}, \forall \omega \in E, f \succsim_{\omega} g \implies f \succsim_E g$.

Proof. Since \succsim_{Ω} satisfies STP, it is possible to define a family $(\succsim_E)_{E \subseteq \Omega}$ of preferences over Savage acts in the following way: for all $E \subseteq \Omega$ and all $F, G \in \mathcal{A}_{\Omega}$, $F \succsim_E G \iff \exists H \in \mathcal{A}_{\Omega}$ such that $F_E H \succsim_{\Omega} G_E H$. The two conditions hereafter hold for all E :

$$\left\{ \begin{array}{l} (1) \forall F, G \in \mathcal{A}_{\Omega}, F \succsim_E G \text{ and } F \succsim_{E^c} G \implies F \succsim_{\Omega} G, \\ (2) \forall F, G \in \mathcal{A}_{\Omega}, (\forall \omega \in E, F(\omega) \succsim_{\Omega} G(\omega)) \implies F \succsim_E G. \end{array} \right.$$

Indeed, if $F \succsim_E G$ and $F \succsim_{E^c} G$, then by definition and STP, $F_E G \succsim_\Omega G_E G$ and $F_E F \succsim_\Omega F_E G$, and, by A1.2, $F \succsim_\Omega G$. In addition, if $F(\omega) \succsim_\Omega G(\omega) \forall \omega \in E$, then $F_E H(\omega) \succsim_\Omega G_E H(\omega)$ for all $\omega \in \Omega$ and all H and, by A1.3, $F_E H \succsim_\Omega G_E H$ so that $F \succsim_E G$. Then, *ex post* preferences over feasible acts can be defined by: $\forall E \subseteq \Omega, \forall f, g \in \mathcal{A}_\mathcal{O}, f \succsim_E g \iff \varphi(f) \succsim_E \varphi(g)$. Th3.1-2 follow from (1) and (2). ■

As a result of Theorem 3, observable behavior which violates STP and therefore generates dynamic inconsistency over \mathcal{O} still admits a dynamically consistent representation over Ω . Consequently, dynamic consistency appears to rather be a property of the TM's representation than a property of the DM's behavior. Furthermore, the dynamically consistent representation constructed by the TM also delivers an *epistemic rationalization* of observable dynamic inconsistencies, which involves two families of preferences, one for each epistemic operator: $(\succsim_A^K)_{A \subseteq \mathcal{O}}$ and $(\succsim_A^B)_{A \subseteq \mathcal{O}}$.

Proposition 11 *There exists two families $(\succsim_A^K)_{A \subseteq \mathcal{O}}$ and $(\succsim_A^B)_{A \subseteq \mathcal{O}}$ of preference relations over feasible acts such that for all $A \subseteq \mathcal{O}$:*

- (P11.1) $\forall f, g \in \mathcal{A}_\mathcal{O}, \forall A \in \mathcal{M}_\mathcal{O}, f \succsim_A^K g \iff f \succsim_A^B g$.
- (P11.2) $\forall f, g \in \mathcal{A}_\mathcal{O}, f \succsim_A^K g$ and $f \succsim_{A^c}^B g \implies f \succsim_\mathcal{O} g$.
- (P11.3) $\forall f, g \in \mathcal{A}_\mathcal{O}, \forall \omega \in K_\Omega(A), f \succsim_\omega g \implies f \succsim_A^K g$.
- (P11.4) $\forall f, g \in \mathcal{A}_\mathcal{O}, \forall \omega \in B_\Omega(A), f \succsim_\omega g \implies f \succsim_A^B g$.

Proof. Let the preferences in the families $(\succsim_A^K)_{A \subseteq \mathcal{O}}$ and $(\succsim_A^B)_{A \subseteq \mathcal{O}}$ be defined for each observable event $A \subseteq \mathcal{O}$ by:

$$\left\{ \begin{array}{l} (1) \forall f, g \in \mathcal{A}_\mathcal{O}, f \succsim_A^K g \iff f \succsim_{K_\Omega(A)} g, \\ (2) \forall f, g \in \mathcal{A}_\mathcal{O}, f \succsim_A^B g \iff f \succsim_{B_\Omega(A)} g. \end{array} \right.$$

If A is measurable, then $K_\Omega(A) = B_\Omega(A)$ and preferences \succsim_A^K and \succsim_A^B coincide. At last, P11.2 follows from Th3.1 and P6.3 while P11.3-4 follow from Th3.2. ■

Epistemic rationalization, as exhibited in Proposition 11, relies on the following idea: conditional upon learning that observable event A is necessarily realized (resp. possible), (the TM thinks that) the DM maximizes \succsim_A^K (resp. \succsim_A^B) subject to feasibility constraints. Then, the DM's behavior as represented by the TM depends only on feasibility constraints and available information if one accepts the idea that information now consists, not only of some observable event, but also of one of the two modalities, knowledge or belief, through which the observable event is processed. More precisely, if an observable event is measurable, then \succsim_A^K and \succsim_A^B coincide. Besides, for any feasible acts f, g and observable event A , if the DM prefers f to g , both when she knows A and when she believes A^c , then she prefers f to g on \mathcal{O} as well. At last, if she prefers f to g at each state where she knows (resp. believes) A , then she prefers f to g when she knows (resp. believes) A as well.

Moreover, epistemic rationalization can be used to produce updating rules. If each local preference relation \succsim_ω is CEU wrt u and ν_ω (as in Theorem 2), then $\succsim_\mathcal{O}, \succsim_A^K$ and \succsim_A^B , for all $A \subseteq \mathcal{O}$, are also CEU wrt u and capacities ν_0, ν_A^K and ν_A^B defined for all observable events $A, A' \subseteq \mathcal{O}$ such that $\mathbb{P}[K_\Omega(A)] > 0$ by:

$$\begin{cases} (1) \nu_0(A') = \sum_{\omega \in \Omega} \nu_\omega(A') \mathbb{P}[\{\omega\}], \\ (2) \nu_A^K(A') = \sum_{\omega \in \Omega} \nu_\omega(A') \mathbb{P}[\{\omega\} | K_\Omega(A)], \\ (3) \nu_A^B(A') = \sum_{\omega \in \Omega} \nu_\omega(A') \mathbb{P}[\{\omega\} | B_\Omega(A)]. \end{cases}$$

Ellsberg Urn 6 The table given hereafter computes ν_0 , $\nu_{\{r,b\}}^K$ and $\nu_{\{r,b\}}^B$ under the additional assumption that $P(\{R\}) = P(\{BG\}) = 1/3$ and $P(\{B\}) = P(\{G\}) = 1/6$:

A	$\{r\}$	$\{b\}$	$\{g\}$	$\{r, b\}$	$\{r, g\}$	$\{b, g\}$
$\nu_0(A)$	$\frac{1}{3}$	$\frac{1}{6} + \frac{1}{3}p$	$\frac{1}{6} + \frac{1}{3}q$	$\frac{1}{2} + \frac{1}{3}p$	$\frac{1}{2} + \frac{1}{3}q$	$\frac{2}{3}$
$\nu_{\{r,b\}}^K(A)$	$\frac{2}{3}$	$\frac{1}{3}$	0	1	$\frac{2}{3}$	$\frac{1}{3}$
$\nu_{\{r,b\}}^B(A)$	$\frac{2}{5}$	$\frac{1}{5} + \frac{2}{5}p$	$\frac{2}{5}q$	$\frac{3}{5} + \frac{2}{5}p$	$\frac{2}{5} + \frac{2}{5}q$	$\frac{3}{5}$

Assume that the DM follows the Ellsberg choices. It must be the case that $p < \frac{1}{2}$ and $q < \frac{1}{2}$. To apply the above updating rule at the upper decision node (udn) of the dynamic decision trees discussed in Introduction, different cases must be distinguished for each tree.

$$\begin{cases} \text{Case 1K:} & \text{at udn of situation 1, } \{r, b\} \text{ is necessarily realized,} \\ \text{Case 1B:} & \text{at udn of situation 1, } \{r, b\} \text{ is possible,} \\ \text{Case 2K:} & \text{at udn of situation 2, } \{r, b\} \text{ is necessarily realized,} \\ \text{Case 2B:} & \text{at udn of situation 2, } \{r, b\} \text{ is possible.} \end{cases}$$

It is only in case 2B that the DM chooses to bet on blue and, in all other cases, she bets on red. More specifically, cases 1K and 2K both illustrate P11.2 and the fact that choices are here the same illustrates P11.3. In contrast, choices in cases 1B and 2B are not the same since feasibility constraints differ: due to the ambiguous state BG that belongs to $B_\Omega(\{r, b\})$, the outcome when the ball is green affects behavior at udn.

4.4 Representation and realizability

Theorem 2 provides a standard functional representation of observable preferences $\succsim_{\mathcal{O}}$. This section rather derives a representational framework in studying the appropriate way to represent the DM's observable preferences $\succsim_{\mathcal{O}}$ within the TM's state space Ω . Such an approach relies upon a modified version ψ of the hermeneutic mapping φ . While the latter maps feasible acts onto Savage acts, the former maps feasible acts onto realizable acts defined as follows: a *realizable act* is a term $(E_i, x_i)_{i=1}^n$ made of an orthopartition $(E_i)_{i=1}^n$ over the lattice \mathcal{R} of realizable events and a corresponding family of pairwise distinct outcomes $(x_i)_{i=1}^n$, i.e. each x_i is the outcome associated to the realizable event E_i . The set of realizable acts is denoted $\mathcal{A}_{\mathcal{R}}$.

Each feasible act $f \in \mathcal{A}_{\mathcal{O}}$ can also be written in the form $(A_i, x_i)_{i=1}^n$ with $(A_i)_{i=1}^n$ a standard partition over \mathcal{O} and $(x_i)_{i=1}^n$ a corresponding family of pairwise distinct outcomes, i.e. $f(o) = x_i, \forall o \in A_i, \forall i \in [1, n]$. For each feasible act $f \in \mathcal{A}_{\mathcal{O}}$ of the form $(A_i, x_i)_{i=1}^n$, let $\psi(f)$ be the realizable act defined by $(K_\Omega(A_i), x_i)_{i=1}^n \in \mathcal{A}_{\mathcal{R}}$. The TM interprets each x_i as a DM's certainty equivalent of f when she thinks that A_i is realized for sure. While the

primitive hermeneutic mapping φ has been used to set up the ortholattice of realizability, the modified mapping ψ is used in the next theorem as the one tool to represent observable preferences within the ortholattice.

Theorem 4 *There exists a family $(\succ_E^{\mathcal{R}})_{E \subseteq \Omega}$ of preferences over $\mathcal{A}_{\mathcal{R}}$ such that for any Savage event $E \subseteq \Omega$:*

(Th4.1) $\forall f, g \in \mathcal{A}_{\mathcal{O}}, f \succ_{\mathcal{O}} g \iff \psi(f) \succ_{\Omega}^{\mathcal{R}} \psi(g)$.

(Th4.2) ψ is an order isomorphism from $(\mathcal{A}_{\mathcal{O}}, \succ_{\mathcal{O}})$ onto $(\mathcal{A}_{\mathcal{R}}, \succ_{\Omega}^{\mathcal{R}})$.

(Th4.3) $\forall a, b \in \mathcal{A}_{\mathcal{R}}, a \succ_E^{\mathcal{R}} b$ and $a \succ_{E^c}^{\mathcal{R}} b \implies a \succ_{\Omega}^{\mathcal{R}} b$.

(Th4.4) $\forall a, b \in \mathcal{A}_{\mathcal{R}}, \forall \omega \in E, a \succ_{\{\omega\}}^{\mathcal{R}} b \implies a \succ_E^{\mathcal{R}} b$.

Proof. Let χ be the function mapping realizable acts onto feasible acts and defined for any $a = (E_i, x_i)_{i=1}^n \in \mathcal{A}_{\mathcal{R}}$ by $\chi(a)(o) = x_i$ for all $o \in K_{\Omega}^{-1}(E_i)$ and all i . Preferences $\succ_{\Omega}^{\mathcal{R}}$ are then defined by: $\forall a, b \in \mathcal{A}_{\mathcal{R}}, a \succ_{\Omega}^{\mathcal{R}} b \iff \chi(a) \succ_{\mathcal{O}} \chi(b)$. Note that χ is the inverse mapping of ψ , hence the result. In addition, for any Savage event $E \subseteq \Omega$ and $a, b \in \mathcal{A}_{\mathcal{R}}$, let $a \succ_E^{\mathcal{R}} b \iff \chi(a) \succ_E \chi(b)$. Then, Th4.3-4 follow directly from Th3.1-2. ■

In the face of the Ellsberg choices, the TM is led to introduce nonobservable states. Nevertheless, such enrichments are not fully arbitrary: Theorem 4 shows that the primitive objects $(\Omega, \varphi, \succ_{\Omega})$ are constrained by nonrefutation to induce a lattice of realizability that is isomorphic to the algebra of observability, not only in terms of events (Proposition 7), but also in terms of preferences (Theorem 4). Moreover, Theorem 4 explains how dynamic consistency can be reformulated within the ortholattice of realizability.

Ellsberg Urn 7 By construction, $\succ_{\Omega}^{\mathcal{R}}$ rationalizes the Ellsberg choices, if $\succ_{\mathcal{O}}$ also does:

$$a_1 \succ_{\Omega}^{\mathcal{R}} a_2 \quad \text{and} \quad b_1 \succ_{\Omega}^{\mathcal{R}} b_2$$

where:

$$\left\{ \begin{array}{ll} a_1 = ((\{R\}, \{B, G, BG\}), (1, 0)) \text{ is a bet on red} & \implies \text{no state ambiguity.} \\ a_2 = ((\{B\}, \{R, G\}), (1, 0)) \text{ is a bet on blue} & \implies \text{state ambiguity.} \\ b_1 = ((\{B, G, BG\}, \{R\}), (1, 0)) \text{ is a bet on not-red} & \implies \text{no state ambiguity.} \\ b_2 = ((\{R, G\}, \{B\}), (1, 0)) \text{ is a bet on not-blue} & \implies \text{state ambiguity.} \end{array} \right.$$

The TM finally thinks that the DM prefers the bets that are not affected by state ambiguity.

Finally, the specific normative approach to theory-making that is undertaken here produces a concrete recommendation on generalizing the SEU assumption: while models of ambiguity typically weaken the Savage axioms, this approach rather weakens the Savage framework itself. Indeed, it ends up justifying the existence of an orthogonality relation over the state space (Definition 4) and a preference relation over the corresponding set of realizable acts (Theorem 4) and suggests the following interpretative rule:

- (a') Realizable events are nothing more than observable events.
- (b') Realizable preferences represent observable behavior.

In this context, the usual Savage framework corresponds to the assumption that the DM perceives no state ambiguity, i.e. that any two distinct states are orthogonal. Yet the Ellsberg choices precisely reveal the DM's perception of ambiguous states and thus the inadequacy of the Savage framework. They do no longer contradict the existence of a probability measure on the state space and are therefore rationalized in a dynamically consistent way.

5 Appendix

5.1 Order isomorphisms

An algebra \mathcal{F} over Ω is a subset $\mathcal{F} \subseteq 2^\Omega$ such that $\forall E, F \in \mathcal{F}$:

- (1) $\emptyset \in \mathcal{F}$,
- (2) $E^c \in \mathcal{F}$,
- (3) $E \cup F \in \mathcal{F}$.

Let \mathcal{M}_Ω and $\mathcal{M}_\mathcal{O}$ be algebras over Ω and \mathcal{O} respectively. An algebra isomorphism is a bijective application $L : \mathcal{M}_\mathcal{O} \rightarrow \mathcal{M}_\Omega$ that preserves complements and reunions and such that $L(\emptyset) = \emptyset$. Let \mathcal{A}_Ω^M and $\mathcal{A}_\mathcal{O}^M$ stand respectively for the sets of measurable Savage acts and measurable feasible acts. Given two binary relations \succsim_Ω and $\succsim_\mathcal{O}$ on \mathcal{A}_Ω and $\mathcal{A}_\mathcal{O}$, an application $\varphi : \mathcal{A}_\mathcal{O}^M \rightarrow \mathcal{A}_\Omega^M$ is said to be an *order isomorphism* between $(\mathcal{A}_\mathcal{O}^M, \succsim_\mathcal{O})$ and $(\mathcal{A}_\Omega^M, \succsim_\Omega)$ if:

- (1) $\forall f, g \in \mathcal{A}_\mathcal{O}^M, f \succsim_\mathcal{O} g \iff \varphi(f) \succsim_\Omega \varphi(g)$,
- (2) $\forall F \in \mathcal{A}_\Omega^M, \exists f \in \mathcal{A}_\mathcal{O}^M, \varphi(f) \sim_\Omega F$.

In Proposition 2, this definition applies to algebras $\mathcal{M}_\mathcal{O} = 2^\mathcal{O}$ and $\mathcal{M}_\Omega = 2^\Omega$.

5.2 Lattices and lattice acts

Let \wedge and \vee be binary operations on some subset $\mathcal{L} \subseteq 2^\Omega$. Then, $(\mathcal{L}, \wedge, \vee)$ is called a *lattice* if for all $E, F, G \in \mathcal{L}$:

- (1) $E \wedge E = E$ and $E \vee E = E$,
- (2) $E \wedge F = F \wedge E$ and $E \vee F = F \vee E$,
- (3) $E \wedge (F \wedge G) = (E \wedge F) \wedge G$ and $E \vee (F \vee G) = (E \vee F) \vee G$,
- (4) $E \wedge (E \vee F) = E \vee (E \wedge F) = E$.

A lattice $(\mathcal{L}, \wedge, \vee)$ induces (and is in fact equivalent to) an ordered structure (\mathcal{L}, \leq) defined for all $E, F \in \mathcal{L}$ by $E \leq F$ if $E = E \wedge F$. A lattice $(\mathcal{L}, \wedge, \vee)$, with a least element 0 and a greatest element 1, equipped with a mapping $(E \in \mathcal{L} \rightarrow E^\perp \in \mathcal{L})$ that satisfies the three conditions hereafter for all $E, F \in \mathcal{L}$ is called an *ortholattice* $(\mathcal{L}, \wedge, \vee, \perp, 0, 1)$:

- (1) $E^\perp \vee E = 1$ and $E^\perp \wedge E = 0$,
- (2) $E^{\perp\perp} = E$,
- (3) $E \leq F \implies F^\perp \leq E^\perp$.

An ortholattice isomorphism between two ortholattices is a bijective mapping that preserves operations and complements. Given an ortholattice $(\mathcal{L}, \wedge, \vee, \perp, 0, 1)$, an orthopar-

tion is defined as a family of elements $(E_i)_{i=1}^n$ such that for all $i \in [1, n]$, $E_i^\perp = \bigvee_{j \neq i} E_j$. In addition, a *lattice act* is defined as a term $a = (E_i, x_i)_{i=1}^n$ made of an orthopartition $(E_i)_{i=1}^n$ over \mathcal{L} and a corresponding family of pairwise distinct outcomes $(x_i)_{i=1}^n$. Their set is denoted $\mathcal{A}_{\mathcal{L}}$. Given two binary relations $\succsim_{\mathcal{R}}$ and $\succsim_{\mathcal{O}}$ on $\mathcal{A}_{\mathcal{R}}$ and $\mathcal{A}_{\mathcal{O}}$, an application $\psi : \mathcal{A}_{\mathcal{O}} \rightarrow \mathcal{A}_{\mathcal{R}}$ is said to be an *order isomorphism* between $(\mathcal{A}_{\mathcal{O}}, \succsim_{\mathcal{O}})$ and $(\mathcal{A}_{\mathcal{R}}, \succsim_{\Omega}^{\mathcal{R}})$ if:

- (1) $\forall f, g \in \mathcal{A}_{\mathcal{O}}, f \succsim_{\mathcal{O}} g \iff \psi(f) \succsim_{\Omega}^{\mathcal{R}} \psi(g)$,
- (2) $\forall a \in \mathcal{A}_{\mathcal{R}}, \exists f \in \mathcal{A}_{\mathcal{O}}, \psi(f) \sim_{\mathcal{R}} a$.

5.3 Orthospaces

Consider an orthogonality relation \perp on Ω assumed to be antireflexive and symmetric. For each $E \subseteq \Omega$, let $E^\perp = \{\omega \in \Omega, \forall e \in E, \omega \perp e\}$. Then, the following are standard properties:

- (1) $E \subseteq E^{\perp\perp}$.
- (2) $E \subseteq F \implies F^\perp \subseteq E^\perp$.
- (3) $E^\perp = E^{\perp\perp\perp}$.
- (4) $(E \cup F)^\perp = E^\perp \cap F^\perp$.

Consider the subset $\mathcal{R} = \{E \subseteq \Omega, E^{\perp\perp} = E\}$. It has the structure of an ortholattice with respect to the following operations:

- (1) $E \vee F = (E \cup F)^{\perp\perp}$.
- (2) $E \wedge F = E \cap F$.
- (3) Least element is \emptyset , greatest element is Ω .
- (4) Orthocomplementation is given by $(E \rightarrow E^\perp)$.

6 References

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