DUAL GARSIDE STRUCTURE OF BRAIDS AND FREE CUMULANTS OF PRODUCTS

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ABSTRACT. We count the n-strand braids whose normal decomposition has length at most 2 in the dual braid monoid B_n^{+*} by reducing the question to a computation of free cumulants for a product of independent variables, for which we establish a general formula.

1. INTRODUCTION

A Garside structure on a group G is a generating family S that gives rise to distinguished decompositions of a particular type, leading in good cases to an automatic structure on G and, from there, to solutions of the word and conjugacy problems of G [7]. In the case of the n-strand Artin braid group B_n, two Garside structures are known: the so-called classical Garside structure, in which the distinguished generating family is a copy of the symmetric group $\mathfrak{S}_n$ [8, Chapter 9], and the so-called *dual Garside structure*, in which the distinguished generating family is a copy of the family $\mathrm{NC}(n)$ of all size n noncrossing partitions [1].

Whenever a finite Garside structure S is given on a group G, natural counting problems arise, namely counting how many elements of the group G or of the submonoid generated by S have length (at most) ℓ with respect to S. Call S-normal the distinguished decompositions associated with a Garside structure S. As S-normal sequences happen to be geodesic, the above question amounts to counting S-normal sequences of length ℓ. And, because S-normality is a purely local property, the central question is to determine S-normal sequences of length two, the general case then corresponding to taking the ℓth power of the incidence matrix associated with length two.

Initially motivated by the investigation of the logical strength of certain statements involving the standard braid ordering [5], the above mentioned counting questions in the case of the classical Garside structure of braids have been addressed in [6], leading to nontrivial results involving Solomon's descent algebra and to natural conjectures, like the one established in [10] using the theory of quasi-symmetric functions.

The aim of this paper is to address similar questions in the case of the dual Garside structure of braids and to obtain an explicit determination of

the generating function for the number of normal sequences of length two (here Cat_n denotes the nth Catalan number):

Proposition 1.1. *Let $b_{n,2}^*$ be the number of braids of length at most 2 in the dual braid monoid B_n^{+*}. Then the function $R(z) = 1 + \sum_{n \geqslant 1} b_{n,2}^* z^n$ is connected with $M(z) = 1 + \sum_{n \geqslant 1} \mathrm{Cat}_n^2 z^n$ by the equality*

$$(1.1) \qquad\qquad R(zM(z)) = M(z).$$

This formula, which inductively determines the numbers $b_{n,2}^*$, will be deduced from a general formula for computing the free cumulants of a product of independent random variables.

Free cumulants where invented by Roland Speicher in order to compute with free random variables [12]. In particular, free cumulants give a simple way of computing the free additive convolution of two probability measures on the real line, namely the free cumulants of the sum of two free random variables are the sums of the free cumulants of the variables. Free cumulants also appear when one tries to compute the free cumulants of a product of free random variables, although in a more complicated way.

Here is the general formula we establish (the notations are standard and explained in section 3):

Theorem 1.2. *Let $X_1, \ldots, X_k$ be a family of commuting independent random variables, and let $R_n^{(i)}$ be the free cumulants of X_i. Then the free cumulants of the product $X_1 X_2 \cdots X_k$ are given by*

$$(1.2) \qquad\qquad R_n = \sum_{\pi_1 \vee \cdots \vee \pi_k = \mathbf{1}_n} \prod_i R_{\pi_i}^{(i)},$$

the sum being over all k-tuples of noncrosssing partitions in $\mathrm{NC}(n)$ whose join is the largest partition.

Free cumulants are specially designed to deal with highly noncommuting objects, therefore it may come as a surprise that there is also a simple formula for computing free cumulants of a product of independent commuting random variables, in terms of the free cumulants of the factors. Actually the formula also holds, with appropriate modifications, for classical and Boolean cumulants.

The organization of the paper is as follows. In Section 2, we recall the description of the dual Garside structure of braid groups and raise the induced counting questions. In Section 3, we review basic definitions about free cumulants and establish Theorem 1.2. This part can be read independently of the rest of the paper. In Section 4, we apply the result of Section 3 to braids and conclude with further questions and one additional result about the determinant of the involved incidence matrix.

2. The dual Garside structure of braids

Braid groups. For $n \geqslant 1$, the n-strand braid group B_n is the group defined by the presentation

$$(2.1) \qquad B_n = \left\langle \sigma_1, ..., \sigma_{n-1} \; \middle| \; \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } |i - j| \geqslant 2 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{for } |i - j| = 1 \end{array} \right\rangle.$$

The group B_n is both the group of isotopy classes of n-strand geometric braids, the mapping class group of an n-punctured disk, and the fundamental group of the configuration space obtained by letting the symmetric group act on the complement of the diagonal hyperplanes in $\mathbb{C}^n$ [3].

Garside structures. A *Garside structure* in a group G is a subset S of G such that every element of G admits a decomposition of a certain syntactic form in terms of the elements of S, namely a symmetric S-normal decomposition in the following sense.

Definition 2.1. [7] Assume that G is a group and S is included in G.

(i) A finite S-sequence $(s_1, ..., s_d)$ is called *S-normal* if, for $i < d$, every element of S left-dividing $s_i s_{i+1}$ left-divides s_i, where "f left-divides g" means "$f^{-1} g$ lies in the submonoid $\widehat{S}$ of G generated by S".

(ii) A pair of finite S-sequences $((s_1, ..., s_d), (t_1, ..., t_e))$ is called *symmetric S-normal* if $(s_1, ..., s_d)$ and $(t_1, ..., t_e)$ are S-normal and, in addition, the only element of S left-dividing s_1 and t_1 is 1.

(iii) We say that $(s_1, ..., s_d)$ (resp. $((s_1, ..., s_d), (t_1, ..., t_e))$) is a *decomposition* for an element g of G if $g = s_1 \cdots s_d$ (resp. $g = t_e^{-1} \cdots t_1^{-1} s_1 \cdots s_d$) holds in G.

Every group G is trivially a Garside structure in itself, and the notion is interesting only when S is small, typically when G is infinite and S is finite or, at least, is properly included in G. Under mild assumptions, the existence of a finite Garside structure implies good properties for the group G such as the existence of an automatic structure or the decidability of the word and conjugacy problems.

Whenever S is a finite generating family in a group G, it is natural to consider the numbers

$$N_d^S = \#\{g \in G \mid \|g\|_S = d\},$$

where $\|g\|_S$ refers to the minimal length of an S-decomposition. In the case of a Garside structure, symmetric S-normal decompositions are (essentially) unique, and they are geodesic. Therefore, N_d^S identifies with the number of length d symmetric S-normal sequences. In such a context, the submonoid $\widehat{S}$ of G generated by S is the family of all elements of G whose symmetric S-normal decomposition has an empty denominator, that is, of all elements that admit an S-normal decomposition. Then, it is also natural to introduce the number

$$N_{d,+}^S = \#\{g \in \widehat{S} \mid \|g\|_S = d\},$$

which, by the above remark, is the number of S-normal sequences of length d. It then follows from the definition that a sequence $(s_1, ..., s_d)$ is S-normal if and only if every length two subsequence is S-normal, and the basic question is therefore to investigate the numbers

$$(2.2) \qquad N_{2,+}^S = \#\{(s_1, s_2) \in S^2 \mid (s_1, s_2) \text{ is } S\text{-normal}\}.$$

The classical Garside structure of B_n. In the case of the braid group B_n, two Garside structures are known. The first one, often called *classical*, involves permutations. By (2.1), mapping σ_i to the transposition $(i, i{+}1)$ induces a surjective homomorphism $\mathrm{pr}_n : B_n \to \mathfrak{S}_n$. Considering the positive braid diagrams in which any two strands cross at most once ("simple braids") provides a set-theoretic section for pr_n, whose image is a copy S_n of $\mathfrak{S}_n$ inside B_n. The family S_n is a Garside family in B_n [8, Chapter 9], the submonoid of B_n generated by S_n being the submonoid B_n^+ generated by $\sigma_1, ..., \sigma_{n-1}$ [9]. The associated numbers $N_{d,+}^{S_n}$ have been investigated in [6]. In particular, writing $b_{n,d}$ for $N_{d,+}^{S_n}$, it is shown that the numbers $b_{n,2}$ are determined by the induction

$$(2.3) \qquad b_{n,2} = \sum_{i=0}^{n-1} (-1)^{n+i+1} \binom{n}{i}^2 b_{i,2}$$

and that the double exponential series $\sum b_{n,2} z^n / (n!)^2$ is the inverse of the Bessel function $J_0(\sqrt{z})$.

The dual Garside structure of B_n. It is known since [4] that, for every n, there exists an alternative Garside structure on B_n, namely the *dual* Garside structure S_n^*, whose elements are in one-to-one correspondence with the noncrossing partitions of $\{1, 2, ..., n\}$. The question then naturally arises of determining the derived numbers $N_d^{S_n^*}$ and $N_{d,+}^{S_n^*}$. Here we shall concentrate on the latter, hereafter denoted $b_{n,d}^*$, and specifically on $b_{n,2}^*$. By (2.2), we have

$$(2.4) \qquad b_{n,d}^* = \#\{(s_1, ..., s_d) \in (S_n^*)^d \mid \forall i < n \, ((s_i, s_{i+1}) \text{ is } S_n^*\text{-normal})\}.$$

In order to compute the numbers $b_{n,d}^*$, we shall describe the correspondence between S_n^* and noncrossing partitions and interpret the S_n^*-normality condition in terms of the latter.

We recall that a set partition of the set $\{1, ..., n\}$ is called noncrossing if there is no quadruple $1 \leqslant i < j < k < l \leqslant n$ such that i and k belong to some part of the partition, whereas j and l belong to another part. The set of noncrossing partitions of $\{1, ..., n\}$, denoted by $\mathrm{NC}(n)$, is a poset for the reverse refinement order: for two partitions π and π' one has $\pi \leqslant \pi'$ if and only if each part of π is included in some part of π'. With this order, $\mathrm{NC}(n)$ is a lattice, with largest element $\mathbf{1}_n$ the partition with only one part, and smallest element $\mathbf{0}_n$, the partition into n singletons. We will denote by $\pi_1 \vee \cdots \vee \pi_d$ the join of a family of noncrossing partitions.

Definition 2.2. [4] For $1 \leqslant i < j$, put $a_{i,j} = \sigma_i...\sigma_{j-2}\,\sigma_{j-1}\,\sigma_{j-2}^{-1}...\sigma_i^{-1}$ in B_n. The *dual braid monoid* B_n^{+*} is the submonoid of B_n generated by all elements $a_{i,j}$, and S_n^* is the family of all left-divisors of Δ_n^* in B_n^{+*}, with $\Delta_n^* = \sigma_1\sigma_2\cdots\sigma_{n-1}$.

Note that $\sigma_i = a_{i,i+1}$ holds for every i, hence B_n^{+*} includes B_n^+, a proper inclusion for $n \geqslant 3$.

Proposition 2.3. [4] *The family S_n^* is a Garside structure on the group B_n.*

It is convenient to associate with the n-strand braid $a_{i,j}$ a graphical representation as the chord (i,j) in a disk with n marked vertices on the border as shown on the right.
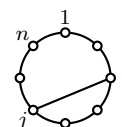
Then the correspondence between the elements of S_n^* and noncrossing partitions stems from the following observation:

Lemma 2.4. [1] *For P a union of disjoint polygons in the n-marked disk, say $P = P_1 \cup \cdots \cup P_d$, let $a_P = a_{P_1}\cdots a_{P_d}$, with $a_{P_k} = a_{i_1,i_2}a_{i_2,i_3}\cdots a_{i_{n_k-1},i_{n_k}}$ where $(i_1,...,i_{n_k})$ is a clockwise enumeration of the vertices of P_k.*
(i) The braid a_P only depends on P and not on the order of enumeration.
(ii) Mapping P to a_P establishes a bijection between unions of disjoint polygons in the n-marked disk and elements of S_n^*.

It is standard to define a bijection between noncrossing partitions of $\{1,...,n\}$ and unions of disjoint polygons in the n-marked disk, yielding the announced correspondence. Noncrossing partitions can be embedded into the symmetric group, using geodesics in the Cayley graph [2]: this amounts to mapping a union of polygons to the product of the cycles obtained by enumerating their vertices in clockwise order. Note that, although noncrossing partitions may be viewed as particular permutations, the associated braids need not coincide: for instance, the cycle (13), which corresponds to the partition $\{\{1,3\},\{2\}\}$, is associated in S_3^* with the braid $a_{1,3}$, that is, $\sigma_1\sigma_2\sigma_1^{-1}$, whereas it associated with $\sigma_1\sigma_2\sigma_1$ in S_3.

For π a noncrossing partition of $\{1,...,n\}$, we shall denote by a_π the associated element of S_n^*. For instance, for $\pi = \{\{1\},\{2,8\},\{3,5,6\},\{4\},\{7\}\}$, we find $a_\pi = a_{2,8}a_{3,5}a_{5,6}$, as shown on the right.
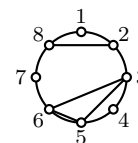
Under the above correspondence, Δ_n^* corresponds to the (unique) n-gon in the n-marked disk, hence to the (noncrossing) partition $\mathbf{1}_n$ with one part. Owing to (2.4), the numbers $b_{n,d}^*$ are determined by

$$(2.5) \quad b_{n,d}^* = \#\{(\pi_1,...,\pi_d) \in \mathrm{NC}(n)^d \mid \forall i < n\,((a_{\pi_i}, a_{\pi_{i+1}}) \text{ is } S_n^*\text{-normal})\},$$

and we have to recognize when the braids associated with two partitions make an S_n^*-normal sequence.

By construction, the Garside structure S_n^* is what is called bounded by the element Δ_n^*, that is, it exactly consists of the left-divisors of Δ_n^* in B_n^{+*}. In this case, the normality condition takes a simple form.

Lemma 2.5. [7, Chapter VI] *Assume that S is a Garside structure that is bounded by an element Δ in a group G. Then, for s, t in S, the pair (s, t) is S-normal if and only if the only common left-divisor of ∂s and t in the monoid $\widehat{S}$ is 1, where ∂s is the element satisfying $s \cdot \partial s = \Delta$.*

We are thus left with the question of recognizing in terms of noncrossing partitions (or of the corresponding unions of polygons) when a braid $a_{i,j}$ left-divides the braid a_π, and what is the partition π' satisfying $a_{\pi'} = \partial a_\pi$.

Lemma 2.6. [1] *Assume that π lies in $\mathrm{NC}(n)$.*
(i) *For $1 \leqslant i < j \leqslant n$, the braid $a_{i,j}$ left-divides a_π in B_n^{+*} if and only if the chord (i, j) lies inside the union of the convex hulls of the polygons associated with π.*
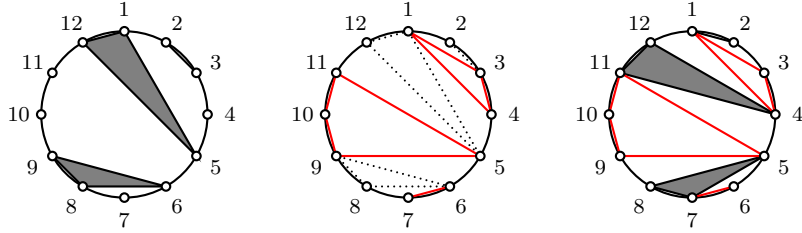(ii) *We have $\partial a_\pi = a_{\overline{\pi}}$, where $\overline{\pi}$ is the Kreweras complement of π.*

FIGURE 1. The polygons associated with a partition π and with its image under the complement map ∂ and ∂^2: the red partition is the Kreweras complement of π, and repeating the operation leads to the image of the initial partition under a rotation by $2\pi/n$, corresponding to conjugating under Δ_n^* in B_n^{+*}.

Putting pieces together, we obtain:

Proposition 2.7. *For all n and d, the number $b_{n,d}^*$ of n-strand braids that have length at most d with respect to the Garside structure S_n^* is given by*

$$(2.6) \qquad b_{n,d}^* = \#\{(\pi_1, ..., \pi_d) \in \mathrm{NC}(n)^d \mid \forall i < n \, (\overline{\pi_i} \wedge \pi_{i+1} = \mathbf{0}_n)\}.$$

Proof. By Lemma 2.6(i), saying that 1 is the only common left-divisor of a_π and $a_{\pi'}$ in B_n^{+*} amounts to saying that the convex hulls of the polygons associated with π and π' have no chord in common, hence are disjoint, so, in other words, that the meet of π and π' in the lattice $\mathrm{NC}(n)$ is the minimal partition $\mathbf{0}_n$. By Lemma 2.6(ii), this condition has to be applied to $\overline{\pi_i}$ and π_{i+1} for every i. $\qquad \square$

3. Free cumulants

In view of Proposition 2.7, we have to count sequences of noncrossing partitions satisfying lattice constraints involving adjacent entries. We shall derive partial results from a general formula expressing the free cumulants of a product of independent random variables.

Noncrossing partitions and free cumulants. We recall some basic facts, more details can be found in [12].

Given a sequence of indeterminates $T_1, ..., T_l, ...$ and a noncrossing partition π in $\mathrm{NC}(n)$, one defines

$$(3.1) \qquad T_\pi = \prod_{p \in \pi} T_{|p|},$$

where the product ranges over all parts p of π and $|p|$ is the number of elements of p in $\{1, ..., n\}$.

Two sequences of indeterminates $M_1, ..., M_l, ...$ and $R_1, ..., R_l, ...$ are related by the moment-cumulant formula if, for every n, one has

$$(3.2) \qquad M_n = \sum_{\pi \in \mathrm{NC}(n)} R_\pi.$$

It is easy to see that this equation can be inverted and the R_l can be expressed as polynomials in the M_l. In fact, introducing the generating series

$$M(z) = 1 + \sum_{l=1}^{\infty} z^l M_l, \qquad R(z) = 1 + \sum_{l=1}^{\infty} z^l R_l,$$

we can recast the relation (3.2) in the form

$$(3.3) \qquad R(zM(z)) = M(z).$$

If the M_l are the moments of a probability measure (or a random variable), then the quantities R_l are called the *free cumulants* of the probability measure (or random variable).

Free cumulants of a product of independent random variables. We establish now the general formula for free cumulants of products of independent random variables stated as Theorem 1.2 in the introduction.

Proof of Theorem 1.2. Let $M_n^{(i)}$ be the moments of X_i. Let us write the moment-cumulant formula for each of the variables in the product $X_1 X_2 \cdots X_k$. As the nth moment of $X_1 X_2 \cdots X_k$ is

$$M_n = M_n^{(1)} \cdots M_n^{(k)},$$

one has

$$M_n = M_n^{(1)} \cdots M_n^{(k)} = \sum_{\pi_1, ..., \pi_k \in \mathrm{NC}(n)} \prod_{i=1}^{k} R_{\pi_i}^{(i)}.$$

Let us now decompose the sum in the right hand side according to the value of $\pi = \pi_1 \vee \cdots \vee \pi_k$. Since $\pi_i \leqslant \pi$ holds, each part of π_i is included in some part

of π. Let p be a part of π. The intersections $\pi_{i,p} := \pi_i \cap p$ form a partition of the set p. If we identify p with $\{1, ..., |p|\}$ by the only increasing bijection, then the sets $\pi_{i,p}$ form noncrossing partitions of $\{1, ..., |p|\}$. Furthermore one has

$$\pi_{1,p} \vee \cdots \vee \pi_{n,p} = \mathbf{1}_{|p|},$$

whence

$$M_n = \sum_{\pi \in \mathrm{NC}(n)} \prod_{p \in \pi} \left[\sum_{\pi_{1,p} \vee \cdots \vee \pi_{n,p} = \mathbf{1}_{|p|}} \prod_i R^{(i)}_{\pi_{i,p}} \right].$$

Defining the sequence Q_n by

$$Q_n = \sum_{\pi_1 \vee \cdots \vee \pi_n = \mathbf{1}_n} \prod_i R^{(i)}_{\pi_i},$$

we obtain

$$M_n = \sum_{\pi \in \mathrm{NC}(n)} \prod_{p \in \pi} Q_{|p|} = \sum_{\pi \in \mathrm{NC}(n)} Q_\pi.$$

Then it follows from (3.2) that the quantities Q_n are the free cumulants of the sequence of moments $M_1, ..., M_n, ...$.   □

The above argument is closely related to an argument in [15]. The first author would like to thank Roland Speicher for pointing out this reference.

Corollary 3.1. *The number of k-tuples $(\pi_1, ..., \pi_k)$ in $\mathrm{NC}(n)^k$ satisfying $\pi_1 \vee \cdots \vee \pi_k = \mathbf{1}_n$ is the nth free cumulant of the variable $X_1^2 \cdots X_k^2$ where $X_1, ..., X_n, ...$ are independent centered semi-circular variables of variance 1.*

Classical and Boolean cumulants. Cumulants can also be defined using the lattice of all set partitions of $\{1, ..., n\}$ (this is the classical case, studied by Rota, Schützenberger...), or the lattice of interval partitions (these are the Boolean cumulants, see [14]). In both cases it is immediate to check that the proof of Theorem 1.2 goes through and gives a formula for computing the corresponding cumulants of a product.

4. Back to braids

We now apply the result of Section 3 to braids.

Incidence matrices. For every n, there exists a binary relation on $\mathrm{NC}(n)$ that encodes S_n^*-normality. We introduce the associated incidence matrix.

Definition 4.1. For $n \geqslant 2$, we let A_n^* be the $\mathrm{Cat}_n \times \mathrm{Cat}_n$-matrix whose entries are indexed by noncrossing partitions, and such that $(A_n^*)_{\pi, \pi'}$ is 1 (*resp.* 0) if $\overline{\pi} \wedge \pi' = \mathbf{0}_n$ holds (*resp.* fails).

For instance, if the partitions of $\{1, 2, 3\}$ are enumerated in the order

 , the matrix A_3^* is $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. It

follows from the properties of $\mathbf{0}_n$ and $\mathbf{1}_n$ that the column of $\mathbf{0}_n$ and the row of $\mathbf{1}_n$ in A_n^* contain only ones, whereas the row of $\mathbf{0}_n$, its $\mathbf{0}_n$-entry excepted, and the column of $\mathbf{1}_n$, its $\mathbf{1}_n$-entry excepted, contain only zeroes.

Proposition 4.2. *For all n and $d \geqslant 1$, the number $b_{n,d}^*$ is the sum of all entries in the matrix $(A_n^*)^{d-1}$; in particular, $b_{n,2}^*$ is the number of positive entries in A_n^*.*

Proof. For π in $\mathrm{NC}(n)$ let $b_{n,d}^*(\pi)$ be the number of S_n^*-normal sequences of length d whose last entry is π. By Proposition 2.6, a length d sequence $(\pi_1, ..., \pi_{d-1}, \pi)$ contributes to $b_{n,d}^*(\pi)$ if and only if $(\pi_1, ..., \pi_{d-1})$ contributes to $b_{n,d-1}^*(\pi_{d-1})$ and (π_{d-1}, π) contributes to $b_{n,2}^*(\pi)$, that is, $(A_n^*)_{\pi_{d-1}, \pi} = 1$ holds. We deduce

$$b_{n,d}^*(\pi) = \sum_{\pi' \in \mathrm{NC}(n)} b_{n,d-1}^*(\pi') \cdot (A_n^*)_{\pi', \pi}.$$

From there, an obvious induction shows that $b_{n,d}^*(\pi)$ is the πth entry in $(1, 1, ..., 1) (A_n^*)^{d-1}$, and the result follows by summing over all π. $\square$

(Note that, for $d = 1$, Proposition 4.2 gives $b_{n,1}^* = \mathrm{Cat}_n$, which is indeed the sum of all entries in the size Cat_n identity-matrix.)

Using Theorem 1.2, we can now complete the proof of Proposition 1.1.

Proof of Proposition 1.1. By Proposition 4.2, $b_{n,2}^*$ is the number of positive entries in the matrix A_n^*, that is, the number of pairs (π, π') in $\mathrm{NC}(n)^2$ satisfying $\overline{\pi} \wedge \pi' = \mathbf{0}_n$. As the Kreweras complement is bijective, this number is also the number of pairs (π, π') in $\mathrm{NC}(n)^2$ satisfying $\pi \wedge \pi' = \mathbf{0}_n$. By complementation, the latter is also the number of pairs satisfying $\pi \vee \pi' = \mathbf{1}_n$. By Corollary 3.1, this number is the nth free cumulant of $X_1^2 X_2^2$ where X_1, X_2 are independent centered semi-circular variables of variance 1. The moments of the latter are the squares of the Catalan numbers, so (3.3) gives the expected result. $\square$

Further questions. It is easy to compute the numbers $b_{n,d}^*$ for small values of n and d, see Table 1.

A natural question is to ask for a description of the columns in Table 1 beyond the first two ones. The characterization of Proposition 1.1 does not extend to $d \geqslant 3$: for instance, we have

$$b_{n,3}^* = \{(\pi_1, \pi_2, \pi_3) \in \mathrm{NC}(n)^3 \mid \overline{\pi_1} \wedge \pi_2 = \mathbf{0}_n \text{ and } \overline{\pi_2} \wedge \pi_3 = \mathbf{0}_n\}:$$

d	1	2	3	4	5	6	7
$b_{1,d}^*$	1	1	1	1	1	1	1
$b_{2,d}^*$	2	3	4	5	6	7	8
$b_{3,d}^*$	5	15	83	177	367	749	1 515
$b_{4,d}^*$	14	99	556	2 856	14 122	68 927	334 632
$b_{5,d}^*$	42	773	11 124	147 855	1 917 046	24 672 817	
$b_{6,d}^*$	132	6 743	266 944	9 845 829	356 470 124		

TABLE 1. The number $b_{n,d}^*$ of n-strand braids of length at most d in the dual braid monoid B_n^{+*}: the first column ($d = 1$) contains the Catalan numbers, whereas the second column contains the sequence specified in Proposition 1.1, which is A168344 in [13].

replacing $\wedge$ and $\mathbf{0}_n$ with $\vee$ and $\mathbf{1}_n$ is easy, but the Kreweras complement cannot be forgotten in this case.

On the other hand, describing the rows in Table 1 leads to further natural questions. By Proposition 4.2, the generating series of the numbers $b_{n,d}^*$ is rational for every n, and $b_{n,d}^*$ can be expressed in terms of the d-power of the eigenvalues of the matrix A_n^*. For instance, one easily finds $b_{3,d}^* = 6 \cdot 2^d - 2d - 5$ for every d, as well as $b_{3,d}^*(\pi) = 2^{d+1} - 1$ for $\pi \neq \mathbf{0}_n, \mathbf{1}_n$ (as above, we write $b_{n,d}^*(\pi)$ for the number of braids with a normal form finishing with π). Very little is known for $n \geqslant 4$.

It would be of interest to compute, or at at least approximate, the spectral radius of the matrix A_n^*. Here are the first values.

n	1	2	3	4	5	6	7
$\rho(A_n^*)$	1	1	2	4.83...	12.83...	35.98...	104.87...

TABLE 2. Spectral radius of the incidence matrix A_n^*

A determinant. Although we are not able to compute explicitly the eigenvalues of the matrix A_n^*, we can give a closed formula for its determinant.

Theorem 4.3. *For every n, we have*

$$|\det(A_n^*)| = \prod_{k=2}^{n} \operatorname{Cat}_{k-1}^{\binom{2n-k-1}{n-1}}.$$

Before proving this formula, we need an auxiliary result involving Möbius matrices. By definition, if X is a finite poset, the associated Möbius matrix μ is the inverse of the order matrix ζ indexed by the elements of X and given by

$$\zeta(x, y) = \begin{cases} 1 & \text{for } x \geq y, \\ 0 & \text{otherwise.} \end{cases}$$

If the elements of X are ordered according to a linear extension of the partial order of X, then ζ is a lower triangular matrix, with ones on the diagonal. In particular, its determinant is 1. The same is true for μ.

Lemma 4.4. *Assume that X is a lattice, and φ is a complex valued function on X. Let Φ be the matrix defined by $\Phi(x, y) = \varphi(x \wedge y)$. Then we have*

$$\det(\Phi) = \prod_{x \in X} \hat{\varphi}(x),$$

where $\hat{\varphi}$ is given by $\hat{\varphi}(x) = \sum_{y \leq x} \mu(x, y)\varphi(y)$.

Note that, under the above assumptions, we have $\varphi(x) = \sum_{y \leq x} \hat{\varphi}(y)$ by definition of the Möbius matrix.

Proof of Lemma 4.4. First observe that $\zeta(x \wedge y, z) = \zeta(x, z)\zeta(y, z)$, therefore

$$\Phi(x, y) = \varphi(x \wedge y) \quad = \quad \sum_{z \in X} \zeta(x \wedge y, z)\hat{\varphi}(z)$$

$$= \quad \sum_{z \in X} \zeta(x, z)\zeta(y, z)\hat{\varphi}(z).$$

Multiplying the matrix Φ on the left by μ, one obtains

$$\mu * \Phi(x, y) \quad = \quad \sum_{u} \mu(x, u)\Phi(u, y)$$

$$= \quad \sum_{u} \sum_{z} \mu(x, u)\zeta(u, z)\zeta(y, z)\hat{\varphi}(z)$$

$$= \quad \zeta(y, x)\hat{\varphi}(x),$$

the last equality coming from $\sum_{u} \mu(x, u)\zeta(u, z) = \delta_{xz}$. It follows that the matrix $\mu * \Phi$ is upper diagonal, with the numbers $\hat{\varphi}(x)$ on the diagonal. $\square$

Proof of Theorem 4.3. We apply Lemma 4.4 to $X = \mathrm{NC}(n)$ and φ defined by $\varphi(\pi) = 1$ for $\pi = \mathbf{0}_n$ and $\varphi(\pi) = 0$ for $\pi \neq \mathbf{0}_n$. We obtain $\hat{\varphi}(\pi) = \mu(\pi, \mathbf{0}_n)$. It is known that the Möbius function for the lattice of noncrossing partitions is multiplicative: if π is made of p_k parts of size k for $k = 1, 2, ...$, then one has

$$\mu(\pi, \mathbf{0}_n) = \prod_{k}((-1)^{k-1}\mathrm{Cat}_{k-1})^{p_k}.$$

The matrix A_n^* coincides with the corresponding matrix Φ up to a permutation of the columns (given by taking the Kreweras complement). We deduce

$$|\det(A_n^*)| = \prod_{k} \mathrm{Cat}_{k-1}^{a_{k,n}},$$

with

$$a_{k,n} = \sum_{\pi \in \mathrm{NC}(n)} p_k(\pi).$$

In order to compute $a_{k,n}$, we introduce the generating functions

$$f_{n,k}(y) = \sum_{\pi \in \mathrm{NC}(n)} y^{p_k} \quad \text{and} \quad f(z,y,k) = \sum_{n \geq k} z^n f_{n,k}(y).$$

Then one has

$$a_{k,n} = f'_{n,k}(1),$$

whereas $f_n(1) = \mathrm{Cat}_n$ and

$$\phi(z) := f(z,1,k) = \frac{1 - \sqrt{1 - 4z}}{2z}.$$

Let π be a noncrossing partition, such that the size of the part containing 1 is l. The complement of this part consists of l (possibly empty) noncrossing partitions $\pi_1, ..., \pi_l$, whose sizes add up to $n - l$. Moreover, we find

$$\begin{aligned} p_k(\pi) &= p_k(\pi_1) + \cdots + p_k(\pi_l) & \text{for } l \neq k, \\ &= p_k(\pi_1) + \cdots + p_k(\pi_l) + 1 & \text{for } l = k. \end{aligned}$$

It follows from these considerations that the generating function $f(z,y,k)$ satisfies the equation

$$f(z,y,k) = 1/(1 - zf(z,y,k)) + (y - 1)z^k f^k(z,y,k).$$

In particular, for $y = 1$, we recover

(4.1) $$\phi(z) = 1/(1 - z\phi(z)).$$

Differentiating with respect to y and evaluating at $y = 1$, we find

$$f_y(z,1,k) = zf_y(z,1,k)/(1 - z\phi(z))^2 + z^k \phi^k(z).$$

By (4.1), one has

$$1 - z/(1 - z\phi(z))^2 = 2 - \phi(z),$$

whence

$$\sum_{n \geq k} a_{n,k} z^n = f_y(z,1,k) = z^k \phi^k(z)/(2 - \phi(z)).$$

It follows that the generating series $h_k := f_y(z,1,k)$ satisfy the inductive rules $h_{k+1} = h_k - zh_{k-1}$. By Pascal's relation for binomial coefficients, the series $g_k := \sum_{n \geq k} z^n \binom{2n-k-1}{n-1}$ satisfy the same induction rules. It remains to check $g_k = h_k$ for $k = 0$ and 1, which is an easy exercise left to the reader, to conclude that $g_k = h_k$ holds for every k. $\qquad\square$

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