Monetary Policy Rules and Economic Stability When Agents Must Learn

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Preface

This thesis is the result of three years of research, both at the Mathematics institute of the University of Warwick and at the department of economics at New York University. I started this research project in December 2001. I was interested in the role of expectations in economic models. Given the mathematical difficulties that the topic presented, I decided to broaden my knowledge of applied dynamical systems and I joined the Interdisciplinary Program at the Mathematics institute. During my first year I attended lectures on dynamical systems and real analysis and in the second year I started working on my own project.

I started analyzing macroeconomic models where aggregate expectations had a major role to determine production and consumption decisions. I was interested in modelling the learning behavior of economic agents and its effects on expectations. I then further specialized in economic models of monetary policy and how learning behavior of market participants and central banks would affect monetary policy design.

In particular I studied non-linear two-dimensional dynamical systems, and showed how the non-linearities complicate the problem of monetary policy design, by introducing multiple equilibria in the model. I also focused on stochastic models of learning, using results from stochastic approximation theory to describe the dynamics of the agents' learning process.

I think that with the papers of my thesis I have fulfilled my main goals. Nevertheless, this work has been made possible by the joint efforts of many colleagues and professors. I would like to thank my three advisors (formal and not!) that taught me almost everything I know about mathematical economics: Jess Benhabib, Jim Bullard and Sebastian Van Strien. Sebastian Van Strien introduced me with infinite patience to the world of dynamical systems. The little understanding that I have of this complex field is because of him. I also thank him for having been supporting me from the start. I also thank Jess Benhabib for making it possible to visit the Economics department at NYU and especially for his constant advice on economics

and nonlinear dynamics. Jim Bullard has also helped me enormously with my dissertation. I began my collaboration with him as I joined the graduate internship program at the Federal Reserve Bank of St Louis. During my visit there I learnt a lot about how to write a potentially successful paper and how to develop successfully original ideas and suggestions.

I also thank Marc Giannoni, Petra Geraats, Mark Gertler, Bruce Preston and Tom Sargent for very useful discussions that improved significantly the quality of my work. Any remaining errors in the thesis are mine.

Declaration

Chapter 1 is largely expository, with the exception of the simultaneous equation model presented in the first section. Otherwise I declare that to the best of my knowledge the material in this thesis is the original work of the author except where stated explicitely in the text.

Introduction

In most economic models used for theoretical exploration or policy analysis, there is a crucial role for agents' expectations about future outcomes. Generally, it is assumed that economic agents take their decisions according to rationality principles and that they have a fairly accurate knowledge about the economic environment. In other words, they are assumed to know the *model* of the economy (Rational Expectations Hypothesis).

The latter assumption is somewhat extreme, given the evident lack of agreement, even among professionals, about the correct model of the economy. In this thesis I maintain the hypothesis that agents take their decisions rationally, i.e. in order to maximize their utilities given their budget constraints, but I assume that each agent has to learn about the economic environment.

More specifically, I consider economic models for monetary policy analysis. The goal is to study how the introduction of learning in these models can affect the design of monetary policy. Policy recommendations that might be sound under Rational Expectations, might lead to disastrous results under learning. I also use learning as a selection device. Some economic models fail to predict a unique Rational Expectations Equilibrium. Nevertheless, a REE is a sensible prediction of the model only if it can be shown that it is the result of some learning process of the economic agents. REE that are unstable under learning are not plausible equilibria.

The thesis is composed of two main essays. The first essay consider a simple class of non-linear monetary models that appear to have multiple REE. The goal of the essay is to compare the performance of two different policy rules. One policy rule states that the central bank has to set the nominal interest rate by responding to its forecast of future inflation. The other rule states that the central bank should set the interest rate according to current and past values of inflation. Higher inflation or expected inflation would trigger an increase in the nominal interest rate, consistently with the goal of the central bank to stabilize inflation around some target value.

The analysis shows that under the forecast-based policy rule there exist many REE equilibria that are stable under learning. Moreover, the market participants' learning process can generate learning equilibria, which would not exist under the assumption of rational expectations. The existence of different equilibria that can be reached depending on the initial conditions of the economic system is clearly destabilizing, given that only one equilibrium is generally consistent with the central bank's objectives. In the second part of the essay I show that adopting a policy rule that responds to current and past inflation leaves only one equilibrium that is stable under learning, and this equilibrium is the inflation target that the bank actually wants to achieve. I therefore conclude that policy rules that react to current and past inflation should be preferred.

The second essay considers a class of linear monetary models. The goal of the paper is to show under what conditions central banks' transparency can enhance economic stability. central bank's transparency is related to the amount of information about policy decisions that the bank is willing to share with the public. In the essay I show that under plausible assumptions about the model of the economy, lack of knowledge about the policy rule and its effects on the economy can generate instability, even in the case where, under rational expectations, there is a unique and stable equilibrium. I show that improved transparency can shrink the set of policy rules that lead to instability. Nevertheless, if agents are uncertain about the true model of the economy, even in the case of perfect transparency of the central bank, some policy rules that perform well under RE, generate instability. The conclusion is that on one hand transparency helps stabilizing expectations but on the other hand many rules generate instability independently on how transparent is the central bank implementing the policy.

Chapter 1

Stochastic Approximation and Learning Dynamics: an Introduction

In order to study the learning behavior of the economic agents, I make use of well known results from stochastic approximation theory, discussed in the book "Learning and Expectations in Macroeconomics" by George Evans and Seppo Honkapohja (EH in the sequel). The material in these section is meant to illustrate the mathematical results used in the thesis and it is almost entirely taken from that book and complementary articles therein quoted. This section shows the main results. These results refer to linear or locally linearized models and have only local validity, as explained below. Some results in the first essay make use of extensions, whose references are mentioned in the text.

The economic models considered can be written in compact notation as

$$A_0 Y_t = A_1 + A_2 E_t Y_{t+1} + A_3 X_t \tag{1.1}$$

where $Y_t \in \mathbb{R}^n$is a vector of endogenous variables (i.e. output, inflation and the interest rate), E_t denotes the average expectation operator, that includes information up to time t. The matrices A_1, A_2 are the time independent coefficients of the model, and $X_t \in \mathbb{R}^n$ represents a vector of exogenous variables. I assume that X_t represents stochastic shocks following an AR(1) process. Therefore

$$X_t = HX_{t-1} + \zeta_t$$

where H is the matrix of autocorrelation coefficients and $\zeta_t \in \mathbb{R}^n$ is an i.i.d. random vector of shocks, with zero mean. Economic agents ignore the *true model* of the economy and are endowed with a model (or Perceived Law of Motion, PLM), that they use for prediction. The model can be expressed as

$$\Omega_0 Y_t = \Omega_1 + \Omega_2 X_t + \eta_t. \tag{1.2}$$

This is a parametric model with constant coefficients $\Omega = [\Omega_0, \Omega_1, \Omega_2]$, $\eta_t \in \mathbb{R}^n$ is a perceived i.i.d. shock, with zero mean. The parameters are re-estimated every period, as new information arrives. The estimated model is

$$\hat{\Omega}_{0,t-1} Y_t = \hat{\Omega}_{1,t-1} + \hat{\Omega}_{2,t-1} X_t \tag{1.3}$$

where the coefficients $\hat{\Omega}_{t-1} = [\hat{\Omega}_{0,t-1}, \hat{\Omega}_{1,t-1}, \hat{\Omega}_{2,t-1}]$ are re-estimated every period. $\hat{\Omega}_{t-1}$ denotes the $t-1$ estimate of the coefficients. In order to estimate the coefficients, the agents use standard econometric techniques. Notice that this is a system of simultaneous equations. In order to estimate a model like (1.3), the agents need to impose restrictions on the matrices in Ω. These restrictions consist of requiring that some coefficient in the matrices is zero. More discussion about estimation and identification of simultaneous equations can be found in Ch. 3 where I consider an application to a specific model. In the second essay I somewhat extend the standard framework used in the economic literature, by allowing the agents to use not only the well known Recursive Least Squares estimator, that would lead to inconsistent estimates of (1.3), but also a Recursive Instrumental Variables estimators, depending on the existence of simultaneity in the equations of their model. According to this more general class of estimators, the coefficients' estimates are updated according to the following algorithm

$$\xi_t = \xi_{t-1} + \delta_t \overline{R}_{t-1}^{-1} Q_t \left(Y_t - (\xi_{t-1}' U_t) \right)' \tag{1.4}$$

$$\overline{R}_t = \overline{R}_{t-1} + \delta_t \left(Q_t U_t' - \overline{R}_{t-1} \right)$$

where ξ_t is the $(n+1) \times 1$ real valued vector of the estimated coefficients in Ω_t that have not been restricted to zero. Consider the following real valued matrices: $\overline{R}_t$ of dimension $(n+1)^2 \times (n+1)^2$, Q_t of dimension $(n+1)^2 \times (n+1)$ and U_t.of dimension $(n+1) \times 1$. The first matrix is

$$\overline{R}_t = \left(I_n \otimes R_t^h \right)$$

where R_t^h is the $n \times n$ precision matrix associated to each single equation h (where $h = 1, ...n$). The precision matrix is an estimate of $E_{t\to\infty} Q_t U_t'$. Also

$$Q_t = (I_n \otimes V_t)$$

is a matrix containing the instruments V_t (which includes a constant), while U_t is the matrix or regressors that might contain current endogenous variables. Finally, δ_t is a nonincreasing sequence of real valued gains, whose properties are defined below. In the case of recursive least squares, we simply set $U_t = V_t = (1, X_t)$ and $R_t^h = R_t$. Given their estimate in (1.3), the agents' expectation for the next period value of Y_t becomes

$$E_t Y_{t+1} = \hat{\Omega}_{0,t-1}^{-1} \hat{\Omega}_{1,t-1} + \Omega_{0,t-1}^{-1} \hat{\Omega}_{2,t-1} H X_t \tag{1.5}$$

where I assume that the $t-$information set includes current observations of X_t. Notice that the agents use $t-1$ estimates of the coefficients to form expectations at time t: this is an assumption that simplifies the analysis of convergence. Substituting the expectation (1.5) in (3.8) I obtain the Actual Law of Motion (ALM) for the variables

$$\begin{aligned} Y_t &= A_0^{-1} A_1 \left(\hat{\Omega}_{0,t-1}^{-1} \hat{\Omega}_{1,t-1} + \Omega_{0,t-1}^{-1} \hat{\Omega}_{2,t-1} H X_t \right) + A_0^{-1} A_2 X_t \quad (1.6) \\ &= A_0^{-1} A_1 \hat{\Omega}_{0,t-1}^{-1} \hat{\Omega}_{1,t-1} + A_0^{-1} \left(A_1 \Omega_{0,t-1}^{-1} \hat{\Omega}_{2,t-1} H + A_2 \right) X_t \\ &= T'(\xi_{t-1}) (1, X_t)' \end{aligned}$$

where the function $T(.)$ denotes the mapping between the PLM and the Actual Law of Motion (ALM) of the economic system. The value of ξ such that $ALM = PLM$, i.e. $T(\xi^*) = \xi^*$, is the Rational Expectations Equilibrium. Stability under learning means that the sequence of estimates ξ_t converges to ξ^*. Substituting (1.6) in (3.17) we get :

$$\xi_t = \xi_{t-1} + \delta_t \overline{R}_{t-1}^{-1} Q_t \left(T'(\xi_{t-1}) V_t - (\xi_{t-1}' U_t) + e_t' \right)' \tag{1.7}$$

$$\overline{R}_t = \overline{R}_{t-1} + \delta_t \left(Q_t U_t' - \overline{R}_{t-1} \right)$$

where e_t' denotes a vector of i.i.d. normally distributed observational shocks with zero mean and standard deviation σ_e that make the learning process non-trivial. Observation shocks capture the fact that the agents might have observations on Y_t that contain errors. Notice again that I assume that in taking expectations at time t the agents observe the current values of the variables but use time $t-1$ parameters' estimates. This assumption is common in the literature and it is made to simplify the analysis.

1.1 Convergence

In order to study the convergence properties of (3.20), we need to use results from stochastic approximation theory. Let $\theta_t \in \mathbb{R}^d$ (where in the case of the example above $d = 2n + n^4$). Also, let $Z_t \in \mathbb{R}^k$ (in our case $k = 4n$) be the vector of observable variables.

Let us rewrite the system as:

$$\theta_t = \theta_{t-1} + \delta_t \Phi(\theta_{t-1}, Z_t) \tag{1.8}$$

where

$$\theta_t = \left(\xi_t, vec(\overline{R}_t)\right).$$

Here $\Phi(.) : \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}^d$ is a function describing how the vector θ_t is updated. The vector of state variables Z_t evolves according to

$$Z_t = B(\theta_{t-1}) Z_{t-1} + CW_t \tag{1.9}$$

where the coefficient matrix $B(\theta_{t-1})$ is real time varying and dependent on θ_t and the matrix C is real and time invariant. Also, W_t is a random disturbance term. Further properties of (1.9) are discussed below.

Considering the example of the previous section, we can define

$$Z_t = [1, Y_t, X_t, e_t]$$

$$W_t = [1, \zeta_t, e_t]$$

$$B(\theta) = \begin{bmatrix} 0 & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times 1} & 0_{n\times n} & T'(\xi) & 0_{n\times n} \\ 0_{n\times 1} & 0_{n\times n} & H & 0_{n\times n} \\ 0_{n\times 1} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0_{1\times n} & 0_{1\times n} \\ 0_{n\times 1} & 0 & 0_{n\times n} \\ 0_{n\times 1} & I_{n\times n} & 0_{n\times n} \\ 0_{n\times 1} & 0 & I_{n\times n} \end{bmatrix}.$$

Let us consider the point θ^* where the economy is at the REE, i.e. $\theta^* = (\xi^*, \bar{R}^*)$ such that $T(\xi^*) = \xi^*$ and $\bar{R}^* = E_{t\rightarrow\infty} Q_t U_t'(\xi^*)$. Assume for the moment that θ^* is an equilibrium point for the system (1.8): that is, assume that $\theta_t \rightarrow \theta^*$ as $t \rightarrow \infty$. Let $D \subset R^d$ be an open set around θ^*. Then consider the following set of assumptions:

A.1.1 δ_t is a positive, nonstochastic, decreasing sequence satisfying

$$\sum_{t=1}^{\infty} \delta_t = \infty, \text{ and } \sum_{t=1}^{\infty} \delta_t^2 < \infty$$

A.1.2 $lim_{t \to \infty} \sup [1/\delta_t - 1/\delta_{t-1}] < \infty$

A.2. For any compact $Q \subset D$, there exist C and q such that for every $\theta \in Q$,

$$|\Phi(\theta, Z)| \leq C(1 + |z|^q)$$

A.3.1 For any compact $Q \subset D$, the function $\Phi(\theta, Z)$ is continuously differentiable with bounded partial derivatives in every Q.

A.3.2 For any compact $Q \subset D$ and for every $\theta, \theta' \in Q$ and $z_1, z_2 \in \mathbb{R}^k$ the function $\Phi(\theta, Z)$ satisfies

(i) $|\partial\Phi(\theta, z_1)/\partial z - \partial\Phi(\theta, z_2)/\partial z| \leq L_1 |z_1 - z_2| (|1 + |z_1|^{p_1} + |z_2|^{p_1}|)$, for some p_1 and some constant L_1.

(ii) $|\partial\Phi(\theta, z)/\partial z - \partial\Phi(\theta', z)/\partial z| \leq L_2 |\theta - \theta'| (1 + |z|^{p_2})$, for some p_2 and some constant L_2.

B.1. W_t is i.i.d with $|W_t| \leq C$.

B.2 For any compact $Q \subset D$

$$\sup_{\theta \in Q} |B(\theta)| \leq \rho_Q < 1$$

for some matrix norm $||$, and $B(\theta)$ satisfies Lipschitz conditions on Q.

Given the assumptions above, the local properties of the system (1.8), (1.9) can be studied by checking the stability conditions of an ordinary differential equation. This can be obtained from the asymptotic expected value of the function Φ. Let $\theta \in Q$, define

$$\bar{Z}_t(\theta) = B(\theta)\bar{Z}_{t-1} + CW_t$$

and consider the behavior of the mean of $\Phi(\theta, \bar{Z}_t(\theta))$

$$h(\theta) = \lim_{t \to \infty} E\Phi(\theta, \bar{Z}_t(\theta)). \quad (1.10)$$

The associated ODE is then defined as

$$\dot{\theta} = h(\theta)$$

As shown below, it is possible to study the local behavior of (1.9) by focusing on the local stability of (1.10). The following two Lemmas describe the existence and the properties of (1.10).

Lemma 1 *Assume B.2 and let $\theta \in Q$. Then $\bar{Z}_t(\theta)$ tends in the limit to an L_p-integrable random variable $\bar{Z}_\infty(\theta)$.*

Proof. see EH, Lemma 6.1, pag.129. ∎

Lemma 2 *Assume A.2. Then $h(\theta)$ exists. Assume A.3. Then $h(\theta)$ is locally Lipschitz*

$$|h(\theta) - h(\theta')| \leq |\theta - \theta'|$$

for every $\theta, \theta' \in Q$.

Proof. see EH, Lemma 6.2, p.129. ∎

I briefly introduce the main steps to 'connect' the ODE with the original system. The algorithm (1.8) can be written in the following form:

$$\theta_{n+1} = \theta_n + \delta_{n+1}h(\theta_n) + f_n \tag{1.11}$$

where f_n is the approximation error between the algorithm and the associated differential equation. Here I change notation from t to n, given that (1.11) denotes a standard discretization of the differential equation (1.10). The next step is to develop bounds on the expression $\sum f_n$. We can rewrite (1.11) as

$$f_n = \delta_{n+1}\left[\Phi\left(\theta_n, Z_{n+1}\right) - h(\theta_n)\right].$$

More specifically, as it becomes clear below, we need bounds on sums of expressions of the form

$$f_n(\phi) = \phi(\theta_{n+1}) - \phi(\theta_n) - \delta_{n+1}\phi'(\theta_n)h(\theta_n) \tag{1.12}$$

where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is twice continuously differentiable with bounded second derivatives. Let Q be defined as above and let

$$\tau(Q) = \inf\left(n : \theta_n \notin Q\right)$$

denote the first time τ that θ_n leaves Q (if $\theta_n \in Q$ for every n then $\tau = \infty$). The following Lemma provides bounds for the sum of the approximation error, provided $\theta_n \in Q$.

Lemma 3 *For all initial values $z \in R^k$ and $a \in R^k$ there exist constants B_1 and s such that*

$$E_{x,a}\left[\sup_n I_{(n\leq\tau(Q))}\left|\sum_{k=0}^{n-1} f_k(\phi)\right|\right]^2 \leq \tilde{B}_1(1+|z|^s)\left[\left(1+\sum_{k=n+1}^{\infty}\delta_k^2\right)\sum_{k=n+1}^{\infty}\delta_k^2\right]$$

where $E_{x,a}$ denotes the expectation over the joint distribution of $(Z_n,\theta_n)_{n\geq 1}$ and I denotes the indicator function. Moreover, if $\tau(Q)=\infty$ then $\sum f_n(\phi)$ converges almost surely.

Proof. *see Evans and Honkapohja (1998), pp. 80-83. The proof considers a more general case where Z_t follows a Markov Process dependent on θ_{t-1}. The main theorems needed for the proof are Doob's inequality and the Martingale Convergence Theorem.* ∎

The above result puts a bound on the sum of the approximation errors, provided θ_n stays in Q.

After the above preliminary results, we proceed to analyze the asymptotic behavior of the algorithm (1.8). The goal is to show that if θ^* is locally stable in the ODE, then the algorithm (1.8) converges to θ^* with some positive probability.

Let θ^* a locally asymptotically stable point of the ODE. Then it can be shown that on the domain of attraction D of θ^* there exists a twice continuously differentiable Lyapunov function $U(\theta):\mathbb{R}^d\to\mathbb{R}$ with the following properties[1]

(i) $U(\theta^*)=0$, $U(\theta)>0$ for all $\theta\in D,\theta\neq\theta^*$;
(ii) $U'(\theta)\,h(\theta)<0$ for all $\theta\in D,\theta\neq\theta^*$;
(iii) $U(\theta)\to\infty$ if $\theta\to\partial D$ or $|\theta|\to\infty$.

where ∂D is the boundary of D.

Let $K(c)=\{\theta:U(\theta)\leq c\}, c>0$, the compact set defined by the C-contour of U and $\tau(c)=\inf(n:\theta_n\notin K(c))$. Let $0<c_1<c_2$ and $K(c_2)\subset D$ and choose ϕ such that

a) ϕ coincides with $U(.)$ on $K(c_2)$;
b) $\inf_{\theta_n\notin K(c_2)}\phi(\theta)=c_2$.

[1] See Proposition 5.9, EH(2001).

If $\tau(c_2) < \infty$, from (1.12) we get

$$\phi\left(\theta_{\tau(c_2)}\right) - \phi\left(\theta_0\right) = \sum_{k=0}^{\tau(c_2)-1} \delta_{k+1}\phi'(\theta_k)h(\theta_k) + \sum_{k=0}^{\tau(c_2)-1} f_k(\phi).$$

When $\theta_0 = a \in K(c_1)$, we have that $c_2 - c_1 = \inf_{\theta_n \notin K(c_2)} \phi(\theta) - c_1 \leq \phi\left(\theta_{\tau(c_2)}\right) - \phi\left(\theta_0\right)$. On the other hand, since $\phi'(\theta) = U'(\theta)$ on $K(c_2)$ we have $\phi'(\theta_k)h(\theta_k) \leq 0$. Thus

$$(c_2 - c_1) I_{(\tau(c_2)<\infty)} \leq I_{(\tau(c_2)<\infty)} \left| \sum_{k=0}^{\tau(c_2)-1} f_k(\phi) \right| \leq \sup_n I_{n \leq \tau(c_2)} \left| \sum_{k=0}^{n-1} f_k(\phi) \right|$$

which, after squaring both sides and taking the expected value, gives

$$(c_2 - c_1)^2 E\left(I_{(\tau(c_2)<\infty)}\right) \leq E\left(\sup_n I_{n \leq \tau(c_2)} \left| \sum_{k=0}^{n-1} f_k(\phi) \right| \right)^2$$

which implies

$$P\left[I_{(\tau(c_2)<\infty)}\right] \leq (c_2 - c_1)^{-2} E\left[\sup_n I_{n \leq \tau(c_2)} \left| \sum_{k=0}^{n-1} f_k(\phi) \right| \right]^2$$

Since, the conditional distribution of (Z_{n+k}, θ_{n+k}) given $Z_n = z$, $\theta_n = a$ is equal to the conditional distribution of (Z_n, θ_n) given $Z_0 = z$, $\theta_0 = a$ and with γ_n replaced with γ_{n+k}, the result holds also for any $n > 0$. Hence, using Lemma (3)

$$E_{z,a}\left[\sup_n I_{n \leq \tau(c_2)} \left| \sum_{k=0}^{n-1} f_k(\phi) \right| \right]^2 \leq \tilde{B}_1(1 + |z|^s) \left[\left(1 + \sum_{k=n+1}^{\infty} \delta_k^2\right) \sum_{k=n+1}^{\infty} \delta_k^2\right]$$

for some $\tilde{B}_1$ and s. Hence

$$P\left[I_{(\tau(c_2)<\infty)}\right] \leq B_1(1 + |z|^s) \left[\left(1 + \sum_{k=n+1}^{\infty} \delta_k^2\right) \sum_{k=n+1}^{\infty} \delta_k^2\right]$$

for some B_1 and s. Finally, let $P_{n,z,a}$ denote the probability distribution of $(Z_k, \theta_k)_{k \geq n}$, with $Z_n = z$ and $\theta_n = a$. The results above lead to the following Theorem (see Evans and Honkapohja, 1998)

Theorem 4 *Let θ^* be an asymptotically stable equilibrium point of the ODE (1.10). Suppose Assumptions A 1.1., A 2., A 3. and B are satisfied and $D = int(K(c))$ for some $c > 0$. Suppose that for some $0 < c_1 < c_2$, we have $K(c_2) \subset D$.*

(i) Then there exist B_1 and s such that for all $a \in K(c_1)$, $n \geq 0$, z

$$P_{n,z,a}\left(\theta_t \text{ leaves } K(c_2) \text{ in finite time}\right) \leq B_1(1+|z|^s)\left[\left(1+\sum_{k=n+1}^{\infty}\delta_k^2\right)\sum_{k=n+1}^{\infty}\delta_k^2\right]$$

(ii) Suppose that $0 < c_1 < c_2$, we have $K(c_2) \subset D$. Then for all $a \in K(c_1)$, $n \geq 0$, z

$$P_{n,z,a}\left(\theta_t \text{ leaves } K(c_2) \text{ in finite time or } \theta_t \to \theta^*\right) = 1$$

(iii) for any compact $Q \subset D$, there exist constants B_2 and s such that for all $a \in K(c_1)$, $n \geq 0$, z

$$P_{n,z,a}\left(\theta_t \to \theta^*\right) = 1 - B_2(1+|z|^s)\left[\left(1+\sum_{k=n+1}^{\infty}\delta_k^2\right)\sum_{k=n+1}^{\infty}\delta_k^2\right]$$

Proof. see EH (1998). (ii) Let $\nu(c) = \inf(n, \theta_n \in K(c))$. Pick $0 < c_1 < c_2$. From the properties of the U function we have that on D there exists $\alpha > 0$ such that $-\phi'(\theta)\,h(\theta) \geq \alpha$ for all θ such that $c_1 \leq \phi(\theta) \leq c_2$. Assume for a contradiction that $\nu(c_1) = \infty$, so that θ_n does not reach $K(c_1)$ in finite time. It then follows from (1.12) that on the set $\{\nu(c_1) = \tau(c_2) = \infty\}$ we have, for all $m > n$

$$\begin{aligned}\sum_{k=n}^{m} f_k(\phi) &= (\phi(\theta_m) - \phi(\theta_n)) - \sum_{k=n}^{m}\gamma_{k+1}\phi'(\theta_k)\,h(\theta_k) \qquad (1.13)\\ &\geq -(c_2 - c_1) + \alpha\left(\sum_{k=n}^{m}\gamma_{k+1}\right)\end{aligned}$$

but since $\sum_{k=n}^{m}\gamma_{k+1} \to \infty$ as $m \to \infty$, (1.13) implies that $\sum_{k=n}^{m} f_k(\phi) \to \infty$. This contradicts the above result that $\sum_{k=n}^{m} f_k(\phi)$ converges almost surely on $\{\tau(c_2) = \infty\}$. Hence $\nu(c_1) < \infty$ on $\{\tau(c_2) = \infty\}$. This means that θ_n enters $K(c_1)$ in finite time. It remains to show that $\theta_n \to \theta^*$.

We need to show that $\limsup \phi(\theta_n) < c$ a.s. on $\{\tau(c_2) = \infty\}$, for every $0 < c < c_2$, z and $a \in K(c)$. Again we proceed by contradiction. Fix

$0 < c_1 < c < c_2$. By the previous result, whenever θ_n leaves $K(c)$ it returns in finite time to $K(c_1)$ a.s. on $\{\tau(c_2) = \infty\}$. Consider the set $\{\tau(c_2) = \infty, \limsup \phi(\theta_n) > c\}$ and let $\nu_1 = \nu(c_1), \tau_1 = \inf_{n>\nu 1}(\theta_n \notin K(c))$, $\nu_k = \inf_{k>\tau_k - 1}(\theta_n \in K(c_1))$, $\tau_k = \inf_{k>\nu_k}(\theta_n \notin K(c))$. All these are finite by assumption and $\nu_k \geq k$. From (1.12) and using $\phi'(\theta_n) h'(\theta_n) \leq 0$, we have that for all k

$$0 < (c - c_1) \leq \phi(\theta_{\tau_n}) - \phi(\theta_{\nu_n}) \leq \sum_{k=\nu_n}^{\tau_n - 1} f_k(\phi).$$

But then $\sum_{k=\nu_n}^{\tau_n-1} f_k(\phi)$ cannot be a Cauchy sequence, contradicting the result that it converges a.s. on $\{\tau(c_2) = \infty\}$. We have come to a contradiction. This implies that for every c $\limsup \phi(\theta_n) < c$ and thus $\lim_{t\to\infty} \phi(\theta_n) = 0$. Hence, from the properties of ϕ on D, $\theta_n \to \theta^*$ a.s. on $\{\tau(c_2) = \infty\}$.

(iii) follows from (i) and (ii). ■

The Theorem states that if $\theta_n \in K(c_1)$ at any time n than the probability that θ_t leaves $K(c_2)$ at time $t > n$ is bounded by an expression that goes to zero as $t \to \infty$. Nevertheless, the result shows that at any point in time n there is the possibility that a large shock drives $\theta_{t>n}$ outside the basin of attraction $K(c_2) \subset Q$ so that the algorithm does not converge to θ^*.

In order to guarantee convergence we modify the algorithm (1.8) to include a so called projection facility. The projection facility eliminates the possibility that large shocks send θ_t outside the basin of attraction $K(c_2) \subset Q$. It simply states that if $\theta_t \in int(K(c_2))$ the algorithm is followed. Otherwise θ_t is projected to some point in $K(c_1)$. In other words, the algorithm dictates:

$$\theta_t = \begin{cases} \theta_{t-1} + \delta_t \Phi(\theta_{t-1}, Z_t) & \text{if } [\theta_{t-1} + \delta_t \Phi(\theta_{t-1}, Z_t)] \in int(K(c_2)) \\ \bar{\theta} \in K(c_1) & \text{if } [\theta_{t-1} + \delta_t \Phi(\theta_{t-1}, Z_t)] \notin int(K(c_2)) \end{cases}. \tag{1.14}$$

That captures the idea that the agents have a shared prior about the interval of the possible values for θ^*, i,e. they are 'confident' that $\theta^* \in K(c_1)$. If the recursive estimator gives them a value that is implausible (i.e. too far from the confidence interval) on the basis of their prior information, they discard it and use instead a value that lie in the interval. Corollary 6.8 in EH (2001) proves that modifying the algorithm with such projection facility guarantees convergence with probability 1.

Of course imposing a projection facility has been subjected by much criticism. The most compelling one is that under the assumption of decentralized market is somewhat implausible that the economic agents would coordinate on a shared prior about the true parameter value θ^*. Nevertheless, this assumption is less restrictive than rational expectations. Also, in the simulations that I show in Ch. 3, I do not employ a projection facility.

The next step is to show that (i) the algorithm cannot converge to a point that is not an equilibrium point of the ODE, i.e. the REE, and (ii) the algorithm will not converge to an unstable equilibrium point of the ODE. The following theorem summarizes the result

Theorem 5 *Consider the algorithm with assumptions A and B. Suppose a some point $\theta^* \in D$, we also have the validity of the conditions (i) $\Phi\left(\theta^*, \bar{Z}_t(\theta^*)\right)$ has a covariance matrix that is bounded below by a positive definite matrix, and (ii) $E\Phi\left(\theta^*, \bar{Z}_t(\theta^*)\right)$ is continuously differentiable in θ, in a neighborhood of θ^* and the derivatives converge uniformly in this neighborhood as $t \to \infty$. Then (a) if $h(\theta^*) \neq 0$ or (b) if $\partial h(\theta^*)/\partial\theta$ has an eigenvalue with a positive real part, then*

$$P(\theta_t \to \theta^*) = 0$$

Proof. see Ljung (1977). ■

The intuition for the result is that as long as there is randomness in the system, i.e. assumption (i), θ_t will leave any unstable point of the ODE.

The results above allow us to study the local stability under learning of a REE by checking only the local stability of the associated ODE. The results have been extended for a special class of nonlinear models. The following section briefly sums up the results, to be found in EH(2001), EH(1995). These results are used in Ch. 2.

1.2 Learning and Stability of Nonlinear Models

In Ch. 2 I study the dynamics of an univariate nonlinear model that can be described as

$$y_t = H\left(E_t G\left(y_{t+1}, e_{t+1}\right), e_t\right) \tag{1.15}$$

where $y_t \in \mathbb{R}$, e_t denotes a one dimensional random shock i.i.d. distributed, with mean zero and bounded support. Also $H : \mathbb{R}^2 \to \mathbb{R}$, $G : \mathbb{R}^2 \to \mathbb{R}$ are assumed to be twice continuously differentiable on some open rectangle. As above E_t is the expectation operator, with respect to time t information. Finally, it is assumed that G is observable to the agents. In the chapter I mainly consider the deterministic case where $e_t = 0$ for every t. In the nonstochastic case (1.15) becomes a one dimensional dynamical system. I study the possible fixed points of the system and I investigate the existence of cycles of second order. In terms of the economic model, fixed points and cycles are defined perfect foresight equilibria. That means that the agents are assumed to know with certainty the future evolution of the variable y_t, i.e. $E_tG(y_{t+1}, 0) = G(y_{t+1}, 0)$. This is a stronger assumption than rational expectations, but is commonly used when analyzing nonlinear economic models. Some results in Ch. 2 concern the stability under learning of these equilibria.

This section focuses on the case where e_t is not zero, even if with a small support. In this case, the theorems described above can be used to study the local stability under learning of the stochastic REE of (1.15). Let us define first a *rational noisy 2-cycle* as a stochastic process of the form

$$\begin{aligned} y_t &= y_1(e_t) \text{ for } t \bmod 2 = 1 \\ y_t &= y_2(e_t) \text{ for } t \bmod 2 = 0 \end{aligned}$$

where the two functions y satisfy

$$\begin{aligned} y_1(e_t) &= H(E_tG(y_2(e_{t+1}), e_{t+1}), e_t) \text{ for } t \bmod 2 = 1 \\ y_2(e_t) &= H(E_tG(y_1(e_{t+1}), e_{t+1}), e_t) \text{ for } t \bmod 2 = 0. \end{aligned}$$

Notice that this includes a steady state where $y_1(e_t) = y_2(e_t)$. Also, notice that while y_t is a stochastic process, the expectations follow a deterministic cycle

$$E_tG(y_{t+1}(e_{t+1}), e_{t+1}) = \begin{cases} \bar{\theta}_2 & \text{if } t \bmod 2 = 1 \\ \bar{\theta}_1 & \text{if } t \bmod 2 = 0 \end{cases}$$

where

$$\bar{\theta}_i = E_tG(y_i(e_{t+1}), e_{t+1}).$$

I can re-express $y_1(e_t) = H(\bar{\theta}_2, e_t)$ and $y_2(e_t) = H(\bar{\theta}_1, e_t)$, so that the rational noisy cycle can be defined by $(\bar{\theta}_1, \bar{\theta}_2)$ such that

$$\begin{aligned} \bar{\theta}_1 &= EG(H(\bar{\theta}_2, e_t), e_t) \\ \bar{\theta}_2 &= EG(H(\bar{\theta}_1, e_t), e_t) \end{aligned}$$

where E denotes the unconditional mean and we are using the fact that the shocks are i.i.d. Note that by setting the noise equal to zero we revert to the deterministic case. EH(1995) prove the *existence* of noisy cycles for the case where the support of the shock is 'small'. I omit the details for brevity.

1.2.1 Recursive Learning and Convergence

The main assumption is that the agents believe that the economy is in a noisy cycle and attempt to estimate the mean value of $G(y_t, e_t)$ at different points in the cycle. A natural estimator of (θ_1, θ_2) is given by *separate sample means for each stage of the cycle* (recall that G is assumed to be observable). Putting the estimator in recursive form we obtain

$$\begin{pmatrix} \theta_{1,s} \\ \theta_{2,s} \end{pmatrix} = \begin{pmatrix} \theta_{1,s-1} \\ \theta_{2,s-1} \end{pmatrix} + \delta_s \begin{pmatrix} G(y_{1,s}, e_{1,s}) - \theta_{1,s-1} \\ G(y_{2,s}, e_{2,s}) - \theta_{2,s-1} \end{pmatrix} \tag{1.16}$$

$$\theta_{i,s} = EG(y_{2s+i}, e_{2s+i}) \tag{1.17}$$

where s is a positive number such that $t = 2s + i$, for $i = 1, 2$. Again detail for the derivation of (1.16) can be found in EH(2001). The system under learning dynamics is given by the equations (1.16),(1.17) and (1.15) and it defines a stochastic recursive algorithm. It can be re-expressed as

$$\theta_s = \theta_{s-1} + \delta_s \left(M\left(\theta_{s-1}, \delta_s, e_s\right) - \theta_{s-1}\right) \tag{1.18}$$

where $\theta_s = (\theta_{1,s},\ \theta_{2,s})$, $e_s = (e_{1,s},\ e_{2,s})$ and $M(.) = [M_1, M_2]$

$$\begin{aligned} M_1 &= G\left(H\left(\theta_{2,s-1}, e_{1,s}\right), e_{1,s}\right) \\ M_2 &= G\left(H\left(\theta_{1,s-1} + \delta_s \left(G\left(H\left(\theta_{2,s-1}, e_{1,s}\right), e_{1,s}\right) - \theta_{1,s-1}\right), e_{2,s}\right), e_{2,s}\right). \end{aligned}$$

Having put the learning algorithm in the form (1.8), it is possible to study the convergence of θ_s by analyzing the associated differential equation. The equation is

$$\dot{\theta} = \lim_{s \to \infty} EM\left(\theta, \delta_s, e_s\right) - \theta$$

Taking individual components in turn one obtains

$$\dot{\theta} = T(\theta) - \theta \tag{1.19}$$

where

$$\begin{aligned} T(\theta) &= \left(R(\theta_2), R(\theta_1)\right) \text{ and} \\ R(\theta_i) &= E\left(G\left(H\left(\theta_i, e_t\right), e_t\right)\right). \end{aligned}$$

Using the above results from stochastic approximation theory it is possible to determine the behavior of the stochastic algorithm from the behavior of the associated ODE. Evaluating the stability of the ODE at the noisy REE, we obtain the following stability condition

Proposition 6 *Consider an REE noisy cycle of order 2 of the model (1.15), with expectation parameters $\bar{\theta}$. Then the ODE (1.19) is stable if an only if*

$$\xi = R'(\theta_1)\, R'(\theta_2) < 1. \tag{1.20}$$

Condition (1.20) is also referred to as E-Stability in the economic literature. The following proposition describes the convergence result for the stochastic algorithm

Proposition 7 *Consider an REE noisy 2-cycle of the model (1.15), and with expectation parameters $\bar{\theta}$. Suppose that $\bar{\theta}$ is a locally stable point of (1.19). Then $\bar{\theta}$ is locally stable under adaptive learning. If instead $\bar{\theta}$ is an unstable point of (1.19) then θ_s converges to $\bar{\theta}$ with probability zero.*

In this section I omit the proofs. The proof of the proposition can be found in EH (1995). As mentioned above, in Ch.2 I consider the deterministic case where $e_t = 0$. In that case the stability condition becomes

$$\begin{aligned} \xi &= F'(\bar{y}_1)F'(\bar{y}_2) < 1 \\ \text{where } F(y) &= H\left(G\left(y,0\right),0\right) \end{aligned} \tag{1.21}$$

and $\bar{y}_1, \bar{y}_2$ denote the points on the cycle. By continuity, noisy cycles with sufficiently small noise are also stable, provided (1.21) holds. This is the result mostly used in Ch. 2.

Another class of rational expectations equilibria that is studied in Ch.2 is that of *sunspot equilibria.* The definition of sunspot equilibrium involves the basic idea that economic agents in the model condition their expectations on some (random) variable s_t which otherwise does not have any influence on the model economy. Let us assume that the extraneous random variable s_t is a two-state Markov chain with a constant transition matrix $\prod = (\pi_{i,j})$, $0 < \pi_{i,j} < 1$, for $i,j = 1,2$. Moreover, $s_t = 0$ in state 1 and $s_t = 1$ in state 2.

Definition 8 *A two-state stationary sunspot equilibrium (SSE) is a process $y_t = y_1^*$, if $s_t = 0$ and $y_t = y_2^*$ if $s_t = 1$, such that*

$$\begin{aligned} y_1^* &= \pi_{11} F\left(y_1^*\right) + \left(1-\pi_{11}\right) F\left(y_2^*\right) \\ y_2^* &= \left(1-\pi_{22}\right) F\left(y_1^*\right) + \pi_{22} F\left(y_2^*\right) \end{aligned}$$

where $F(.)$ is defined as in (1.21).

Under suitable regularity conditions the following two results hold

1. Let $\bar{y}_1, \bar{y}_2$ denote the points on a two-period cycle, with $\bar{y}_1 \neq \bar{y}_2$. Then there exist SSEs y_1^*, y_2^* near $\bar{y}_1, \bar{y}_2$ with π_{11}, π_{22} close to zero.

2. Let $\bar{y}_1, \bar{y}_2$ denote two distinct steady states, with $\bar{y}_1 \neq \bar{y}_2$. Then there exist SSEs y_1^*, y_2^* near $\bar{y}_1, \bar{y}_2$ with π_{11}, π_{22} close to one.

3. Let $\bar{y}$ denote a steady state. Then there exist SSEs y_1^*, y_2^* near $\bar{y}$ provided $F'(\bar{y}) > 1$.

Such an SSE is also called an $\epsilon' - SSE$, to denote that it lies in a neighborhood $0 < \epsilon' < \epsilon$, for some small ϵ, of the deterministic equilibrium. The results are stated very informally. The interested reader should check EH (1994), EH (2001). Since the sunspot equilibria are close to the deterministic equilibria, they can be showed to "inherit" their stability properties. Hence the sunspots are going to be stable provided (1.21) holds.

More generally, a general condition for the existence of sunspots is the following.

Proposition 9 *For two points y_1^*, y_2^*, assume that $F(y_1^*) < F(y_2^*)$. There exist $0 < \pi_{i,j} < 1$ such that y_1^*, y_2^* is an SSE with transitional probabilities $\pi_{i,j}$ if and only if the points y_1^*, y_2^*, both lie in the open interval $(F(y_1^*), F(y_2^*))$.*

An example is given in the Appendix of Ch.2.

Assume that the agents do not know the states y_1^*, y_2^* and needs to learn about them. Assume,without loss of generality that the agents know the transition matrix of s_t. Concerning local stability under learning, we need to define a recursive algorithm. This is given by

$$\phi_{l,t} = \phi_{l,t-1} + t^{-1}\psi_{l,t-1}q_{l,t-1}^{-1}(y_{t-1} - \phi_{l,t-1} + \epsilon_{t-1}) \tag{1.22}$$

$$q_{l,t} = q_{l,t-1} + t^{-1}(\psi_{l,t-1} - q_{l,t-1})$$

which gives the following actual law of motion:

$$\begin{aligned} y_t &= \psi_{1,t}[\pi_{11}F(\phi_{1,t}) + (1-\pi_{11})F(\phi_{2,t})] + \\ &\quad + \psi_{2,t}[(1-\pi_{22})F(\phi_{1,t}) + \pi_{22}F(\phi_{2,t})] \end{aligned} \tag{1.23}$$

where $\phi_{l,t}$ denotes the estimate of the l state of the sunspot equilibrium value of y_t. The algorithm works as follows. The agents observe a sunspot Markov process s_t. They are assumed to know the transition matrix. They update recursively each estimate of y_l (for $l = 1, 2$) depending on the current state of the sunspot: the variable $\psi_{l,t}$ is equal to one if the sunspot process is in state l and zero otherwise. Also, $q_{l,t}$ represents the fraction of observations in state l over the whole sample up to time $t-1$. Finally $\phi_{l,t}$ represents the estimates of y_l in the two states of the sunspots and ϵ_t is a measurement error i.i.d, with bounded support. In order to study the stability properties of the algorithm let us define $\theta_t' = (\phi_{1,t}, \phi_{2,t}, q_{1,t}, q_{2,t})$ and $Z_t' = (\psi_{1,t}, \psi_{2,t}, \epsilon_t)$ and the functions

$$\begin{aligned} H_l(\theta_{t-1}, Z_t) &= \psi_{l,t-1} q_{l,t-1}^{-1}(y_{t-1} - \phi_{l,t-1} + \epsilon_{t-1}) \quad \text{for } l = 1, 2 \\ H_j(\theta_{t-1}, Z_t) &= \psi_{l,t-1} - q_{l,t-1} \quad \text{for } j = 1, 2. \end{aligned}$$

This allows us to put the system in the standard form (1.19). The associated ODE becomes

$$\begin{aligned} h_1(\theta) &= \bar{\pi}_1 q_1^{-1}(\pi_{11} F(\phi_1) + (1 - \pi_{11}) F(\phi_2) - \phi_1) \\ h_2(\theta) &= \bar{\pi}_2 q_2^{-1}(\pi_{22} F(\phi_1) + (1 - \pi_{22}) F(\phi_2) - \phi_2) \\ h_3(\theta) &= \bar{\pi}_1 - q_1 \\ h_4(\theta) &= \bar{\pi}_2 - q_2 \end{aligned} \tag{1.24}$$

where $\bar{\pi}_1, \bar{\pi}_2$ is the limiting distribution of the states of the Markov chain. Clearly at the equilibrium point $\bar{\pi}_1 = q_1, \bar{\pi}_2 = q_2$. Also, in the ODE the last two equations always converge, independently of ϕ, i.e. $\bar{\pi}_1 \to q_1, \bar{\pi}_2 \to q_2$. Therefore the ODE is stable provided

$$\frac{d(\phi_1, \phi_2)'}{d\tau} = T(\phi_1, \phi_2) - (\phi_1, \phi_2)'$$

is stable at the point $\phi_1 = y_1^*, \phi_2 = y_2^*$, where T is given by the right hand side of the first two equation of the associated ODE (1.24). The following propositions follow from the above result and are used in Ch.2.

Proposition 10 *(i) The learning rule (2.13) converges locally to an SSE* y_1^*, y_2^* *provided the eigenvalues of* $DT(y_1^*, y_2^*)$ *have real parts less than one.*

(ii) Suppose that $DT(y_1^*, y_2^*)$ *has one eigenvalue with real part greater than one. The learning dynamics converges to the SSE with probability zero.*

Proposition 11 *(i) let* $\bar{y}_1, \bar{y}_2$ *denote the points on a two-period cycle with* $F'(\bar{y}_1) F'(\bar{y}_2) \neq 0$. *Then all* $\epsilon' - SSE$ *relative to* $\bar{y}_1, \bar{y}_2$ *are locally stable under learning if and only if (1.21) holds.*

(ii) Given two distinct steady states $\bar{y}_1, \bar{y}_2$. *Then all* $\epsilon' - SSE$ *relative to* $\bar{y}_1, \bar{y}_2$ *are locally stable under learning if and only if* $F'(\bar{y}_1) < 1, F'(\bar{y}_2) < 1$ *hold.*

(iii) Let $\bar{y}$ *be a steady state that is not stable under learning, i.e.* $F(\bar{y}) > 1$. *Then all* $\epsilon' - SSE$ *relative to* $\bar{y}$ *are locally unstable under learning. Let* $F(\bar{y}) < -1$, *then there exist* $\epsilon' - SSE$ *relative to* $\bar{y}$ *that are locally stable under learning.*

Proof. The result follow from the definition of the T map and from the continuity of eigenvalues of DT. See EH (2001), Proposition 12.9, 12.10, EH (2001a). ■

1.3 Some Key Definitions

This section defines some concepts that are commonly used in economics and that might be unfamiliar to the reader with a more mathematical background.

Taylor Rule. A Taylor rule is an approximation to actual monetary policy decisions taken by the Federal Reserve. Originally proposed by John Taylor, it states that the central bank decides the short term nominal interest rate according to a time invariant rule. The rule dictates that if the inflation rate is higher than a certain value decided by the central bank, called the inflation target, then the interest rate should be raised. The opposite happens if inflation is lower than target. Moreover, the rule states that if output is higher than its long term trend, then the interest rate should be raised.

An active Taylor rule is a Taylor rule that states that the interest rate should be changed more than proportionally, than the change in the inflation rate from target (i.e. it states that the central bank should increase the real interest rate).

The Taylor Principle states that if a policy rule is active, then the economy is stable. What does 'stable' mean in the class of models considered in the Thesis? I need to define two more concepts.

Determinate REE. A rational expectations equilibrium is said to be determinate if it is unique. We say that the equilibrium is locally determinate if there do not exist other equilibria in a small neighborhood of it. An economic model, as the ones included in the thesis, can be expressed as a one dimensional difference equation in the economic variable y, i.e.

$$y_{t+1} = F(y_t)$$

where a REE is a real valued sequence $(y_0, ...y_n)$ for $n = 0...\infty$, such that, given $(y_0, ...y_n)$; a) economic agents optimize their utility, b) their resource constraint is satisfied. Condition for local uniqueness of the equilibrium is that $F'(y^*) > 1$, where $y^* = F(y^*)$. That means that the only equilibrium is $(y^*, ...y^*)$, a constant value for the economic variable y. Why? Suppose we pick $y_0 \neq y^*$, then the y_t will diverge from y^*. Such (locally) explosive behavior of increasing or decreasing y_t cannot be considered an equilibrium, since we do not observe such a behavior in economic variables. Take consumption: if consumption is growing without bound, this is unlikely to be an equilibrium because soon the agents in the economy will not have resources to pay for it. Hence, this cannot be an equilibrium. Since the agents in the economy are assumed to have perfect foresight, they will not select such a path for consumption. Also, negative consumption could be ruled out with a similar argument (i.e. the agents wouldn't be maximizing their utility with such a low consumption!).

Naturally, the arguments used above make more sense if we could show that there does not exists any bounded trajectory for y_t, other than constant y^*. This amounts to showing global determinacy. If there exist other bounded trajectories for y_t, then there might be other equilibria, provided a) and b) are not violated. If $F'(y^*) < 1$ there are infinite sequences satisfying a) and b) and converging to the steady state. Each of these sequences is an equilibrium. Under REE, a determinate steady state is a stable equilibrium because the economy stays forever at the steady state. In an indeterminate equilibrium we get instability, because arbitrary changes in expectations can give equilibria where y changes over time. For a more precise formulation of the above concepts, see Farmer (1999).

Learnability.I use this term in the thesis, because it is commonly adopted by economists. It simply means that the algorithms described above converge to the REE.

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Chapter 2

Forecast-Based v. Backward-Looking Policy Rules: a "Global" Analysis

2.1 Introduction

In the monetary policy literature forecast-based Taylor rules are widely used and recommended for policy analysis[1]. However, recent contributions have shown that forecast-based policy rules might have important drawbacks.

First, they might lead to indeterminate equilibria, as shown by Woodford (2002) and Bullard and Mitra (2001). Second, Levin, Wieland and Williams (2001) show that their performance might not be robust to uncertainty about the model of the economy. Third, they are not consistent with a conventional intertemporal loss function, as proved by Svensson (2001).

Moreover, Evans, Honkapohja and Marimon (2001), and Carlstrom and Fuest (2001a) show the existence of learnable sunspot equilibria if the inflation target equilibrium is indeterminate.

In general, it is a well known result that adopting Taylor rules might generate instability in the economic system. On one side, a nonlinear Taylor rule (consistent with a zero bound on the interest rate) implies the existence of two steady states, see Benhabib et al. (2001a). One is the inflation target and the other is a low inflation/deflation equilibrium. On the other side,

[1] See, among the others, Batini and Nelson (2001).

Benhabib et al. (2001b) show that in a simple model where money enters in the production function, linear and nonlinear Taylor rules might lead to cycles or even chaotic behavior.

The purpose of this paper is to compare the performance of forecast-based and backward-looking Taylor rules in a class of simple monetary models. In the course of the paper I focus on those policy rules that satisfy the 'Taylor Principle' and guarantee local determinacy of the inflation target steady state. I therefore restrict the analysis to the best-case scenario where local indeterminacy does not appear. I then consider the possibility of *learnable global equilibria* and how the choice of the Taylor rule affects their existence or changes their stability.

In particular, I study the existence of: a) learnable cycles and sunspots around the *determinate* inflation target equilibrium and, b) learnable liquidity traps implied by the zero bound condition on the policy rule - and so far considered a mere theoretical curiosity, e.g. in McCallum (2001).

I show that forecast-based Taylor rules have a destabilizing effect on the economy: they lead to learnable cycles and sunspots, even in the case where the inflation target equilibrium is locally unique and stable under learning. Moreover, the economy can converge to a liquidity trap that is stable under learning. In contrast, adopting a backward-looking Taylor rule stabilizes the economy. In fact, equilibrium cycles disappear, while sunspot equilibria and the liquidity trap become unstable under learning (i.e. not robust to expectational mistakes).

I do not analyze the welfare properties of the equilibria. The analysis is conducted from the point of view of the central banker: any equilibrium other than the inflation target is considered a 'bad' equilibrium, in the sense that the policy goal is systematically missed. As it is common in this class of models, the inflation target equilibrium does not necessarily Pareto dominate the other equilibria, if the welfare criterion is based on the agent's intertemporal utility function.

The paper is structured as follows. The second section quickly reviews the model and introduces the learning algorithm. The third section analyzes stability under learning of steady states, cycles and sunspots under a forecast-based Taylor rule and discusses the main results. The fourth section compares the performance of forecast-based and backward-looking Taylor rules.

2.2 The Model

I consider a simple model of the economy with flexible prices, a discrete time version of the model of Benhabib et al. (2001a). The model has three main components.

The representative agent's problem. Agents take consumption, production and saving decisions in order to maximize their utility from consumption and money balances under the following budget constraint:

$$\begin{aligned} &\max_{c_t, M_t^{np}, M_t^p B_t} \sum_{t=0}^{\infty} \beta^t U\left(c_t, \frac{M_t^{np}}{P_t}\right) \\ \text{sub } M_t + B_t + P_t c_t &= M_{t-1} + R_{t-1} B_{t-1} + P_t f\left(\frac{M_t^p}{P_t}\right) \end{aligned} \tag{2.1}$$

where $M_t = M_t^{np} + M_t^p$ represents the sum of non-productive (M_t^{np}) and productive (M_t^p) money balances, B_t denotes bonds, R_t is the gross nominal return on bonds. Also, $f(.)$ is the production function, which depends on productive real money balances and a fixed input $\overline{y}$, c_t denotes real consumption and P_t is the price level. Finally, β is the discount rate and $U(.)$ and $f(.)$ satisfy all the standard conditions[2]. I further assume that $U_{cm} > 0$, that is consumption and money balances are Edgeworth complements[3], where U_x is the partial derivative with respect to the argument x. In order to keep the analysis simple, the model abstracts from endogenous labor supply and capital accumulation.

Policy rules. The Central Bank takes policy decisions consistent with a nonlinear Taylor rule:

$$R_t = \rho(\pi_{t+i}) = 1 + (R^* - 1)\left(\frac{\overline{\pi}_{t+i}}{\pi^*}\right)^{\frac{A}{(R^*-1)}} \tag{2.2}$$

where $\overline{\pi}_{t+i}$ denotes a measure of inflation: in the case of a (perfect foresight) forecast-based Taylor rule we have that $\overline{\pi}_{t+i} = E_t\pi_{t+1} = \pi_{t+1}$. In the case of a backward-looking policy rule, I set $i = 0$, while $\overline{\pi}_t$ is a weighted average of current and past inflation rates, to be defined below. Also, π^* is the inflation target chosen by the monetary authority and R^* is the gross nominal interest rate consistent with the steady state Fisher relation:

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[2] See Benhabib (2001a).

[3] See Benhabib et al. (2001b).

$$\beta^{-1}\pi^* = R^*.$$

Optimality conditions. The representative agent chooses sequences of money, bonds and consumption so as to maximize intertemporal utility. Appendix A discusses the optimization problem in detail. The first order conditions of the representative agent's problem (2.1) give the following Euler equation:

$$U_c\left(c_t, m_t^{np}\right) = \frac{\beta U_c\left(c_{t+1}, m_{t+1}^{np}\right)}{\pi_{t+1}} R_t \tag{2.3}$$

where $m = M/P$ denotes real money balances and an (implicit) money demand equation for both production and consumption money balances:

$$f_{m^p}\left(m_t^p\right) = \frac{R_t - 1}{R_t} \tag{2.4}$$

$$\frac{U_{m^{np}}\left(c_t, m_t^{np}\right)}{U_c\left(c_t, m_t^{np}\right)} = \frac{R_t - 1}{R_t}. \tag{2.5}$$

The assumptions about utility and production function guarantee that the demand for money gives a negative relation between real money demand and the nominal interest rate. Finally, in equilibrium markets clear and $c_t = y_t$.

Role for Money. I consider two cases. First, money enters only in the utility function ($M^p = 0$) and output is constant: $f(.) = \bar{y}$ (i.e. it is an endowment economy). Second, money enters only in the production function ($M^{np} = 0$)[4] and the agents gain utility from consumption only.

Perfect Foresight Solution. Independently of the assumptions about the role of money and the policy rule, the perfect foresight (reduced-form) solution of the model can be expressed as a first order nonlinear difference equation:

$$\pi_{t+1} = F\left(\pi_t\right). \tag{2.6}$$

The particular solutions of the model (i.e. the form of the function $F(.)$) for the different versions of the model are discussed in detail in the next sections.

Calibration. Concerning the model with money in the production function, I follow Benhabib et al. (2001b) and specify a Cobb-Douglas utility function and a CES production function:

[4] In the sequel, I drop the superscript to avoid complications in the notation.

$$f(m_t) = [(1-a)\, m_t^{\mu} + a\bar{y}^{\mu}]^{\frac{1}{\mu}} \tag{2.7}$$

$$U(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}. \tag{2.8}$$

Given those functional forms, I calibrate the model by using the equilibrium

money demand: the benchmark calibration is shown in Table 1[5]. In order to allow comparisons between the two models and match the data, I assume that in the model where money enters in the utility function the functional form is CES:

$$U(c_t, m_t) = \frac{\left([(1-a)\, m_t^{\mu} + a c_t^{\mu}]^{\frac{1}{\mu}}\right)^{(1-\sigma)}}{1-\sigma}. \tag{2.9}$$

This specification allows the money demand implied by (2.4) and (2.5) to be exactly the same. Hence, the two models have the same calibration as shown in Table 1.

Table1 Benchmark Parametrization

β	μ	μ^{LR}	μ^{SR}	π^*	R^*	$\bar{y}$	a	A
0.996	-9	-3.5	-50	1.0103	1.0143	1	0.000350	1.522

The parameter μ is chosen to be consistent with both long run and short run log-elasticity of the money demand: μ^{LR} is the parameter consistent with the long run log-elasticity of demand and μ^{SR} is consistent with the log-elasticity in the short run[6]. The parameters σ and α are 'free' parameters[7].

2.2.1 Backward dynamics and learning

In this paper I am mainly concerned with real time learning behavior and thus with the following (reduced-form) decision rule:

$$\pi_t = EG(\pi_{t+1}) \tag{2.10}$$

[5] For details on how to calibrate these models, see Benhabib et al. (2001b).

[6] Notice that as I change the value of μ, I need also to change the value of a, in order to keep the model close to the data, see Benhabib et al. (2001b).

[7] In the simulations below I consider values for σ that range from 1 to 3.5, that are most commonly used in macroeconomic literature. Woodford uses $\sigma = 6.3$ which is somewhat higher.

where E is the (possibly not rational) expectations operator. Notice that this decision rule is consistent with the perfect foresight solution (2.6). In fact, under perfect foresight $\pi_{t+1} = G^{-1}(\pi_t) = F(\pi_t)$. Depending on which model is considered and on the parameters, G might not be a function but a correspondence. In fact, in the next section I show the existence of two 'branches' of G: this implies that for any future value of inflation, the decision rule gives two choices for current inflation.

The existence of different 'branches'[8] of G poses the problem of how current inflation is decided, given the expectations about the future. In this paper I mainly focus on the local stability of the perfect foresight equilibria: I assume therefore that the expectational errors are 'small' and that market participants are somewhat coordinated in a neighborhood of the perfect foresight equilibrium[9]. Nevertheless, I also consider the dynamics under learning and show the existence of other equilibria that depend on which branch of G is selected.

According to (2.10), π_t depends only on future expected values of π_{t+1}. In order to model learning behavior, I follow Guesnerie and Woodford (1991) and Evans and Honkapohja (2001a). The agents are assumed to have an (asymptotically) correct model of the economy: their perceived law of motion of the economy corresponds to the actual law of motion, *if* the learning process converges to the perfect foresight equilibrium. More specifically, the agents believe that the system is at the equilibrium, even though they do not know *which* equilibrium (steady state, cycle or sunspot) and *what values of inflation* correspond to the equilibrium.

Concerning the monetary authority, for the case of forecast-based Taylor rules I maintain the assumption of perfect foresight. This assumption is restrictive but it is imposed to simplify the analysis. Nevertheless, relaxing this assumption might have consequences on my conclusions. The analysis of learning on part of the central bank is left as an issue for further research.

Learning steady states and cycles. Let us first consider the deterministic case where the agents face two possible equilibria: steady states and (period-two) cycles. They expect $\pi_t = \pi_1$ for odd-t and $\pi_t = \pi_2$ for even-t, where $\pi_1 = \pi_2 = \tilde{\pi}$ if the system is in steady state. (Notice that, as shown in Figure 1, each branch of G has a steady state.such that inflation is constant and $G(\pi) = \pi$.) Nevertheless, they do not observe π_1 and π_2; they

[8]For an example where the branches are in the forward-looking map, see Christiano and Harrison (1999).

[9]Another way to express this is that the agents have *strong priors* about the equilibrium values of π.

estimate them recursively, updating every period their information about the state of the system. Given that at the cycle:

$$\pi_1 = G_i(\pi_2)\,;\ \pi_2 = G_j(\pi_1)$$

where i and j denote the (possibly different) branches of G, the agents estimate the two states of the cycle by averaging the past data $G(\pi_{t-i})$ for even and odd periods separately[10]. Hence, their forecast will be $EG_i(\pi_{t+1}) = \theta_{2,t}$ if $t+1$ is odd and $EG_j(\pi_{t+1}) = \theta_{1,t}$ if $t+1$ is even, where $\theta_{l,t}$ is an estimate of π_l (for $l = 1, 2$). In order to update their estimates of the equilibrium values of inflation, market participants make use of the adaptive algorithm[11]:

$$\begin{bmatrix} \theta_{1,s} \\ \theta_{2,s} \end{bmatrix} = \begin{bmatrix} \theta_{1,s-1} \\ \theta_{2,s-1} \end{bmatrix} + \alpha_s \begin{bmatrix} G_i(\pi_{2,s}) - \theta_{1,s-1} \\ G_j(\pi_{1,s}) - \theta_{2,s-1} \end{bmatrix} \tag{2.11}$$

where s is such that $t = 2s + l$ and $\pi_{l,s} = \pi_{2(s-1)+l}$, allowing the agent to consider the data *in successive pairs.* This learning rule has the desirable property of being consistent with both steady states and period-two cycles[12]. In fact, if the perfect foresight equilibrium is a steady state, the two estimates of the states π_1 and π_2 converge to a single constant[13]. Also, the algorithm allows the agents to learn cycles[14] on different branches of G.

Given (2.10) and (2.11) the system can be expressed in terms of the expectations variable as[15]:

$$\begin{bmatrix} \theta_{1,s} \\ \theta_{2,s} \end{bmatrix} = \begin{bmatrix} \theta_{1,s-1} \\ \theta_{2,s-1} \end{bmatrix} + \alpha_s \begin{bmatrix} G_i(\theta_{2,s-1}) - \theta_{1,s-1} \\ G_j(\theta_{1,s}) - \theta_{2,s-1} \end{bmatrix} \tag{2.12}$$

[10] This learning mechanism implies a small deviation from rationality, since the agents are assumed to have a well specified model of the economy (the Minimum State Variable solution).

[11] For details, see Evans and Honkapohja (1995, 2001a).

[12] In the sense that $\theta_{1t} \to \pi_1$ and $\theta_{2t} \to \pi_2$ as $t \to \infty$.

[13] Note that when considering a deterministic environment I assume that agents use a fixed gain algorithm: $\alpha_s = \alpha \in (0,1)$. In the case of stochastic environment, the fixed gain algorithm does not converge, i.e. see Sargent (1999). I therefore assume for simplicity that $\alpha_s = s^{-1}$.

[14] The simple models considered above can also generate perfect foresight equilibrium cycles of order higher than two. These cycles could not be detected by the learning rule above and therefore they are not learnable in this framework, but it would be straightforward to modify the algorithm to include the possibility of learning higher order cycles. Nevertheless, this extension does not seem interesting because the results are not likely to be robust to behavioral and learning heterogeneity. On this point, see Bullard and Duffy (1998).

[15] The dynamical system is obtained by substituting the first equation for $\theta_{1,s}$ in the second equation. After this substitution, the system becomes $\theta_s = H(\theta_{s-1})$, where $\theta_s = (\theta_{1,s}, \theta_{2,s})$.

which gives a two dimensional dynamical system at each branch i, j. Learning behavior is determined by the behavior of this dynamical system.

Learning sunspots. I also consider the possibility that agents learn to believe in sunspots. In fact, as shown in the next sections, under the hypothesis of rational expectations there exist sunspot equilibria. In order to assess their learnability, let us assume that the agents include in their perceived law of motion the possibility of being at a two-state sunspot equilibrium, generated by a non-fundamental exogenous 'sunspot' process s_t. Following Evans and Honkapohja (2001a) I assume agents estimate recursively the states π_1 and π_2 (depending on the current realization of s_t) by using the adaptive learning rule:

$$\theta_{l,t} = \theta_{l,t-1} + t^{-1}\psi_{l,t-1}q_{l,t-1}^{-1}(\pi_{t-1} - \theta_{l,t-1} + \epsilon_{t-1}) \tag{2.13}$$

$$q_{l,t} = q_{l,t-1} + t^{-1}(\psi_{l,t-1} - q_{l,t-1})$$

which gives the following actual law of motion:

$$\begin{aligned} \pi_t &= \psi_{1,t}[z_{11}G_i(\theta_{1,t}) + (1 - z_{11})G_j(\theta_{2,t})] + \\ &\quad + \psi_{2,t}[(1 - z_{22})G_i(\theta_{1,t}) + z_{22}G_j(\theta_{2,t})] \end{aligned} \tag{2.14}$$

where, again, i and j denote the (possibly) different branches of G. The algorithm works as follows. The agents observe a sunspot Markov process s_t with two states and a transition matrix:

$$\begin{bmatrix} z_{11} & 1 - z_{11} \\ 1 - z_{22} & z_{22} \end{bmatrix}.$$

They are assumed to know the transition matrix. They update recursively each estimate of π_l (for $l = 1, 2$) depending on the current state of the sunspot: the variable $\psi_{l,t}$ is equal to one if the sunspot process is in state l and zero otherwise. Also, $q_{l,t}$ represents the fraction of observations in state l over the whole sample up to time $t-1$. Finally θ_{lt} represents the estimates of π_l in the two states of the sunspots and ϵ_t is a measurement error i.i.d, with bounded support.

Notice that the learning algorithm (2.13) is able to detect cycles and steady states, provided that the agents learn also about the transition probabilities[16]. In other words, the learning rule (2.11) is a special case of (2.13).

[16] If z_{11} and z_{22} are equal to one we have a steady state, while if z_{11} and z_{22} are equal to zero we have a cycle.

For simplicity I do not consider this extension, but it should be clear that the learning rules are kept distinct only for expository purposes.

2.3 Forecast-based Taylor rule

2.3.1 Local Stability of Steady States

Given the dynamical system (2.12), the aim of this section is to verify local determinacy and learnability of the steady states, under both the assumption of money in the utility function and money in the production function.

Money in the production function

By using the equilibrium condition in the goods market, I get the following decision rule for the representative agent:

$$U_c(y_t) = E\left\{\beta U_c(y_{t+1})\left[\frac{R_t}{\pi_{t+1}}\right]\right\} \tag{2.15}$$

where $E(.)$ denotes the expectation operator. Combining the expression:

$$m_t = f^{-1}(y_t)$$

obtained from the production function and the money demand function (2.4) gives a negative relation between output and the interest rate:

$$y_t = y(R_t),\ y' < 0 \tag{2.16}$$

Combining (2.15), (3.5), (2.16) with $i = 1$, I obtain:

$$U_c(y(\rho(\pi_{t+1}))) = E\left\{\beta U_c(y(\rho(\pi_{t+2})))\left[\frac{\rho(\pi_{t+1})}{\pi_{t+1}}\right]\right\}. \tag{2.17}$$

Assuming perfect foresight, the solution is a well defined one-dimensional map[17] which takes the form of (2.6). This can be shown by rewriting:

$$U_c(y(\rho(\pi_{t+2}))) = \beta^{-1}U_c(y(\rho(\pi_{t+1})))\left[\frac{\rho(\pi_{t+1})}{\pi_{t+1}}\right]^{-1} \tag{2.18}$$

[17] As noted already in Benhabihb et al. (2001), the solution displays nominal indeterminacy.

and noting that the function $U_c(y(\rho(\pi)))$ is invertible and its inverse is well defined. Linearizing the forward looking map around the active and passive steady states gives the coefficient[18]:

$$\widehat{\pi}_{t+2} = \left[1 + \frac{1}{-\sigma\epsilon_y}\left(\frac{1}{\epsilon_\rho} - 1\right)\right]\widehat{\pi}_{t+1} \tag{2.19}$$

where $\widehat{\pi}$ describes deviations from the steady state, ϵ_ρ is the elasticity of R with respect to π, $-\sigma = \frac{U_{cc}y}{Uc} < 0$ and $\epsilon_y = \frac{y'R}{y} < 0$. Notice that for consistency with the assumptions about the utility function described in the previous section, I assume that the intertemporal elasticity of substitution is constant.

The following proposition describes the steady states and their stability under perfect foresight and learning dynamics (all the proofs are in the Appendix).

Proposition 12 *Consider the map G implied by (2.17), such that $\pi_t = F^{-1}(\pi_{t+1}) = G(\pi_{t+1})$. Then:*

(i) The system (2.12) has two steady states, π^* and $\overline{\pi}$. At π^* monetary policy is active and $\epsilon_\rho(\pi^*) > 1$. At $\overline{\pi}$ monetary policy is passive and $\epsilon_\rho(\overline{\pi}) < 1$.

(ii) Provided $\epsilon_\rho(\pi^*)$ is such that $\epsilon_\rho(\pi^*) > \frac{1}{1-2|\sigma\epsilon_y|}$ both the active and passive steady states are locally determinate in the perfect foresight dynamics and stable under learning dynamics for any $\alpha \in [0,1]$.

Remark 13 *By adding small noise and letting $\alpha_t = t^{-1}$ we have that both fixed points are strongly E-Stable in the sense of Evans and Honkapohja (2001) and thus they are robust to overparametrized perceived laws of motion (i.e. if the agents' model were consistent with an n-cycle, it would still converge to the fixed point)*[19].

The Proposition states that local determinacy and learnability of the inflation target equilibrium are achieved provided that the Taylor rule is 'sufficiently active' at the inflation target. Under the benchmark calibration (with $\mu = -9$) in Table 1, condition (ii) is satisfied for $\sigma \in (1, 2.4)$. Notice that provided $0 < |\sigma\epsilon_y| < 1/2$ it is possible to find a sufficiently active Taylor rule that guarantees determinacy. Nevertheless, a policy rule that satisfy the Taylor Principle might not achieve local stability, i.e. $\epsilon_\rho > 1$ is not a

[18]The result is also in Benhabib et al. (2001b)

[19]See Proposition 12.2 of Evans and Honkapohja (2001).

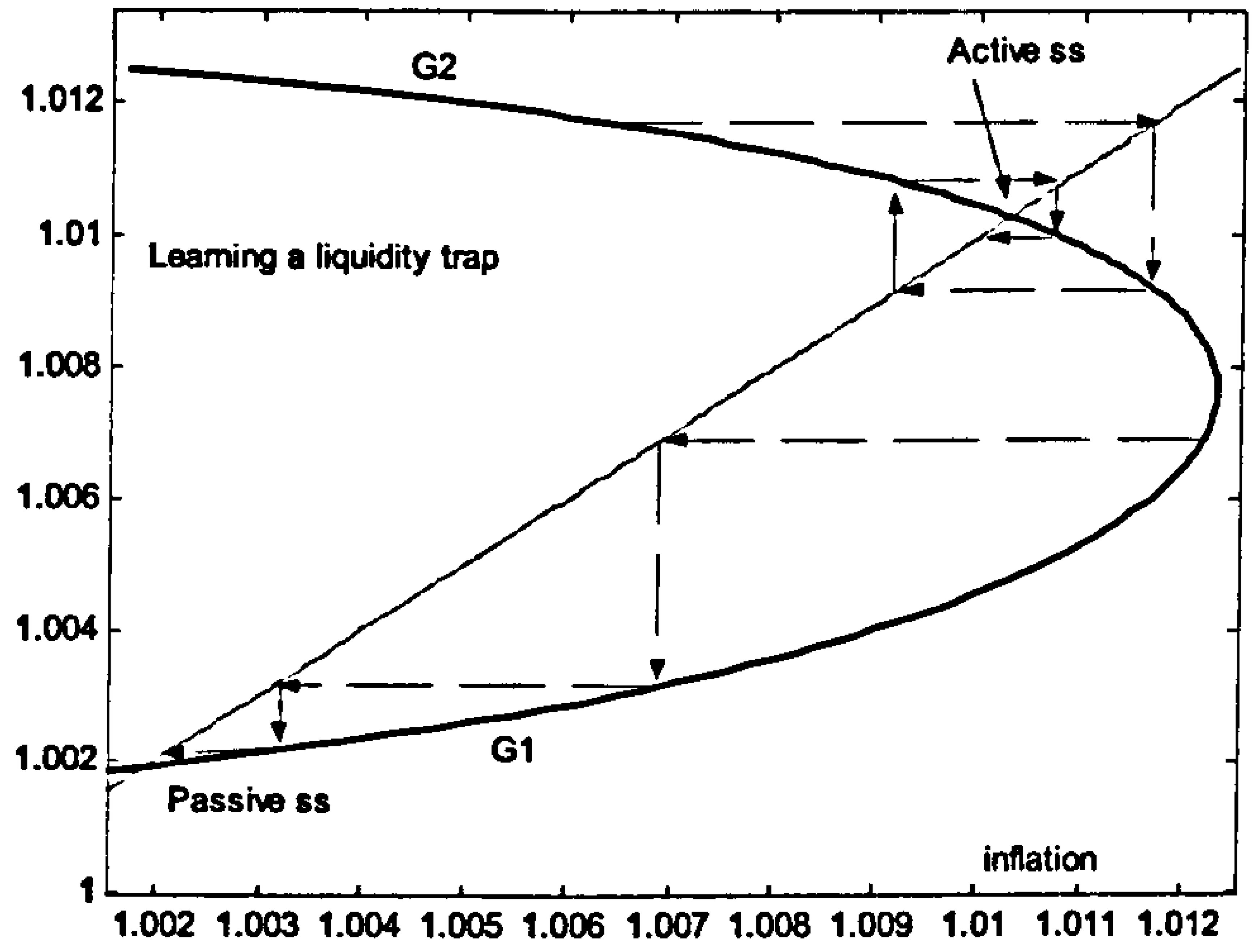

Figure 2.1:

sufficient condition for local determinacy and learnability. For example, the smaller the intertemporal elasticity of substitution $1/\sigma$, the more aggressive the Taylor rule should be.

As mentioned in the introduction, I focus on the case where the inflation target is locally determinate. Concerning the steady states, forecast based rules can be destabilizing even in the case where the equilibrium corresponding to the inflation target is locally determinate and learnable. In fact, for suitable initial conditions (depending on the shocks hitting the economy and the state of expectations) the economy can be driven into a liquidity trap. We should expect the existence of a "stability corridor": if the economy experiences small shocks (i.e. in expectations), than the Taylor rule drives the economy back to the inflation target. On the contrary, large shocks might lead to a liquidity trap, as shown in Figure 2.1.

In the next section I show that, if we take into account the existence of equilibrium cycles and sunspots, then the stability corridor is rather small. In fact, small shocks can drive the economy to sunspot equilibria arbitrarily close to the active steady state.

Money in the utility function

In this case output is equal to a constant endowment $\overline{y}$. Using the Euler equation (2.3) and the goods market condition:

$$c_t = \overline{y}$$

I obtain the decision rule:

$$U_c(\overline{y}, m_t) = \frac{\beta U_c(\overline{y}, m_{t+1})}{\pi_{t+1}} R_t. \tag{2.20}$$

From the demand equation (2.5) I get the following (real) money demand equation:

$$m_t = m(R_t) \;\; m' < 0 \tag{2.21}$$

Finally, using (2.20), (2.21), and (3.5) to substitute for R and m, I obtain the following reduced-form solution in π:

$$U_c(\overline{y}, m(\rho(\pi_{t+1}))) = \frac{\beta U_c(\overline{y}, m(\rho(\pi_{t+2})))}{\pi_{t+1}} \rho(\pi_{t+1}). \tag{2.22}$$

Once again, the solution is a well defined one dimensional dynamical system which takes the form of (2.6).

Let us assume that $U_{cm} > 0$: under this assumption, we know from the continuous time version of this model that the active steady state is locally determinate[20]. Linearizing the model (2.22) I get the following coefficient:

$$\widehat{\pi}_{t+2} = \left[1 + \frac{1}{\epsilon_{cm}\epsilon_m}\left(\frac{1}{\epsilon_\rho} - 1\right)\right]\widehat{\pi}_{t+1} \tag{2.23}$$

where $\widehat{\pi}$ defines deviations from steady state, $\epsilon_{cm} = \frac{U_{cm}m}{U_c} > 0$ and $\epsilon_m = \frac{m'R}{m} < 0$.

It is immediate from (2.23) that the active steady state $(\epsilon_\rho(\pi^*) > 1)$ is always determinate, while the determinacy of the passive steady state $(\epsilon_\rho(\overline{\pi}) < 1)$ depends crucially on the parameters of the model. I also consider the existence and learnability of equilibrium stationary sunspots ϵ–close to the deterministic steady states (defined $\epsilon' - SSEs$ in the sequel). Local stability and learnability of the steady states are described as follows:

[20] See Benhabib et al. (2001a).

Proposition 14 *Consider the map G implied by (2.22). The map has the same steady states as (2.17). Consider the deterministic system (2.12);*

(i) the active steady state is determinate and learnable for every $\alpha \in (0,1)$;

(ii) for $\epsilon_p(\overline{\pi})$ such that $\epsilon_p(\overline{\pi}) > \frac{1}{1+2|\epsilon_{cm}\epsilon_m|}$ the passive steady state is indeterminate and non-learnable for every $\alpha \in (0,1)$;

(iii) for $\epsilon_p(\overline{\pi})$ such that $\epsilon_p(\overline{\pi}) < \frac{1}{1+2|\epsilon_{cm}\epsilon_m|}$ the passive steady state is determinate and learnable for every $\alpha \in (0,1)$.

Consider the stochastic system (2.13), (2.14);

(iv) for $\epsilon_p(\overline{\pi})$ such that $\frac{1}{1+|\epsilon_{cm}\epsilon_m|} > \epsilon_p(\overline{\pi}) > \frac{1}{1+2|\epsilon_{cm}\epsilon_m|}$ there exist learnable $\epsilon' - SSEs$ around the (indeterminate) passive steady state.

Proposition (14) shows that also in the case, more common in the literature, where money enters in the utility function, forecast-based Taylor rules may lead to economic instability, even if the active steady state is locally determinate and learnable. In fact, *the economy can either fall into a liquidity trap or converge to a sunspot equilibrium where inflation fluctuates around the passive steady state.* A sufficiently aggressive rule at the inflation target leads to a very passive response at the liquidity trap (as it is apparent from the policy rule (3.5)). Hence, a sufficiently aggressive rule verifies the conditions for a learnable liquidity trap or a learnable sunspot. Concerning the empirical relevance of this result, learnability of the liquidity trap steady state or of the sunspots around it obtains for $\mu = -9$ and $\sigma > 2$, under the benchmark calibration in Table 1. If we consider a more aggressive Taylor rule (i.e. $A = 2$, which captures the estimated Taylor rule for the post-Volker era[21]), learnability of the passive steady state or sunspots around it is verified for an even larger parameter space. Hence, the economy can fall into a liquidity trap for very plausible parameter values.

Summing up, these two results qualify the findings of Bullard and Mitra (2001) and McCallum (2001). These results where derived in somewhat different model environments and so should not be expected to apply in the current setting[22]. First, $\epsilon_p > 1$ (the Taylor Principle) is not a sufficient condition to stabilize the economy. The conditions in Proposition (14) show that a 'too active' policy rule[23] at the inflation target (which implies a too passive policy at the liquidity trap) might be destabilizing because it leads to

[21] See Clarida et al. (1999).

[22] This shows that the nature of the model does matter for results in this area, which is perhaps not surprising.

[23] Note that the optimal monetary policy literature advocates more active Taylor rules (i.e. ϵ_p) than our benchmark case, see Taylor (1999).

learnable liquidity traps. Also, from Proposition (12) we know that a 'too' cautious active policy rule leads to an indeterminate and non-learnable inflation target equilibrium, while the liquidity trap is learnable. The conclusion is that uncertainty about the correct specification of the model with respect to the role of money balances makes forecast-based Taylor rules potentially destabilizing.

Second, regardless of how the model is formulated with respect to the role of money, liquidity traps are, in some cases, robust to expectational mistakes[24].

2.3.2 Stability of Cycles and Sunspots: Money in the Production Function

The results discussed in the previous section are based on linearization and therefore have only local validity. As Shown already in Benhabib et al. (2001b, 2002) there may be other equilibria under perfect foresight. In this section I consider the local stability of these equilibria. The scope of the analysis is to assess the theoretical relevance of the model's *global indeterminacy.* I therefore proceed to evaluate the learnability of other perfect-foresight equilibria like cycles and sunspots. I focus on the case where money enters in the production function for two main reasons. First, as argued in Benhabib et al. (2002) this is likely to be the most relevant case, because a large part of money demand in the U.S. comes from firms. Second, this is the case in which the Taylor rule is potentially more destabilizing, because cycles and sunspots exist *close* the inflation target equilibrium.

Concerning the model with money in the production function, under the benchmark calibration (with $\mu = -9$) the map under perfect foresight has a cycle of period 2, for (approximately) $\sigma \in (1, 2.42)$[25]. The economy fluctuates between two states of higher and lower inflation, with respect to the inflation target. Figure 2.2 shows the period-two cycle for a given value of σ, under both F and G.

The thick lines correspond to the branches of the (backward) map G; the thin line describes the second iterate of the map $\pi_{t+2} = F(\pi_{t+1})$ in forward dynamics and the dashed line shows the first iterate of the map in forward dynamics. As shown in Figure 1, the correspondence G -implied by the map (2.17)- is made up of two 'branches'. The first, G_1, intersects the 45 degree

[24]The result also contrasts with Honkapohja and Mitra (2001). They analyze Markov sunspots equilibria and show that none is learnable in a class of models with money in the utility function and nominal rigidities.

[25]See Benhabib et al. (2001b).

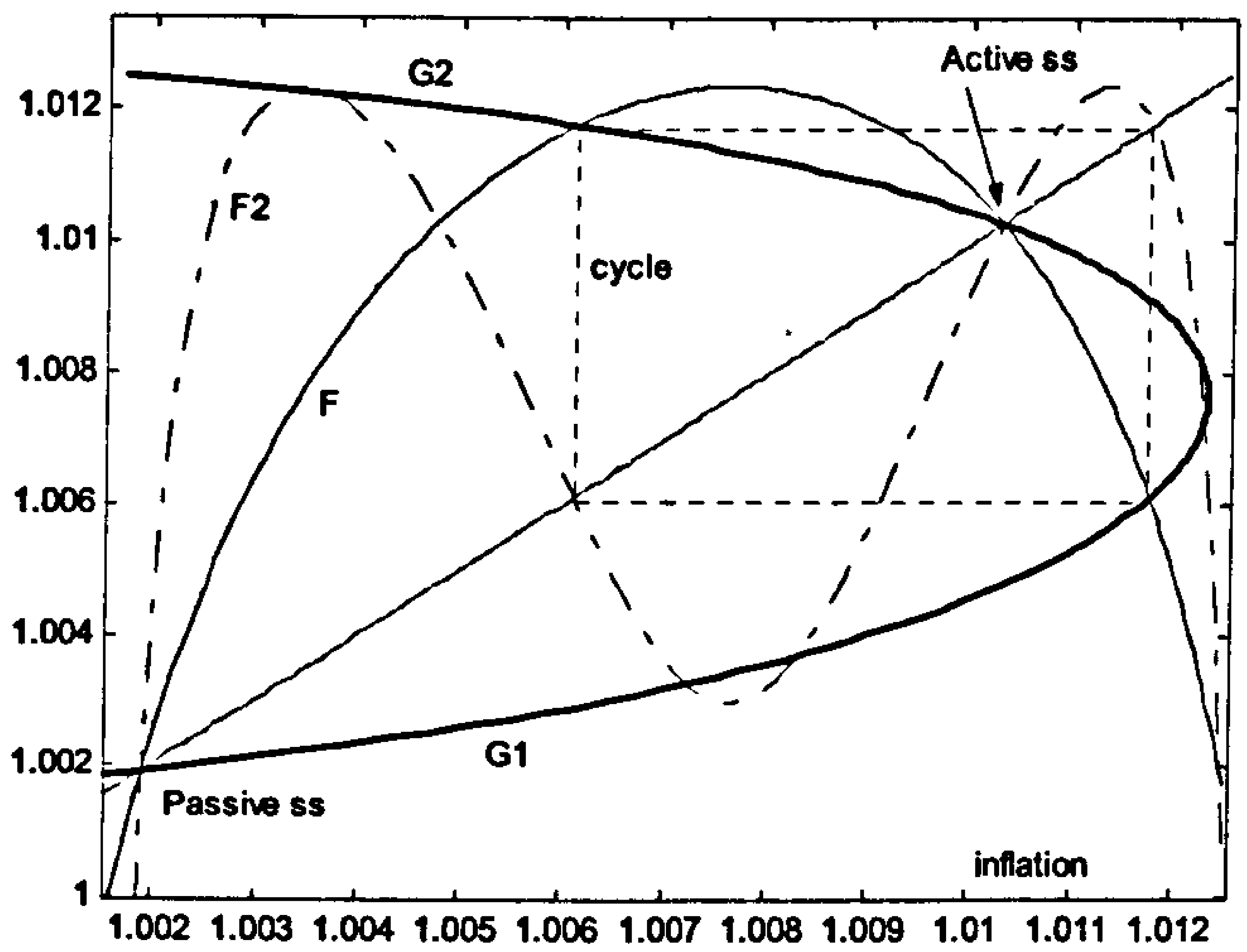

Figure 2.2:

line at the passive steady state while G_2 intersects the 45 degree line at the active steady state.

Backward (G) and forward (F) maps are equal at the fixed point and at the equilibrium cycle (shown by the dotted line). Notice that each point of the cycle rests on a different branch of the backward map. Nevertheless, it is immediate to see that at the cycle:

$$G_1'(\pi_H) = \frac{1}{F'(\pi_L)};\ G_2'(\pi_L) = \frac{1}{F'(\pi_H)}$$

where π_H corresponds to the point of the cycle where inflation is higher than the inflation target, while π_L is the point where inflation is lower than the target. Hence, local stability of the cycle under learning can be studied from the local properties of F.

Let us consider Figure 3. It describes the map F_σ^2: the different curves show how the cycle changes for varying σ.

From the picture we know that the map F_σ^2 admits a cycle for certain values of σ. Consider $\sigma \in (1.5, 2.5)$: there exists $\overline{\sigma}$ such that for $\sigma < \overline{\sigma} < 2.5$, F_σ^2 has four fixed points. For $\sigma > \overline{\sigma}$ F_σ^2 has two fixed points.

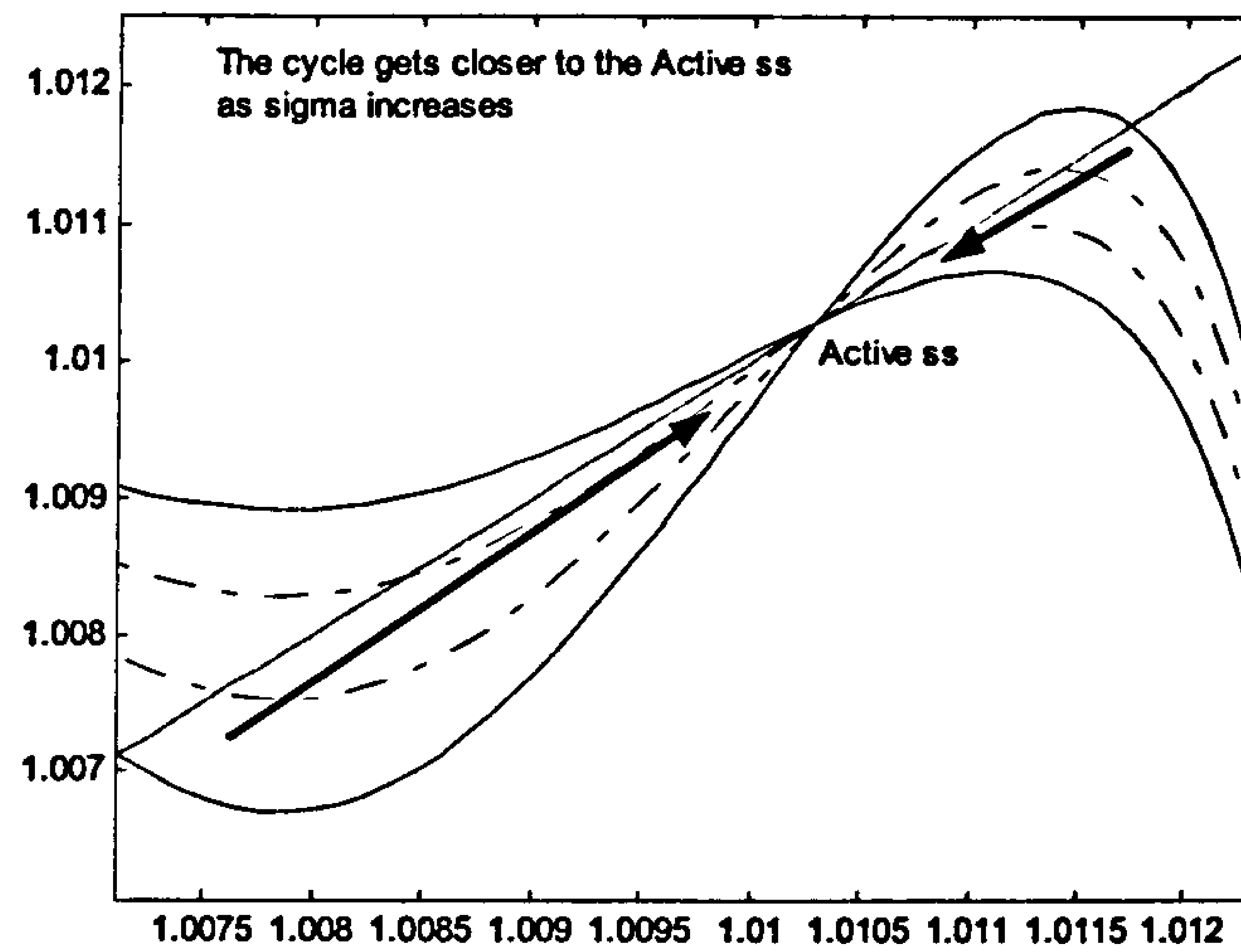

Figure 2.3:

From the observation of Figures 2.2 and 2.3 and from the form of the policy rule[26], it is possible to conclude the following:

1. at the point on the cycle where inflation is high (π_H), monetary policy is always active;
2. the degree of aggressiveness of the monetary rule at this point decreases as the two-period cycles gets closer to the active steady state and subsequently vanishes by "merging" with the fixed point;
3. as σ increases, the cycle gets closer to the fixed point and eventually disappears. Instead, for low values of σ, the point of the cycle where inflation is low (π_L) is close to the passive steady state and thus the policy rule is passive. But, as $\sigma \rightarrow \overline{\sigma}$ we have that $\epsilon_\rho(\pi_L) \rightarrow \epsilon_\rho(\pi^*)$ and it exists σ_0 such that for $\sigma < \sigma_0$, $\epsilon_\rho(\pi_L) > 1$.

Given the existence of a cycle, using well known results from Azariadis and Guesnerie (1986), Grandmont (1987) and Evans and Honkapohja

[26]Notice that changes in σ do not affect the policy rule.

(2001a) it is possible to show that sunspot equilibria exist *close* to the period two cycle and to the determinate active steady state[27]. Finally, notice that the qualitative features of the map, described above, are robust to small changes in the parameters. The following result is thus robust to small deviations from benchmark parameters.

Proposition 15 *Let G be given by (2.17).*

(i) Given the deterministic system (2.12), under the benchmark calibration, there exist values $\widehat{\sigma}$ *and* $\widetilde{\sigma}$, $(\widehat{\sigma} < \widetilde{\sigma})$ *such that for* $\sigma < \widehat{\sigma}$ *the cycle is stable under learning for every* $\alpha \in (0,1)$. *For* $\sigma > \widetilde{\sigma}$ *the cycle is unstable under learning for every* $\alpha \in (0,1)$. *For* $\sigma \in [\widehat{\sigma}, \widetilde{\sigma}]$ *there exists* $\widehat{\alpha}_\sigma \in (0,1)$ *such that for* $\alpha < \widehat{\alpha}_\sigma$ *the cycle is stable under learning, while for* $\alpha > \widehat{\alpha}_\sigma$ *the cycle is unstable under learning*[28].

Given the stochastic system (2.13) and (2.14);

(ii) for $\sigma < \widehat{\sigma}$ *there exist learnable* $\epsilon' -SSEs$ *relative to the deterministic cycle;*

(iii) for any $1 < \sigma < \widetilde{\sigma}$, *let* $\pi_1(\sigma) \in (\pi^*, \pi_H(\sigma))$ *and* $\pi_2(\sigma) < \pi_L(\sigma)$, *where* $\pi_H(\sigma), \pi_L(\sigma)$ *represent the equilibrium cycle, for given* σ. *Then there exist* z_{11}, z_{22} *such that the sunspot equilibrium* $(\pi_1(\sigma), \pi_2(\sigma), z_{11}, z_{22})$ *is learnable (whether or not the cycle is learnable).*

Remark 16 *Given the standard calibration a more active policy rule (say* $A = 2$*) implies the existence of learnable cycles for a wider range of* σ *(in the simulations cycles are learnable for values of* $\sigma \approx 2.5$*)*[29].

Remark 17 *Using Proposition 12.7 and 12.9 in Evans and Honkapohja (2001a) it is possible to show that sunspots constructed on the active and passive steady state exists and are learnable if and only if* $|G_2'(\pi_1(\sigma))| < 1$ *and* $|G_1'(\pi_2(\sigma))| < 1$ *which is verified under the condition in Proposition (12).*

Remark 18 *It is straightforward to show that under the benchmark calibration the inflation target steady state is determinate and learnable. In fact, as* σ *decreases both the stability conditions for the learnability of cycles and steady states are satisfied.*

[27] See Appendix B for details.

[28] Evans and Honkapohja (1995) show the existence of noisy REE cycles that are learnable if the deterministic cycle is learnable, provided the learning parameter is decreasing over time (i.e. s^{-1} is used instead of α). Hence, the results in Proposition (15) are robust to the introduction of small noise.

[29] Also, more active Taylor rule imply period three cycles and chaos for a wider range of parameters than found in Benhabib et al. (2001b).

The Proposition leads to two interesting results. First, it shows that *the inflation target and the equilibrium cycle and sunspots around it are both learnable.* The implication for monetary policy is that small shocks are sufficient to destabilize the economy. Small deviations from the active steady state might lead the market participants to coordinate on a equilibrium cycle. Second, very active policy rules (i.e. $A > 1.5$), that are commonly obtained from optimal monetary policy design, can be destabilizing, given that they imply learnable cycles and sunspots for a wider range of parameters.

Empirical validity. Consider the two following examples. Concerning the stability of cycles, Proposition (15) implies that if π_L is not too close to the inflation target, than the cycle is stable under learning, at least for some values of the parameter α. Under the benchmark calibration[30] we have that $\widehat{\sigma} \approx 2$ and $\widetilde{\sigma} \approx 2.182$. For example, we have that for $\sigma = 2.18$, $\pi_H = 4.7\%$ while $\pi_L = 3.2\%$, and $\epsilon_p(\pi_L) = 1.18$, which implies a mildly active policy. This equilibrium is learnable for $\alpha < 0.21$. Given that the target is 4.2%, the cycle is 'close' to the active steady state.

Concerning the stability of sunspots, result (iv) is new in the literature. I construct the sunspots on different branches of G -see Appendix B. For example, it can be shown that for $\sigma = 1.9$, the SSE given by $\pi_1 = 1.0104$ (4.25% in annual terms), $\pi_2 = 1.0075$ (3%), $z_{11} = 0.2$, $z_{22} = 0.3$ is learnable. Note that $\pi_1 \approx \pi^*$. Moreover, assuming $\sigma = 2.35$, for which the cycle is indeterminate and non-learnable, the SSE given by $\pi_1 = 1.046$ (6%), $\pi_2 = 1.0087(3.5\%)$, $z_{11} = 0.19$, $z_{22} = 0.25$ is learnable and 'close' to the deterministic cycle. Note that in both the inflation states monetary policy is active.

Dynamics under learning. By considering explicitly the functional forms of the model, it is also possible to give an intuition about the dynamics under learning. Figure 2.1 shows the convergence process of (2.10) to the active steady state and the liquidity trap, under the hypothesis of perfect foresight, to give an idea of how the economy can fall into a liquidity trap. A shock can start the economy on the deflationary path, depending on how the agents react to their expectations (i.e. which branch of G we are considering). In this case the economy converges monotonically to the new equilibrium. Instead, convergence to the central bank's target implies fluctuations in the inflation rate.

[30] For small changes of the other parameters, such as the degree of policy aggressiveness, $\widehat{\sigma}$ and $\widetilde{\sigma}$ also change.

Proposition (15) concerns the learnability of equilibrium cycles obtained under perfect foresight. But learning dynamics also generates other equilibria along bifurcation parameter values[31] of the system (2.12). As the cycle loses stability and σ increases, it is likely that a *saddle-node bifurcation* occurs. Hence, given $\overline{\sigma}$ as the bifurcation parameter, for some $\sigma < \overline{\sigma}$ the fixed point of the second iterate of (2.12) is surrounded by two other fixed point (i.e. two cycles of period 2). One of them is stable, and the other is unstable. For $\sigma > \overline{\sigma}$ the fixed point disappears (as we also see it from the graph).

Also, the cycle looses stability as α increases for given $\sigma \in [\widehat{\sigma}, \widetilde{\sigma}]$. In this case a *period-doubling bifurcation* might occur, generating a period four cycle that can be either stable or unstable. Note that *these other fixed points do not exist under perfect foresight.* They are generated by the learning behavior. Hence, forecast-based rules have the potential to further de-stabilize the economy if we take into account the dynamics under learning[32].

This result is important in its simplicity because it shows the possibility of learnable equilibria, other than the inflation target, for any parameter value of σ for which the cycle exists[33]. Even though we have local determinacy and learnability of the inflation target equilibrium, the agents learning process can still converge to a sunspot equilibrium that is 'close' to it. This also means that the 'stability corridor' mentioned in the previous section is indeed extremely small if it exists at all[34]. Hence, forecast-based Taylor rules are likely to be destabilizing, under this model of the economy[35].

[31] Note that, given the standard parametrization of the model, the eigenvalues at the fixed points are real (and with opposite sign), while the eigenvalues at the cycle are complex when stable and real-saddle (with one eigenvalue greater than one) when unstable. In particular, the eigenvalues are real *before* becoming unstable. Also, for $\sigma \in (\widehat{\sigma}, \widetilde{\sigma})$ the cycle loses stability as α increases and, again, the change in stability involves real eigenvalues. The ustable cycle is a saddle with one eigenvalues less than minus one.

[32] These bifurcations occur only if extra conditions are satisfied, see Wiggins (1990). More precisely, the problem can be reduced to a one dimensional bifurcation through a center manifold reduction. But we do not go further in the analysis.

[33] Sunspot equilibria may seem a more empirically plausible explanation for the instability generated by the Taylor rule. In fact, the regular behavior implied by deterministic cycles is not observed in the data.

[34] Christiano and Rostagno (2001) and Benhabib et al. (2001c) propose to shift to a money growth rule if the observed inflation deviates consistently from target, so to avoid other equilibria, like a liquidity trap. This solution would not be helpful in our case because cycles and sunspot equilibria can be very close to the active fixed point. Thus, such a scheme would eliminate the stabilization properties of the Taylor rule.

[35] Simulations show that the result is unchanged if I consider different inflation targets. For example the (implicit) inflation target for the European Central Bank would be $\pi^* = 2\%$: Proposition (3) holds also in this case.

2.3.3 Money in the utility function

Because of our uncertainty about the 'correct' model of the economy, it is useful to consider the behavior of the forecast-based Taylor rule when money enters in the utility function[36]. In order to allow comparisons between the two models and match the data I assume that, in the model where money enters in the utility function the functional form is CES:

$$U(c_t, m_t) = \frac{\left([(1-a)\, m_t^{\mu} + a c_t^{\mu}]^{\frac{1}{\mu}}\right)^{(1-\sigma)}}{1-\sigma}. \tag{2.24}$$

This specification allows the money demand implied by (2.16) and (2.21) to be exactly the same. Hence, the two models have the same calibration as shown in Table 1.

We know from Proposition (14) that the stability of the fixed point, for a given Taylor rule, depends on the elasticities $\epsilon_{cm}\epsilon_m$. This is also true for the global dynamics of the system. It is easy to show that $\epsilon_{cm}\epsilon_m$ is increasing in both σ and μ. Figures 2.4 and 2.5 show the map under different values of σ and μ. First, notice that if I use values for μ close to the short run value, no cycle exists, as shown in Figure 2.5. This is good news, but we know from Proposition (14) that for $\epsilon_{cm}\epsilon_m$ not too low, there exist learnable sunspots around the indeterminate and non-learnable passive steady state.

For example, if we fix $\mu = -9$ and choose $\sigma > 2$ than the conditions for the existence of learnable sunspots are verified. Hence, the economy can fall into a liquidity trap for very plausible parameter values.

A period two cycle emerges *as both μ and σ increase.* Given the uncertainty about these parameters I consider a choice of $\mu < \mu^{LR}$ and $\sigma < 4$ as plausible parameter values. For example, for $\mu = -3.5$ and $\sigma = 3.1$ (and the other parameters at their benchmark values) there exists a cycle around the passive steady state, as shown in Figure 2.4. The result holds for higher μ and it is robust to parameter perturbation. For lower μ the values of σ required to get the cycle are not interesting. But also in this case, a too active monetary policy is further destabilizing. For example, if $A = 2$, then the cycle appears for $\mu = -9$ and $\sigma \gtrsim 1.5$. Given that $A = 2$ has been suggested to describe the policy stance of the post-Volker period[37], the existence of cycles holds for a large portion of the 'plausible' parameter space.

[36] Carltrom and Fuest (2001a) point out the fact that the way money enters in the utility functions has important implications for the stability properties of the Taylor rule. I consider the standard approach in this section and a different trading environment in the extensions.

[37] See Clarida et al. (2001).

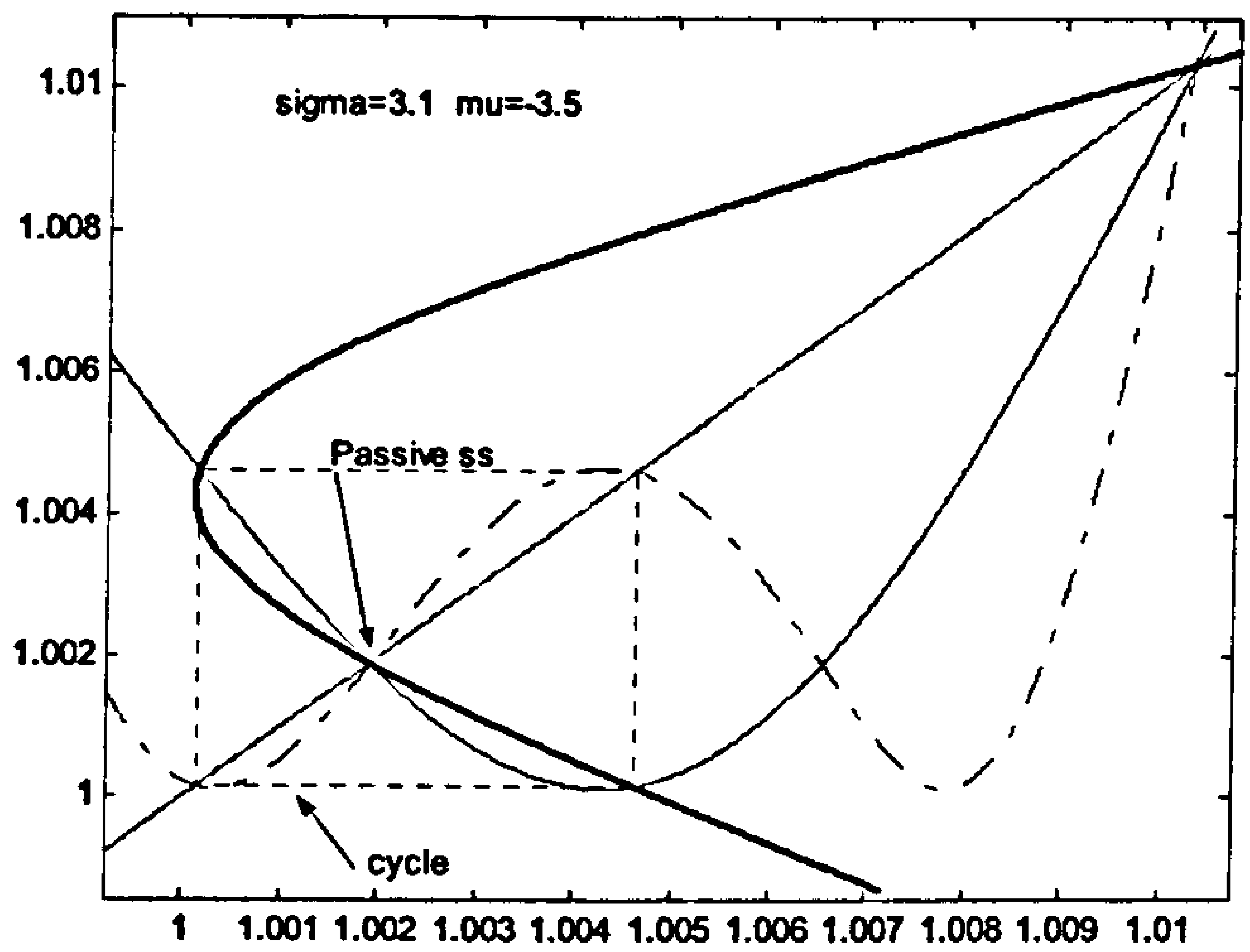

Figure 2.4:

The results are described in the following Proposition. Again, the results remain valid for small deviations from the benchmark parametrization.

Proposition 19 *Let G given by (2.22) and under the benchmark parametrization but with $A = 2$, such that for $\sigma > \overline{\sigma}'$ a period-two cycle exists.*

Under the deterministic system (2.12);

(i) for $\overline{\sigma}' < \sigma < \widetilde{\sigma}'$ the cycle is not learnable under rule (2.11) for any value of $\alpha \in (0,1)$;

(ii) for $\sigma \in (\widetilde{\sigma}', \widehat{\sigma}')$, $\widetilde{\sigma}' < \widehat{\sigma}'$, there exists an $\widehat{\alpha}'$ such that for every $\alpha < \widehat{\alpha}'$, the cycle is learnable;

(iii) for $\sigma > \widehat{\sigma}'$ the cycle is learnable for every $\alpha \in (0,1)$.

Under the stochastic system (2.13) and (2.14);

(iv) for $\sigma > \widetilde{\sigma}'$ there exist learnable SSE ϵ-close to the cycle. For $\sigma > \widehat{\sigma}'$ the sunspots are robust to agent's overparametrization.

(v) there exist learnable 'liquidity trap' sunspots ϵ-close to the two fixed points.

Remark 20 *Under the benchmark parametrization $\widetilde{\sigma}' \approx 2.23$ and $\widehat{\sigma}' \approx 2.89$. Again, changing the parameters affects $\widetilde{\sigma}'$, $\widehat{\sigma}'$. It is straightforward to verify*

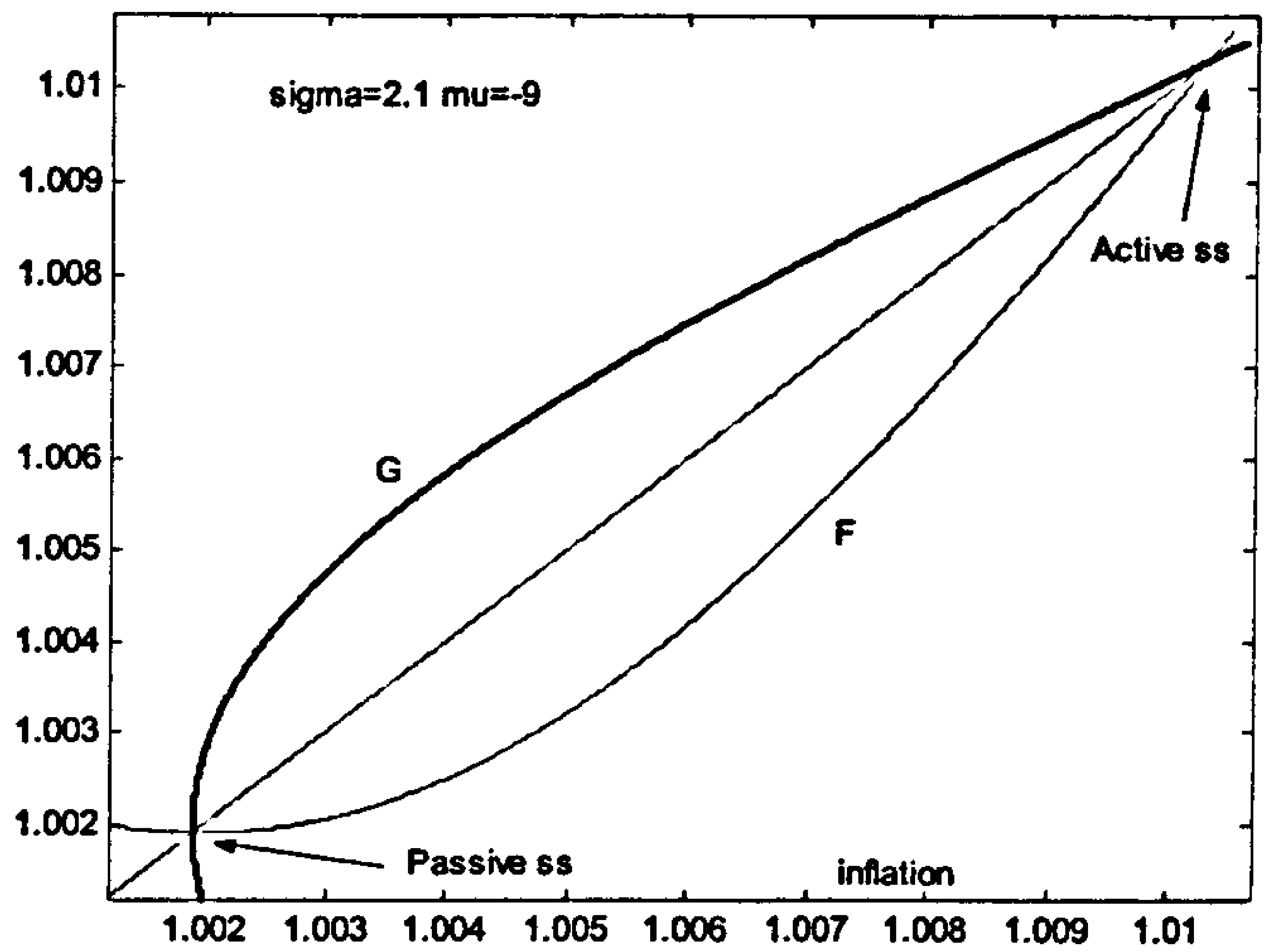

Figure 2.5:

that under the benchmark parametrization, the inflation target, the liquidity trap, the cycle and the sunspots are all locally stable under learning.

Remark 21 *Even if a formal proof is omitted for brevity, it can be shown that also for the case of money in the utility function there exist learnable sunspots that are 'close' to an indeterminate and non-learnable cycle. Assume that $\sigma = 1.8$ and $A = 2$, while the other parameters are set according to Table I. Let us consider the following equilibrium: $\pi_1 = 1.002$ (0.08%), $\pi_2 = 0.9975$ (−0.1%), with $z_{11} = 0.15$, $z_{22} = 0.15$. This is relatively close to the indeterminate and non-learnable cycle where high inflation is $\pi_H = 1.001$ (0.04%) and low inflation is $\pi_L = 0.997$ (−0.12%). The economy fluctuates among almost zero inflation and negative inflation. It can be shown numerically, by using (54) in the Appendix, that this sunspot is learnable. Hence, also for the case of money in the utility function I conjecture the existence of learnable sunspots for all values of σ such that the cycle exists.*

The results are obtained with the same procedure as for the model with money in the production function and the proof is therefore omitted. Also in this model, forecast-based Taylor rules can be destabilizing: the economy can converge to equilibria in which inflation fluctuates below the target.

Concluding, uncertainty about the correct model of the economy and the parameter σ makes a forecast-based Taylor rule a hazardous choice. In fact, if the 'true σ', which the Central Bank is not likely to observe, is sufficiently low and the economy is better approximated by a model with money in the production function, then forecast-based Taylor rules are going to be destabilizing. On the other side, if the true σ is sufficiently high and the 'correct' model has money in the utility function we get the same conclusion. Hence, *whatever is the 'estimate' of σ that it is used for calibration, there exist a model of the economy for which the forecast-based Taylor rule generates instability.*

2.4 Backward-Looking Taylor Rule

In this section I argue that the instability generated by a forecast-based Taylor rule can be completely eliminated by shifting to a Taylor rule that reacts to current and past inflation. In fact, I show that such a policy rule improves on the forward looking rule in two important ways.

First, for plausible calibrations, the economy does not have equilibrium cycles. Second, a backward-looking policy rule reverses the stability properties of the liquidity trap and sunspots, leaving a unique learnable equilibrium: the inflation target.

Let us consider the case where the central bank responds to inflation with some inertia. This is justified in Woodford (2002) and Benhabib et al. (2001a) as a more plausible description of central banks' operating rules[38].

In this case the central bank is assumed to react to an exponential moving average of current and past inflation rates that takes the form:

$$\overline{\pi}_t = (1-\delta)\sum_{j=0}^{\infty}\delta^j \pi_{t-j} \tag{2.25}$$

where $0 < \delta < 1$ defines how much weight is given to current inflation, as is easily seen by re-writing (2.25) as:

$$\overline{\pi}_t = (1-\delta)\pi_t + \delta\overline{\pi}_{t-1}. \tag{2.26}$$

[38] "In practice, monetary policy will never involve feedback from an *instantaneous* rate of inflation [...], because available inflation measures will always be time-averaged over at least a period such as a month". Woodford (2002), Chapter 2, p. 44.

By taking $\overline{\pi}_t$ as the inflation measure to which the central bank reacts, we can write the Taylor reaction function as:

$$R_t = 1 + (R^* - 1)\left(\frac{\overline{\pi}_t}{\pi^*}\right)^{\frac{A}{(R^*-1)}}. \tag{2.27}$$

The next sub-sections evaluate the properties of this rule under the two models used in the previous sections.

2.4.1 Money in the production function

Under the Taylor rule (2.27), the reduced-form model becomes[39]:

$$U_c\left(y\left(\rho\left(\overline{\pi}_t\right)\right)\right) = E\left\{\beta\left(1-\delta\right)U_c\left(y\left(\rho\left(\overline{\pi}_{t+1}\right)\right)\right)\left[\frac{\rho\left(\overline{\pi}_t\right)}{\overline{\pi}_{t+1} - \delta\overline{\pi}_t}\right]\right\}. \tag{2.28}$$

Local stability and learnability conditions are considered in the following Proposition:

Proposition 22 *Let G be given by (2.28). Given the system (2.12), assume;*

$$\sigma < \max\left\{-\frac{\left(\epsilon_\rho\left(\pi^*\right)\epsilon_y\left(\overline{\pi}\right)\right)^{-1}}{(1-\delta)}, -\frac{1}{2\epsilon_y\left(\overline{\pi}\right)}\left(1+\frac{1}{\epsilon_\rho\left(\pi^*\right)}\right)\right\} \tag{2.29}$$

(i) Then the inflation target is determinate and learnable for every α, while the passive steady state is indeterminate and non-learnable for every α. Moreover, condition (2.29) is verified for every parameter value that gives a locally determinate inflation target under the forecast-based Taylor rule.

(ii) given the benchmark calibration, for $\mu \in \left(\mu^{SR}, \mu^{LR}\right)$, no cycles exist around the active steady state;

Given the system (2.13), (2.14);

(iii) There exist $\epsilon' - SSEs$ around the indeterminate passive steady state but they are unstable under learning.

(iv) for any σ, there exists δ^* such that (2.29) holds;

[39] In both models, the solution for the backward looking Taylor rule is a non invertible (in both the forward and backward dynamic) one dimensional difference equation. Also, by computing the map numerically, it is possible to notice that both steady state belong to the same branch. The second branch either does not exist or it exists for values of the inflation rate that do not make economic sense.

Remark 23 *Note that under the (implausible) case where $\delta = 0$, statements (i)-(iii) hold for values of $\mu \in (\mu^{SR}, \mu^{LR})$ and for $3.5 > \sigma > 1$, provided*[40] $\frac{A}{R^*} < 2.3$.

Remark 24 *From condition (2.29), the set of parameters which lead to local determinacy under the backward-looking Taylor rule is larger than for the forecast-based Taylor rule. Also, the set of parameters for which (i), (ii) and (iii) hold increases for more backward-looking policy rules ($\delta \to 1$).*

Remark 25 *By simulating the map numerically, it can be shown that no cycles exist for the mentioned parameter values.*

The Proposition says that if the policy rule is sufficiently backward-looking, then it guarantees a unique learnable equilibrium, i.e. a unique equilibrium that it is robust to expectational mistakes. Hence, *it is always possible to find a policy rule that is sufficiently backward-looking to stabilize the economy*[41].

2.4.2 Money in the utility function

The reduced-form under the backward-looking rule (2.27) becomes:

$$U_c(\overline{y}, m(\rho(\overline{\pi}_t))) = \frac{\beta U_c(\overline{y}, m(\rho(\overline{\pi}_{t+1})))(1-\delta)}{\overline{\pi}_{t+1} - \delta \overline{\pi}_t} \rho(\overline{\pi}_t). \tag{2.30}$$

The stability properties of the Taylor rule are discussed in the following Proposition.

Proposition 26 *Let G be given by (2.30). Given the system (2.11) we have that for any parameter value;*

(i) the active steady state is determinate and learnable for any $\alpha \in (0, 1)$;

(ii) the passive steady state is indeterminate and non-learnable for any $\alpha \in (0, 1)$;

(iii) no cycles exist around the active steady state;

[40] Notice that the last condition must hold only in the worse case where $\mu = \mu^{LR}$ and $\sigma = 3.5$. With these parameter values the elasticity of output with respect to the interest rate at the passive steady state is very high in absolute value. Under the standard parametrization the condition becomes $\frac{A}{R^*} < 12$!

[41] Also a contemporaneous Taylor rule guarantees a unique equilibrium, for plausible parameter values.

Given the system (2.13), (2.14);

(iii) there exist $\epsilon' - SSE$ sunspot equilibria around the passive steady state but they are not learnable.

Remark 27 *By simulations it is possible to show that no cycles exist, given that the map G is monotonic.*

The proof of the Proposition (26) follows the same steps as for Proposition (25) and it is therefore omitted for brevity[42]. Also for the case of money in the utility function, if the central bank responds to a weighted average of current and past inflation rates there exists a unique equilibrium that is stable under learning: the inflation target.

Concluding, from the Propositions (25) and (26) it is apparent that more backward-looking Taylor rules should be preferred to forecast-based rules for the following reasons: 1) they eliminate the possibility that adverse shocks might lead the economy to a liquidity trap, where the inflation target is systematically missed; 2) they reduce possible sources of instability in the economic system, generated by equilibrium fluctuations around the inflation target; 3) they increase model robustness, given that the parameter values for which we obtain a unique learnable equilibrium is increased. Notice also that the results are obtained allowing uncertainty about what the correct model of the economy should be, at least concerning the role of money: in fact the results are robust to both specifications of the model.

Finally, the results obtained with this model seem to be in contrast with Bullard and Mitra (2001). They find that a policy rule that reacts to *lagged inflation* is potentially destabilizing, i.e. could give rise to local indeterminacy or instability under learning. Instead, under rule (2.25) the Central Bank reacts also to *current* inflation. This might explain the different conclusions[43].

[42] The interested reader should see Eusepi (2002).

[43] It could be argued that having the central bank reacting to lagged inflation rates is more plausible, given its information constraints. Nevertheless, the model under perfect foresight is solved under the assumption that both the Central Bank and the private sector observe current inflation. Rule (2.25) is consistent with this assumption.

2.5 Extensions

2.5.1 Cash In Advance Constraint (CIA)

Let us consider a different trading environment as in Carlstrom and Fuest (2001b). They show that under CIA timing the stability of the model under a Taylor rule dramatically changes. Under a forecast-based Taylor rule determinacy is achieved by a passive policy stance. The goal of this paragraph is to show that, on the contrary, the previous results are robust to a modification in the trading environment.

In the simplest case of an endowment economy, the budget constraint becomes:

$$M_{t+1} = M_t + \tau + B_{t-1}R_{t-1} - B_t - P_t c_t + P_t y$$

and the CIA constraint is:

$$A_t = M_t + \tau + B_{t-1}R_{t-1} - B_t$$

where A_t denotes the liquidity available to the representative agent in the current period. Substituting back the CIA constraint in the utility functional (defined over current consumption and real liquidity) and solving the optimization under the resource constraint we obtain a money demand equation:

$$a_t = H(R_t) \quad H' < 0$$

and the following reduced-form:

$$U_c(c, H(R_t)) = U_c(c, H(\rho(R_{t+1}))) \frac{\beta}{\pi_{t+1}} \rho(R_{t+1}). \tag{2.31}$$

In the following Proposition I compare the stability properties of forecast-based and backward-looking Taylor rules. As above, I consider the *most favorable* case where the rules lead to local determinacy and I study the possibility of learnable equilibria other than the inflation target. Where needed, the functional form for the utility function is CES. This leads to a demand for money that is equivalent to the case of money in the production function.

Proposition 28 *Let G be given by the CIA model. Assume:*

$$|\epsilon_{cm}\epsilon_{H}| > \frac{1}{2}\left(\frac{1}{\epsilon_{\rho}(\pi^*)} + 1\right) \tag{2.32}$$

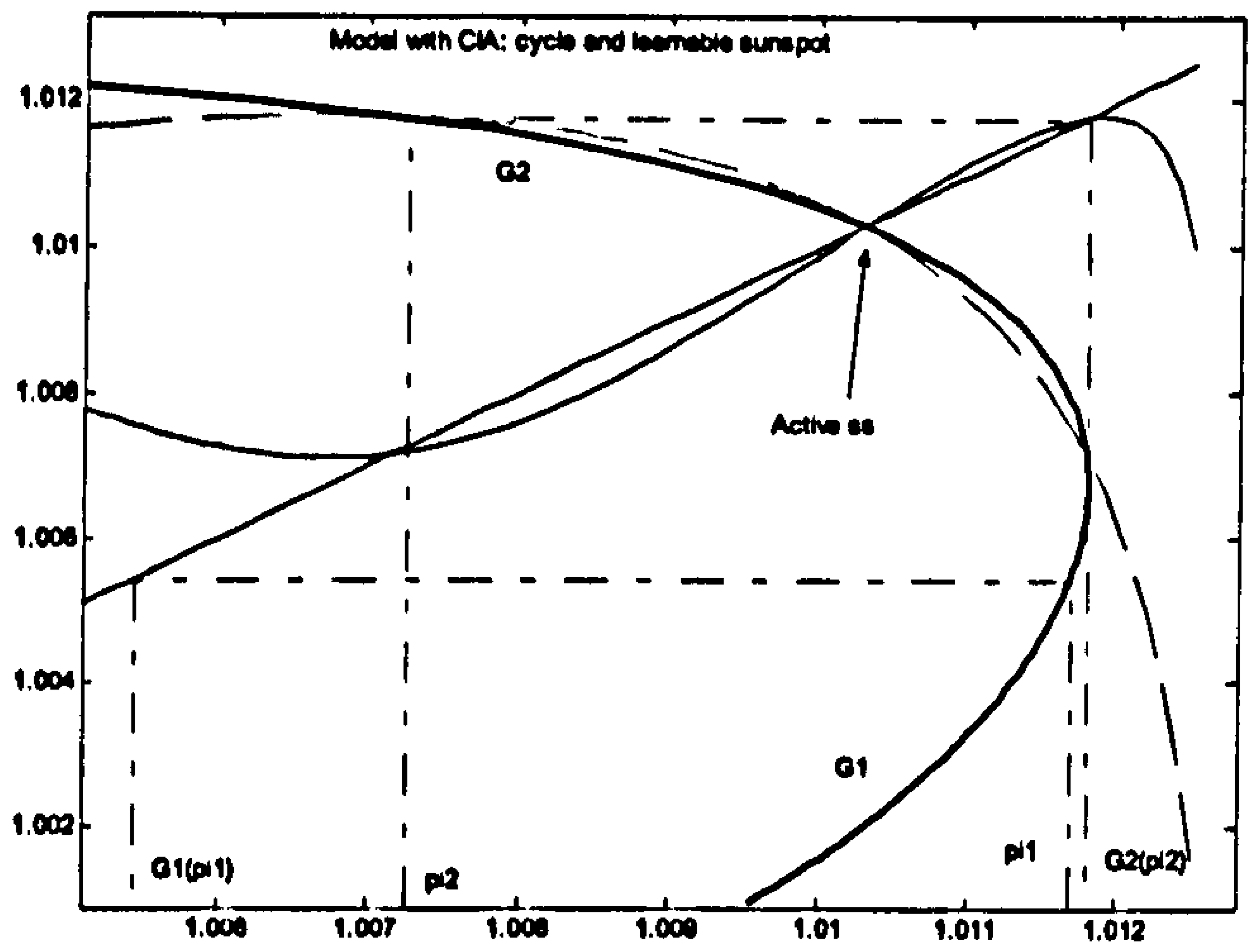

Figure 2.6:

where ϵ_{II} denotes the elasticity of the demand for money. Under the forecast-based Taylor rule;

(i) the active steady state is determinate and learnable for every α, under the learning rule (2.11);

(ii) the passive steady state is determinate and learnable for every α;

(iii) there exists $\widehat{\epsilon_p}(\pi^*)$ consistent with (2.32) such that for $\epsilon_p(\pi^*) > \widehat{\epsilon_p}(\pi^*)$ there exist learnable cycles and $\epsilon' - SSE$ around the active steady state, under the learning rules (2.11) and (2.13) respectively;

Under the backward-looking Taylor rule;

(iv) the active steady state is determinate and learnable for every α and no cycles or sunspots exist around it for $\mu \in \left(\mu^{SR}, \mu^{LR}\right)$, under the learning rules (2.11) and (2.13) respectively;

(v) the passive steady state is indeterminate and non-learnable for every α under rule (2.11); there exist $\epsilon' - SSE$ sunspots around it but they are not learnable under (2.13).

(vi) for any value of $\epsilon_{cm}\epsilon_{II}$, it exists δ^* such that (iv) and (v) hold;

Remark 20 *A forecast-based policy rule that is aggressive enough to satisfy (2.32) exists provided the elasticity of money demand is not too low (i.e.*

$\mu \to \mu^{LR}$). Condition (2.32) is verified for quite high values of A. Under the most favorable condition ($\sigma \to 1$ and $\mu \to \mu^{SR}$) there exists $A < 3$ for which the condition holds.

Remark 30 *As for Proposition 5, it is possible to show the existence of learnable sunspots around indeterminate and non-learnable cycles. Consider the following example, as shown in Figure 2.6. Set $\mu = \mu^{SR}$, $\sigma = 1.01$, $A = 3.2$ and the other parameters as in Table I. Then the sunspot equilibrium $\pi_1 = 1.0117$ (4.7%), $\pi_2 = 1.0068$ (2.7%) and $z_{11} = 0.027$, $z_{22} = 0.19$ is learnable. It is relatively close to the indeterminate and non-learnable cycle where high inflation $\pi_H = 1.0118$ (4.8%) and low inflation $\pi_L = 1.0071$ (2.8%)*[44].

Remark 31 *Figures 2.7,2.8 make clear the result in (iv), for a given parametrization. It is immediate to see that the cycle appears only as the active steady state becomes indeterminate and thus (2.32) is violated.*

Remark 32 *Also in this case, the properties of the backward-looking Taylor rule are preserved for a wider set of parameter values than implied by condition (2.32): in other words, a backward-looking Taylor rule is stabilizing also for parameter values that imply indeterminacy under a forecast-based rule. Also, a more backward-looking rule (higher δ) increases the parameter space for which (iv) and (v) hold —see the proof of (vi).*

Figures 2(6), 2(8) give an example of the possible equilibria in the two cases. It is apparent that also in a different trading environment, forecast-based Taylor rules generate instability[45]: a too passive policy generates local indeterminacy, while a too active policy leads to multiple learnable equilibria around the locally determinate and stable inflation target equilibrium. Instead, shifting to a sufficiently backward looking rule guarantees a unique equilibrium. Hence, changing the trading environment does not reverse my results.

[44] In this case the stability conditions are calculated numerically, given that the map is not invertible both in backward and forward dynamics.

[45] We know from Carlstrom and Fuest (2001b) that a passive forward-looking Taylor rule achieves local determinacy. Nevertheless, it can be shown that a too passive Taylor rule leads to learnable 'inflationary traps'. The economy converges to a sunspot equilibrium where inflation fluctuates at hyperinflationary levels.

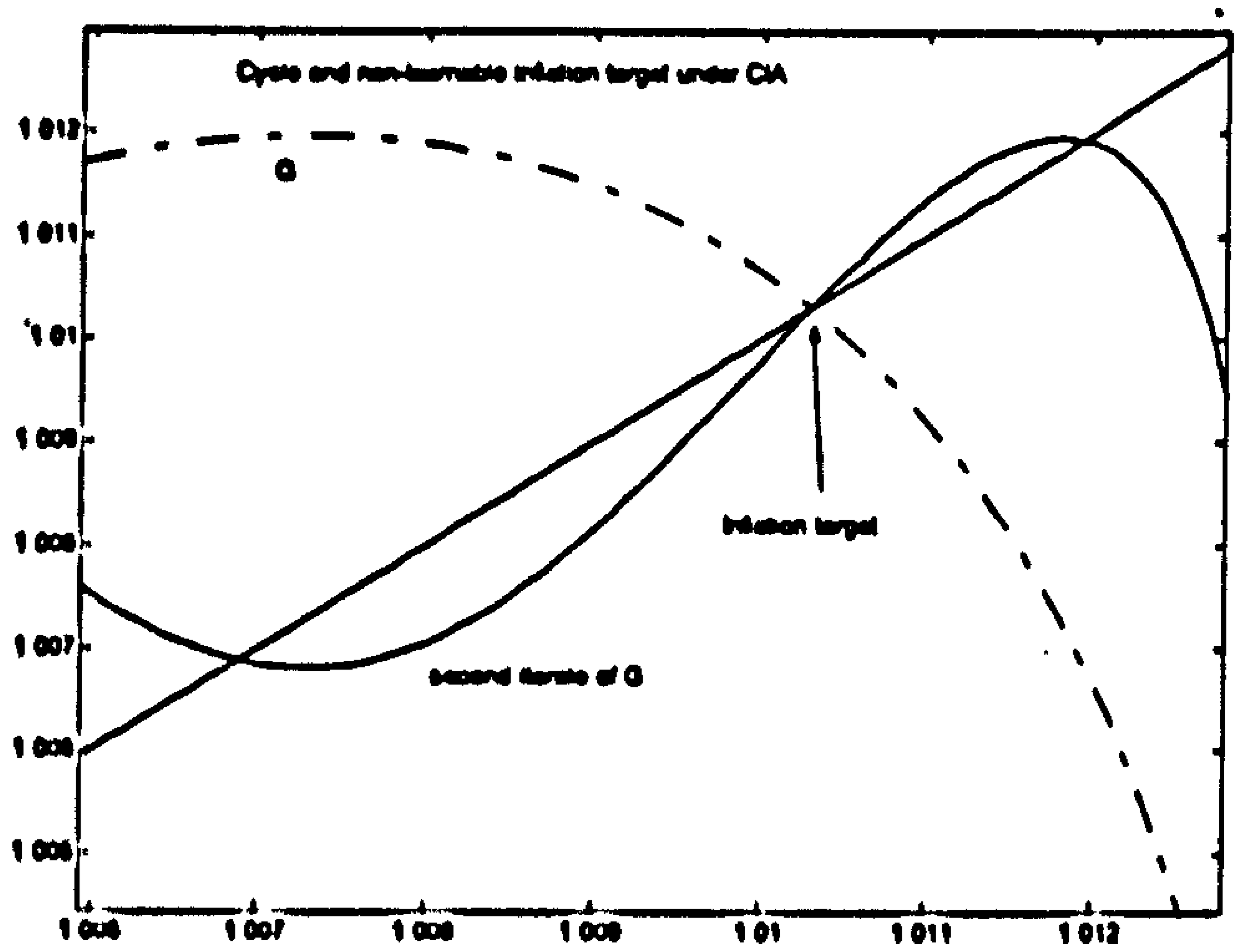

Figure 2.7:

2.5.2 The model with sticky prices

Let us consider endogenous labor supply, monopolistic competition and price stickiness. In this section I show that a *contemporaneous* Taylor rule ($\delta = 0$) maintains the 'good' properties described in the previous sections. I analyze the latter case for analytical simplicity, but it is also the least favorable parametrization to study the stability properties of a backward-looking Taylor rule, if prices are flexible. Nevertheless, Benhabib et al. (2002) find, in a continous time version of the sticky price model, that instability can arise under perfect foresight, even if backward-looking rules are adopted. A study of the discrete time model with $\delta > 0$ implies a three dimensional system and it is left to further research[46].

I use a simplified version of the standard model considered in the literature on Taylor rules, see Woodford (1999)[47]. The consumer-producer solves the following intertemporal problem:

[46] It is not immediate to extend the result in continous time to the discrete time model, because of timing issues. Also, there is the question of the learnability of the other equilibria that are discussed in Benhabib et al. (2002).

[47] The model is a discrete time version of Benhabib et al. (2001a,b).

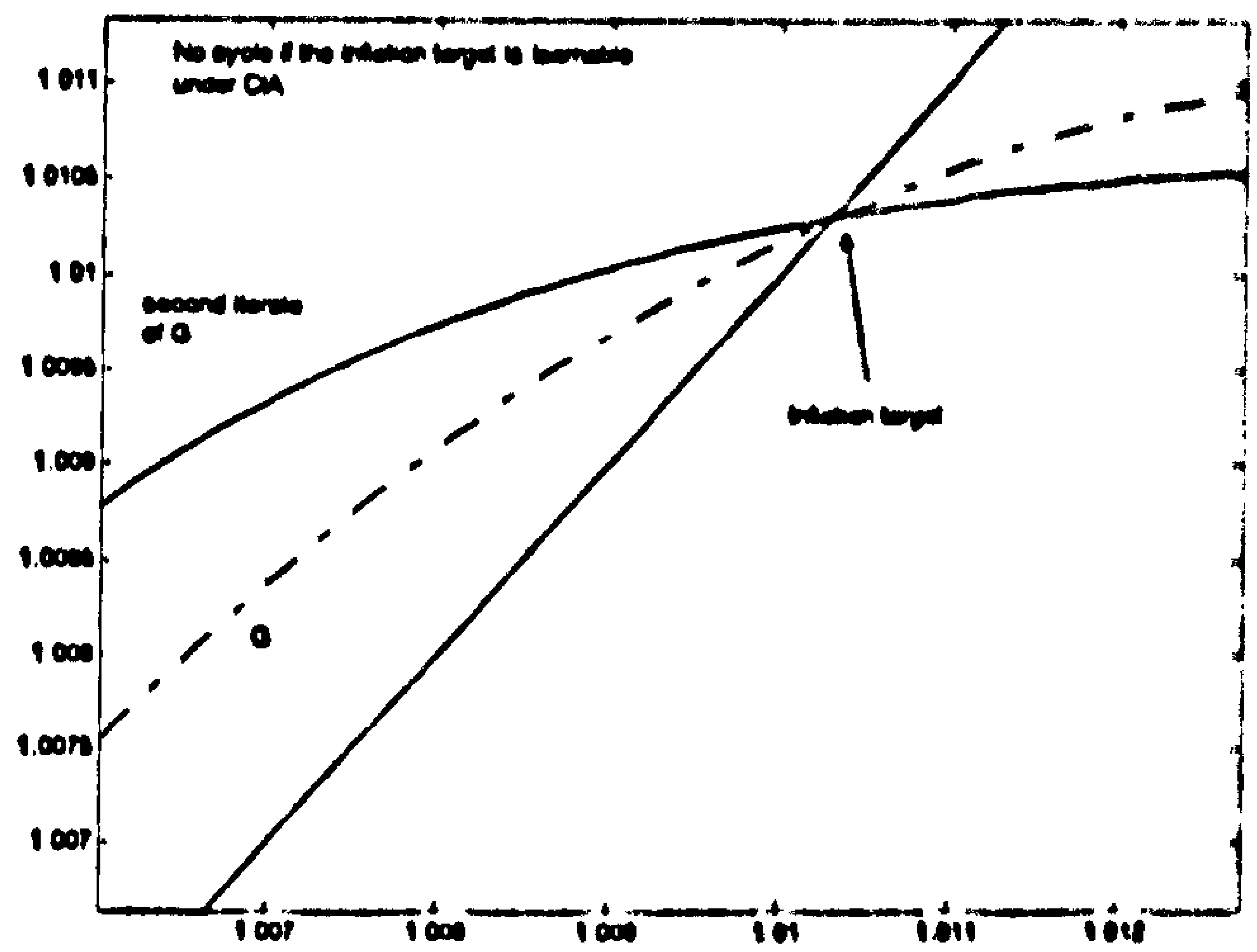

Figure 2.8:

$$\max_{a_t, m_t, h_t, p_t^j} \sum_{t=0}^{\infty} \beta^t \left[U(c_t, m_t) + V(h_t) - \frac{\theta}{2} \left(\frac{p_t^j}{p_{t-1}^j} - \pi^* \right)^2 \right] \tag{2.33}$$

$$a_t + c_t = \frac{R_{t-1}}{\pi_t} a_{t-1} + \frac{1 - R_{t-1}}{\pi_t} m_{t-1} + \frac{p_t^j}{p_t} Y^d d\left(\frac{p_t^j}{p_t}\right) - \tau_t$$

where $V(.)$ represents the disutility from labor and θ captures the costs of price revisions. Finally τ is a transfer from the government. As in the previous sections, I consider both cases of money in the production function and money in the utility function.

Money in the utility function

Given the equilibrium condition:

$$f(h_t, m_t) = Y^d d\left(\frac{p_t^j}{p_t}\right) = c_t \tag{2.34}$$

and the problem (3.3) we obtain the following first order conditions:

$$U_c(c_t, m_t) = \beta \frac{U_c(c_{t+1}, m_{t+1}) R_t}{\pi_{t+1}} \tag{2.35}$$

$$U_m(c_t, m_t) = U_c(c_t, m_t) \frac{(R_t - 1)}{R_t} \tag{2.36}$$

$$\begin{aligned} \theta(\pi_t - \pi^*)\pi_t &= \theta(\pi_{t+1} - \pi^*)\pi_{t+1} + \\ &+ c_t U_c(c_t, m_t) \eta \left(\frac{1+\eta}{\eta} - \frac{z'(h_t)}{f'(h_t) U_c(c_t, m_t)} \right) \end{aligned} \tag{2.37}$$

which define respectively the consumption Euler equation, the money demand and the price setting equation (which generates a version of the Phillips curve). From (2.34), (2.35) and (2.36) it is possible to express c_t as a function of m_t and R_t. By substituting it in both the Euler equation and the Phillips curve, and by using the contemporaneous Taylor rule, I obtain a two dimensional dynamical system in π_t and c_t. It is possible to show that the system is well defined under perfect foresight, under the functional form specification described above[48]. But, as in the case of flexible prices, the backward dynamics is not well defined. For this reason, I restrict the analysis to the local stability of the system under learning.

Money in the production function

In this case the first order conditions become:

$$U_c(c_t) = \beta \frac{U_c(c_{t+1}) R_t}{\pi_{t+1}} \tag{2.38}$$

$$U_c(c_t) \frac{(R_t - 1)}{R_t} = \frac{z'(h_t)}{f_h(h_t, m_t)} f_m(h_t, m_t) \tag{2.39}$$

$$\begin{aligned} \theta(\pi_t - \pi^*)\pi_t &= \theta(\pi_{t+1} - \pi^*)\pi_{t+1} + \\ &+ U_c(c_t) c_t \eta \left(\frac{1+\eta}{\eta} - \frac{z'(h_t)}{f'(h_t) U_c(c_t)} \right) \end{aligned} \tag{2.40}$$

[48]Notice that the money demand equation that we obtain from the FOC has the same specification that the one obtained under flexible prices, thus allowing for the same calibration.

By using the equilibrium condition $c_t = f(h_t, m_t)$ and equations (2.38) and (2.39) it is possible to express h_t and m_t both as a function of c_t and R_t. By substituting these expressions into the Euler equation and the Phillips curve and using the Taylor rule we obtain a two dimensional dynamical system in π_t and c_t. Also in this case, it is possible to show that it is well defined in forward dynamics[49] but not in backward dynamics.

Learning and Simulation Results

Unlike in the previous sections, it is not possible to give analytical results. Therefore, I study the stability properties of the system by numerical simulations. I consider a grid of plausible parameters' values: it is described in the Table 2 below[50]. The other parameters are left at benchmark values.

Table 2

Param.	Min	Max
η	-50	-3.5
σ	1.001	5
θ	35	350
A	1.4	3

Cycles and complex behavior. Under perfect foresight there is no evidence of local bifurcations and cycles around the active and passive steady states, in both cases of money in the production function and money in the utility function, for the parameter values described in the Table 2. As in the continuous time case[51], the active steady state is a *source* and the passive steady state is a *saddle*. For no parameter values the active steady state or the passive steady state reverse their stability so that I exclude local bifurcations. Under perfect foresight there is global indeterminacy as in Benhabib et al. (2001), so that, after a small shock, the economy might converge to a liquidity trap. Are these equilibria robust to expectational mistakes?

Learnability of the steady states. The model under backward dynamics can be represented as follows:

[49] This is achieved by imposing the restriction that the dis-utility from labor is described by a linear function. If this restriction is not satisfied, then also the forward map is a correspondence.

[50] Notice that these parameter values include the case where $U_{cm<0}$.

[51] Described in Benhabib et al. (2001b).

$$
\begin{aligned}
c_t &= G_1(c_{t+1}, \pi_{t+1}) \\
\pi_t &= G_2(c_{t+1}, \pi_{t+1}).
\end{aligned}
$$

Having excluded the existence of cycles under perfect foresight, I simplify the analysis by restricting to the possibility of learning steady states. Hence, under the (asymptotically correct) agents' perceived low of motion, the economy evolves as $(c_t, \pi_t) = (\tilde{c}, \tilde{\pi})$, where the $\tilde{}$ denotes a steady state value. Since the steady state values are unknown, the agents use the following simple rule to estimate them recursively:

$$
\begin{aligned}
\theta_t^c &= \theta_{t-1}^c + \alpha\left(G_1\left(\theta_{t-1}^c, \theta_{t-1}^\pi\right) - \theta_{t-1}^c\right) \\
\theta_t^\pi &= \theta_{t-1}^\pi + \alpha\left(G_2\left(\theta_{t-1}^c, \theta_{t-1}^\pi\right) - \theta_{t-1}^\pi\right)
\end{aligned}
\tag{2.41}
$$

where θ^c is the estimate for $\tilde{c}$, θ^π is the estimate for $\tilde{\pi}$, and α is the learning parameter, as in the last sections. In order to study the learnability of the steady states, I linearize the backward map around the active and passive steady state and I evaluate the stability under learning for various parameter values (including α). The linearized system can be written as[52]:

$$
\begin{bmatrix} \hat{c}_t \\ \hat{\pi}_t \end{bmatrix} = DGE_{t-1} \begin{bmatrix} \hat{c}_{t+1} \\ \hat{\pi}_{t+1} \end{bmatrix}
$$

and local stability under learning depends on:

$$
J = \begin{bmatrix} 1 + \alpha(DG_{11} - 1) & \alpha DG_{12} \\ \alpha DG_{21} & 1 + \alpha(DG_{22} - 1) \end{bmatrix}. \tag{2.42}
$$

Simulations show that the active steady state is learnable for every $\alpha \in (0, 1)$. On the other side, *the passive steady state is never learnable* for any $\alpha \in (0, 1)$. The liquidity trap that the model predicts under the assumption of perfect foresight is not robust to learning. Moreover, using a result from Honkapohja and Mitra (2001) I can exclude, from the numerical simulations, the existence of learnable sunspots around the passive steady states[53].

In contrast, different results are obtained under forward looking Taylor rules. From simulations I obtain the following results:

[52] Note that this stability condition is weaker than E-stability.

[53] The result states that learnablility of sunspot equilibria around an indeterminate steady state is excluded if DG has at least one eigenvalue that has real part greater than one, as is verified in this model for all the plausible parameter values.

1) under perfect foresight, there are parameter values for which the stability of the active steady state changes in both models with money in the utility function and money in the production function;

2) under learning, there are parameter values for which the stability of the active steady state under learning changes as α changes. In both cases the system is likely to undergo a Hopf bifurcation[54], thus generating instability even in the case of a locally determinate or learnable inflation target;

3) liquidity traps are non learnable for the model with $\theta > 0$. But in the case that prices are fixed for a *finite* time[55], the model can be approximated (asymptotically) by the solution of (3.3) with $\theta = 0$, and liquidity traps become learnable under both elastic labor supply and monopolistic competition.

Summing up, under more general assumptions about the market structure and market frictions the results discussed in the previous section seem to hold, for plausible parameter values. Taking into account more sources of uncertainty about the 'correct' model of the economy does not seem to change the predictions of the simple version of the model.

2.6 Conclusions

The paper shows that forecast-based Taylor rules can lead to economic instability, even in the case of local learnbility of the inflation target equilibrium and after excluding equilibria that are not learnable. I study a very simple model with flexible prices, under different assumptions about the role of money. I find that, whether money enters in the utility function or the production function, forecast-based Taylor rules are destabilizing in the following sense. Even if the inflation target (or active) steady state is locally determinate and stable under learning, there exist other 'global' equilibria that are also stable under learning. In particular, the economy can be driven into a liquidity trap, under both model specifications. Also, the economy can converge to learnable cycles and sunspots that can be either close to the inflation target or to the liquidity trap equilibrium, depending on the role of

[54] In this case, for nearby parameters we have that the active steady state is surrounded by an attracting closed curve. In different models, Grandmont et al. (1998) and Bloise (2001) show the possibility of constructing sunspots that might be learnable. DeVilder (1995) shows, under the assumption of perfect foresight, that a Hopf bifurcation might lead to higher period cycles that also might be learnable.

[55] see Carltrom and Fuest (2000).

money. These results show that the multiplicity of equilibria generated by the forecast-based Taylor rule seem to be *robust*, at least to expectational mistakes. On the other hand, more backward-looking policy rules stabilize the economy in the following sense. The equilibria that exist under forecast-based Taylor rules either cease to exist or are not robust to expectational mistakes. In fact, backward-looking Taylor rules lead to a unique learnable equilibrium: the inflation target. Deterministic cycles, sunspot equilibria or the liquidity trap cease to exist or are not stable under learning. The results are derived under two particularly unappealing assumptions. First, the model is derived under the assumption that the agents observe current economic variables, but under learning dynamics the agents form their expectations using $t-1$ information. Future research should address this inconsistency, indeed very common in the literature on bounded rationality. Second, in the paper I assume that the central bank has perfect foresight while the private sector is boundedly rational. Future research should eliminate this asymmetry and assess the consequences on the results of this paper.

2.7 Appendix A: Model Solution

Solving (2.1), with respect to M_t^{np}, c_t, M_t^p and B_t gives the following first order conditions,

$$\beta^t U_{m^{np}}(c_t, m_t^{np}) = P_t(\lambda_t - \lambda_{t+1}) \quad (2.43)$$

$$\beta^t U_c(c_t, m_t^{np}) = P_t \lambda_t \quad (2.44)$$

$$\lambda_t f_{m^p}(m_t^p) = \lambda_t - \lambda_{t+1} \quad (2.45)$$

$$\lambda_t = \lambda_{t+1} R_t \quad (2.46)$$

where λ_t denotes the Lagrange multiplier. By combining (2.44) with (2.46) we get equation (2.3). Using (2.43) and (2.43) I get:

$$f_{m^p}(m_t^p) = \frac{U_{m^{np}}(c_t, m_t^{np})}{U_c(c_t, m_t^{np})}.$$

Finally, using (2.46) and (2.43) I get

$$\beta^t U_{m^{np}}(c_t, m_t^{np}) = \lambda_t \left(1 - \frac{1}{R_t}\right). \quad (2.47)$$

Substituting (2.44) gives (2.4) and (2.5).

2.8 Appendix B: Proofs

The proofs refer to results in the book by Evans and Honkapoja mentioned in Ch.1. The results are also described by Propositions 10 and 11 in Ch. 1.

Proof of Proposition (12)

(i) I just re-state the results in Benhabib et al. (2001a, 2001b). Notice that the fixed points of F and (2.12) are exactly the same.

(ii) In order to verify the local stability of the system (2.12), under the backward map implied by (2.17), these conditions on the Jacobian need to be verified:

$$|D| = \left|(1-\alpha)^2\right| < 1 \quad (2.48)$$

$$|T| = \left|G'(\tilde{\pi})^2 \alpha^2 - 2\alpha + 2\right| < |1 + D| \tag{2.49}$$

where $\tilde{\pi}$ denotes the fixed point. It is straightforward to show that (2.48), (2.49) imply the following necessary and sufficient condition for the learnability of the steady state:

$$\left|G'(\pi)\right| < 1.$$

We do not need to know explicitly on which branch of G the fixed point is. In fact, I use the fact that around the fixed point:

$$G'(\pi) = \frac{1}{F'(\pi)}.$$

Hence, determinacy implies learnability and indeterminacy implies nonlearnability. Given that the condition in the Proposition implies determinacy of both steady states (as it is easy to see from (2.19)), (ii) is proved.

Proof of Proposition (14)

(i)-(iii) the proof follows the same steps as in Proposition (12) and it is therefore omitted.

(iv) this follows directly from Evans and Honkapohja (2001b): if $G'(\pi) < -1$ then there exist E-Stable (and therefore learnable under (2.13) and (2.14)) stationary sunspot equilibria around the fixed point. This condition is verified exactly for $-(\frac{1}{\epsilon_p(\pi)} - 1) < \epsilon_{cm}\epsilon_m < -(\frac{1}{\epsilon_p(\pi)} - 1)/2$.

Proof of Proposition (15)

(i) first, notice again that although we cannot provide an analytical expression for G_l we can use 1) the fact that locally the map is conjugate to a linear map, and; 2) its relation with the forward looking map F.

The conditions for stability of the cycle under learning are:

$$|D| = \left|(1-\alpha)^2\right| < 1 \tag{2.50}$$

$$|T(\sigma)| = \left|\nu(\sigma)\alpha^2 - 2\alpha + 2\right| < |1 + D| \tag{2.51}$$

where:

$$\begin{aligned}
\nu(\sigma) &= \frac{1}{F'(\pi_H(\sigma))\, F'(\pi_L(\sigma))} \\
&= \Pi_{i=1}^2 \left[1 + \frac{1}{-\sigma\epsilon_{yi}(\sigma)}\left(\frac{1}{\epsilon_{pi}(\sigma)} - 1\right)\right]^{-1}
\end{aligned}$$

and where $-\sigma, \epsilon_{y_i}$ and ϵ_{p_i} are the elasticities computed at the two values of π over the cycle. From (2.50), (2.51) it is straightforward to obtain a necessary and a sufficient condition for local stability of deterministic cycles of period 2 [56]. The sufficient condition for stability under learning, for every value of α is:

$$\begin{aligned} & \left|G_2'(\pi_H(\sigma))\, G_1'(\pi_L(\sigma))\right| \\ = & \ |\nu(\sigma)| < 1 \end{aligned}$$

which implies $|1/v(\sigma)| = |F'(\pi_H(\sigma))\, F'(\pi_L(\sigma))| > 1$. I checked numerically the existence of $\hat{\sigma}$ such that for $\sigma < \hat{\sigma}$ the condition is verified.

From (2.50), (2.51), the necessary (but not sufficient) condition for stability is:

$$v(\sigma)^{-1} < 0.$$

I then check numerically the existence of $\widetilde{\sigma}$ such that for $\sigma \in (\widehat{\sigma}, \widetilde{\sigma})$ the necessary condition is satisfied. Now it is clear from (2.50), (2.51) that in this case there exists an $\widehat{\alpha}_\sigma$ of low enough value such that the stability conditions are verified. Finally, I verify numerically that for $\sigma \in [\widetilde{\sigma}, \overline{\sigma})$ we have $v(\sigma) > 1$. Hence, for $\sigma > \widetilde{\sigma}$ we have instability under learning, for every parameter of α and this completes the proof of (i).

(ii) first using Proposition 12.5 of Evans and Honkapohja (2001a) the existence of sunspots ϵ-close to the deterministic cycle is guaranteed, provided $G_2(\pi_H(\sigma))G_1(\pi_L(\sigma)) \neq 1$. Since we know that this condition is violated only in the case where the cycle is destroyed ($\sigma = \overline{\sigma}$), then it is possible to construct stationary sunspots on the deterministic cycle for all $\sigma < \overline{\sigma}$.

Second, by using Proposition 12.7 and 12.9 in Evans and Honkapohja (2001a) we have that necessary and sufficient condition for learnability of sunspots with the algorithm (2.11) is that:

$$G_2'(\pi_H(\sigma))G_1'(\pi_L(\sigma)) < 1.$$

But we know from (i) that this condition holds for every $\sigma < \widetilde{\sigma}$. From Proposition 12.2 of Evans and Honkapohja (2001a)[57], strong E-stability is obtained for $|G_2'(\pi_H(\sigma))G_1'(\pi_L(\sigma))| < 1$. This is verified if $\sigma < \widehat{\sigma}$.

[56] The result is obtained by Guesnerie and Woodford (1991), p. 118 for a similar learning algorithm.

[57] pg. 303.

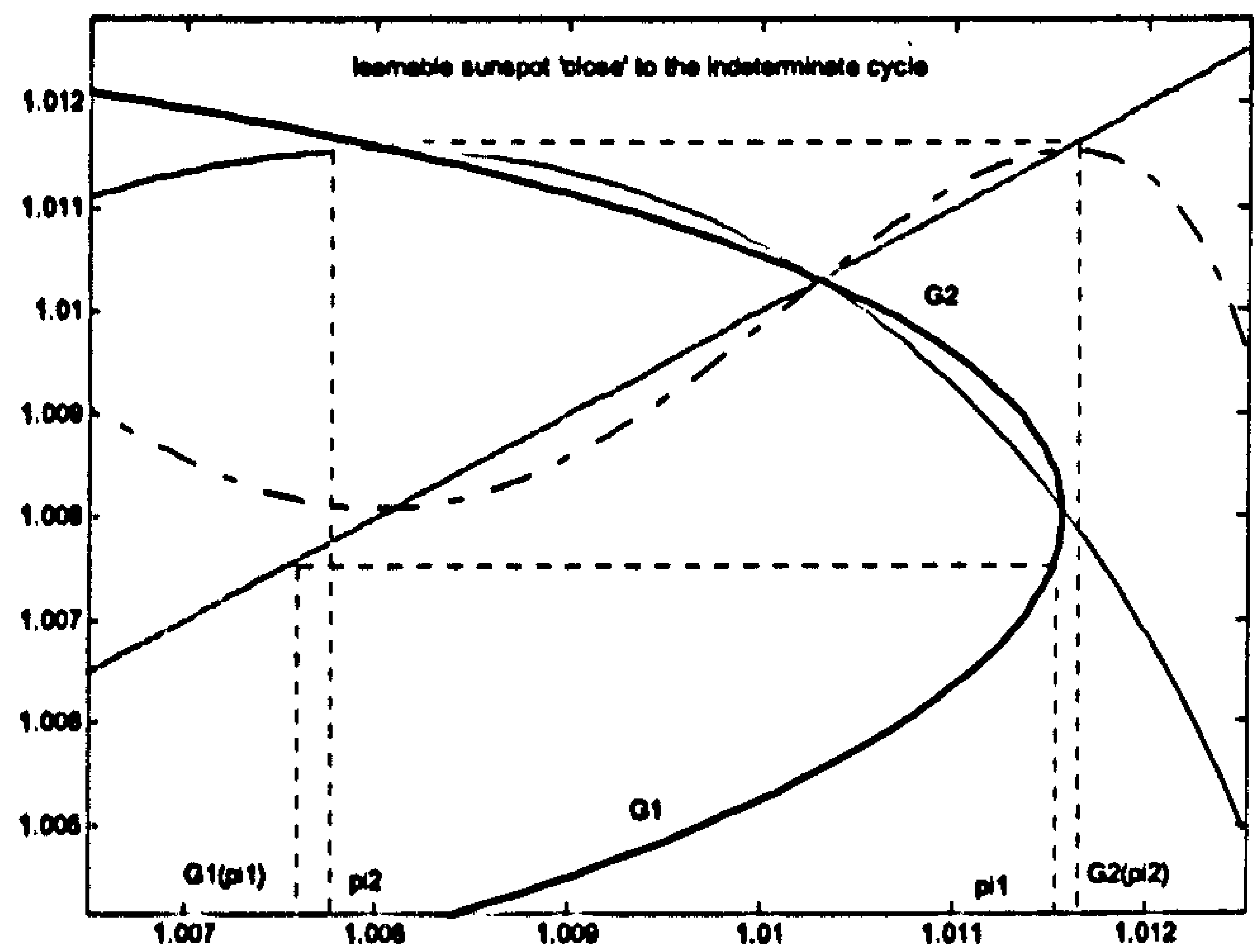

Figure 2.9:

(iii) If we consider explicitly the map in its backward dynamics, it is actually possible to show that sunspot equilibria exist outside a neighborhood of the deterministic cycle, both in the cases where the cycle is learnable and not. The result depends on the existence of the two branches G_1 and G_2.

These sunspots can be constructed by using the global methods of Grandmont (1987): a sunspot equilibrium exists if and only if $\pi_1(\sigma)$ and $\pi_2(\sigma)$ are included in the open interval formed by their images:

$$G_1(\pi_2(\sigma)) < \pi_1(\sigma) < \pi_2(\sigma) < G_2(\pi_1(\sigma)) \tag{2.52}$$

Figure 2.9 gives an example: as $\sigma = 2.35 > \widehat{\sigma}$, the cycle is indeterminate in forward dynamics, and non-learnable. By continuity of the eigenvalues, also sunspots in the neighborhood of the cycle are not learnable. But it is possible to construct sunspots relatively close to the cycle by choosing the appropriate branch of G.

The Figure shows how to construct such a cycle: it is easy to verify that condition (2.52) is satisfied[58].

[58] The existence of such suspots for every σ depends on the existence of the two branches and it can be shown numerically.

Following Evans and Honakpohja (2001), the E-stability conditions are given by:

$$|T(\sigma)| = \left|(z_{11} + z_{22} - 1)G_1'(\pi_1(\sigma))G_2'(\pi_2(\sigma))\right| < 1 \qquad (2.53)$$

$$\left|-z_{11}G_1'(\pi_1(\sigma)) - z_{22}G_2'(\pi_2(\sigma))\right| < |1 + D(\sigma)| \qquad (2.54)$$

For $\sigma < \widehat{\sigma}$ the conditions are satisfied because $|G_1'(\pi_1(\sigma))G_2'(\pi_2(\sigma))| < 1$: it is sufficient to choose the states inflation values close to the cycle.

Let us consider the case $\sigma > \widehat{\sigma}$. First notice that it is possible to show numerically that $-1 < G_2'(\pi_2(\sigma)) = \frac{1}{F'(G_2(\pi_2(\sigma)))} < 0$ because $G_2(\pi_2(\sigma)) > \pi^*$, for every σ. Fix $\pi_2(\sigma)$ such that $\pi_2(\sigma) \leq \pi_L$ (i.e. the low inflation rate at the equilibrium cycle).

Second, fix $\pi_1(\sigma)$ such that $G_1'(\pi_1(\sigma)) > 0$. This value exists for every admissible σ: in fact, as $\pi_1(\sigma) \to \overline{\pi}$, $G_1'(\pi_1(\sigma)) = \frac{1}{F'(G_1(\pi_1(\sigma)))}$ turns positive and decreases in absolute value, given that $F'(\overline{\pi}) > 1$ for every σ. Moreover, I checked numerically[59] (and it is clear from the picture, for a given σ) that $0 < G_1'(\pi^*) < 1$, for every σ. Hence, it is possible to choose $\pi_1(\sigma) \in (\pi^*, \pi_H(\sigma))$ such that $0 < G_1'(\pi^*) < 1$. But this implies that both (2.53) and (2.54) are satisfied and the sunspot is learnable.

Proof of Proposition (5):

(i) The linearization gives the following difference equation:

$$\widehat{\pi}_{t+1} = \left[\frac{\left(-\sigma\epsilon_y - \frac{(1-\delta)\epsilon_p + \delta}{(1-\delta)\epsilon_p}\right)}{\left(-\sigma\epsilon_y - \frac{1}{(1-\delta)\epsilon_p}\right)}\right]\widehat{\pi}_t. \qquad (2.55)$$

Given that at the active ss $\epsilon_p > 1$, local uniqueness is clearly guaranteed by (2.29). Learnability follows as in Proposition (12). On the other hand, for $\epsilon_p < 1$ the result is reversed.

(ii) The only bifurcation exists at $|\sigma| = -\frac{1}{2\epsilon_y(\pi)}\left(1 + \frac{1}{\epsilon_p(\pi^*)}\right)$; it is straightforward to check that the system undergoes a flip bifurcation (it exists a σ^* such that $G_{\sigma^*}'(\pi^*) = -1$). It is possible to show numerically, by using results from Devaney (1989) that the bifurcation is sub-critical: this means that as the active steady state is determinate (and thus (2.29) holds), no cycle around it exists.

[59] I checked numerically that for $\sigma \geq 1.001$ and $\sigma \leq 2.5$ ($\geq \overline{\sigma}$, given that the cycle does not exists for that values of σ) the derivative has values ranging from 0.15 to 0.85.

Lemma *At each* $\mu \in \left(\mu^{SR}, \mu^{LR}\right)$ *there exists* $\sigma(\mu) = -\frac{1}{2\epsilon_y(\mu)}\left(1 + \frac{1}{\epsilon_p}\right)$ *such that a period-doubling bifurcation occurs. For* $\sigma < \sigma(\mu)$ *the active steady state is determinate and for* $\sigma > \sigma(\mu)$ *it is indeterminate. (2) For* $\sigma > \sigma(\mu)$ *and sufficiently close to the bifurcation point a period two cycle exists. For any* $\sigma < \sigma(\mu)$ *no cycle exists.*

Proof of Lemma. From Devaney (1989) the condition for a period doubling bifurcation is:

$$e = \left(\frac{1}{2}\left(\frac{\partial^2 F}{\partial \pi^2}\right)^2 + \frac{1}{3}\left(\frac{\partial^3 F}{\partial \pi^3}\right)\right) \neq 0 \text{ at } (\pi^*, \sigma(\mu)).$$

Moreover, given

$$d = \frac{\partial F_{\sigma(\mu)}}{\partial \sigma} \text{ at } \pi^*$$

we have that if $\frac{d}{e} > 0$ then the bifurcation is sub-critical (the cycle exists when the fixed point is stable). It is trivial to verify from the linearized equation that $d < 0$. It is also possible to show numerically[60] that $e < 0$ for $\mu \in \left(\mu^{SR}, \mu^{LR}\right)$. Moreover, from the linearized equation we have that at $\sigma = \sigma(\mu)$ F' is equal to -1. Hence (1) and (2) follow. This proves that no cycles exist around the active steady state.

iii) The linearized equation implies that under condition (2.29), we have:

$$G'(\overline{\pi}) > 1$$

where G represent the backward map under the contemporaneous rule. This is the condition for the non-learnability of sunspots equilibria, as in Propositions 12.6 and 12.10 in Evans and Honkapohja (2001a).

(iv) trivial from (2.29). This completes the proof.

Proof of Proposition (7):

(i) After linearization, the local behavior of the system is determined by the linear difference equation:

$$\widehat{\pi}_{t+1} = \left[\frac{\left(\epsilon_{cm}\epsilon_m - \frac{(1-\delta)\epsilon_p + \delta}{(1-\delta)\epsilon_p}\right)}{\left(\epsilon_{cm}\epsilon_m - \frac{1}{(1-\delta)\epsilon_p}\right)}\right]\widehat{\pi}_t \qquad (2.56)$$

[60] The Maple program is available on request.

which gives determinacy (and learnability) under active monetary policy, for any parameter values. By converse, passive policy imply indeterminacy (and non-learnability).

(ii) follows the same steps as (ii) in Proposition (22).

(iii) it is apparent that $\frac{\left(\epsilon_{cm}\epsilon_m - \frac{(1-\delta)\epsilon_p+\delta}{(1-\delta)\epsilon_p}\right)}{\left(\epsilon_{cm}\epsilon_m - \frac{1}{(1-\delta)\epsilon_p}\right)}$ is always positive. Hence, at the passive steady state $\frac{\left(\epsilon_{cm}\epsilon_m - \frac{(1-\delta)\epsilon_p+\delta}{(1-\delta)\epsilon_p}\right)}{\left(\epsilon_{cm}\epsilon_m - \frac{1}{(1-\delta)\epsilon_p}\right)} > 1$ and Proposition 12.6 and 12.10 in Evans and Honkapohja (2001a) allow to exclude learnability of the sunspots equilibria.

Proof of Proposition (28)

(i)-(v) In the case of backward-looking, the linearization leads to:

$$\widehat{\pi}_t = \left[1 + \frac{(1-\delta)(\epsilon_p - 1)}{\epsilon_{cm}\epsilon_H\epsilon_p(1-\delta) - \delta}\right]\widehat{\pi}_{t+1}.$$

In the case of forward looking:

$$\widehat{\pi}_{t+2} = \frac{\left(\epsilon_{cm}\epsilon_H + \frac{1}{\epsilon_p}\right)}{(\epsilon_{cm}\epsilon_H + 1)}\widehat{\pi}_{t+1}.$$

The proof follows following the same steps as for Proposition (15), Proposition (19) and Proposition (22). Concerning (iv): it is possible to show numerically, as I do for the Lemma, that the change of stability of the inflation target leads to a subcritical bifurcation for $\mu \in \left(\mu^{LR}, \mu^{SR}\right)$. Figure 7 shows the bifurcation cycle for a given value of μ.

(vi) It is easy to verify that (iv) and (v) hold if:

$$|\epsilon_{cm}\epsilon_H| > \frac{(\epsilon_p - 1)}{2\epsilon_p} - \frac{\delta}{\epsilon_p(1-\delta)}.$$

Then, by letting $\delta \to 1$ the statement is proved.

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Chapter 3

Does Central Bank Transparency Matter for Economic Stability?

3.1 Introduction

Monetary policy design is a difficult task, given the lack of consensus about a 'correct' model of the economy. Two main sources of model uncertainty are the monetary transmission mechanism and the market expectations' formation process. Recent monetary theory[1] has proposed the adoption of simple instrument rules, dictating that the interest rate should respond to deviations of inflation and output gap from their targets. Simple rules are claimed to be 'robust' to uncertainty about the true model.

On one hand, many studies have attempted to design an optimal simple rule that is robust to different hypothesis about the impact of monetary policy on the economy[2]. On the other hand, a growing literature on bounded rationality has focused on the robustness of simple policy rules to small expectations mistakes[3] on the part of central banks and the private sector.

[1] For example, see Woodford (2003).

[2] Among the others, Hansen and Sargent (2000) or Onatski and Stock (2000).

[3] See for example Bullard and Mitra (2002), Evans and Honkapohja (2002), Howitt (1992) and Preston (2002), Sargent and Williams (2002), Sargent (1999).

A given policy rule is robust to expectational mistakes if it gives a satisfactory performance also when expectations are out of the (rational expectations) equilibrium, as a result of a change in policy or a structural change in the economy. The criterion used to evaluate the performance of a rule is whether it induces stability under learning. Thus we are interested in whether economic agents who correct their expectations over time, as new data are available, will converge to the rational expectation equilibrium (REE).

This paper explores the effects of central bank transparency on the performance of simple policy rules. I consider the hypothesis that the way a given policy rule is *implemented* might affect its robustness to expectational mistakes. In particular, central banks' transparency might affect economic stability: if market participants have access to some information about central bank actions, this can improve their predictions and stabilize the economic system. Some degree of *transparency and credibility might improve the private sector's learning process, affecting the stability under learning.*

Nevertheless, knowing the central bank policy actions does not mean that market participants fully understand the 'true' model of the economy, especially in the case of decentralized markets, where agents ignore each others' tastes, production possibilities and expectations. That implies that even a well understood policy rule might lead to economic instability, where the agents forecasting process does not converge to the REE.

Following Faust and Svensson (2000), I define transparency as the degree to which the central bank's intentions can be inferred by market participants. For example, a transparent central bank should provide monetary policy reports that explain and motivate its policy choices and should publish inflation and output forecasts, used for policy decisions.

In the context of simple policy rules, I model transparency as the public's knowledge about the policy rule. In particular, knowledge about the *form* of the policy rule and about the main variables to which the central bank is responding when the interest rate is set. Still, I allow for the possibility that the central bank might not be fully transparent. This is captured by imperfect knowledge about the *coefficients* of the policy rule, i.e. how the interest rate reacts to the economic variables. A discretionary element of policy (modelled as white noise) makes more difficult for the public to infer the policy coefficients.

Current contribution to policy design in a bounded rationality framework assume that economic agents learn by recursive updating of an unrestricted VAR. This approach is not suitable for the analysis of transparency because it does not allow the agents to use prior information about the monetary policy rule. In the paper, learning with unrestricted VAR corresponds to the case of no transparency or secrecy on the part of the central bank. The agents have no information about the central bank decision process.

I propose a framework where market participants have a model of the economy that includes the monetary policy rule and its effects on output and inflation. The agents estimate recursively their model using recursive instrumental variable estimators. In order to study the conditions for stability under learning in this framework, I apply the results from stochastic approximation theory, elaborated by Marcet and Sargent (1989) and Evans and Honkapohja (2001), to study the convergence properties of recursive simultaneous equation estimation.

In this paper monetary policy is conducted by setting the interest rate according to a Taylor rule. The economy is described by a simple forward looking microfunded model with nominal rigidities, on the lines of Woodford (2003) and Benhabib et al. (2001). Given the uncertainty about the impact of monetary policy on the economy, I consider different hypotheses. I first analyze a version of the model where monetary policy has immediate effects on output and inflation. This is the most common assumption in the literature, but empirical evidence shows that monetary policy has little immediate effect on real activity and inflation. For this reason, following Woodford (2002) I consider a more general version of the model where there are delays in the effects of monetary policy. I assume that expenditures and pricing decisions are made *in advance*, and thus depend on old information about the economic conditions.

I show that in the case of a monetary policy rule that reacts to *current* inflation and output gap or in the case that expectations about current and future interest rates do not affect output gap and inflation, stability under learning is not affected by transparency. This seems to imply that some rules are destabilizing *per se*, independently of the way they are implemented. In other words, instability occurs even if every agent in the economy understands how monetary policy is conducted. In this case, even *expected and fully credible* changes in the policy rule lead the economy to instability induced by self-fulfilling expectations. I also show that the cost channel of monetary policy modifies the stability properties of Taylor rule. For a Taylor rule to induce a learnable equilibrium, the interest rate should react to some degree to the output gap, even in the case of perfect transparency of the central bank. This implies that the Taylor Principle, stating that a Taylor rule is stabilizing if it reacts aggressively to deviations of the inflation rate from target, is not a *sufficient condition* to guarantee stability. It also implies that determinacy of the minimum state variable solution of a model under rational expectations is not enough to guarantee the stability of the equilibrium under learning, in contrast with the findings of McCallum (2002). A similar result is found in Preston (2003), in a different model environment.

Conversely, under plausible assumptions about the monetary transmission mechanism, *I show that lack of transparency can induce instability even if desirable policies are adopted.*

In the second part of the paper I attempt an evaluation of the effects of central bank transparency on the volatility and persistence of inflation output and the interest rate. I also propose an estimation approach for the monetary model with learning. The implementation is left for further research.

The results of the paper might suggest an alternative explanation for the observed response of the economy to a monetary policy shock observed in the US data. The response is more dramatic in the 70's and it is extremely reduced in the 90's. This might be due to changes in the public's understanding of monetary policy, rather than changes in the policy rule.

The paper is organized as follows. The first section introduces the model and discusses its stability under Rational Expectations. The second and third sections describe the model and its solution. In sections 4-6 I discuss the stability result under different hypothesis about the model environment. Finally, in section 7 I describe some simulation results and the plans for future research.

3.2 A Simple Model

I consider a simple model of the economy, on the lines of Woodford (2003) and Benhabib et al. (2001). The model is fully forward-looking, explicitly microfunded and displays sticky prices. In order to keep the analysis simple I abstract from capital accumulation. The economy is populated by a continuum of identical consumers/producers.

Each agent j produces a differentiated good (Y^j) in a monopolistically competitive market. Assuming a fixed capital stock, labor and money services are the production inputs. Therefore, output is produced according to

$$Y_t^j = f\left(h_t^j, \frac{M_t^j}{P_t}\right) \tag{3.1}$$

where M_t^j denotes nominal money balances and P_t is an index of the price level. In the paper I also consider the case where money gives direct utility to the consumer, because it facilitates transactions. The production function f satisfies the standard conditions. Each agent consumes a composite good C_t^j, obtained by some aggregation of each single differentiated goods produced. The aggregate demand for each good depends on the aggregate income and the relative price of the good

$$Y_t^j = D_t \left(\frac{P_t^j}{P_t} \right) Y_t \tag{3.2}$$

where Y_t is aggregate output, P_t^j is the price of good j and the function D_t is assumed to be decreasing in the price and satisfies the following two conditions: $D_t(1) = 1$ and $\theta_t = D_t'/D_t < -1$ for every t. The parameter θ_t, measuring the elasticity of demand is assumed to vary over time according to an exogenous process. This implies time-varying mark-ups for the producers and introduces a source for supply shocks in the economy.

The economy is represented by a continuum of consumers-producers that seek to maximize the value of the sum of future expected utilities of the form

$$E_{t-1}^j \left\{ \sum_{s=t}^{\infty} \beta^{s-t} U \left(C_s^j, \frac{M_s^j}{P_s} \right) - V\left(h_s^j\right) + \frac{\gamma_s}{2} \left(\frac{P_s^j}{P_{s-1}^j} - \Pi^* \right)^2 \right\} \tag{3.3}$$

where M_t^j denotes nominal money balances held by agent j and Π^* is the steady state gross inflation rate. Also, $U(.,.)$ is the utility function from consumption and money balances and $V(.)$ denotes the disutility from labor. Moreover, $U(.,.)$ is assumed to be increasing, twice differentiable and strictly concave and $V(.)$ is increasing, twice differentiable and strictly convex. In particular, I consider the case in which $U(.,.)$ is non-separable in consumption and real money balances[4]. This might be considered the most empirically relevant case, given that the marginal benefit of additional real balances increases as consumption (i.e. total transactions) increases[5].

The last term in (3.3) denotes the cost of changing the current price. Assuming $\gamma_t = \bar{\gamma}(1+\theta_t) < 0$ for every t implies price stickiness (but not sticky inflation). The choice of convex adjustment costs follows Rotemberg

[4]The case where the utility function is separable is the most analyzed in the literature about policy rule.

[5]For details, see Woodford (2002).

(1982) and it is dictated by the necessity to keep the non-linear model as simple as possible. The linearized solution of the model takes the same form that would be obtained if a Calvo pricing scheme is used (even if the parameters have a different interpretation). Notice that the adjustment cost term has a plausible behavioral interpretation in an economic environment where gross inflation is close to Π^*, which is the case I consider in the paper.

The expectation operator $\hat{E}^j_{t-1}$ denotes the subjective beliefs of agent j about the probability distribution of the model's state variables. Given the assumption that the agents do not know the true model of the economy, the "^" denotes non-rational expectations. Notice that agents take decisions for time t consumption and production, on the basis of $t-1$ information. This can be interpreted in two ways: either the agents plan their consumption, production and asset holding in advance or they act on the basis of old information.

I assume that financial markets are incomplete, and the only non-monetary asset that is possible to trade is a one period riskless bond. The agents' flow budget constraint is

$$M^j_t + B^j_t \leq \left(1 + i^m_{t-1}\right) M^j_{t-1} + \left(1 + i_{t-1}\right) B^j_{t-1} + P^j_t f\left(h^j_t, M^j_t\right) - T_t - P_t C^j_t \tag{3.4}$$

where B^j_tdenotes the riskless bond, i^m_t denotes the interest paid on money balances and i_t denotes the interest paid on the bond. Also T_t denotes lump-sum taxers from the government.

3.2.1 Monetary and Fiscal Policy

I assume that the government is committed to a zero debt fiscal policy. As a consequence, taxes evolve as follows

$$T_t = \left(1 + i^m_{t-1}\right) M_{t-1} + P_t G_t - M_t$$

where G_t defines government expenditures. I further assume that G_t is an exogenous AR(1) process.

Monetary policy is conducted according to a *simple* interest rate rule. I consider two possible ways of implementing the interest rule.

1. Quantity adjustments. The Central Bank decides the target interest rate and then implement it trough quantity adjustment in the money supply. In this case, i^m_t is fixed to zero.

2. Adjustments of the interest paid on the monetary base. In this case I assume that the central bank changes i_t^m as the target interest rate is modified, in order to keep a constant spread: i.e. $\hat{\imath}_t = \hat{\imath}_t^m$.

The policy rule determines the interest rate as a function of the current state of the economy, or estimates of it. In the course of the paper I consider different rules that are commonly considered in the literature. In order to keep the analysis simple, this paper does not consider *inertial* rules[6]. This case would require a separate study. In its general form the policy rule can be expressed as

$$i_t = \bar{\imath} + \phi_\pi E_{t-1}^{CB}(\pi_t - \pi^*) + \phi_x E_{t-1}^{CB}(x_t - x^*) + \epsilon_t \tag{3.5}$$

where x_t denotes the output gap, as defined below. In order to keep the analysis simple, I do not explicitly consider what decision process leads to a specific choice of the policy coefficients[7]. Forecasts by the central bank, E_{t-1}^{CB}, might be different from the private sector's. Notice that I consider the hypothesis that the bank's rule is 'operational' in the sense of McCallum (1999). Also, ϵ_t can be seen as a control error or some discretionary component, modelled as white noise.

3.3 Model Solution

The solution of the model gives a sequence of consumption, labor, and money balances that maximizes (3.3) subject to (3.4) and a market clearing condition. In equilibrium, agents will take identical production, consumption and saving decisions. I further assume that they have the same beliefs about the economy (even though they might not be aware of that). Appendix A describes the model solution in both cases where money enters in the utility function or in the production function. Assuming that consumption, money and pricing decisions are predetermined $t-1$ periods in advance, the log-linearized model is described by the following equations. The demand side of the economy is

$$x_t = -\tilde{\sigma} E_{t-1}^{PS}\left[\eta_1 \hat{\imath}_t - \eta_2 \hat{\imath}_t^m - \pi_{t+1} - \eta_2\left(\hat{\imath}_{t+1} - \hat{\imath}_{t+1}^m\right) - \hat{r}_t^n\right] + E_{t-1}^{PS} x_{t+1} \tag{3.6}$$

[6] See Woodford (2002).

[7] For example, the optimal non-inertial rules, under both commitment and discretion, in Giannoni and Woodford (2002).

where $\tilde{\sigma}, \eta_1$ and $\eta_2 = \eta_1 - 1$ depend on the coefficients of the demand for money and the utility function. E_{t-1}^{PS} denotes the expectation operator for the private sector. Also, $x_t = \hat{Y}_t - \hat{Y}_t^e$, is the output gap, defined as output deviations from the *efficient level* of output, obtained in the case of fully flexible prices and in the absence of mark-up shocks. Finally, $\hat{r}_t^n$ is the *natural* rate of interest, assumed to be evolving as an AR(1) process.

If monetary frictions are modeled by having money in the utility function, real money balances affect intertemporal consumption decisions. As shown in (3.6), expected high interest rates on nonmonetary assets, and therefore expected low money balances, stimulate consumption today relatively to next period's. Notice that this holds only in the case where the interest differential between monetary and nonmonetary assets is allowed to vary. This is the case where the policy rule is implemented through adjustment in the supply of money (in this case $\hat{i}_t^m = 0$). If the central bank keeps a fixed spread between monetary and nonmonetary asset, then the IS equation is equivalent to (3.6) with $\eta_1 = 1$ and $\eta_2 = 0$. Only the expected current interest rate on nonmonetary assets affects the evolution of the output gap. In the case of money in the production the IS equation is equivalent to (3.6) with $\eta_1 = 1$ and $\eta_2 = 0$, independently of how monetary policy is implemented.

Independently of how money enters in the model, the key parameter that describes the demand effects of monetary policy is $\tilde{\sigma}$: it is proportional to the intertemporal marginal rate of substitution of consumption. As shown in the Appendix, the value of $\tilde{\sigma}$ is higher in the case of monetary frictions, thus magnifying the effects of interest rate changes on the output gap.

The supply side of the economy is described by the following Phillips curve

$$\pi_t = E_{t-1}^{PS} \beta \pi_{t+1} + \kappa E_{t-1}^{PS} \left[\hat{x}_t + \eta_3 \left(\hat{i}_t - \hat{i}_t^m \right) \right] + u_t \tag{3.7}$$

where κ measures the inflation response to the output gap and η_3 measures the supply-side effects of monetary policy. An increase in interest rates differentials increases the opportunity cost of holding money and thus increases the marginal cost of production. Equation (3.7) with $\hat{i}_t^m = 0$ has the same functional form of the supply curve obtained from a simple model including the cost channel of monetary policy[8]. The parameter that captures the size of the cost channel is η_3: it is different from zero in both cases of money in the production function and money in the utility function[9].

[8] See for example Christiano and Eichenbaum (1992) and Ravenna and Walsh (2003).

[9] Assuming money as a productive asset is a simple and coherent way to evaluate the

The equation includes a cost-push shock u_t, that depends on shocks to the mark-up of firms. The shock is assumed to be generated by an AR(1) process.

The model of the economy is given by equations (3.6), (3.7), and the policy rule (3.5). It can be written in matrix term as follows

$$V_t = A_0 + A_1^{PS} E_{t-1}^{PS} V_t + A_1^{CB} E_{t-1}^{CB} V_t + A_2^{PS} E_{t-1}^{PS} V_{t+1} + A_3 X_t \tag{3.8}$$

$$X_t = H X_{t-1} + \zeta_t$$

$$H = \begin{bmatrix} \rho_r & 0 \\ 0 & \rho_u \end{bmatrix}$$

where $V_t = \begin{pmatrix} x_t & \pi_t & i_t \end{pmatrix}'$, $X_t = \begin{pmatrix} \hat{r}_t^n & u_t \end{pmatrix}'$and ζ_t is a vector of i.i.d. shocks. In order to close the model I need to specify how expectations are formed.

3.3.1 The Expectation Formation Mechanism: Methodology

In this paper I follow the "Euler Approach" to econometric learning, as defined in Evans, Honkapohja and Mitra (2002) and widely used and discussed in Evans and Honkapohja (2001) and Marcet and Sargent (1989). It predicts that agents' behavior is based on equations (3.6) and (3.7), derived from the Euler equations, under the assumption of possibly non fully rational expectations[10]. The agents are modelled as econometricians. They are endowed with beliefs about the law of motion of the main economic variables. Their Perceived Law of Motion (PLM) includes all the relevant variables and is asymptotically correctly specified. As a result, the agents will eventually learn to make rational predictions, if the learning process converges to the Rational Expectations Equilibrium.

As noted in Preston (2002), the Euler equations are not the optimal decision rules given the assumed beliefs and microfundations. In fact,

the agents should take into account not only the flow budget constraint but also their intertemporal budget constraint. This results on decision rules that depend on infinite horizon forecasts. My choice is based on the

cost channel of monetary policy, given the evidence that the highest fraction of money demand comes from firms.

[10]A recent contribution by Preston (2002) proposes a different approach where the agents decision rules depend on long horizon forecasts. It would be possible to extend my analysis in that framework. This is left for future research.

analytical simplicity of the Euler approach. Nevertheless, the decision rules of the agents *converge asymptotically to the optimal decision rule*, under the assumption that their initial wealth is zero.

I assume that market participants are atomistic and they are *not coordinated* on some shared belief on the model of the economy. Also, I assume that they cannot observe aggregate expectations about the macroeconomic variables. As a consequence, their *model* of the economy cannot be the true aggregate model represented by (3.6) and (3.7). Their Perceived Law of Motion (PLM) of the economy might include contemporaneous variables, like the output gap and the interest rate, but does not include aggregate expectations.

This implies that, even if the model is asymptotically correct; a) during the learning process the agents' model is misspecified; b) its coefficients *will not be policy invariant*. Adopting a new policy will require the agents to adjust their model, irrespective to their knowledge of how monetary policy is conducted. No matter how precise it is the knowledge about the policy rule, the agents are still uncertain about the economic environment and thus they cannot properly calculate the effects of the monetary policy on the main economic variables such as output, inflation and the interest rate[11]. Still, once the learning process has converged the model delivers the same forecasts as the true model.

In conclusion, knowledge of the policy rule does not eliminate the problem of stability under learning. It is then possible to evaluate whether transparency has effects on the stability under learning of a given policy rule.

3.4 The Model With Current Information

I consider first the version of the model which is mostly used for policy analyses, including stability under learning. Under the assumption of no delays, the model can be expressed in matrix notation as

$$V_t = \hat{A}_1 + \hat{A}_2 E_t^{PS} V_{t+1} + \hat{A}_3 X_t \tag{3.9}$$

for suitable matrices.

[11]This is because the information available to the agents is not enough to recover all the policy-invariant parameters that define the economy. In other words, the model that they estimate is still subject to the Lucas critique, since the parameters change with the monetary policy rule.

3.4.1 VAR Learning: The Case of No Transparency

As I mentioned in the section above, in the case of no transparency, the public is not given enough reliable information to use the policy rule to predict interest rate movements and their impact on output gap and inflation. In this case, I model the agents' prediction process following Evans and Honkapohja (2001) and Marcet and Sargent (1989). I assume that each agent has the same PLM

$$V_t = \Omega_{0,t-1} + \Omega_{1,t-1} X_t + e_t \tag{3.10}$$

where output gap, inflation and the interest rate depend on exogenous shocks. Also, e_t denotes a vector of perceived i.i.d. shocks. The PLM is linear and include all the variables that are included in the MSV solution of the model. Thus the model is consistent with the REE. Nevertheless, it is misspecified during the learning process. The agents estimate recursively the coefficients of their linear model using recursive least squares (RLS). They assume the model to have fixed coefficients. The coefficients are updated according to the following algorithm

$$\Omega_t = \Omega_{t-1} + \delta_t R_{t-1}^{-1} X_t \left(V_t - X_t' \Omega_{t-1} + o_t\right)' \tag{3.11}$$

$$R_t = R_{t-1} + \delta_t \left(X_t X_t' - R_{t-1}\right)$$

where $\Omega_t = \begin{pmatrix} \Omega_{0,t} & \Omega_{1,t} \end{pmatrix}'$, R_t is the precision matrix and δ_t is a decreasing sequence of gains, satisfying certain properties[12]. The updating equation includes an observational i.i.d. error,o_t that makes the learning process non trivial. As it is well known from Evans and Honakapohja (2001), stability of the REE under learning obtains if the E-Stability conditions are met. Inserting (3.10) into (3.9) gives the Actual Law of Motion (ALM) of the economic system

$$V_t = T' \left(\Omega_{0,t-1}, \Omega_{1,t-1}\right) W_t$$

where $W_t = \begin{pmatrix} 1 & X_t \end{pmatrix}'$. The E-Stability condition requires that the mapping between PLM and ALM to be locally stable at the REE, where $T(\Omega^*) = \Omega^*$. It is apparent why during the learning process the agents' model is misspecified. The ALM implies a model with time-varyng coefficients. The PLM is a correctly specified model of the economy only asymptotically, if the learning process converges to the REE.

The following proposition defines the conditions for learnability under reduced-form learning. In order to obtain clear analytical results I impose

[12] See Evans and Honkapohja (2001).

assumptions on some of the parameters, that are not contradicted by standard calibrations, as showed in Table I.

Table I

W (2003)	Calibration				
$\sigma = 6.3$	$\kappa = 0.024$	$\beta = 0.99$	$\eta_1 = 1.56$	$\eta_3 = 0.89$	$\chi = 0.02$
CGG (1999)	Calibration				
$\sigma = 1$	$\kappa = 0.3$	$\beta = 0.99$	-	-	-

Proposition 33 *Assume that* $\frac{\eta_3}{\sigma} \leq 1$. *Assume that the private sector's learning process is described by (3.10) and (3.11).*

(i) Then the REE is stable under learning if and only if

$$\kappa(\phi_\pi - 1) + \phi_x(1-\beta) - \eta_3\kappa\phi_x > 0 \tag{3.12}$$

Proof. see Appendix B. ∎

The proposition shows a 'qualified' version of the results obtained by Howitt (1992) and Bullard and Mitra (2002) and Preston (2002), for the case of monetary frictions. If the policy rule is *too passive* or it prescribes an *excessive reaction* to the output gap, then the REE is unstable under learning. If the REE is unstable under learning, there will be self-fulfilling expectations leading to potentially explosive behavior of output and inflation.

In the analysis above I have assumed that the agents do not have information concerning the monetary policy rule. With knowledge about future policy actions the agents can improve their forecasts and this might affect the stability properties of a given policy rule. This amounts to asking the following question: is a policy rule violating (3.12) *inherently* destabilizing or is it the way the policy is *implemented* that affects its performance?

3.4.2 A Transparent Central Banker

When agents have information about how monetary policy is conducted and they are willing to use it to improve their forecasts. Consider the most plausible case where market participants know the *form* of the policy rule but not the exact value of the parameters. The policy rule that they estimate is

$$i_t = \psi^j_{0,t-1} + \psi^j_{\pi,t-1}\pi_t + \psi^j_{x,t-1}x_t + e^3_t \tag{3.13}$$

where the constant captures the long-run objectives of the central bank, i.e. the inflation target, and the coefficients describe how aggressive the policy is in responding to inflation and output gap deviations from target. The initial parameters $\begin{pmatrix} \psi_{0,0} & \psi_{\pi,0} & \psi_{x,0} \end{pmatrix}$ can be interpreted as the *initial level of credibility* of the central bank, depending on how close they are to the true parameter values. The agents might use the information about the policy rule to improve their forecast of inflation and output.

In order for this information to be useful for prediction, the agents need a model to identify the effects of monetary policy on output and inflation. Since, in a decentralized market, agents do not have specific information about other market participants' tastes and expectations, their model does not include the average opinion and does not correspond to the true model. On the other hand, it explicitly includes the effects of monetary policy on output and inflation. The agents PLM for output will then be

$$x_t = b^j_{01,t-1} + \gamma^j_{1,t-1} i_t + b^j_{11,t-1} r^n_t + e^1_t \tag{3.14}$$

where each agent j can observe the current interest rate and the demand shock r^n_t. The equation for output gap takes into account the possible effects of monetary policy on the current output gap, and movements in the natural rate of interest. The inflation equation can take the form

$$\pi_t = b^j_{02,t-1} + \gamma^j_{2,t-1} x_t + b^j_{11,t-1} u_t + e^2_t \tag{3.15}$$

where the agents take into account that monetary policy has its effects on the inflation process because it affects the output gap. The equation includes also the cost-push shock. Aggregating across different agents, and using the fact that expectations are identical, it is possible to write the PLM in a more compact notation

$$\Gamma_{t-1} V_t = B_{0,t-1} + B_{1,t-1} X_t + e_t \tag{3.16}$$

which gives a *system of simultaneous equations*. Notice that the last equation of the system, corresponding to the interest rate, depends on the form of the policy rule.

Given that the agents estimate a model with the unique purpose of prediction, I should discuss what are the incentives to use information about the policy rule. Assuming that the agents do not take the into account the effects of their learning process on the aggregate variables, if the model is exactly identified, they should be indifferent between reduced form and structural estimation[13].

Nevertheless, the recursive updating of the estimator allows them to use *prior information* about the coefficients of the estimated policy rule, thus making structural estimation more suitable. In the case of perfect transparency and credibility, the agents know the value of the policy coefficients, and the PLM (3.16) allows them to use this information for prediction. Notice also that by estimating the policy rule (3.13) *they actually estimate an equation that is well specified at any point in time*, and not only asymptotically, as it is the case for the other equations[14].

The agents are assumed to estimate recursively the system (3.16). As it is well known, Least squares estimation would lead to inconsistency. Hence, I assume that they update the coefficients of their model by using Recursive Instrumental Variables (RIV). In order for the model (3.16) to be estimated, it needs to be *identified*, i.e. we need as many instruments as many endogenous variables. Recall that an exogenous variable of the model can be used as an instrument if it is not included in the equation. In (3.14) we have one endogenous variable, the interest rate, and one instrument available, the cost-push shock u_t. In the inflation equation there is also one endogenous variable, and the instrument available is the demand shock $\hat{r}_t^n$. Finally, estimation of the Taylor rule (3.13) requires two instruments, since both x_t and π_t enter the equation. Both the demand and supply shocks can be used as instruments, because they do not appear in the equation. Given the instruments, the recursive version of the estimator can be showed to be[15]

$$\theta_t = \theta_{t-1} + \delta_t \overline{R}_{t-1}^{-1} Q_t \left(V_t - (\theta_{t-1}' Z_t)' + o_t \right) \tag{3.17}$$

$$\overline{R}_t = \overline{R}_{t-1} + \delta_t \left(Q_t Z_t' - \overline{R}_{t-1} \right)$$

where

$$\theta_t = \begin{pmatrix} b_{01,t} & b_{11,t} & \gamma_{11,t} & b_{02,t} & \gamma_{21,t} & b_{22,t} & \psi_{0,t} & \psi_{\pi,t} & \psi_{x,t} \end{pmatrix}'$$

Consider the matrices $\overline{R}_t$, Q_t and Z_t. The first matrix is

[13] For example, Dhrymes (1978).

[14] In fact, it is well known that during the learning process the agents' model is misspecified because the ALM has time varying coefficients. This does not not hold for the estimated Taylor rule.

[15] In order to simplify the convergence analysis I assume that the gain matrix $\bar{R}$ appears lagged in the updating equation.

$$\overline{R}_t = \left(I_3 \otimes R_t^h\right)$$

where R_t^h is the matrix gain associated to each single equation (with $h = y, \pi, i$ respectively). Also

$$Q_t = (I_3 \otimes W_t)$$

is the matrix of instruments and $W_t = \begin{pmatrix} 1 & X_t \end{pmatrix}'$. Finally

$$Z_t = \begin{bmatrix} Z_t^x & 0_{3\times1} & 0_{3\times1} \\ 0_{3\times1} & Z_t^\pi & 0_{3\times1} \\ 0_{3\times1} & 0_{3\times1} & Z_t^i \end{bmatrix}$$

is the matrix of regressors, where

$$Z_t^x = \begin{pmatrix} 1 & r_t^n & i_t \end{pmatrix}'; \; Z_t^\pi = \begin{pmatrix} 1 & x_t & u_t \end{pmatrix}'; \; Z_t^i = \begin{pmatrix} 1 & x_t & \pi_t \end{pmatrix}'.$$

Using the PLM (3.16), the aggregate expectation is

$$E_t V_{t+1} = \Gamma_{t-1}^{-1} B_{0,t-1} + \Gamma_{t-1}^{-1} B_{1,t-1} H X_t$$

where to simplify I assume, without loss of generality, that the agents know the matrix H. Inserting the PLM in (3.9), we get the Actual Law of Motion

$$V_t = A_1 + A_2 \left(\Gamma_{t-1}^{-1} B_{0,t-1} + \Gamma_{t-1}^{-1} B_{1,t-1} H X_t\right) + A_3 X_t + A_4 \epsilon_t \tag{3.18}$$

that can be re-written

$$V_t = \tilde{T}'(\theta_{t-1}) W_t + A_4 \epsilon_t. \tag{3.19}$$

Given (3.19), the REE equilibrium is defined as the fixed point of the map $\tilde{T}(.)$, which satisfy[16]

$$\tilde{T}(\theta^*) = \theta^*.$$

Inserting (3.19) in (3.17) I obtain the following stochastic dynamical system

$$\theta_t = \theta_{t-1} + \delta_t \overline{R}_{t-1}^{-1} Q_t \left(\tilde{T}'(\theta_{t-1}) W_t - (\theta_{t-1}' Z_t)'\right) + \delta_t \overline{R}_{t-1}^{-1} Q A_4 \epsilon_t \tag{3.20}$$

and

$$\overline{R}_t = \overline{R}_{t-1} + \delta_t \left(Q_t Z_t' - \overline{R}_{t-1}\right).$$

The following Proposition describes the conditions for stability under learning.

[16] Notice that the REE can be expressed in the form of (3.16). In fact, $\Omega_1^* = \Gamma^{-1^*} B_1^*$ and $\Omega_0^* = \Gamma^{-1^*} B_0^*$

Proposition 34 *Assume that $\frac{\eta_3}{\sigma} \leq 1$. Assume that the private sector's learning process is described by (3.16) and (3.17).*

(i) In the case where $U_{cm} = 0$. The REE is stable under learning if and only if (3.12) is Satisfied. Monetary policy transparency does not affect local stability under learning.

(ii) In the case with $U_{cm} > 0$, (or money enters in the production function), under Woodford (2002) and Clarida et al. (1999) calibration, the stability condition is not affected by policy transparency.

(iii) Assume full transparency, i.e. each agents knows the coefficients of the policy rule. The stability conditions are unchanged.

Proof. see Appendix B. ■

The Proposition shows that under the current assumptions about the model of the economy, being transparent does not help. Notice that if $U_{cm} = 0$, expectations about the future interest rate are not used for prediction. This does not mean that information about the policy rule is not useful to forecast future output and inflation, since those depend on the future interest rate. As stated in (ii), transparency does not help even if the agents need to predict the future interest rate. This leads to the following conclusions. Condition (3.12) does not depend on the way the monetary authority implements the policy rule. Even if *every agent* understands how monetary policy is conducted, a policy rule that violates (3.12) would lead to economic instability. Instability is determined by the fact that the agents in the economy are not coordinated on the REE, and do not fully understand the true model of the economy.

More generally, this result seems to suggest that improved predictability does not necessarily improve stability. Even full knowledge about the policy rule does not improve stability under learning. Even if a *fully credible* central bank *announces* a policy rule that violates (3.12), the outcome will be destabilizing. This conclusion suggests that some policy options are destabilizing per se, without the possibility for the central bank to improve on their performance.

3.5 The general case with delays

As mentioned in the introduction, there is evidence from empirical studies and central bank practices that the assumptions made in the previous section about the monetary transmission mechanism and the policy rule are at odds with the facts. Considering the monetary transmission mechanism, VAR evidence from Rotemberg and Woodford (1997) and Boivin and Giannoni (2002) shows that output and inflation respond to a monetary shock with lags. Concerning monetary policy rules, Orphanides (2003) shows that the Federal Reserve makes active use of forecasts about current and future values of inflation and output gap. The next section investigates the implications of these assumptions on the effects of central bank transparency.

3.5.1 The Case of a Non-Transparent Central Bank

As above, the agents only know the sets of variables that appear in the MSV reduced form solution of the model under rational expectations and they conjecture a linear relationship between output, inflation, interest rate and these variables. The central bank does not disclose information about the relevant variables to which reacts, its forecasts or the policy rule coefficients. Therefore, the agents forecast current and future interest rates by using an unrestricted VAR, which the interest rate to the $t-1$ observations of the exogenous processes. The PLM becomes

$$V_t = \Omega_{0,t-1}^h + \Omega_{1,t-1}^h X_{t-1} \tag{3.21}$$

where $h = PS$ denotes the aggregate PLM of the private sector and $h = CB$ denotes the forecast of the central bank. The following proposition describes the stability conditions under learning.

Proposition 35 *Assume* $\frac{\eta_3}{\bar{\sigma}} \leq 1$. *Given the model (3.8) and the PLM (3.21).*

(i) The REE is locally stable under learning if the following conditions are satisfied:

$$\kappa(\phi_\pi - 1) + (1-\beta)\phi_x - \kappa\eta_3\phi_x > 0 \tag{3.22}$$

and

$$\phi_x > \hat{\phi}_x = \frac{\bar{\sigma}\kappa\left[\phi_\pi\left(\frac{\bar{\sigma}+\eta_3(2-\beta)}{\bar{\sigma}}\right)-1\right]+\left[\kappa\bar{\sigma}-(1-\beta)\right](2-\beta)}{\left[1+\kappa\eta_3\right]\bar{\sigma}} \tag{3.23}$$

(ii) there exist learning equilibria, where inflation fluctuates around the inflation target, even in the case (3.12) is satisfied and the REE is locally determinate and unique.

Proof. see Appendix C. ■

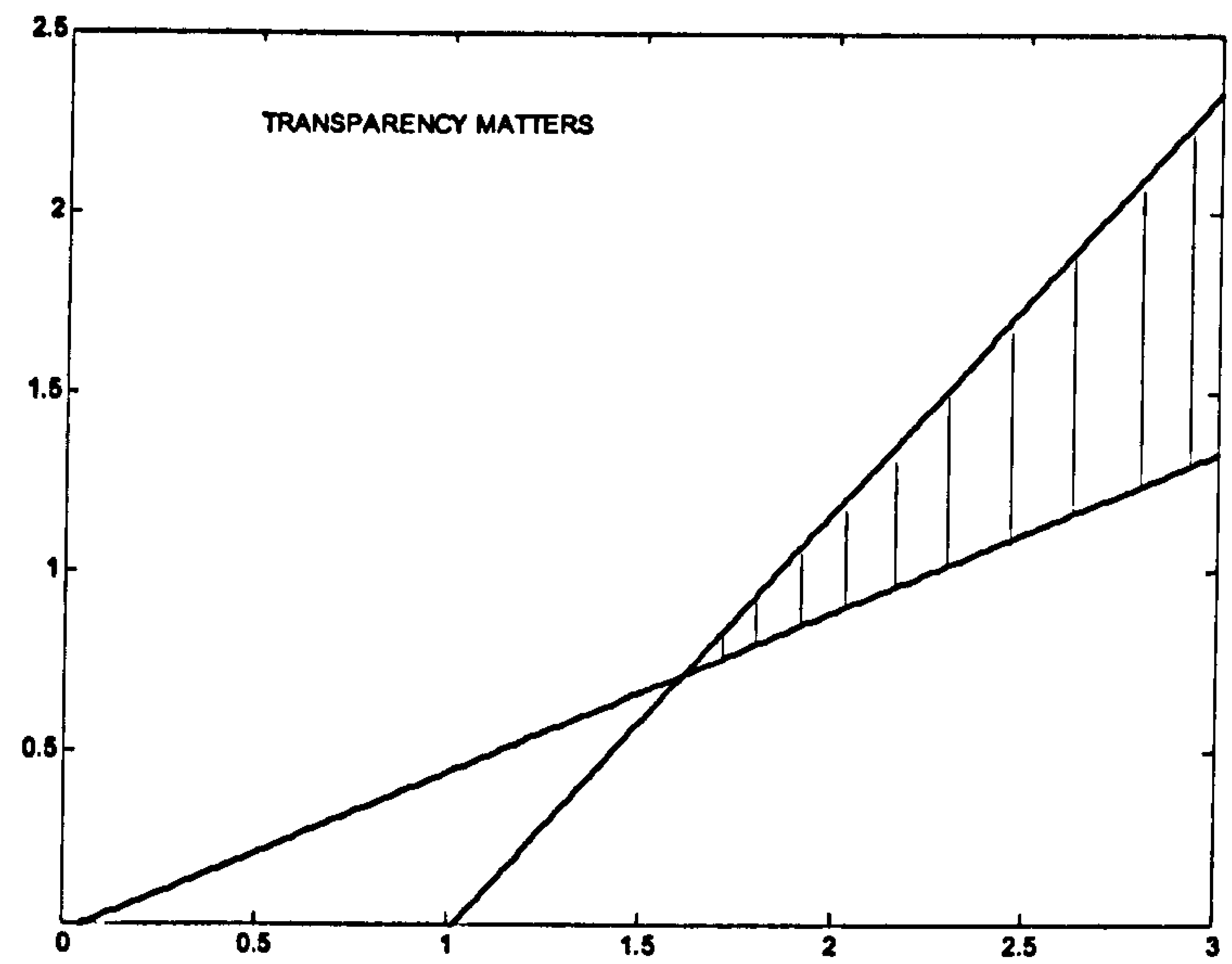

Figure 3.1:

Predetermined economic decisions and less information about the state of the economy affect the stability conditions under learning. The choice of ϕ_x becomes crucial for stability. If the policy rule reacts too much to the output gap, (3.22) is violated and instability occurs. But (3.23) states that the policy rule should react to some degree to the output gap.

The stability condition is modified if we consider the model with small real balance effects (i.e. η_3 small). In this case, (3.22) approximates the Taylor principle but (3.23) requires a positive value for $\hat{\phi}_x$, even if $\eta_3 = 0$. Using the Woodford calibration with $\phi_\pi = 2$, $\hat{\phi}_x \simeq 0.05$. Using the Clarida et al. calibration $\hat{\phi}_x \simeq 0.9$. Figure 3.1 shows the case for this latter calibration.

Condition (3.22) is represented by the solid line, while condition (3.23) is represented by the dotted line. In the area below the solid line the equilibrium is determinate, under rational expectations. The shaded area shows the combination of policy parameters for which we obtain stability, if the central bank is not transparent. It is immediate to see that determinacy is obtained for a wider set of parameters.

Also, from (ii) in the proposition we know that for values of ϕ_x that is close to the bifurcation value, the *learning process induces additional fluctuations in inflation, output and interest rate, even in the case of local learnability of the REE.* Notice that under rational expectations the unique equilibrium is the inflation target. It is the learning behavior that generates other equilibria where inflation fluctuates. In the next section I consider whether some degree of transparency can avoid these outcomes or if they inherently depend on the learning process.

Summing up, in order to get stability (at least locally) the central bank would need to increase its response to the output gap. But we know that central banks do not have accurate data about this variable, and that excessive response to this variable might lead to destabilizing policies[17]. Hence, on one side the central bank has to respond aggressively to the output gap, in order to coordinate expectations. On the other side this might not be an option, because of the scarce reliability of output gap estimates. In the next section, I consider whether improved transparency can promote economic stability.

3.5.2 The Case of a Transparent Central Bank

In this case, I assume that the agents make use of information about the central bank decision making. The central bank is assumed to be more transparent about its policy decisions. For example, it is clear about its goals and how to achieve them and it publishes its forecasts about the current inflation rate and output gap. This means that these forecasts are in the agents' information set, at the time they form their expectations. On the other side, I allow for the possibility that the bank is not fully transparent, i.e. it has an incentive to misrepresent the economic conditions. Hence, I consider the possibility that the central bank reports its forecast with an i.i.d. error, i.e. $\hat{E}_{t-1}^{CB}\pi_t = E_{t-1}^{CB}\pi_t + e_{\pi,t-1}$ and $\hat{E}_{t-1}^{CB}x_t = E_{t-1}^{CB}x_t + e_{x,t-1}$. The agents are assumed to estimate over time the following policy rule

$$i_t = \psi_{0,t-1} + \psi_{\pi,t-d}\hat{E}_{t-1}^{CB}\pi_t + \psi_{x,t-1}\hat{E}_{t-1}^{CB}x_t + e_t \tag{3.24}$$

which, assuming a potential i.i.d. observation errors, can be estimated consistently using recursive instrumental variables, by regressing i_{t-1} on $\hat{E}_{t-2d}^{CB}\pi_{t-1}$ and $\hat{E}_{t-2d}^{CB}x_{t-1}$. At the end of each period the agents update their estimate of the Taylor rule according to

$$\psi_t = \psi_{t-1} + \delta_t R_{\psi,t-1}^{-1} W_{t-1}\left(i_t - \psi_{t-1}' Z_{t-1}\right) \tag{3.25}$$

[17]See, Orphanides (2001) and Bullard and Eusepi (2003).

$$R_{\psi,t} = R_{\psi,t-1} + \delta_t \left(W_{t-1} Z'_{t-1} - R_{\psi,t-1} \right)$$

where

$$\psi_t = \begin{bmatrix} \psi_{0,t-1} & \psi_{x,t} & \psi_{\pi,t} \end{bmatrix}'; \; Z_{t-1} = \begin{bmatrix} 1 & \hat{E}^{CB}_{t-1} x_t & \hat{E}^{CB}_{t-1} \pi_t \end{bmatrix}'.$$

Notice that, in order to form expectations at time t, the agents are assumed to use $t-1$ estimates of the coefficients. I also allow the agents to use a less efficient but simpler and more robust learning rule, i.e. the Stochastic Gradient Algorithm. The updating equation is defined as follows

$$\psi_t = \psi_{t-1} + \delta_t W_{t-1} \left(i_t - \psi'_{t-1} Z_{t-1} \right) \tag{3.26}$$

The model (3.8) can be re-expressed as follows

$$\begin{aligned} \tilde{V}_t &= B_0 + B_1 \psi'_{t-1} E^{CB}_{t-1} \tilde{V}_t + B_2 E^{PS}_{t-1} \tilde{V}_t + & (3.27) \\ & + B_3 \psi'_{t-1} E^{CB}_{t-1} \tilde{V}_{t+1} + B_4 E^{PS}_{t-1} \tilde{V}_{t+1} + B_5 X_{t-1} + \tilde{\zeta}_t & (3.28) \\ i_t &= \bar{\imath} + \phi_\pi E^{CB}_{t-1} (\pi_t - \pi^*) + \phi_x E^{CB}_{t-1} (x_t - x^*) + \epsilon_t \end{aligned}$$

where $\tilde{V}_t = \begin{bmatrix} x_t & \pi_t \end{bmatrix}'$. Notice that I use the fact that the private sector can observe the central bank forecast, i.e. $E^{PS}_{t-1} \left(E^{CB}_{t-1} \right) = \hat{E}^{CB}_{t-1}$ and $E^{PS}_{t-1} \left(E^{CB}_t \right) = E^{PS}_{t-1} \left(E^{CB}_{t-1} \left(E^{CB}_t \right) \right) = \hat{E}^{CB}_{t-1}$. The latter equality can be justified in two ways; a) the central bank publishes forecasts of current and future inflation, b) the central bank makes available its forecasting procedures. In both ways, the bank makes it easier for the public to predict its future policy moves.

Since contemporaneous variables dot not enter in the true model, the agents estimate the behavior of output gap and inflation by using unrestricted VAR estimation. Their PLM is therefore

$$\tilde{V}_t = \tilde{\Omega}^h_{0,t-1} + \tilde{\Omega}^h_{1,t-1} X_{t-1} + e_t \tag{3.29}$$

where, again, $h = PS, CB$. The coefficients are updated using either RLS or SG. The following Proposition describes local stability under learning, under RLS and SG learning. To simplify the analysis I assume that the central bank and the private sector use the same learning rule.

Proposition 36 *Consider the case $U_{cm} = 0$.*

(i) In the case of Recursive Instrumental Variable or RLS learning, some degree of transparency implies REE stability, provided that the Taylor Principle is satisfied: full transparency is NOT needed for stability. The REE is locally unique;

(ii) In the case of SG learning, perfect transparency implies REE stability, provided that the Taylor Principle is satisfied. In the case of partial transparency stability is achieved provided the Taylor principle is satisfied and the eigenvalues of $M_x\tilde{\Omega}^{*\prime}$ are positive, where $M_x = E_{\lim t\to\infty} X_t X_t'$.

Consider the case $U_{cm} > 0$ (or money in the production function).

(iii) Transparency implies that the set of locally learnable Taylor rules is larger than in the case of no transparency; Nevertheless, pure inflation targeting leads to instability and learning equilibria. A sufficient condition for uniqueness and stability under learning, other than $\phi_\pi > 1$, is

$$\frac{\eta_3 + 1}{\phi_x} < \frac{\phi_\pi}{\phi_x} < \frac{\tilde{\sigma}}{\kappa\eta_3} \tag{3.30}$$

As $\beta \to 1$ condition (3.30) becomes also necessary.

Proof. see Appendix C. ■

The result shows that under a more plausible model environment transparency matters. It is important to remark that full transparency is not needed for stability. That is what makes the result appealing, given that full transparency (i.e. ϵ_t is observable by the public and no errors in expectations) is not observed in reality and it is not advocated by monetary theory, as showed in Faust and Svensson (2002).

Nevertheless, including expectations in the policy rule and having the agents taking decisions based on older information restricts the set of stabilizing policy rules, as showed in (3.30). The central bank needs to set $\phi_x > 0$ to guarantee stability. Even a *fully transparent policy violating (3.30) would be destabilizing*, as long as the agents have imperfect information about the economic environment. Lack of transparency has the effect of increasing the policy rules that do not induce instability under learning. Condition (3.30) is expressed in terms of the two key structural parameters of the model: $\tilde{\sigma}$ and η_3. If the demand channel of monetary policy is much stronger than the supply channel, stability can be achieved by choosing relatively low values of ϕ_x, which might be desirable, given the poor information available about the output gap. Estimates of η_3 in Ravenna and Walsh (2003) vary from 1.3 to around 5, indicating that a too low response to the output gap might lead to instability for plausible parameter values. Consider the following example. Assume $\phi_\pi = 2$, $\eta_3 = 0.89$ (or 2) and the other parameters at benchmark values. Then, for $\phi_x > 0.004$ (0.015) the REE is locally stable under learning. The Taylor rule prescribes a coefficient much higher than this value. Nevertheless, the central bank should choose ϕ_x much higher

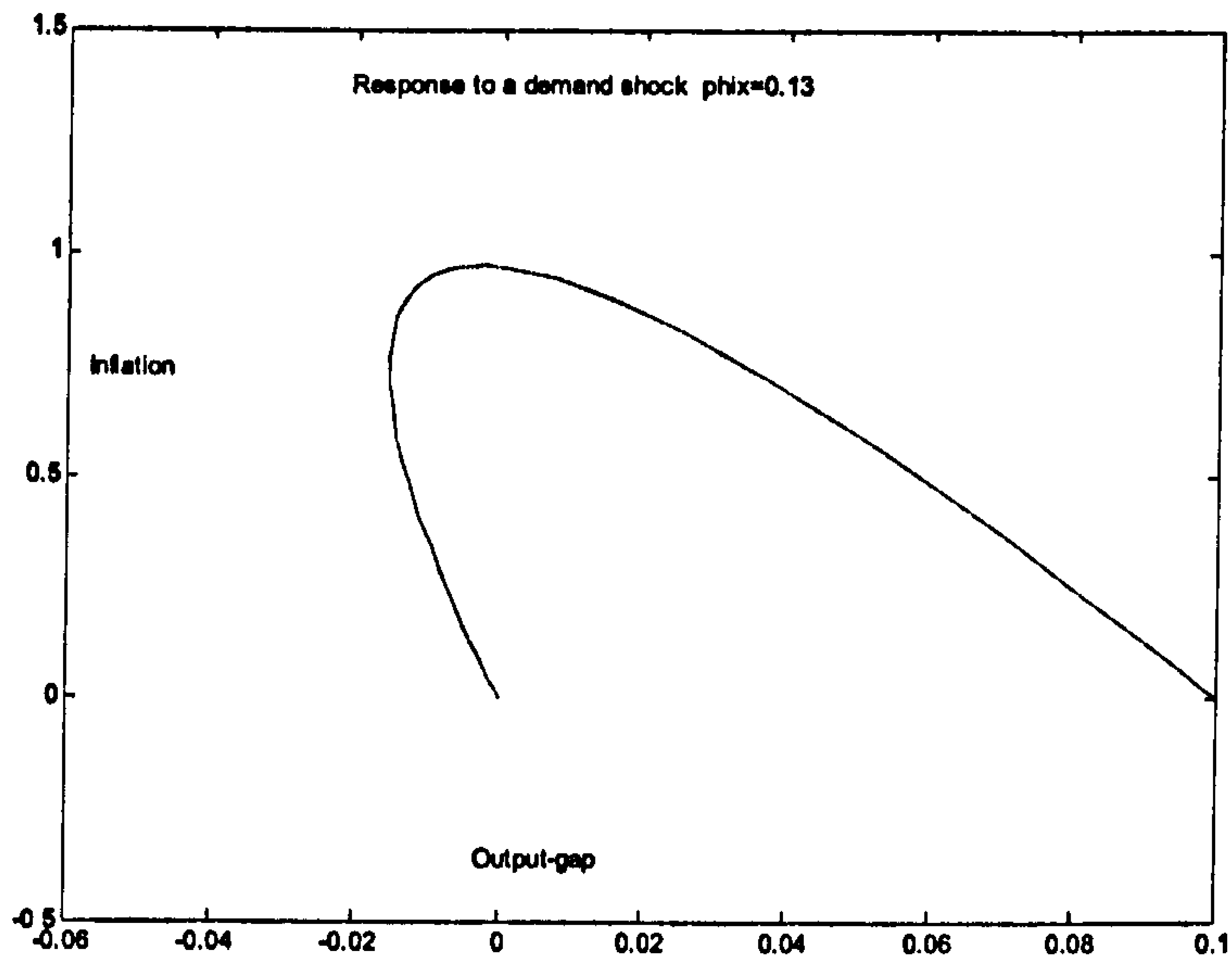

Figure 3.2:

than 0.004, given that for values close to this, the learning behavior generates extra volatility in the main variables. Numerical simulations show that choosing $\phi_x > 0.13$ guarantees a monotone response of the economy to economic disturbances. This is very close to the coefficient of the Taylor, which is $\phi_x = 0.5/4$.

Figures 3.2 and 3.3 show the response of output and inflation after a positive demand shock. In Figure 3.2, which shows the case of $\phi_x = 0.13$, inflation increases as the output gap increases and it is promptly reduced. In Figure 3.3, with $\phi_x = 0.05$, the increase in the output gap is followed by an initial increase in inflation and an oscillatory adjustment of the two variables. Notice that in the latter case the fluctuations in inflation and output gap are more pronounced[18]. Notice that if (3.30) is satisfied the only equilibrium under learning is the REE, by the same argument used for the previous proposition.

[18]The Figures show simulation of the ODE described in the text and in the appendix.

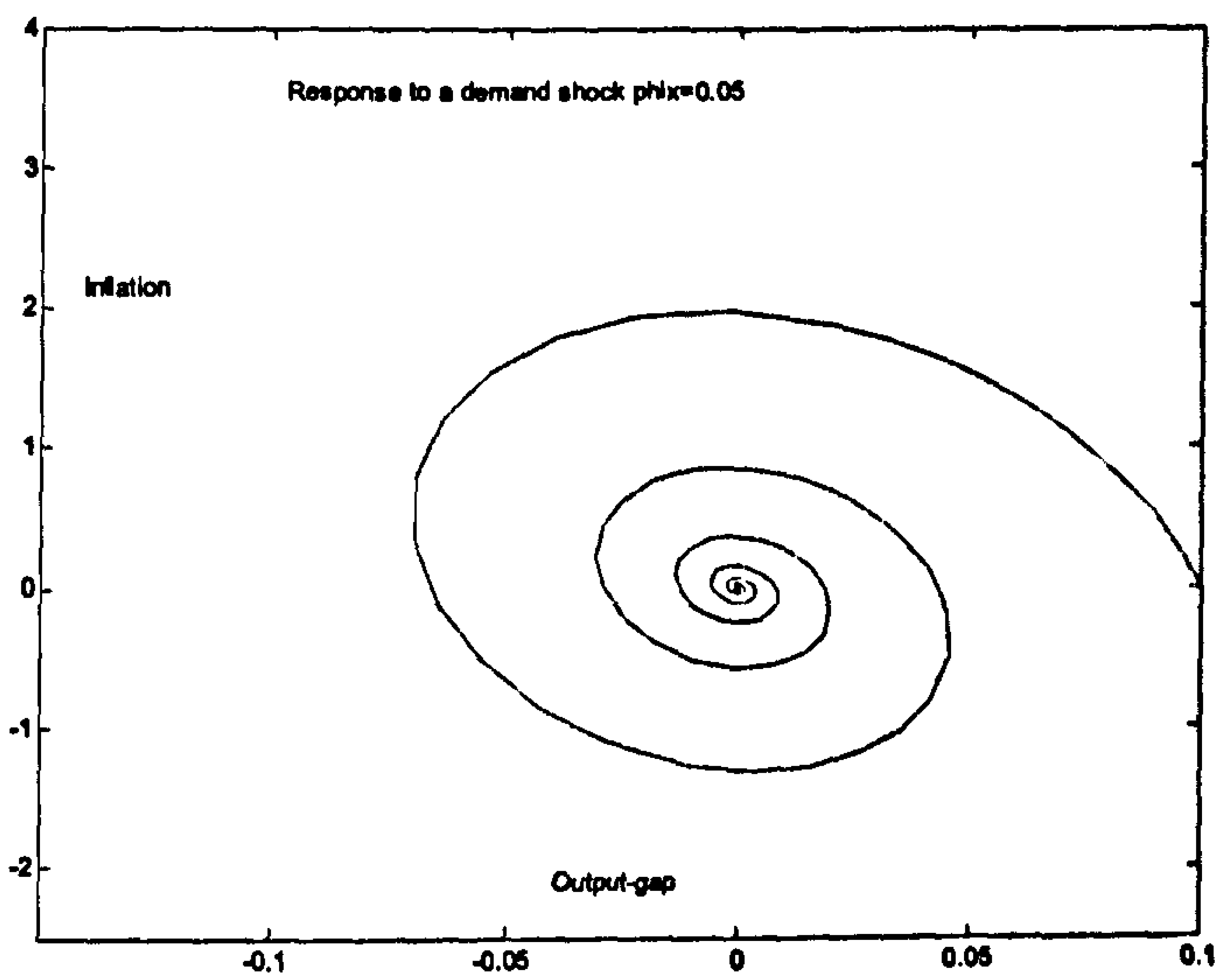
Response to a demand shock phix=0.05
Inflation
Output-gap
4
3
2
1
0
-1
-2
-0.1
-0.05
0
0.05
0.1

Figure 3.3:

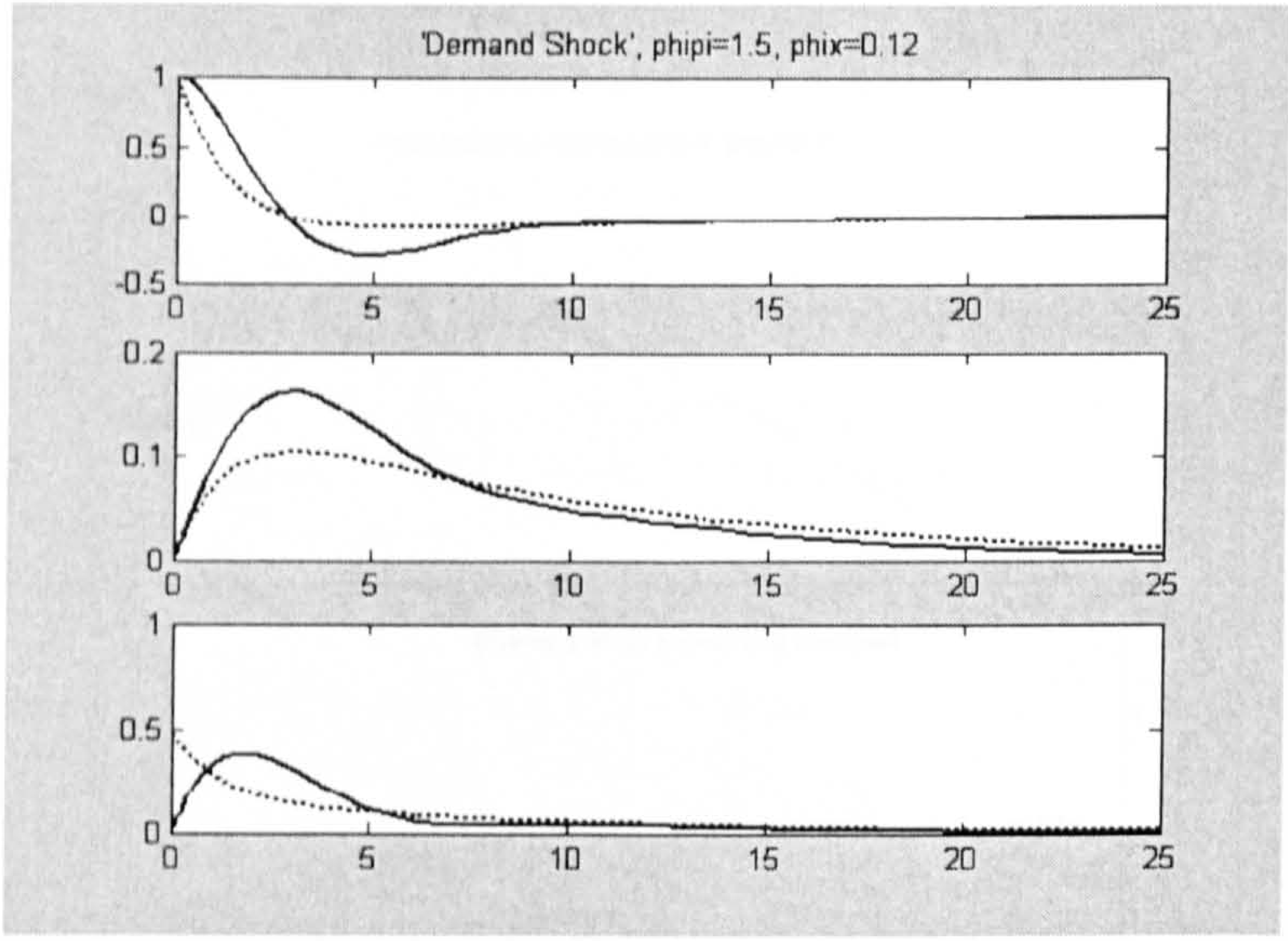

Figure 3.4:

Finally, notice that again *determinacy of the MSV under rational expectations is not sufficient to induce stability under learning*, even in the case where agents are informed about the policy rule. This results is in line with Preston (2003) and in contrast with McCallum (2002).

The results also show that if the public and the central bank update their information less efficiently, than even partial degrees of transparency can affect economic stability. Simulations show that SG learning does not induce instability, for plausible parameter values, but it slows down considerably the learning process, thus increasing substantially the volatility of the economic variables. The following Figures 3.4-3.7 show the effects of lack of transparency on agent's expectations in both cases of a demand and supply shock. The pictures describe in turn: expected output gap, expected inflation and expected interest rate. The solid line describes the response of agent's beliefs under lack of transparency and the dotted line describes the belifs under perfect transparency. It is apparent that, a) lack of transparency increases the volatility of the main economic variables; b) the less responsive the policy rule to the output gap the more volatile the economic variables.

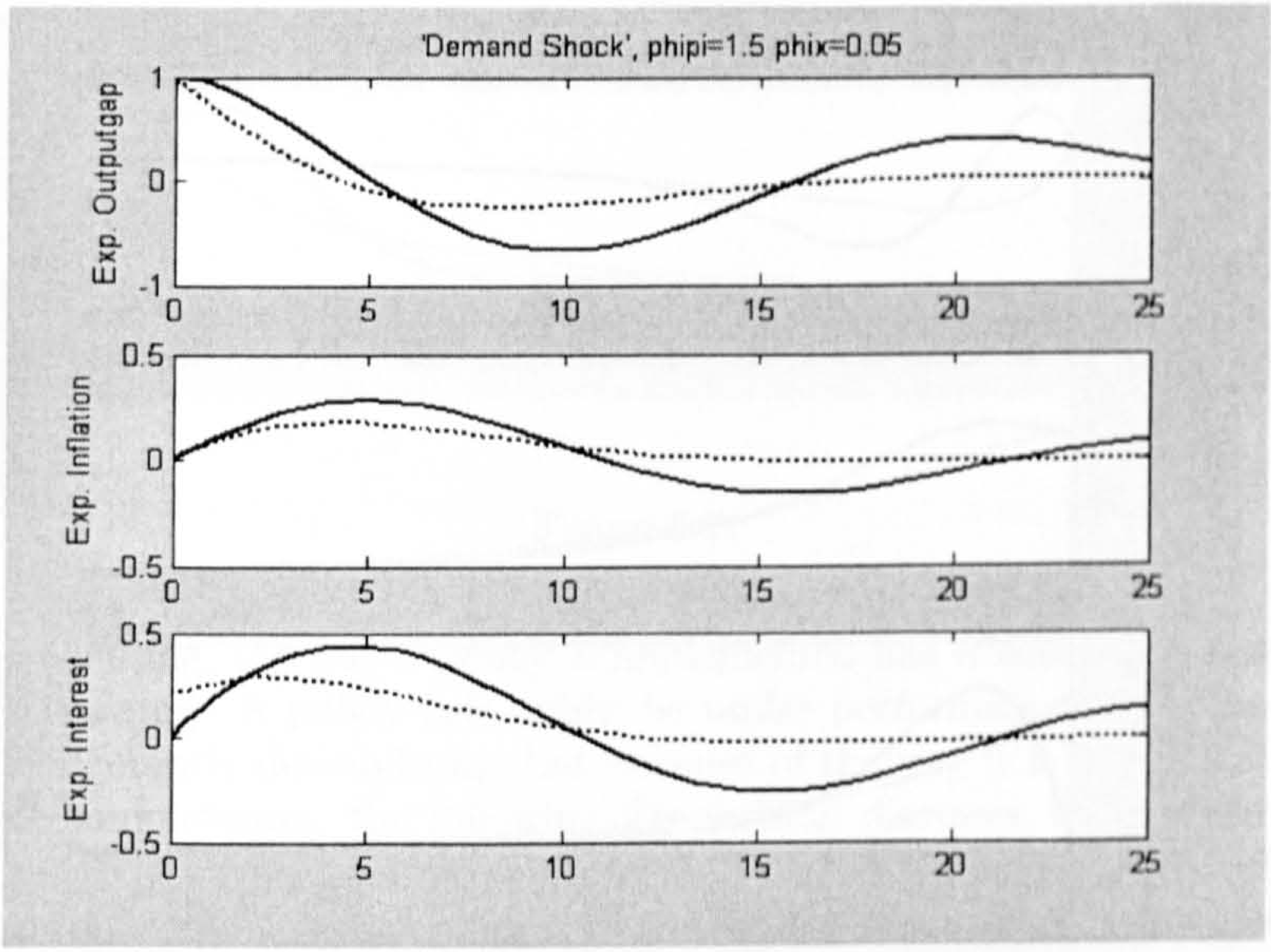
'Demand Shock', phipi=1.5 phix=0.05
Exp. Outputgap
Exp. Inflation
Exp. Interest
1
0
-1
0.5
-0.5
0
5
10
15
20
25

Figure 3.5:

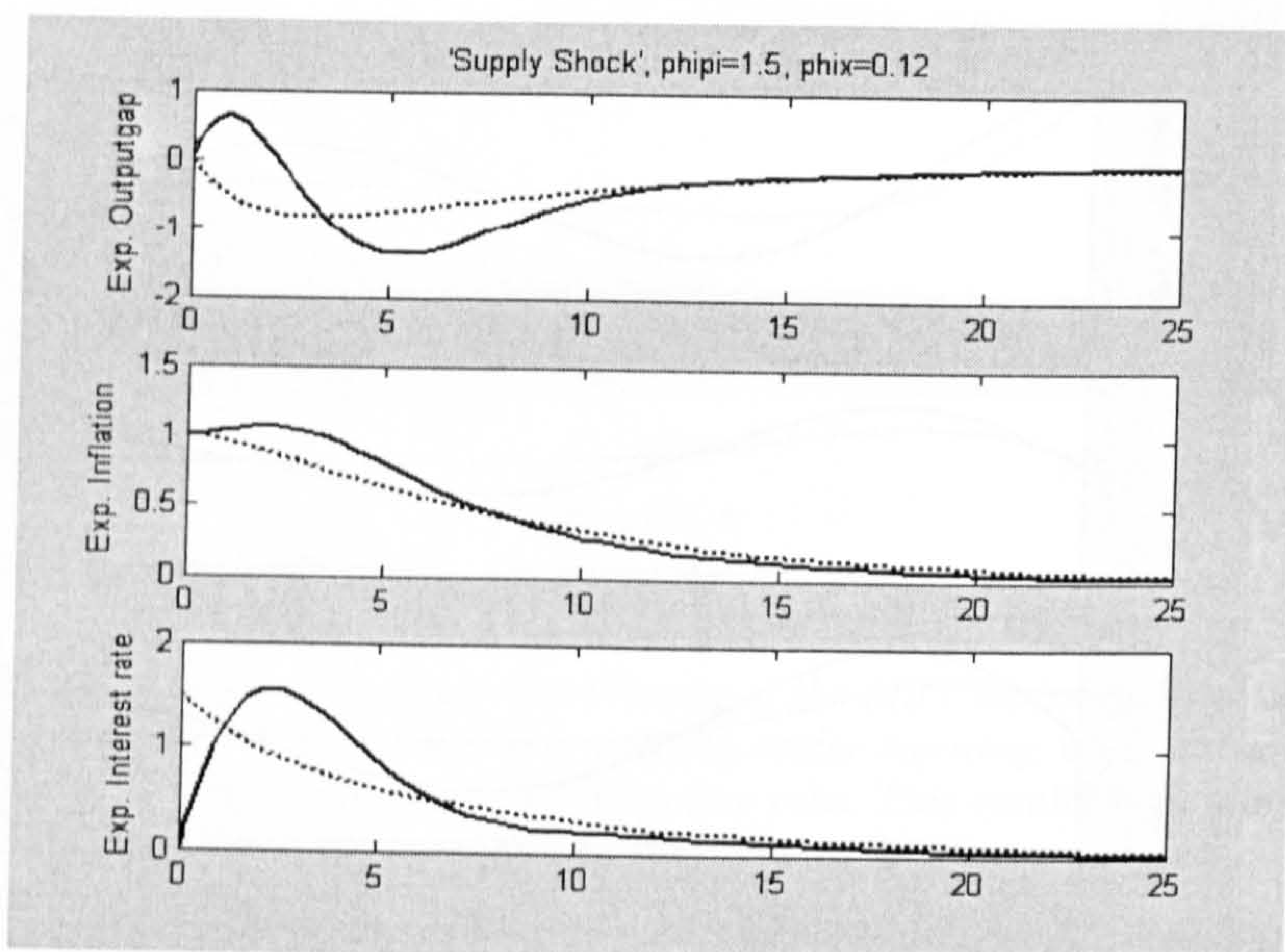
'Supply Shock', phipi=1.5, phix=0.12
Exp. Outputgap
Exp. Inflation
Exp. Interest rate
1
0
-1
-2
1.5
0.5
2
5
10
15
20
25

Figure 3.6:

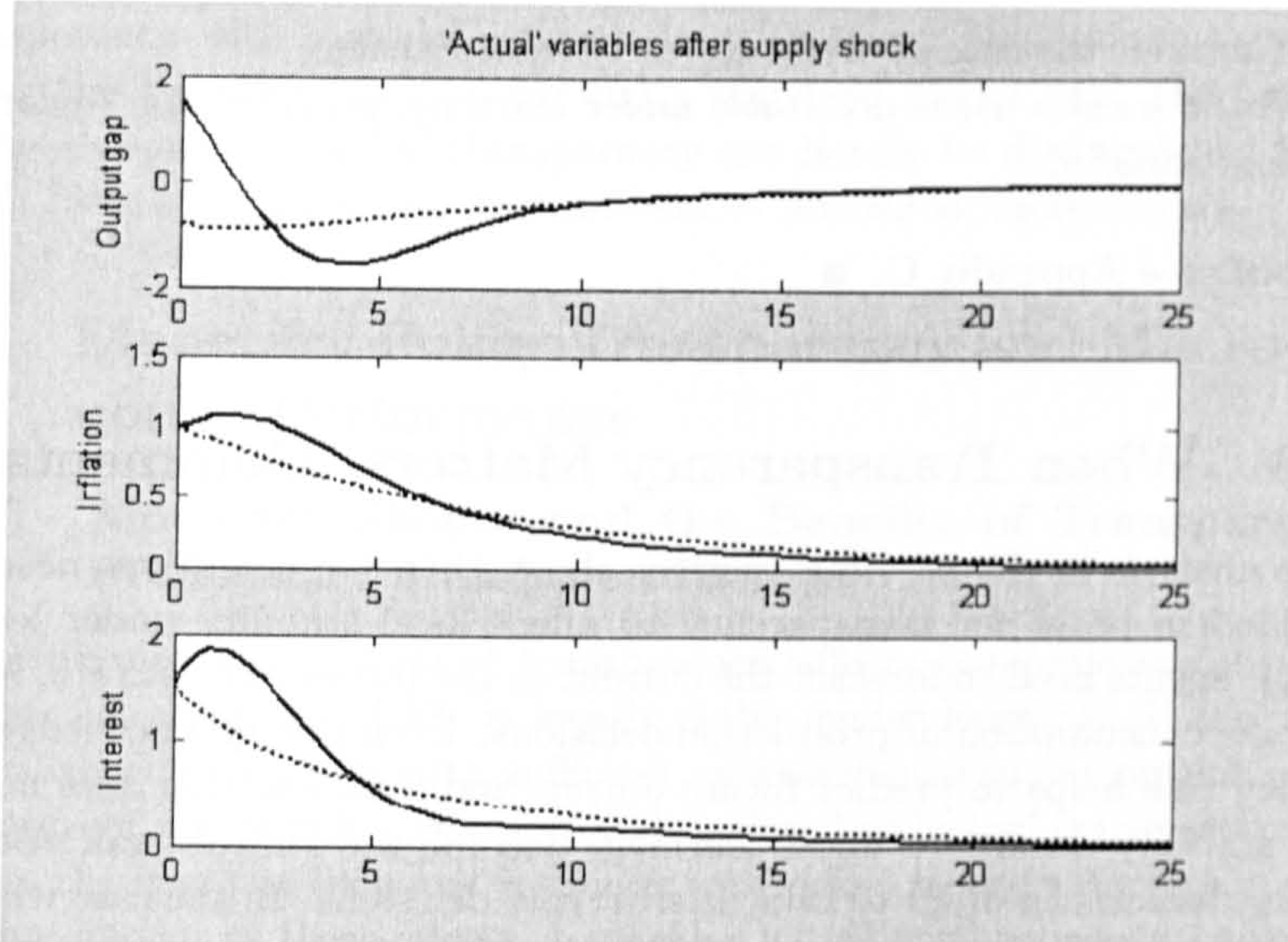

Figure 3.7:

Concluding, the way a policy is implemented has effects on its stability under learning. A policy rule might be under performing not because the rule is inherently destabilizing, but because of the way it is implemented.

For completeness, the following Proposition discusses the performance of other policy rules currently analyzed in the literature. In particular, a policy rule responding to private sector expectations might not solve the problem of instability, at least for this class of policy rules.

Proposition 37 *Assume that private sector's decisions are taken with $t-1$ information and $U_{cm} = 0$.*

(i) if the central bank reacts to current inflation and output gap, transparency does not affect local stability under learning;

Consider the case of no transparency.

(ii) A policy rule of the form: $i_t = \phi_\pi E^h_{t-1}\pi_t + \sigma^{-1} r^n_t$, with $h = PS, CB$ leads to a non-learnable REE;

(iii) Consider the policy rule: $i_t = \phi_\pi E^h_{t-1}\pi_{t+1} + \phi_x E^h_{t-1}x_{t+1} + \phi_r E^h_{t-1}r^n_t + \phi_u E^h_{t-1}u_t$ and assume, without loss of generality that $\rho_r = \rho_u = 0$. Then the stability conditions under learning are the same as in Proposition (35). In particular, if $\phi_x = 0$ the REE is non-learnable for any parameter value.

Consider the case of some degree of transparency;
(iv) the rules above are stable under learning, provided the Taylor Principle is satisfied.

Proof. see Appendix C. ■

3.6 When Transparency Matters. Comments.

The analysis in the previous sections shows that two conditions need to be verified in order for transparency to affect local stability under learning. First, agents need to forecast the current or the future interest rate, in order to take consumption or production decisions. Even though knowledge of the policy rule helps to predict future output and inflation, this does not alter the stability conditions under learning. Second, the central bank also needs to use forecast in order to take interest rate decisions. In the case where the bank reacts to current output and inflation, knowledge about the policy rule doe not change local stability under learning. This is because the learning process of the agents does not affect directly the interest rate dynamics. Hence, the unrestricted VAR coefficients corresponding to the interest rate equation converge for any parameter value. In other words, from the proofs in the Appendix, the characteristic equation of the Jacobian can be written as

$$P(\lambda) = (1+\lambda)\left(\lambda^2 + a_1\lambda + a_0\right)$$

where the eigenvalue corresponding to the interest equation is equal to-1 for any parameter value. It is obvious that under these conditions knowing the policy rule does not affect local stability after learning, even though it might have welfare improving effects in reducing overall volatility. Concluding, necessary and sufficient condition for transparency to matter is a) the central bank responding to forecasts (possibly different from the private sector's and b) the market forecasting the interest rate.

It is also worth mentioning that the response of the economy to a shock would be different, for empirically plausible parameter values, whether the market understands the policy rule or not. In the case of transparency the economy converges back to the steady state monotonically, even though the effects on inflation are likely to be persistent. In the case of no transparency the convergence is oscillatory, with possible negative effects on welfare. I leave a more complete analysis of these effects to further research. The following section attempts a numerical evaluation of the effects of lack of transparency with a model estimated on US data. Simulations show that under the Woodford calibration and a standard Taylor rule, the behavior of the economy under full transparency can hardly be distinguished by the REE. Lack of transparency instead lead to undesired fluctuations.

3.7 Monetary Policy Transparency and Macroeconomic Performance

3.7.1 Monetary Shocks and the Benefits of Transparency: Preliminary Simulation Results

As mentioned above, lack of transparency affects economic stability also in the case where the REE is locally stable under learning. If the monetary authority responds with sufficient aggressiveness to the output gap, it can prevent self-fulfilling inflation or deflation, but lack of transparency still affects the way the economy responds to shocks. In order to give a quantitative impact of these effects, I consider an artificial economy, calibrated using Woodford's estimates. In order to evaluate quantitatively the effects of transparency for a particular economies, we would need an estimated of the structural parameters and the shock disturbances affecting the economy. This analysis is performed in the sections below. This preliminary simulations are conducted including only the policy shock.

I consider a more general Taylor tule with interest smoothing. The rule can be written as

$$i_t = \bar{\imath} + \rho i_{t-1} + (1-\rho)\left[\phi_\pi E_{t-1}^{CB}(\pi_t - \pi^*) + \phi_x E_{t-1}^{CB}(x_t - x^*)\right] + \epsilon_t \quad (3.31)$$

In fact, Bullard and Mitra (2002) show that interest smoothing has a stabilizing effect on the economy. The simulations below show that even in this case lack of transparency has a considerable effect on the dynamic of output, inflation and interest rate. In the simulations I assume $\rho = 0.7$. The following figures show that also in the case of local stability under learning, lack of transparency have important consequences of the dynamics of inflation. The only sizable shock that I consider is the monetary shock ϵ_t. I also include i.i.d. r^n and u_t shocks with negligible standard deviation, in order to make the learning process nontrivial. I assume that $\sigma_\epsilon = 0.0025, \sigma_r = 0.0001, \sigma_u = 0.0001$. I further assume that the agents use the Stochastic Gradient algorithm to update their coefficients. The simulations start with the economy at its REE. I do not assume structural changes in the parameters. Fluctuations from learning depend on the constant revision of the agents' estimates. It is assumed that market participants expect changes in policy and other structural parameters so that they update their estimates using constant gain algorithms, where $\delta_t = \delta$.

Learning Dynamics

Assuming a fixed gain algorithm has the implication that the matrix of coefficients Ω_t described above does not converge to the REE Ω^*. This because learning becomes a persistent process The agents expect structural changes in their model and therefore discount past observations and keep updating their estimates as new observations are available. Nevertheless, provided the gain $\delta_t = \delta$ is 'small', results from Benviste, Metivier and Priouret (1993) show[19] that the matrix of coefficient converges to a time-invariant Gaussian distribution, centered in Ω^*. The variance of the distribution tends to zero as the gain tends to zero. Hence the the stochastic process generated by the model is asymptotically stationary, for large t. This allows us to use the estimation method described below. Also, the asymptotic behavior of the estimated coefficients is a source of extra volatility and fluctuations in the economic variables, with respect to the case of RE.

Simulation Results

In the simulations I use a benchmark case where $\phi_\pi = 1.5$, $\phi_x = 0.5/4$ and $\delta = 0.05$. I then consider the effects of, a) increasing the gain to 0.1 and b) decreasing the coefficient of the output gap to 0.2/4.

Simulation results show that, under the benchmark calibration, if the central bank is fully transparent and credible the dynamics of inflation under learning is virtually identical to the REE. This result resembles the finding of Williams (2002). Instead, lack of transparency implies more volatility and persistence in the inflation process. The tables below reports more specific results on the effects of transparency.

Table II

[19] See also Evans and Honkapohja (2001).

$\phi_x = 0.5/4; \delta = 0.05$	REE	Transparency	No Transparency
$\sigma(x)$	1.7109	1.7118	2.1272
$\sigma(\pi)$	0.2451	0.2456	0.3825
$\sigma(i)$	1.0707	1.07118	1.1595
$corr(x_t, x_{t-1})$	0.3548	0.3574	0.5193
$corr(\pi_t, \pi_{t-1})$	0.3455	0.3516	0.7022
$corr(i_t, i_{t-1})$	0.3553	0.3580	0.4518

The results are mean values, obtained by simulating the economy for 2000 times. Each time the length of the simulation is 2000 periods. Notice how both standard deviation and autocorrelation of the variables increase, with respect of the REE. On the other hand, an higher gain also increases both standard deviation and volatility.

Table III

$\phi_x = 0.5/4; \delta = 0.1$	REE	Transparency	No Transparency
$\sigma(x)$	1.7109	1.7122	2.5005
$\sigma(\pi)$	0.2451	0.2463	0.5157
$\sigma(i)$	1.0707	1.0714	1.2570
$corr(x_t, x_{t-1})$	0.3548	0.3574	0.5995
$corr(\pi_t, \pi_{t-1})$	0.3455	0.3559	0.8277
$corr(i_t, i_{t-1})$	0.3553	0.3581	0.5307

Table IV

$\phi_x = 0.2/4; \delta = 0.05$	REE	Transparency	No Transparency
$\sigma(x)$	2.3728	2.3727	3.0792
$\sigma(\pi)$	0.3741	0.3741	0.6039
$\sigma(i)$	1.1000	1.0999	1.2869
$corr(x_t, x_{t-1})$	0.4165	0.4163	0.6010
$corr(\pi_t, \pi_{t-1})$	0.4119	0.4124	0.7479
$corr(i_t, i_{t-1})$	0.4167	0.4165	0.5707

This might suggest an alternative explanation for the higher volatility and persistence of inflation, at business cycle frequencies, during the 70's. Clarida, et al. (1999) and Boivin and Giannoni (2002) among the others claim that high fluctuations in the pre-Volker were due to a too passive Taylor rule. This implies an indeterminate REE equilibrium and thus undesired fluctuations. This view could be questioned on two grounds.

First, the analysis above seems to suggest that even allowing for perfect knowledge of the policy rule, the indeterminate REE associated to a passive policy rule is not going to be learnable. This rises the question of the plausibility of this equilibrium (unless one would regard the rising inflation in the 70s as a the non-stationary outcome of a non learnable REE).

Second, Orphanides (2003), using real time data, shows that the Taylor rule has been active in the post war period. If this is a plausible description of US monetary policy, then no indeterminate equilibrium would exist.

Assume that in the pre-Volker era the market did not properly understand monetary policy decisions, while from the 80' the public have spent more resources to analyze Fed behavior and Fed have become increasingly transparent about its decisions (especially since the 90s'). Then the results shown above seem to suggest an alternative explanation for undesired fluctuations. A proper evaluation of this hypothesis should be left to further research[20].

3.8 Conclusions

The paper shows that transparency matters for monetary policy design. I consider the case where a class of policy rules is evaluated for robustness to forecasting mistakes of the market participants. I show that in a model where monetary policy has immediate effects on aggregate activity and inflation, knowledge about the policy rule does not enhance the stability of the economic system.

In the more empirically plausible case where expenditure and pricing decisions are predetermined, and therefore monetary policy affects the economy with delays, a transparent implementation of policy rule is crucial for the stability of the economic system. Lack of transparency might generate instability to forecasting mistakes and other equilibria generated by the learning behavior. A more transparent implementation of the rule instead guarantees stability of the unique REE, provided the central bank reacts to some degree to the output gap.

Finally, the paper proposes a method to estimate the learning model with simulated quasi-maximum likelihood methods.

[20] For example, the model should be evaluated including realistic assumptions on the shock processes.

Future research should address the case of inertial policy rule and the role of transparency in this case. Also, in this paper I model delays in the effects of monetary policy with lagged expectations. The role of transparency on learnability should be investigated in the case where delays are captured by lagged variables.

3.9 Appendix A. The Model Solution.

The budget constraint (3.4) can be rewritten as

$$A_t^j + C_t^j = \frac{1+i_{t-1}}{\Pi_t} A_{t-1}^j - \frac{i_{t-1} - i_{t-1}^m}{\Pi_t} m_{t-1}^j + \frac{P_t^j}{P_t} f\left(h_t^j, \frac{M_t}{P_t}\right) - T_t \quad (3.32)$$

where $A_t^j = (B_t + M_t)/P_t$ denotes real wealth and the term $\frac{i_{t-1}-i_{t-1}^m}{\Pi_t}$ is the opportunity cost of holding money, expressed in real terms. Substituting the budget constraint (3.32) in (3.3) the maximization problem becomes

$$\max_{A_t^j, m_t^j, h_t^j, P_t^j}$$

$$E_{t-1}^j \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{c} U\left(\frac{1+i_{t-1}}{\Pi_t} A_{t-1}^j + \frac{i_{t-1}-i_{t-1}^m}{\Pi_t} m_{t-1}^j + \frac{P_t^j}{P_t} Y_t D(\frac{P_t^j}{P_t}) - A_t^j - T_t, m_t\right) \\ +V\left(h_t^j\right) - \frac{\gamma}{2}\left(\frac{P_t^j}{P_{t-1}^j} - \Pi^*\right)^2 \end{array} \right]$$

$$+\lambda_t \left[f\left(h_t^j, m_t^j\right) - Y_t D(\frac{P_t^j}{P_t}) \right]$$

where λ_t is the Lagrangian multiplier. Differentiating with respect to P_t^j, imposing a *symmetric* equilibrium and substituting for the equilibrium condition $Y_t = C_t + G_t = f(h_t, m_t)$, the expression is simplified to

$$E_{t-1}^j \left[\begin{array}{c} U_c\left(Y_t - G_t, m_t^j\right)(Y_t - G_t)(1+\theta_t) - \lambda_t \theta_t + \\ -\gamma_t (\Pi_t - \Pi^*) \Pi_t + \beta \gamma_t (\Pi_{t+1} - \Pi^*) \Pi_{t+1} \end{array} \right] = 0 \quad (3.33)$$

where $\theta_t = D_t'/D_t < -1$ for every t. The f.o.c. for the labor supply is

$$E_{t-1}^j \left[\lambda_t - \frac{V'(h_t^j)}{f_h\left(h_t^j, m_t^j\right)} \right] = 0. \quad (3.34)$$

Substituting for λ_t in (3.33) gives

$$\begin{aligned}
\left(\Pi_t - \Pi^*\right)\Pi_t &= \beta E^j_{t-1}\left(\Pi_{t+1} - \Pi^*\right)\Pi_{t+1} + && (3.35)\\
&\quad - E^j_{t-1}\left[U_c\left(Y_t, m_t\right)\left(Y_t - G_t\right)\frac{\theta_t}{\gamma_t}\left(s_t - \mu_t^{-1}\right)\right] && (3.36)\\
&= \beta E^j_{t-1}\left(\Pi_{t+1} - \Pi^*\right)\Pi_{t+1} + \\
&\quad - E^j_{t-1}\left[\frac{U_c\left(Y_t - G_t, m_t\right)\left(Y_t - G_t\right)}{\bar{\gamma}}\left(\mu_t s_t - 1\right)\right] && (3.37)
\end{aligned}$$

where

$$s_t = \frac{V'(h_t)}{f'\left(h_t, m_t\right)U_c\left(Y_t - G_t, m_t\right)} \tag{3.38}$$

is the average real marginal cost and $\mu_t = \theta_t / \left(1 + \theta_t\right)$ is the desired mark up. Last, the f.o.c. with respect to assets and money balances are

$$E^j_{t-1}\left[U_c\left(C_t, m_t\right)\right] = \beta E^j_{t-1}\left[\frac{U_c\left(C_{t+1}, m_{t+1}\right)\left(1 + i_t\right)}{\Pi_{t+1}}\right] \tag{3.39}$$

which gives the consumption Euler equation, and

$$\begin{aligned}
E^j_{t-1}\left[U_m\left(C_t, m_t\right)\right] &= \beta E^j_{t-1}\left[\frac{U_c\left(C_{t+1}, m_{t+1}\right)\left(i_t - i^m_t\right)}{\Pi_{t+1}}\right] && (3.40)\\
&= E^j_{t-1}\left[\frac{U_c\left(C_t, m_t\right)\left(i_t - i^m_t\right)}{\left(1 + i_t\right)}\right]
\end{aligned}$$

which gives (implicitly) a money demand function for the consumer. There is also an implicit demand function for the producer, which can be shown to be

$$E_{t-1}\left[U_m\left(C_t, m_t\right)\left(\frac{i_t - i^m_t}{1 + i_t}\right)\right] = E_{t-1}\left[\frac{V'\left(h_t\right)}{f_h\left(h_t, m_t\right)}f_m\left(h_t, m_t\right)\right] \tag{3.41}$$

3.9.1 The Linearized Model

Money in the Utility Function

As mentioned in the introduction, I consider separately the case of money in the utility function and money in the production function. In both cases the pricing equation (3.35) can be linearized to get

$$\pi_t = E_{t-1}\beta\pi_{t+1} + \xi E_{t-1}\left(\hat{s}_t + \hat{\mu}_t\right) \tag{3.42}$$

where

$$\xi = \frac{U_c \bar{Y}}{\bar{\gamma}}$$

is a measure of the degree of price stickiness. Notice that the linearization is a good approximation of the non-linear model only for small values of inflation.

In the case of money in the utility function, the real marginal cost is

$$s_t = \frac{V'(f^{-1}(Y_t))}{f'(f^{-1}(Y_t))\, U_c(Y_t - G_t, m_t)}. \tag{3.43}$$

Linearizing (3.43) I obtain

$$\hat{s}_t = \left(\sigma^{-1} + \omega\right) \hat{Y}_t - \sigma^{-1} g_t - \chi \hat{m}_t^j \tag{3.44}$$

where

$$\sigma = -\frac{U_c\left(\bar{Y} - \bar{G}, \bar{m}\right)}{U_{cc}\left(\bar{Y} - \bar{G}, \bar{m}\right)\bar{Y}} > 0$$

is the intertenporal elasticity of substitution of consumption.

$$\chi = \frac{U_{cm}\left(\bar{Y} - \bar{G}, \bar{m}\right)\bar{m}}{U_c\left(\bar{Y} - \bar{G}, \bar{m}\right)} > 0$$

measures the marginal utility of extra consumption, as real balances change and

$$\omega_w = \frac{V''\left(\bar{Y}\right)\bar{Y}}{f'\left(\bar{Y}\right)V'\left(\bar{Y}\right)} > 0$$

defines the elasticity of the marginal disutility of work with respect to output.

$$\omega_f = -\frac{f''\left(\bar{Y}\right)\bar{Y}}{f'\left(\bar{Y}\right)} > 0$$

defines the elasticity of the marginal product of labor with respect to output. Finally, $\omega = \omega_w + \omega_p$.Linearization of the Euler equation (3.39) gives

$$\hat{c}_t^j = -\sigma E_{t-1}^j\left(\hat{\imath}_t - \pi_{t+1}\right) + E_{t-1}^j \hat{c}_{t+1}^j - \chi\sigma E_{t-1}^j\left(\hat{m}_{t+1}^j - \hat{m}_t^j\right). \tag{3.45}$$

Linearizing the money demand gives

$$\hat{m}_t^j = \eta_y \hat{c}_t^j - \eta_i E_{t-1}^j\left(\hat{\imath}_t - \hat{\imath}_t^m\right) \tag{3.46}$$

where I use the fact that each agent knows his consumption at time t, when deciding the amount of money. where $\eta_y > 0, \eta_i > 0$ denote the elasticity of money demand with respect to income and the nominal interest rate. The coefficients can be shown to be

$$\eta_y = \left(\frac{\chi_c + \bar{\Delta}\sigma^{-1}}{\bar{\Delta}\chi + \epsilon_{mm}}\right)$$

$$\eta_i = \left(\frac{1 - \bar{\Delta}}{\bar{\Delta}\chi + \epsilon_{mm}}\right)$$

and

$$\chi_c = \frac{U_{mc}\left(\bar{Y} - \bar{G}, \bar{m}\right)\bar{Y}}{U_c\left(\bar{Y} - \bar{G}, \bar{m}\right)}$$

$$\epsilon_{mm} = \frac{U_{mm}\left(\bar{Y} - \bar{G}, \bar{m}\right)\bar{m}}{U_c\left(\bar{Y} - \bar{G}, \bar{m}\right)}.$$

Substituting (3.46) in (3.45), I get

$$\hat{c}_t^j = -\tilde{\sigma}E_{t-1}^j\left[\eta_1\left(\hat{\imath}_t - \hat{\imath}_t^m\right) - \pi_{t+1} - \eta_2\left(\hat{\imath}_{t+1} - \hat{\imath}_{t+1}^m\right)\right] + E_{t-1}^j\hat{c}_{t+1}^j$$

where $\eta_1 = (1 + \chi\eta_i) > 1$, $\eta_2 = \eta_1 - 1 > 0$, and $\tilde{\sigma} = \frac{\sigma}{(1 - \chi\sigma\eta_y)} > \sigma$. Assuming that the agents understand that their future consumption depend on aggregate output[21], i.e. that $C_{t+1}^j = Y_{t+1} - G_{t+1}$, then their consumption decisions depend on their expectations about future output, inflation, interest rate and government expenditures. Imposing equilibrium in the goods market and aggregating over the agents I obtain the IS equation with real money balances effects

$$x_t = -\tilde{\sigma}E_{t-1}\left[\eta_1\hat{\imath}_t - \eta_2\hat{\imath}_t^m - \pi_{t+1} - \eta_2\left(\hat{\imath}_t - \hat{\imath}_t^m\right) - \hat{r}_t^n\right] + E_{t-1}x_{t+1} \quad (3.47)$$

where $x_t = \hat{Y}_t - \hat{Y}_t^e$, is the output gap, expressed as output deviations from the *efficient* level $\hat{Y}_t^e$. Following Woodford (2003), the efficient level of output can be defined by

$$s\left(Y_t^e, m_t, G_t\right) = \bar{\mu}^{-1}. \quad (3.48)$$

[21] This is not a strong asumption. As mentioned in Evans, Honkapohja and Mitra (2002), running a regression of consumption on the output gap and government expenditures would reveal what I am assuming.

Notice that markup shocks do not affect the efficient level of output. Using the money demand function and linearizing (3.48) I obtain

$$Y_t^e = -\frac{(\sigma^{-1} + \chi\eta_y)}{\epsilon_{mc}}\hat{g}_t$$

where $\epsilon_{mc} = (\sigma^{-1} + \omega - \chi\eta_y)$ denotes the elasticity of the real marginal cost and

$$\hat{Y}_t^e = \hat{Y}_t^n + \frac{1}{\epsilon_{mc}}\hat{\mu}_t$$

where $\hat{Y}_t^n$ is the equilibrium level of output if prices are fully flexible, given a monetary policy that maintains a constant interest rate spread between monetary and nonmonetary assets. The (exogenous) process $\hat{r}_t^n$ is defined as

$$\hat{r}_t^n = (\sigma^{-1} - \eta_y\chi)\left(E_{t-1}\hat{Y}_{t+1}^e - \hat{Y}_t^e\right) - \sigma^{-1}(E_{t-1}g_{t+1} - g_t).$$

In order to simplify the analysis and consistently with the previous literature I assume that $\hat{r}_t^n$ is observable and evolves as an AR(1) process. All variables are in log-levels. This is consistent with the assumption that the agents do not know the long run equilibrium of the variables, and in particular the inflation target. Notice that current expenditures depend on expectations about the current and the future interest rate.

Finally, combining (3.44) with the money demand curve, and using the definition of efficient output I obtain the following Phillips curve

$$\pi_t = E_{t-1}\beta\pi_{t+1} + \kappa E_{t-1}[\hat{x}_t + \eta_3(\hat{\imath}_t - \hat{\imath}_t^m)] + u_t$$

where

$$\begin{aligned}\kappa &= \xi\epsilon_{mc}\\ \eta_3 &= \frac{\eta_i X}{\epsilon_{mc}}\end{aligned}$$

and $u_t = \left(\frac{\kappa}{\epsilon_{mc}}\right)E_{t-1}\hat{\mu}_t$. Notice that the shock is pre-determined at time t.

Money in the Production Function

In this case he real marginal cost is

$$s_t = \frac{V'(h_t)}{f'(h_t, m_t)U_c(Y_t - G_t)}. \qquad (3.49)$$

Defining

$$h_t = H(Y_t, m_t)$$

with $H_y > 0$ and $H_m < 0$, it is possible to rewrite (3.49) as a function of output and money balances only. Hence,

$$s_t = \frac{V'(H(Y_t, m_t))}{f'(H(Y_t, m_t))\, U_c(Y_t - G_t)}.$$

Log-linearization leads to the following expression

$$\hat{s}_t = \left(\sigma^{-1} + \tilde{\omega}\right)\hat{Y}_t + \sigma^{-1}\hat{g}_t - \tilde{\chi}\hat{m}_t$$

where

$$\begin{aligned} \tilde{\omega} &= \tilde{\omega}_w + \omega_{hh} \\ \tilde{\chi} &= \chi_{vv} + \chi_{hh} + \chi_{hm} \end{aligned}$$

$$\tilde{\omega}_w = \frac{V''\left(H\left(\bar{Y}, \bar{m}\right)\right) H_y\left(\bar{Y}, \bar{m}\right)\bar{Y}}{V'\left(H\left(\bar{Y}, \bar{m}\right)\right)} > 0$$

$$\omega_{hh} = -\frac{f_{hh}\left(\bar{Y}, \bar{m}\right) H_y\left(\bar{Y}, \bar{m}\right)\bar{Y}}{f_h\left(\bar{Y}, \bar{m}\right)} > 0$$

$$\chi_{vv} = -\frac{V''\left(H\left(\bar{Y}, \bar{m}\right)\right) H_m\left(\bar{Y}, \bar{m}\right)\bar{m}}{V'\left(H\left(\bar{Y}, \bar{m}\right)\right)} > 0$$

$$\chi_{hh} = \frac{f_{hh}\left(\bar{Y}, \bar{m}\right) H_m\left(\bar{Y}, \bar{m}\right)\bar{m}}{f_h\left(\bar{Y}, \bar{m}\right)} > 0$$

$$\chi_{hm} = \frac{f_{hm}\left(\bar{Y}, \bar{m}\right)\bar{m}}{f_h\left(\bar{Y}, \bar{m}\right)} > 0.$$

The Euler equation for consumption and thus the IS equation do not depend on real money balances. The IS is equivalent to (3.47) with $\eta_1 = 1$ and $\eta_2 = 0$.

The money demand function can be linearized to yield

$$\hat{m}_t = \tilde{\eta}_y \hat{c}_t - \tilde{\eta}_i E_{t-1}\left(\hat{\imath}_t - \hat{\imath}_t^m\right) \tag{3.50}$$

where

$$\tilde{\eta}_y = \left(\frac{\tilde{\omega}_w + \omega_{hm} + \omega_{hh} + \sigma^{-1}}{\chi_{vv} + \chi_{hm} + \chi_{mm} + \chi_{hh} + \chi_{hm}}\right)$$

where

$$\omega_{mh} = \frac{f_{mh}\left(\bar{Y},\bar{m}\right) H_c\left(\bar{Y},\bar{m}\right)\bar{Y}}{f_m\left(\bar{Y},\bar{m}\right)} > 0$$

$$\chi_{mh} = -\frac{f_{mh}\left(\bar{Y},\bar{m}\right) H_m\left(\bar{Y},\bar{m}\right)\bar{m}}{f_m\left(\bar{Y},\bar{m}\right)} > 0$$

$$\chi_{mm} = -\frac{f_{mm}\left(\bar{Y},\bar{m}\right)\bar{m}}{f_m\left(\bar{Y},\bar{m}\right)} > 0$$

and

$$\tilde{\eta}_i = \left(\frac{\Delta^{-1}}{\chi_{vv} + \chi_{hm} + \chi_{mm} + \chi_{hh} + \chi_{hm}}\right).$$

Inserting the demand for money (3.50) in the real marginal cost equation, I obtain

$$\hat{s}_t = \left(\sigma^{-1} + \tilde{\omega} - \tilde{\chi}\tilde{\eta}_y\right)\hat{Y}_t + \tilde{\chi}\tilde{\eta}_i E_{t-1}\left(\hat{\imath}_t - \hat{\imath}_t^m\right) + \left(\sigma^{-1} + \tilde{\chi}\tilde{\eta}_y\right)\hat{g}_t.$$

Proceeding as for the case of money in the utility function, it is possible to express the real marginal cost in terms of deviations of current output from its efficient level

$$\hat{s}_t = \tilde{\epsilon}_{mc}\hat{Y}_t^e + \tilde{\chi}\tilde{\eta}_i E_{t-1}\left(\hat{\imath}_t - \hat{\imath}_t^m\right)$$

where, again, $\tilde{\epsilon}_{mc} = \left(\sigma^{-1} + \tilde{\omega} - \tilde{\chi}\tilde{\eta}_y\right)$. The log-linear Phillips curve is again

$$\pi_t = E_{t-1}^j \beta\pi_{t+1} + \xi E_{t-1}^j \hat{s}_t$$

Substituting the real marginal cost in the Phillips curve we get

$$\pi_t = E_{t-1}\beta\pi_{t+1} + \tilde{\kappa}E_{t-1}\left[\hat{x}_t + \tilde{\eta}_3\left(\hat{\imath}_t - \hat{\imath}_t^m\right)\right] + \tilde{u}_t$$

where

$$\begin{aligned}\tilde{\kappa} &= \tilde{\xi}\tilde{\epsilon}_{mc} \\ \tilde{\eta}_3 &= \frac{\tilde{\eta}_i\tilde{\chi}}{\tilde{\epsilon}_{mc}}\end{aligned}$$

and $\tilde{u}_t$ is defined as above.

3.10 Appendix B. Stability With t Expectations.

The proofs make use of the mathematical results described by Theorems 4 and 5, in CH. 1.

Proof. Proposition (33)

(i) Local stability is determined by the stability of the following differential equation in notional time

$$\frac{d\Omega}{d\tau} = T(\Omega) - \Omega$$

where

$$T(\Omega) = \left(\hat{A}_1 + \hat{A}_2\Omega_0,\ \hat{A}_2\Omega_1 H + \hat{A}_3\right).$$

Following Evans and Honkapohja (2001), E-Stability is obtained, provided the following matrices have eigenvalues with real parts less than one.

$$\hat{A}_2 - I_3 \tag{3.51}$$

and

$$H \otimes \hat{A}_2 - I_3. \tag{3.52}$$

It is enough to verify the condition for (3.51), given that ρ_r, $\rho_u < 1$. The characteristic equation of (3.51) can be factorized in the form

$$P(\lambda) = -(1+\lambda)\left(\lambda^2 + a_1\lambda + a_0\right)$$

where

$$a_1 = \frac{\begin{array}{c}\tilde{\sigma}\kappa\left[\phi_\pi - 1\right] + \tilde{\sigma}\phi_x(1-\beta)\left[1 + (1-\beta)\eta_2\right] + \\ + \tilde{\sigma}\kappa\phi_\pi\left(1 - \frac{\eta_3}{\sigma}\right) - \tilde{\sigma}\kappa\eta_3\phi_x + (1-\beta)\end{array}}{\left(\eta_1 - \frac{\eta_3}{\sigma}\right)\tilde{\sigma}\kappa\phi_\pi + \tilde{\sigma}\phi_x\eta_1 + 1}$$

and

$$a_0 = \frac{\tilde{\sigma}\kappa(\phi_\pi - 1) + \tilde{\sigma}\phi_x(1-\beta) + \tilde{\sigma}\kappa - \tilde{\sigma}\kappa\eta_3\phi_x}{\left(\eta_1 - \frac{\eta_3}{\sigma}\right)\tilde{\sigma}\kappa\phi_\pi + \tilde{\sigma}\phi_x\eta_1 + 1}.$$

The conditions to be satisfied to obtain stability are

$$a_0 > 0, \quad a_1 > 0.$$

Given that $\eta_1 - \eta_3 > 0$ and $\sigma \geq 1$, we have that $a_1 > a_0$. Hence, stability obtains if $a_0 > 0$. This gives condition (3.12).

Also, when the change of stability occurs, a_0 is equal to zero. That means that the eigenvalues are real. Hence, no local Hopf bifurcation occurs and the inflation target is the only equilibrium. ■

Proof. Proposition (34)

I study the convergence properties of the algorithm by using stochastic approximation theory, using results by Evans and Honkapohja (2001) and Marcet and Sargent (1989). In order to apply those results, I need to put the system (3.20) in the following form:

$$\xi_t = \xi_{t-1} + \delta_t \Phi(\xi_{t-1}, S_t) \tag{3.53}$$

$$S_t = G(\xi_{t-1}) S_{t-1} + C\nu_t$$

which is achieved by setting:

$$\xi_t = \begin{bmatrix} \theta_t & vec\left(\bar{R}_t\right) \end{bmatrix}$$

$$S_t = \begin{bmatrix} Y_t & X_t \end{bmatrix}$$

$$G = \begin{bmatrix} 0_{3\times 3} & \tilde{T}(\xi_{t-1}) \\ 0_{2\times 3} & H \end{bmatrix}$$

where θ is the d-dimensional vector of the estimates, S represents the state vector, ν is the disturbance term and C its coefficients. The latter two are trivially identifiable. The local dynamics of this system (local convergence), i.e. the stability of the RE equilibrium depend on the associated ODE:

$$d\theta/d\tau = h(\theta) \tag{3.54}$$

where $h(\theta) = \lim_{t\to\infty} E\Phi(\theta, S_t(\theta))$. An exhaustive survey of this approach with analytical proofs can be found in Evans and Honkapohja (2001). Given that the system can be put in the in form (3.53), it is easy to verify that it satisfies the properties, A.1, A.2, B.1, B.2 in Evans and Honkapohja (2001).

First, I can rewrite the matrices of regressors as:

$$Z_t(\theta_{t-1}) = K'(\theta_{t-1})\, Q_t \tag{3.55}$$

where

$$K'(\theta_{t-1}) = \begin{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \multicolumn{3}{c}{\tilde{T}^{i\prime}(\theta_{t-1})} \end{pmatrix} & 0_{3,3} & 0_{3,3} \\ 0_{3,3} & \begin{pmatrix} 1 & 0 & 0 \\ \multicolumn{3}{c}{\tilde{T}^{x\prime}(\theta_{t-1})} \\ 0 & 0 & 1 \end{pmatrix} & 0_{3,3} \\ 0_{3,3} & 0_{3,3} & \begin{pmatrix} 1 & 0 & 0 \\ \multicolumn{3}{c}{\tilde{T}^{x\prime}(\theta_{t-1})} \\ \multicolumn{3}{c}{\tilde{T}^{\pi\prime}(\theta_{t-1})} \end{pmatrix} \end{bmatrix}.$$

Substituting (3.55) in (3.20) I get

$$\begin{aligned} \theta_t &= \theta_{t-1} + \delta_t \overline{R}_{t-1}^{-1} Q_t \left(\bar{T}(\theta_{t-1}) Q_t - \left(\theta'_{t-1} K'(\theta_{t-1}) Q_t\right)' \right) \\ &\quad + i.i.d. \text{ errors} \end{aligned}$$

$$\overline{R}_t = \overline{R}_{t-1} + \delta_t \left(Q_t \left(K'(\theta_{t-1}) Q_t\right)' - \overline{R}_{t-1} \right)$$

where

$$\bar{T}(\theta_{t-1}) = \begin{pmatrix} \tilde{T}^{\pi\prime}(\theta_{t-1}) & \tilde{T}^{\pi\prime}(\theta_{t-1}) & \tilde{T}^{i\prime}(\theta_{t-1}) \end{pmatrix}$$

Rearranging, the system becomes

$$\begin{aligned} \theta_t &= \theta_{t-1} + \delta_t \overline{R}_{t-1}^{-1} Q_t Q'_t \left(\bar{T}(\theta_{t-1}) - K(\theta_{t-1}) \theta_{t-1} \right) \\ &\quad + i.i.d. \text{ errors} \end{aligned}$$

$$\overline{R}_t = \overline{R}_{t-1} + \delta_t \left(Q_t Q'_t K(\theta_{t-1}) - \overline{R}_{t-1} \right).$$

By taking the asymptotic mean, the convergence properties can be studied by checking the stability of the following ODE

$$\frac{d\overline{R}}{d\tau} = M_Q K(\theta) - \overline{R} \tag{3.56}$$

$$\frac{d\theta}{d\tau} = \overline{R}^{-1} M_Q \left(\bar{T}(\theta) - K(\theta) \theta \right) \tag{3.57}$$

where $M_Q = E_{\lim t \to \infty} (Q_t Q'_t)$. Given that, from the first equation

$$\overline{R} \to M_Q K(\theta)$$

the stability analysis reduces to:

$$\begin{aligned}\frac{d\theta}{d\tau} &= K(\theta)^{-1} M_Q^{-1} M_Q \left(\bar{T}(\theta) - K(\theta)\theta\right) \\ &= K(\theta)^{-1} \bar{T}(\theta) - \theta.\end{aligned}$$

The resulting system is a non-linear function of Γ and B and a complicated expression of the single parameters. Using Matlab Symbolic Toolbox it is possible to show that the linearized system is composed of three independent subsystems

$$\begin{pmatrix} \dot{b}_{01} \\ \dot{b}_{02} \end{pmatrix} = G_0 \begin{pmatrix} b_{01} \\ b_{02} \end{pmatrix} \tag{3.58}$$

$$\begin{pmatrix} \dot{\psi}_0 \\ \dot{\psi}_x \\ \dot{\psi}_\pi \end{pmatrix} = \begin{pmatrix} i - \phi_\pi \pi^* - \phi_y x^* \\ \phi_x \\ \phi_\pi \end{pmatrix} - \begin{pmatrix} \psi_0 \\ \psi_x \\ \psi_\pi \end{pmatrix} \tag{3.59}$$

and

$$\begin{pmatrix} \dot{\gamma}_{11} \\ \dot{\gamma}_{21} \\ \dot{b}_{11} \\ \dot{b}_{22} \end{pmatrix} = G_1 \begin{pmatrix} \gamma_{11} \\ \gamma_{21} \\ b_{11} \\ b_{22} \end{pmatrix}. \tag{3.60}$$

First, (3.59) shows that the estimates of the Taylor rule converge for any parameter value. This is expected, because as I mentioned in the main text, the agents' equation is correctly specified at any point in time. Hence, provided that output and inflation stay bounded, the estimates will converge. Also, it is possible to show that the characteristic equation of (3.58) is the same as for $\hat{A}_2 - I_3$ (even if $G_0 \neq \hat{A}_2 - I_3$, so that the matrices are similar).

(ii) The system (3.60) is more complicated and it is not possible find an analytical solution. Imposing $U_{cm} = 0$ it is possible to show that the eigenvalues of (3.52) and G_1 are the same so that the matrices are similar. For the general case, I had to resort to simulation, as mentioned in the Proposition. Repeated simulations suggest that G_1 and (3.52) are similar for every parameter value.

(iii) The result trivially depends on the fact that the stability of (3.58) and (3.60) do not depend on the estimates of the policy parameters. ■

3.11 Appendix C. Stability With t-1 Expectations.

Proof. Proposition (35)

The model can be written in matrix form as

$$V_t = A_0 + A_1^{PS} E_{t-1}^{PS} V_t + A_1^{CB} E_{t-1}^{CB} V_t + A_2 E_{t-1}^{PS} V_{t+1} + A_3 X_t + \epsilon_t. \quad (3.61)$$

Consider the E-Stability conditions. Inserting the PLMs I get

$$V_t = T\left(\Omega_0^{PS}, \Omega_0^{CB}, \Omega_1^{PS}, \Omega_1^{CB}\right) \begin{bmatrix} 1 \\ X_{t-d} \end{bmatrix} + \epsilon_t \quad (3.62)$$

the mapping between the PLM and ALM is described by the following ODE

$$\frac{d\Omega_0^{PS}}{d\tau} = A_0 + \left(A_1^{PS} - I_3\right) \Omega_0^{PS} + A_1^{CB} \Omega_0^{CB} + A_2 \Omega_0^{PS} \quad (3.63)$$

$$\frac{d\Omega_0^{CB}}{d\tau} = A_0 + \left(A_1^{CB} - I_3\right) \Omega_0^{CB} + A_1^{PS} \Omega_0^{PS} + A_2 \Omega_0^{PS} \quad (3.64)$$

$$\frac{d\Omega_1^{PS}}{d\tau} = A_1^{PS} \Omega_1^{PS} + A_1^{CB} \Omega_1^{CB} + A_2 \Omega_1^{PS} + A_3 H - \Omega_1^{PS} \quad (3.65)$$

$$\frac{d\Omega_1^{PS}}{d\tau} = A_1^{PS} \Omega_1^{PS} + A_1^{CB} \Omega_1^{CB} + A_2 \Omega_1^{PS} + A_3 H - \Omega_2^{PS}. \quad (3.66)$$

Stability under learning is determined by the following independent subsystems

$$\begin{bmatrix} \dot{\Omega}_0^{PS} \\ \dot{\Omega}_0^{CB} \end{bmatrix} = F_1 \begin{bmatrix} \Omega_0^{PS} \\ \Omega_0^{CB} \end{bmatrix} \quad (3.67)$$

$$\begin{bmatrix} vec\dot{\Omega}_1^{PS} \\ vec\dot{\Omega}_1^{CB} \end{bmatrix} = F_2 \begin{bmatrix} vec\Omega_1^{PS} \\ vec\Omega_1^{CB} \end{bmatrix} \quad (3.68)$$

where

$$F_1 = \begin{pmatrix} A_1^{PS} - I_3 + A_2 & A_1^{CB} \\ A_1^{PS} + A_2 & A_1^{CB} - I_3 \end{pmatrix} \quad (3.69)$$

$$F_2 = \begin{pmatrix} I_2 \otimes A_1^{PS} + H \otimes A_2 - I_6 & I_2 \otimes A_1^{CB} \\ I_2 \otimes A_1^{PS} + H \otimes A_2 & I_2 \otimes A_1^{CB} - I_6 \end{pmatrix}. \quad (3.70)$$

In order to extract the stability conditions I follow Honkapohja and Mitra (2002). Stability under learning is obtained if the eigenvalues of F_1 and F_2 have negative real parts. The characteristic equations of associated to the two matrices can be simplified to

$$\begin{aligned} |F_1 - \lambda I_6| &= \begin{vmatrix} A_1^{PS} - I_3(1+\lambda) + A_2 & A_1^{CB} \\ A_1^{PS} + A_2 & A_1^{CB} - I_3(1+\lambda) \end{vmatrix} \\ &= (-(1+\lambda))^2 \left| A_1^{PS} + A_1^{CB} + A_2 - I_3(1+\lambda) \right| \end{aligned} \tag{3.71}$$

and

$$\begin{aligned} |F_2 - \lambda I_{12}| &= \begin{vmatrix} -(1+\lambda) I_6 & (1+\lambda) I_6 \\ I_2 \otimes A_1^{PS} + H \otimes A_2 & I_2 \otimes A_1^{CB} - I_6(1+\lambda) \end{vmatrix} \\ &= (-(1+\lambda))^6 \left| I_2 \otimes A_1^{PS} + H \otimes A_2 + I_2 \otimes A_1^{CB} - (1+\lambda) I_6 \right|. \end{aligned} \tag{3.72}$$

So, determining stability boils down to determinate whether the eigenvalues of the following matrices have negative real part

$$\tilde{A}_1 = A_1^{PS} + A_1^{CB} + A_2 - I_3 = \begin{pmatrix} 0 & \tilde{\sigma} & -\tilde{\sigma} \\ \kappa & \beta - 1 & \kappa\eta_3 \\ \phi_y & \phi_\pi & -1 \end{pmatrix} \tag{3.73}$$

and

$$\tilde{A}_2 = A_1^{PS} + A_1^{CB} + \rho_i A_2. \tag{3.74}$$

for $i = r, u$. Let us consider first (3.73). According to Routh's Theorem, the number of roots of (3.73) with positive real parts is equal to the number of variations of sign in the following scheme

$$-1 \qquad \text{Trace}\,(\tilde{A}_1) \qquad -B_1 + \frac{\text{Det}(\tilde{A}_1)}{\text{Trace}\,(\tilde{A}_1)} \qquad \text{Det}(\tilde{A}_1) \tag{3.75}$$

where

$$\text{Trace}\,(\tilde{A}_1) = \beta - 2 < 0 \tag{3.76}$$

$$\text{Det}(\tilde{A}_1) = -\left[\kappa(\phi_\pi - 1) + (1-\beta)\phi_x\right] + \eta_3\kappa\phi_x < 0 \tag{3.77}$$

provided (3.22) holds (3.78)

$$B_1 = -\kappa\sigma + \sigma\phi_x + (1-\beta) - \eta_3\kappa\phi_\pi \tag{3.79}$$

where B is the sum of all second order principle minors of $\tilde{A}_1$. A pattern of $- - --$ corresponds to all eigenvalues having negative real part. In order to obtain that we need

$$-B_1 \cdot \text{Trace } (\tilde{A}_1) + \text{Det}(\tilde{A}_1) > 0 \tag{3.80}$$

Algebraic manipulations show that (3.80) is positive if $\phi_x > \hat{\phi}_x$ in the Proposition is verified. Consider the matrix $\tilde{A}_2$. It is easy to show that

$$\text{Trace } (\tilde{A}_2) = \rho_i (1 + \beta) - 3 < 0 \tag{3.81}$$

$$\text{Det}(\tilde{A}_2) = -\tilde{\sigma} \left[\kappa (\phi_\pi - \rho_i) + \phi_x (1 - \beta\rho_i) (\eta_1 - \rho_i \eta_2)\right] +$$

$$-\kappa (1 - \rho_i) (\tilde{\sigma}\eta_1 - \eta_3) - (1 - \rho_i) (2 - \beta\rho_i) < 0$$

provided (3.22) holds

$$\begin{aligned} B_2 &= -\kappa\rho_i\sigma + (\eta_1 - \rho_i\eta_2) \sigma\phi_y + 2 - \beta (1 + \rho_i) + \\ & (1 - \rho_i) (1 - \beta\rho_i) - \eta_3\kappa\phi_\pi. \end{aligned} \tag{3.82}$$

Assume (3.22) holds. Since$B_2 \geq B_1$, Det$(\tilde{A}_2) \leq$Det$(\tilde{A}_1)$ and Trace $(\tilde{A}_2) \leq$Trace $(\tilde{A}_1)$, $-B_2$·Trace $(\tilde{A}_2)$+Det$(\tilde{A}_2)>0$, if (3.80) is satisfied. So that provided that $\phi_x > \hat{\phi}_x$, the REE is stable under learning.

(ii) Notice that if (3.80) is not verified the sign pattern becomes $- - + -$, which indicates two eigenvalues with positive real parts. Since, the determinant of $\tilde{A}_1$ does not vanish at $\hat{\phi}_x$, we know that the eigenvalues are complex.

(ii) Determinacy obtains if $\left(I_3 - A_1^{PS} + A_1^{CB}\right)^{-1} A_2^{PS}$has eigenvalues inside the unit circle. The characteristic equation can be factorized to give

$$P(\lambda) = -\lambda \left(\lambda^2 + a_1\lambda + a_0\right)$$

where

$$a_1 = -\frac{\left(\eta_2\kappa\phi_\pi + \kappa\eta_3\phi_x + \kappa + \eta_2\phi_y + \beta\phi_x\eta_1\right)\tilde{\sigma} - \kappa\eta_3\phi_\pi + 1 + \beta}{\left(\kappa\eta_1\phi_\pi + \phi_x\eta_1\right)\tilde{\sigma} + 1 - \kappa\eta_3\phi_\pi}$$

and

$$a_0 = \frac{\beta(1 + \phi_x\eta_2\tilde{\sigma})}{\left(\kappa\eta_1\phi_\pi + \phi_x\eta_1\right)\tilde{\sigma} + 1 - \kappa\eta_3\phi_\pi}.$$

The conditions for determinacy are

$$|a_0| < 1, \quad |a_1| < 1 + a_0$$

The first condition is verified, given that $\eta_1 \tilde{\sigma} > \eta_3$. Notice that the denominator is positive, provided $\eta_1 > \eta_3$ and $\sigma \geq 1$. Also, imposing the condition mentioned in the main text we have that $a_1 < 0$, so that the condition for determinacy becomes

$$-a_1 < 1 + a_0$$

which gives (3.12).

Finally, local uniqueness of the equilibrium comes from the fact that indeterminacy occurs as $-a_1 = 1 + a_0$. Hence the eigenvalues are real at the bifurcation point so that there is no Hopf bifurcation. Moreover, the maximum eigenvalue is positive, so that we can exclude a flip bifurcation. This implies that no other equilibria exist close to the inflation target. ■

Proof. Proposition (36)

(i) Consider first the convergence properties of the policy rule estimates. Assume that the central bank and the private sector have the same expectations. This is without loss of generality, from the results above. Then substituting the expectations in the Taylor rule we get

$$\psi_t = \psi_{t-1} + \delta_t R_{\psi,t-1}^{-1} X_{t-1} \left[\phi' \left(\tilde{\Omega}_{t-1} W_{t-1}\right) + \epsilon_t - \psi_{t-1}' \left(\tilde{\Omega}_{t-1} W_{t-1}\right)\right]$$

$$R_{\psi,t} = R_{\psi,t-1} + \delta_t \left[X_{t-1} \left(\tilde{\Omega}_{t-1} W_{t-1}\right)' - R_{\psi,t-1}\right]$$

where ϕ denotes the true policy coefficients. This can be rearranged to yield

$$\psi_t = \psi_{t-1} + \delta_t R_{\psi,t-1}^{-1} W_{t-1} W_{t-1}' \tilde{\Omega}_{t-1}' \left(\phi - \psi_{t-1} + \epsilon_t\right)$$

$$R_{\psi,t} = R_{\psi,t-1} + \delta_t \left(W_{t-1} W_{t-1}' \tilde{\Omega}_{t-1}' - R_{\psi,t-1}\right).$$

The corresponding ODE is

$$\dot{\psi} = R_{\psi}^{-1} M \tilde{\Omega}' \left(\phi - \psi\right)$$

$$\dot{R}_{\psi} = \left(M \tilde{\Omega}' - R_{\psi}\right)$$

where $M = E_{t \to \infty} W_t W_t'$. Hence, we have that $R_\psi \to M\tilde{\Omega}'$. Substituting in the above we obtain $\phi \to \psi$. Consider the other coefficients. The updating mechanism is

$$\tilde{\Omega}_t' = \tilde{\Omega}_{t-1}' + \delta_t R_{t-1}^{-1} W_{t-1} \left[\tilde{V}_t - \tilde{\Omega}_{t-1} W_{t-1}\right]' \tag{3.83}$$

$R_t = R_{t-1} + \delta_t \left(W_{t-1}W'_{t-1} - R_{t-1}\right).$

It is possible to express (3.83) as

$$\tilde{\Omega}'_t = \tilde{\Omega}'_{t-1} + \delta_t R_{t-1}^{-1} W_{t-1} \left[T'\left(\tilde{\Omega}_{t-1}, \psi_{t-1}\right) W_{t-1} - \tilde{\Omega}_{t-1} W_{t-1} + \zeta_t\right]' \quad (3.84)$$

where

$$T'\left(\Omega_{t-1}, \psi_{t-1}\right) = \left(B_1 \psi'_{t-1} \tilde{\Omega}_{t-1} + B_2 \tilde{\Omega}_{t-1} + B_3 \psi'_{t-1} \tilde{\Omega}_{t-1} H + B_4 \tilde{\Omega}_{t-1} H + \tilde{B}_5\right)$$

$$\begin{aligned} \tilde{V}_t &= B_0 + B_1 \psi'_{t-1} E_{t-1}^{CB} \tilde{V}_t + B_2 E_{t-1}^{PS} \tilde{V}_t + && (3.85) \\ &\quad + B_3 \psi'_{t-1} E_{t-1}^{CB} \tilde{V}_{t+1} + B_4 E_{t-1}^{PS} \tilde{V}_{t+1} + B_5 X_{t-1} + \zeta_t && (3.86) \\ i_t &= \bar{\imath} + \phi_\pi E_{t-d}^{CB} (\pi_t - \pi^*) + \phi_x E_{t-d}^{CB} (x_t - x^*) + \epsilon_t \end{aligned}$$

It is then possible to rearrange (3.84) into

$$\tilde{\Omega}'_t = \tilde{\Omega}'_{t-1} + \delta_t R_{t-1}^{-1} W_{t-1} W'_{t-1} \left[T\left(\tilde{\Omega}_{t-1}, \psi_{t-1}\right) - \tilde{\Omega}'_{t-1} + \zeta'_t\right].$$

The associated ODE can be calculated as

$$\frac{d\tilde{\Omega}'}{d\tau} = \left[T\left(\tilde{\Omega}, \phi\right) - \tilde{\Omega}'\right]$$

where I use the fact that, a) $R \to M$; b) $\phi \to \psi$. It is straightforward to show that stability under learning depends on the eigenvalues of the following matrix

$$\tilde{B}_1 + \tilde{B}_2 - I_2 = \begin{bmatrix} -\phi_x \tilde{\sigma} & \tilde{\sigma}(1 - \phi_\pi) \\ \kappa(1 + \eta_3 \phi_y) & \beta + \kappa \eta_3 \phi_\pi - 1 \end{bmatrix} \quad (3.87)$$

and

$$\tilde{B}_1 + \rho_i \tilde{B}_2 - I_2 \quad (3.88)$$

where

$$\tilde{B}_1 = B_1 \phi' + B_2; \ \tilde{B}_2 = B_3 \phi' + B_4$$

In order to have negative eigenvalues, both the trace and the determinant should be negative. Consider the case $U_{cm} = 0$. It is straightforward to show that the eigenvalues of the matrix (3.87) are negative provided the Taylor Principle holds. Also it is possible to show that if the matrix (3.88) satisfies this property, then also matrix (3.87) will satisfy it. Furthermore, the determinant of $\tilde{B}_1+\tilde{B}_2-(1+\lambda)\,I_2$ vanishes if the Taylor condition holds with equality. Hence the eigenvalues are real and no Hopf bifurcation occur.

(ii) .In the case of perfect transparency and SG learning it is easy to verify that the associated ODE becomes

$$\frac{d\tilde{\Omega}'}{d\tau} = M\left[\tilde{\Omega}'\tilde{B}_1' + H\tilde{\Omega}'\tilde{B}_2' - \tilde{\Omega}'\right]$$

by vectorizing and transposing we obtain

$$vec\left(\dot{\Omega}\right) = ((M \otimes I_3))\left[\left(I \otimes \tilde{B}_1\right) + \left(H \otimes \tilde{B}_2\right) - I_9\right] vec\,(\Omega)\,.$$

Using the fact that M is diagonal I can re-express the matrix as

$$\begin{array}{l}\left(\tilde{B}_1 + \tilde{B}_2 - I_3\right)\\ m_1\left(\tilde{B}_1 + \rho_r\tilde{B}_2 - I_3\right)\\ m_2\left(\tilde{B}_1 + \rho_u\tilde{B}_2 - I_3\right)\end{array}$$

where I need to adjust notation for the constant and m_1,m_2 are the elements of M on the diagonal. Given the fact that by definition m_1,m_2 are positive, the stability condition is identical to the case with RLS. Consider the case of imperfect transparency. With SG, it is possible to show that the linearized ODE becomes

$$\left[\begin{array}{c}\dot{\psi}\\ vec(\dot{\Omega}')\end{array}\right] = \left[\begin{array}{ll}-M\tilde{\Omega}' & (\phi-\psi)'\otimes M\\ B_1\otimes M\tilde{\Omega}' + B_3\otimes MH\tilde{\Omega}' & CC\end{array}\right]\left[\begin{array}{c}\psi\\ vec\left(\tilde{\Omega}'\right)\end{array}\right]$$

$$CC = B_1\psi'\otimes M + B_2\otimes M + B_3\psi'\otimes MH + B_4\otimes MH - I\otimes M$$

evaluating the equation at the REE coefficients we get

$$\left[\begin{array}{c}\dot{\psi}\\ vec(\dot{\Omega}')\end{array}\right] = BB\left[\begin{array}{c}\psi\\ vec\,(\Omega')\end{array}\right]$$

$$BB = \left[\begin{array}{ll}-M\tilde{\Omega}^{*\prime} & 0\\ B_1\otimes M\tilde{\Omega}^{*\prime} + B_3\otimes MH\tilde{\Omega}^{*\prime} & \tilde{B}_1\otimes M + \tilde{B}_2\otimes HM - I\otimes M\end{array}\right].$$

Stability conditions depend on the eigenvalues of the matrices $-M\tilde{\Omega}^{*\prime}$ and $\tilde{B}_1\otimes M + \tilde{B}_2\otimes HM - I\otimes M$. Hence, from the results in (ii), the Taylor principle it is not sufficient for stability of the REE.

(iii) Consider the case where $U_{cm} > 0$. In this case the trace is negative if $\phi_x\tilde{\sigma} + 1 - \beta - \kappa\eta_3\phi_\pi > 0$. This implies that, also in the case of full transparency, a policy rule that does not react to the output gap is destabilizing. Nevertheless transparency increases the set of rules that are robust to expectational mistakes. In order to show this, notice that (3.80) implies

$$\phi_x\tilde{\sigma} + 1 - \beta - \kappa\eta_3\phi_\pi > \tilde{\sigma}\frac{\kappa + \kappa(\phi_\pi - 1) + \phi_x(1-\beta) - \kappa\eta_3\phi_x}{2-\beta} > 0$$

which is a more stringent condition for stability than in the case of full transparency. Combining the stability conditions for the case of transparency, I obtain the condition (3.30) ■

Proof. Proposition (37)

(i) The characteristic equation in the case of no transparency can be written as

$$P(\lambda) = (1+\lambda)\left(\lambda^2 + a_1 + a_0\right)$$

where

$$\begin{aligned} a_1 &= \phi_x\tilde{\sigma} + (1-\beta) - \kappa\eta_3\phi_\pi \\ a_0 &= \kappa(\phi_\pi - 1) + \phi_x(1-\beta) - \kappa\eta_3\phi_x. \end{aligned}$$

It can be easily verified that the same condition can be found imposing full transparency, following the same steps as in the proof above.

(ii) This is equivalent to setting $\phi_x = 0$ in (3.5).

(iii) I assume $\rho_r = \rho_u = 0$, in order to simplify the proof. Consistently with the findings in the previous proofs, I expect the result would not change for positive autoregressive components. Under the current assumptions, the stability under learning depends on the eigenvalues of (3.73).

(iv) Under the current assumptions, the stability under learning depends on the eigenvalues of (3.87). ■

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