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On the long time behaviour of stochastic vortices systems

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Abstract

In this paper, we are interested in the long-time behaviour of stochastic systems of n interacting vortices: the position in $\mathbb{R}^2$ of each vortex evolves according to a Brownian motion and a drift summing the influences of the other vortices computed through the Biot and Savart kernel and multiplied by their respective vorticities. For fixed n, we perform the rescalings of time and space used successfully by Gallay and Wayne [5] to study the long-time behaviour of the vorticity formulation of the two dimensional incompressible Navier-Stokes equation, which is the limit as $n \to \infty$ of the weighted empirical measure of the system under mean-field interaction. When all the vorticities share the same sign, the $2n$-dimensional process of the rescaled positions of the vortices is shown to converge exponentially fast as time goes to infinity to some invariant measure which turns out to be Gaussian if all the vorticities are equal. In the particular case $n = 2$ of two vortices, we prove exponential convergence in law of the 4-dimensional process to an explicit random variable, whatever the choice of the two vorticities. We show that this limit law is not Gaussian when the two vorticities are not equal.

Keywords : vortices, stochastic differential equation, long-time behavior, Lyapunov function, logarithmic Sobolev inequality.

AMS 2010 subject classifications : 76D17, 60H10, 37A35, 26D10.

1 Introduction

In this work, we are interested in stochastic systems of interacting vortices :

$$X_t^i = X_0^i + \sqrt{2\nu}\, W_t^i + \int_0^t \sum_{\substack{j=1 \\ j \neq i}}^n a_j K(X_s^i - X_s^j)ds,\ 1 \leq i \leq n \tag{1.1}$$

where $K : x = (x_1, x_2) \in \mathbb{R}^2 \to \frac{x^\perp}{2\pi|x|^2}$ $(x^\perp = (-x_2, x_1))$ denotes the Biot and Savart kernel, $(X_0^i)_{i \geq 1}$ are two-dimensional random vectors independent from the sequence $(W^i)_{i \geq 1}$ of independent two-dimensional Brownian motions. For $i \in \{1, \ldots, n\}$, the real number a_i is the intensity or vorticity of the i-th vortex. The Biot and Savart kernel K is singular at

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the origin but locally bounded and Lipschitz continuous on $\mathbb{R}^2 \setminus \{0\}$. Under the assumption $\mathbb{P}(\exists i \neq j \text{ s.t. } X_0^i = X_0^j) = 0$ that will be made throughout the paper, existence and uniqueness results for this $2n$-dimensional stochastic differential equation are given in [19, 15, 14, 2]. Moreover, it is shown in [15] that for $t > 0$, the random vector $X_t = (X_t^1, \ldots, X_t^n)$ has a density $\rho_t(x)$.

The relation between System (1.1) and the vorticity formulation

$$\partial_t w(t, x) = \nu \Delta w(t, x) - \nabla. \left(w(t, x) \int_{\mathbb{R}^2} K(x - y) w(t, y) dy \right) \tag{1.2}$$

of the two dimensional incompressible Navier-Stokes equation is well known and has been studied by several authors. It arises through the propagation of chaos property, stating in particular that after a suitable normalization of the vortex intensities a_i, some weighted empirical measure of the system (1.1) or some variant of it involving a regularized version of the kernel K, converges in law as n tends to ∞ to w_t, see [9, 10] for the regularized case and [16, 4] that deal with the true Biot and Savart kernel.

On the other hand, the long time behavior for the two dimensional vortex equation has been successfully studied in Gallay and Wayne [5], who established the strong convergence of a time-space rescaled version of its solution w_t as $t \to \infty$ to a gaussian density with total mass given by the initial circulation $\int_{\mathbb{R}^2} w_0(x) dx$.

Motivated by that result, our goal in this paper is to explore some of the asymptotic properties of System (1.1) as time tends to ∞. To that end, following the scaling introduced in [5], we define $Z_t^i = e^{-\frac{t}{2}} X_{e^t-1}^i$. The process $(Z_t = (Z_t^1, \ldots, Z_t^n))_{t \geq 0}$ solves

$$Z_t^i = X_0^i + \sqrt{2\nu} B_t^i + \int_0^t \left(\sum_{\substack{j=1 \\ j \neq i}}^n a_j K(Z_s^i - Z_s^j) - \frac{Z_s^i}{2} \right) ds, \ 1 \leq i \leq n \tag{1.3}$$

where $\left(B_t^i = \int_0^{e^t-1} \frac{dW_s^i}{\sqrt{s+1}} \right)_{i \geq 1}$ are independent two dimensional Brownian motions according to the Dambin-Dubins-Schwarz theorem. This consequence of the homogeneity of the Biot and Savart kernel can easily be checked by computing $\frac{X_u^i}{\sqrt{u+1}}$ by Itô's formula and then choosing $u = e^t - 1$. For $t > 0$, Z_t admits the density

$$p_t(z) = e^{nt} \rho_{e^t-1}(z e^{t/2}) \tag{1.4}$$

with respect to the Lebesgue measure on $\mathbb{R}^{2n}$. Moreover, the Fokker-Planck equation

$$\partial_t p_t(z) = \nu \triangle p_t(z) - \sum_{i=1}^n \left(\sum_{\substack{j=1 \\ j \neq i}}^n a_j K(z^i - z^j) \right) \cdot \nabla_{z^i} p_t(z) + \frac{1}{2} \nabla_z.(z p_t(z)), \ t > 0, z \in \mathbb{R}^{2n}. \tag{1.5}$$

holds in the weak sense.

The outline of the paper and the results that we obtain are the following. In Section 2, we consider System (1.3) with vorticities a_i of constant sign, and show in Subsection 2.1 by using Lyapunov-Foster-Meyn-Tweedie techniques the exponential convergence in total variation norm to some invariant law with a positive density with respect to the Lebesgue measure. In Subsection 2.2, we study the more restrictive case of particles having equal vorticities. In that case, we show that the invariant measure is the standard $2n$ dimensional normal law scaled by the viscosity coefficient. Using the Logarithmic Sobolev inequality satisfied by this invariant

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measure, we deduce that the relative entropy of the law of the particle positions with respect to that Gaussian law goes exponentially fast to 0, at an explicit rate independent of n. Although the techniques employed in the proof of these two results are standard, the singularity of the drift requires some specific adaptations by regularization. These results are also translated into the original time-space scale. We have not been able to study by similar techniques the long-time behaviour when the vorticities are allowed to have different signs. In order to gain some insight on the difficulties that the long time behavior raises in that case, we consider in Section 3 the particular case of $n = 2$ vortices with arbitrary intensities. We completely describe the equilibrium law in $\mathbb{R}^4$ in terms of the stationary solution of some related stochastic differential equation. In particular, we prove that, although some linear combinations of the two particles positions are Gaussian under the stationary measure, this stationary measure is not jointly Gaussian unless the intensities are equal. We moreover show exponential convergence to this limiting law in suitable Wasserstein distances. The argument for long-time convergence, which is based on coupling and time reversal techniques, is original to our knowledge and could be of interest in different contexts.

Notation

- For $d \in \mathbb{N}^*$ and $\alpha \geq 1$, let W_α denote the Wasserstein metric on the space of probability measures on $\mathbb{R}^d$ defined by

$$W_\alpha(\mu, \nu) = \inf_{\rho < \frac{\mu}{\nu}} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^\alpha \rho(dx, dy) \right)^{1/\alpha},$$

  where the infimum is computed over all measures ρ on $\mathbb{R}^d \times \mathbb{R}^d$ with first marginal equal to μ and second marginal equal to ν and $|x - y|$ denotes the Euclidean norm of $x - y \in \mathbb{R}^d$.

- To deal with the singularity of the Biot and Savart kernel, we construct smooth approximations of this kernel which coincide with it away from 0. Let φ be a smooth function on $\mathbb{R}_+$ such that

$$\varphi(r) = \begin{cases} \frac{1}{2} \text{ for } r \leq \frac{1}{2} \\ r \text{ for } r \geq 1 \end{cases},$$

  and for $\varepsilon > 0$, $\varphi_\varepsilon(r) = \varepsilon \varphi(r/\varepsilon)$. The kernel $K_\varepsilon(z) \stackrel{\text{def}}{=} \frac{1}{2\pi} \nabla_z^\perp \ln(\varphi_\varepsilon(|z|))$ coincides with K for $|z| \geq \varepsilon$ and is divergence-free, globally Lipschitz continuous and bounded.

2 Case of vorticities $(a_i)_{1 \leq i \leq n}$ with constant sign

In the present section, we assume that either $\forall i \in \{1, \ldots, n\}$, $a_i > 0$ or $\forall i \in \{1, \ldots, n\}$, $a_i < 0$. Let $\mathcal{Z} = \{z = (z^1, \ldots, z^n) \in \mathbb{R}^{2n} : \forall i \neq j, z^i \neq z^j\}$. For any $z \in \mathcal{Z}$, the unique strong solution (Z_t^z) of the SDE (1.3) starting from $(Z_0^1, \ldots, Z_0^n) = z$ is such that $\tau^z = \inf\{t \geq 0 : \exists i \neq j, Z_t^i = Z_t^j\}$ is a.s. infinite. Since the process $(\sqrt{1+t}(Z_{\ln(1+t)}^1, \ldots, Z_{\ln(1+t)}^n))_{t \geq 0}$ solves (1.1), this is in particular a consequence of [19].

2.1 Convergence to equilibrium

Proposition 2.1 *The SDE* (1.3) *admits a unique invariant probability measure. This measure admits a positive density p_∞ with respect to the Lebesgue measure and there exist constants*

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$C, \lambda \in (0, +\infty)$ depending on ν, n and $(a_1, \cdots, a_n)$ such that

$$\forall z \in \mathcal{Z}, \quad \sup_{f: \forall x, |f(x)| \leq 1 + V(x)} \left| \mathbb{E}[f(Z_t^z)] - \int_{\mathbb{R}^{2n}} f(x) p_\infty(x) dx \right| \leq C e^{-\lambda t} (1 + V(z)),$$

where

$$V(z) = \sum_{i=1}^n |a_i| |z_i|^2. \tag{2.1}$$

According to (1.4), the density of the original particle system $(X_s^1, \ldots, X_s^n)$ is

$$\rho_s(x) = (1 + s)^{-n} p_{\ln(1+s)} \left(\frac{x}{\sqrt{1+s}} \right). \tag{2.2}$$

We deduce that

Corollary 2.2 *Let* $x = (x^1, \ldots, x^n) \in \mathcal{Z}$. *Denoting by* X_t^x *the solution of* (1.1) *starting from* $(X_0^1, \ldots, X_0^n) = (x^1, \ldots, x^n)$, *one has*

$$\sup_{f: \forall y, |f(y)| \leq 1 + V(y/\sqrt{1+t})} \left| \mathbb{E}[f(X_t^x)] - \int_{\mathbb{R}^{2n}} \frac{f(y)}{(1+t)^n} p_\infty \left(\frac{y}{\sqrt{1+t}} \right) dy \right| \leq \frac{C}{(1+t)^\lambda} \left(1 + V \left(\frac{x}{\sqrt{1+t}} \right) \right).$$

All the statements but the existence of the positive density p_∞ are a consequence of Theorem 6.1 in [11] which supposes a Lyapunov condition and that all compact sets of $\mathcal{Z}$ are petite sets for some skeleton chain. Let us first check that the function V defined by (2.1) is a Lyapunov function. Denoting by

$$L = \nu \Delta_z - \frac{z}{2} . \nabla_z + \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n a_j K(z^i - z^j) . \nabla_{z^i}$$

the infinitesimal generator associated with (1.3), one has

$$LV(z) = 4\nu \sum_{i=1}^n |a_i| - V(z) + 2 \sum_{j \neq i} a_i a_j K(z^i - z^j) . z^i = 4\nu \sum_{i=1}^n |a_i| - V(z) + \sum_{j \neq i} a_i a_j \frac{(z^i - z^j)^\perp . (z^j - z^i)}{2\pi |z^i - z^j|^2}$$

$$= 4\nu \sum_{i=1}^n |a_i| - V(z)$$

where we used the oddness of the Biot and Savart kernel K for the second equality. Hence

$$LV(z) \leq \left(4\nu \sum_{i=1}^n |a_i| \right) 1_{\{\sum_{i=1}^n |a_i| |z_i|^2 \leq 8\nu \sum_{i=1}^n |a_i|\}} - \frac{V(z)}{2}.$$

and condition (CD3) in Theorem 6.1 [11] is satisfied. So it only remains to check that all compact sets of $\mathcal{Z}$ are petite sets for some skeleton chain to apply this theorem. This is a consequence of Proposition 6.2.8 [12] and of the next Lemma which implies that any skeleton chain is Lebesgue measure-irreducible and that the invariant probability measure admits a positive density.

Lemma 2.3 *The semi-group* $(P_t)_{t \geq 0}$ *defined by* $P_t f(z) = \mathbb{E}[(f(Z_t^z))]$ *is Feller and for any* $z \in \mathcal{Z}$ *and any* $t > 0$, Z_t^z *admits a positive density with respect to the Lebesgue measure.*

Proof . Let us check by probabilistic estimations that the semi-group $(P_t)_{t\geq 0}$ is Feller and first that for $f : \mathbb{R}^{2n} \to \mathbb{R}$ bounded and going to 0 at infinity, so does $P_t f$.

Let $R_t^z = \sum_{i=1}^n |a_i| |Z_t^{z,i}|^2$. By Itô's formula,

$$dR_t^z = 2\sqrt{2\nu} \sum_{i=1}^n |a_i| Z_t^{z,i}.dB_t^i - R_t^z dt + 4\nu \sum_{i=1}^n |a_i| dt. \tag{2.3}$$

$$d\frac{1}{1+R_t^z} = -\frac{2\sqrt{2\nu}}{(1+R_t^z)^2} \sum_{i=1}^n |a_i| Z_t^{z,i}.dB_t^i + \frac{R_t^z}{(1+R_t^z)^2}dt - \frac{4\nu \sum_{i=1}^n |a_i|}{(1+R_t^z)^2}dt + \frac{8\nu}{(1+R_t^z)^3} \sum_{i=1}^n a_i^2 |Z_t^{z,i}|^2 dt.$$

The stochastic integral $\left(\int_0^t \frac{1}{(1+R_s^z)^2} \sum_{i=1}^n |a_i| Z_s^{z,i}.dB_s^i \right)_{t\geq 0}$ is a locally in time square integrable martingale. Denoting by $\bar{a} = \max_{1\leq i\leq n} |a_i|$, remarking that $\sum_{i=1}^n a_i^2 |Z_t^{z,i}|^2 \leq \bar{a} R_t^z$ and taking expectations, we deduce that

$$\frac{d}{dt}\mathbb{E}\left[\frac{1}{1+R_t^z}\right] \leq \mathbb{E}\left[\frac{R_t^z}{(1+R_t^z)^2} + \frac{8\nu\bar{a}R_t^z}{(1+R_t^z)^3}\right] \leq (1+8\nu\bar{a})\mathbb{E}\left[\frac{1}{1+R_t^z}\right].$$

Therefore

$$\mathbb{E}\left[\frac{1}{1+\bar{a}|Z_t^z|^2}\right] \leq \mathbb{E}\left[\frac{1}{1+R_t^z}\right] \leq \frac{1}{1+\sum_{i=1}^n |a_i||z^i|^2}e^{(1+8\nu\bar{a})t} \leq \frac{1}{1+|z|^2 \min_{1\leq i\leq n}|a_i|}e^{(1+8\nu\bar{a})t}. \tag{2.4}$$

For $r \in (0,+\infty)$, since by Markov inequality, $\mathbb{P}(|Z_t^z| \leq r) \leq \mathbb{E}\left[\frac{1+\bar{a}r^2}{1+\bar{a}|Z_t^z|^2}\right]$, we deduce that $\lim_{|z|\to+\infty} \mathbb{P}(|Z_t^z| \leq r) = 0$. As a consequence, for any bounded function f on $\mathbb{R}^{2n}$ going to 0 at infinity, $\lim_{|z|\to\infty} P_t f(z) = 0$.

Let $f : \mathbb{R}^{2n} \to \mathbb{R}$ be continuous and bounded. To check that $z \in \mathcal{Z} \mapsto P_t f(z)$ is continuous, we introduce for $\varepsilon > 0$ a bounded and smooth kernel $K_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ which coincides with the Biot and Savart kernel K on $\{x \in \mathbb{R}^2 : |x| \geq \varepsilon\}$ and define $Z^{z,\varepsilon} = (Z^{z,\varepsilon,1}, \ldots, Z^{z,\varepsilon,1})$ as the solution of the SDE

$$Z_t^{z,\varepsilon,i} = z^i + \sqrt{2\nu}B_t^i + \int_0^t \left(\sum_{\substack{j=1 \\ j\neq i}}^n a_j K_\varepsilon(Z_s^{z,\varepsilon,i} - Z_s^{z,\varepsilon,j}) - \frac{Z_s^{z,\varepsilon,i}}{2} \right)ds, \ 1 \leq i \leq n. \tag{2.5}$$

Since K_ε is Lipschitz continuous, one easily checks that $|Z_t^{z,\varepsilon} - Z_t^{z',\varepsilon}| \leq |z - z'|e^{C_\varepsilon t}$ for some deterministic finite constant C_ε depending on ε but not on $z, z' \in \mathbb{R}^{2n}$. Hence $z \mapsto \mathbb{E}[f(Z_t^{z,\varepsilon})]$ is continuous. To deduce the continuity of $z \mapsto \mathbb{E}[f(Z_t^z)]$, we are going to control the stopping times

$$\tau^{z,\varepsilon} = \inf\{t \geq 0 : \exists i \neq j : |Z_t^{z,i} - Z_t^{z,j}| \leq \varepsilon\}.$$

Notice that for $z \in \mathcal{Z}$, the processes Z^z and $Z^{z,\varepsilon}$ coincide on $[0, \tau^{z,\varepsilon}]$. Therefore,

$$\forall z, z' \in \mathcal{Z}, \ |P_t f(z) - P_t f(z')| \leq |\mathbb{E}[f(Z_t^{z,\varepsilon}) - f(Z_t^{z',\varepsilon})]| + 2\|f\|_\infty \left(\mathbb{P}(\tau^{z,\epsilon} < t) + \mathbb{P}(\tau^{z',\epsilon} < t) \right). \tag{2.6}$$

By continuity of the paths of Z^z, for $z \in \mathcal{Z}$, $\lim_{\varepsilon\to 0} \tau^{z,\varepsilon} = \tau^z = +\infty$ a.s. which implies that $\lim_{\varepsilon\to 0} \mathbb{P}(\tau^{z,\epsilon} < t) = 0$. But we need some uniformity in the starting point z to deduce that $z \in \mathcal{Z} \mapsto P_t f(z)$ is continuous. The following computations inspired from [19] improve the result

therein into a quantitative estimate. By Itô's formula and since

$$\sum_{i\neq j} a_i a_j \frac{Z_t^{z,i} - Z_t^{z,j}}{|Z_t^{z,i} - Z_t^{z,j}|^2} \cdot \left(\sum_{\substack{k=1 \\ k\neq i}}^n a_k \frac{(Z_t^{z,i} - Z_t^{z,k})^\perp}{|Z_t^{z,i} - Z_t^{z,k}|^2} - \sum_{\substack{l=1 \\ l\neq j}}^n a_l \frac{(Z_t^{z,j} - Z_t^{z,l})^\perp}{|Z_t^{z,j} - Z_t^{z,l}|^2} \right)$$

$$= 2 \sum_{i=1}^n a_i \left(\sum_{\substack{j=1 \\ j\neq i}}^n a_j \frac{Z_t^{z,i} - Z_t^{z,j}}{|Z_t^{z,i} - Z_t^{z,j}|^2} \right) \cdot \left(\sum_{\substack{k=1 \\ k\neq i}}^n a_k \frac{Z_t^{z,i} - Z_t^{z,k}}{|Z_t^{z,i} - Z_t^{z,k}|^2} \right)^\perp = 0,$$

one obtains

$$d \sum_{i\neq j} a_i a_j \ln |Z_t^{z,i} - Z_t^{z,j}| = \sqrt{2\nu} \sum_{i\neq j} a_i a_j \frac{Z_t^{z,i} - Z_t^{z,j}}{|Z_t^{z,i} - Z_t^{z,j}|^2} \cdot (dB_t^i - dB_t^j) - \frac{1}{2} \sum_{i\neq j} a_i a_j dt$$

Hence

$$\mathbb{E}\left[\sum_{i\neq j} a_i a_j \ln |Z_{\tau^{z,\varepsilon}\wedge t}^{z,i} - Z_{\tau^{z,\varepsilon}\wedge t}^{z,j}| \right] = \sum_{i\neq j} a_i a_j \ln |z^i - z^j| - \frac{1}{2} \sum_{i\neq j} a_i a_j \mathbb{E}[\tau^{z,\varepsilon} \wedge t] \geq \sum_{i\neq j} a_i a_j \left(\ln |z^i - z^j| - \frac{t}{2} \right).$$

Since $\forall x > 0, \ln(x) < x$, for $\varepsilon \in (0,1)$, the left-hand-side is not greater than

$$\min_{i\neq j} a_i a_j \ln(\varepsilon) \mathbb{P}(\tau^{z,\varepsilon} \leq t) + \sum_{i\neq j} a_i a_j \mathbb{E}\left[\sup_{s\in[0,t]} |Z_s^{z,i} - Z_s^{z,j}| \right]$$

$$\leq \min_{i\neq j} a_i a_j \ln(\varepsilon) \mathbb{P}(\tau^{z,\varepsilon} \leq t) + 2 \sum_{i=1}^n a_i \left(\sum_{\substack{j=1 \\ j\neq i}}^n a_j \right) \mathbb{E}\left[\sup_{s\in[0,t]} |Z_s^{z,i}| \right]$$

$$\leq \min_{i\neq j} a_i a_j \ln(\varepsilon) \mathbb{P}(\tau^{z,\varepsilon} \leq t) + 2 \sum_{i=1}^n \sqrt{|a_i|} \left(\sum_{\substack{j=1 \\ j\neq i}}^n |a_j| \right) \mathbb{E}^{1/4}\left[\sup_{s\in[0,t]} (R_s^z)^2 \right].$$

Hence

$$\mathbb{P}(\tau^{z,\varepsilon} \leq t) \leq \frac{\sum_{i\neq j} a_i a_j \left(\frac{t}{2} - \ln |z^i - z^j| \right) + 2 \sum_{i=1}^n \sqrt{|a_i|} \left(\sum_{j\neq i} |a_j| \right) \mathbb{E}^{1/4}\left[\sup_{s\in[0,t]} (R_s^z)^2 \right]}{\min_{i\neq j} a_i a_j \ln(1/\varepsilon)}$$

$$(2.7)$$

By a standard localisation procedure, one checks that the stochastic integral the differential of which appears in the right-hand-side of (2.3) is a martingale and that

$$\mathbb{E}[R_t^z] = e^{-t} \sum_{i=1}^n |a_i||z^i|^2 + (1 - e^{-t}) 4\nu \sum_{i=1}^n |a_i|.$$

By (2.3), $R_s^z \leq R_0^z + 4\nu \sum_{i=1}^n |a_i| t + 2\sqrt{2\nu} \int_0^s \sum_{i=1}^n |a_i| Z_u^{z,n,i} . dB_u^i$ and by Doob's inequality,

$$\mathbb{E}\left[\sup_{s\in[0,t]} (R_s^z)^2 \right] \leq 2 \left((\sum_{i=1}^n |a_i|(|z^i|^2 + 4\nu t))^2 + 32\nu \int_0^t \sum_{i=1}^n a_i^2 \mathbb{E}[|Z_s^{z,i}|^2] ds \right)$$

$$\leq 2 \left((\sum_{i=1}^n |a_i|(|z^i|^2 + 4\nu t))^2 + 32\nu\bar{a} \left(\sum_{i=1}^n |a_i||z^i|^2 (1 - e^{-t}) + 4\nu \sum_{i=1}^n |a_i|(t + e^{-t} - 1) \right) \right).$$

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Plugging this estimation in (2.7), we deduce that for $z \in \mathcal{Z}$ and $\alpha > 0$ small enough so that the ball $B(z, \alpha)$ centered in z with radius α is contained in $\mathcal{Z}$, one has $\lim_{\varepsilon \to 0} \sup_{z' \in B(z, \alpha)} \mathbb{P}(\tau^{z', \varepsilon} \leq t) = 0$. With (2.6) and the continuity of $z \mapsto \mathbb{E}[f(Z_t^{z, \varepsilon})]$ for fixed $\varepsilon > 0$, we conclude that $z \mapsto P_t f(z)$ is continuous.

Last, by Theorem 2 and Example 2 [15], for $t > 0$, $\sqrt{1 + t} Z_{\ln(1+t)}^z$ admits a density satisfying some Gaussian lower bound. This implies that for $t > 0$, Z_t^z admits a positive density. Notice that one could also deduce the Feller property of the semi-group from the estimations of the fundamental solution of $\frac{\partial}{\partial t} - L$ obtained by partial differential equation's techniques in that paper but we preferred to give a probabilistic argument.

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2.2 Case of particles with equal vorticity

In the present subsection, we assume the existence of $a \in \mathbb{R}^*$ such that $a_i = a$ for all $i \in \{1, \ldots, n\}$. For instance, when w_0 is a non-negative (resp. non-positive) initial vorticity density on $\mathbb{R}^2$ with positive and finite total mass $\|w_0\|_1$, it is natural to choose the initial positions X_0^i i.i.d. according to the probability density $\frac{|w_0|}{\|w_0\|_1}$ and $a = \frac{\|w_0\|_1}{n}$ (resp. $a = -\frac{\|w_0\|_1}{n}$). In this situation, the invariant density turns out to be Gaussian and we explicit the exponential rate of convergence to equilibrium. In fact, both the invariant density and the rate of convergence are the same as for the Ornstein Uhlenbeck dynamics given by (1.3) in the case without interaction: $a_i = 0$ for all $i \in \{1, \ldots, n\}$.

Proposition 2.4 *Assume the existence of $a \in \mathbb{R}^*$ such that $a_i = a$ for all $i \in \{1, \ldots, n\}$. The density $p_\infty(z^1, \ldots, z^n) = \frac{1}{(4\pi\nu)^n} e^{-\frac{1}{4\nu} \sum_{i=1}^n |z^i|^2}$ of the normal law $\mathcal{N}_{2n}(0, 2\nu I_{2n})$ is invariant for the SDE (1.3). Moreover, if $(X_0^1, \ldots, X_0^n)$ has a density p_0 with respect to the Lebesgue measure on $\mathbb{R}^{2n}$ such that the relative entropy $\int_{\mathbb{R}^{2n}} p_0 \ln\left(\frac{p_0}{p_\infty}\right)$ is finite. Then, one has*

$$\forall t \geq 0, \quad \int_{\mathbb{R}^{2n}} p_t \ln\left(\frac{p_t}{p_\infty}\right) \leq e^{-t} \int_{\mathbb{R}^{2n}} p_0 \ln\left(\frac{p_0}{p_\infty}\right).$$

The above rate 1 of exponential convergence is the same as for the Ornstein-Uhlenbeck process obtained when $a = 0$ and depends neither on n nor on ν.
Since, by (2.2), the density ρ_s of the orginal particle system $(X_s^1, \ldots, X_s^n)$ is such that

$$\int_{\mathbb{R}^{2n}} \rho_s \ln\left(\frac{\rho_s}{(1+s)^{-n} p_\infty\left(\frac{\cdot}{\sqrt{1+s}}\right)}\right) = \int_{\mathbb{R}^{2n}} p_{\ln(1+s)} \ln\left(\frac{p_{\ln(1+s)}}{p_\infty}\right),$$

one easily deduces its asymptotic behaviour as $s \to +\infty$.

Corollary 2.5 *Assume that $(X_0^1, \ldots, X_0^n)$ has a density p_0 with respect to the Lebesgue measure on $\mathbb{R}^{2n}$ such that the relative entropy $\int_{\mathbb{R}^{2n}} p_0 \ln\left(\frac{p_0}{p_\infty}\right)$ is finite. Then*

$$\forall s \geq 0, \quad \int_{\mathbb{R}^{2n}} \rho_s \ln\left(\frac{\rho_s}{(1+s)^{-n} p_\infty\left(\frac{\cdot}{\sqrt{1+s}}\right)}\right) \leq \frac{1}{1+s} \int_{\mathbb{R}^{2n}} \rho_0 \ln\left(\frac{\rho_0}{p_\infty}\right) = \frac{1}{1+s} \int_{\mathbb{R}^{2n}} p_0 \ln\left(\frac{p_0}{p_\infty}\right).$$

Remark 2.6 *Assume that $(X_0^1, \ldots, X_0^n)$ is such that $\mathbb{E}(\sum_{i=1}^n |X_0^i|^2) < +\infty$ and let $s > 0$. We have $\mathbb{E}\left(\sum_{i=1}^n |X_s^i|^2\right) = \mathbb{E}(\sum_{i=1}^n |X_0^i|^2) + 2n\nu s$. Moreover, by Theorem 2 and Example 2 [15], $(X_s^1, \ldots, X_s^n)$ admits a density ρ_s bounded by $C_n s^{-n}$ so that*

$$\int_{\mathbb{R}^{2n}} \rho_s \ln\left(\frac{\rho_s}{(1+s)^{-n} p_\infty\left(\frac{\cdot}{\sqrt{1+s}}\right)}\right) \leq \ln(C_n) + n\ln\left(1 + \frac{1}{s}\right) + n\ln(4\pi\nu) + \frac{1}{4\pi\nu(1+s)} \mathbb{E}\left(\sum_{i=1}^n |X_s^i|^2\right) < +\infty.$$

By Proposition 2.4 and Corollary 2.5,

$$\forall t \geq \ln(1+s), \int_{\mathbb{R}^{2n}} p_t \ln\left(\frac{p_t}{p_\infty}\right) \leq (1+s)e^{-t} \int_{\mathbb{R}^{2n}} \rho_s \ln\left(\frac{\rho_s}{(1+s)^{-n} p_\infty\left(\frac{\cdot}{\sqrt{1+s}}\right)}\right)$$

$$\forall t \geq s, \int_{\mathbb{R}^{2n}} \rho_t \ln\left(\frac{\rho_t}{(1+t)^{-n} p_\infty\left(\frac{\cdot}{\sqrt{1+t}}\right)}\right) \leq \frac{1+s}{1+t} \int_{\mathbb{R}^{2n}} \rho_s \ln\left(\frac{\rho_s}{(1+s)^{-n} p_\infty\left(\frac{\cdot}{\sqrt{1+s}}\right)}\right).$$

Notice that $\inf_{s>0}(1+s)\left(\ln(C_n) + n\ln\left(1 + \frac{1}{s}\right) + n\ln(4\pi\nu)\right) + \frac{\mathbb{E}\left(\sum_{i=1}^n |X_0^i|^2\right) + 2n\nu s}{4\pi\nu}$ is attained at a unique point $s_\star > 0$.

Remark 2.7 *Asume that the initial conditions $(X_0^i)_{1 \leq i \leq n}$ are i.i.d. according to some density p_0^1 on $\mathbb{R}^2$ such that $\int_{\mathbb{R}^2} p_0^1 \ln\left(\frac{p_0^1}{p_\infty^1}\right) < +\infty$ where $p_\infty^1(z^1) = \frac{1}{4\pi\nu} e^{-\frac{1}{4\nu}|z^1|^2}$. Then, for all $t \geq 0$, $(Z_t^1, \ldots, Z_t^n)$ is exchangeable and, denoting by $p_t^{1,n}$ the common density of the random vectors Z_t^i, we have*

$$ne^{-t} \int_{\mathbb{R}^2} p_0^1 \ln\left(\frac{p_0^1}{p_\infty^1}\right) = e^{-t} \int_{\mathbb{R}^{2n}} p_0 \ln\left(\frac{p_0}{p_\infty}\right) \geq \int_{\mathbb{R}^{2n}} p_t \ln\left(\frac{p_t}{p_\infty}\right) dz$$

$$= \int_{\mathbb{R}^{2n}} p_t(z) \ln\left(\frac{p_t(z)}{\prod_{i=1}^n p_t^{1,n}(z^i)}\right) dz + \int_{\mathbb{R}^{2n}} p_t(z) \sum_{i=1}^n \ln\left(\frac{p_t^{1,n}(z^i)}{p_\infty^1(z^i)}\right) dz$$

$$\geq 0 + n \int_{\mathbb{R}^2} p_t^1 \ln\left(\frac{p_t^{1,n}}{p_\infty^1}\right).$$

We have $p_t^{1,n}(.) = e^t \rho_{e^t-1}^{1,n}(e^{\frac{t}{2}}.)$, where $\rho_s^{1,n}$ denotes the common density of the vectors X_s^i. By the transport inequality satisfied by the Gaussian density p_∞^1, $\int_{\mathbb{R}^2} p_0^1 \ln\left(\frac{p_0^1}{p_\infty^1}\right) < +\infty$ implies $\int_{\mathbb{R}^2} p_0^1(x)(|\ln(p_0^1(x))| + |x|^2)dx < +\infty$. Hence, by [4], $\rho_s^{1,n}(x)dx$ converges weakly to $w(s,x)dx$ as $n \to \infty$ with $w(s,x)$ denoting the solution of the vorticity formulation (1.2) of the two dimensional incompressible Navier-Stokes equation starting from $w(0,x) = p_0^1(x)$. By the lower semi-continuity of the relative entropy with respect to the weak convergence, we conclude that

$$\forall s \geq 0, \int_{\mathbb{R}^2} w(s,.) \ln\left(\frac{w(s,.)}{(1+s)^{-1} p_\infty^1\left(\frac{\cdot}{\sqrt{1+s}}\right)}\right) \leq \frac{1}{1+s} \int_{\mathbb{R}^2} p_0^1 \ln\left(\frac{p_0^1}{p_\infty^1}\right),$$

which can also be deduced from the proof of Lemma 3.2 [5].

We shall first give formal arguments for Proposition 2.4, which will be made rigorous by replacing K with the regularized kernel K_ε. The results for the system (1.3) will then be justified by suitable passages to the limit as $\varepsilon \to 0$.

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Let us check that p_∞ solves the stationary version of the Fokker-Planck equation (1.5). Since for $i \in \{1, \ldots, n\}$, $\nu \nabla_{z^i} p_\infty + \frac{z^i}{2} p_\infty = 0$, the result follows from

$$\sum_{i=1}^{n} \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} K(z^i - z^j) \right) \cdot \nabla_{z^i} p_\infty = -\frac{p_\infty}{2\nu} \sum_{i \neq j} K(z^i - z^j).z^i = \frac{p_\infty}{2\nu} \sum_{i \neq j} K(z^i - z^j).\frac{z^j - z^i}{2}$$

$$= \frac{p_\infty}{8\pi\nu} \sum_{i \neq j} \frac{(z^i - z^j)^\perp.(z^j - z^i)}{|z^i - z^j|^2} = 0,$$

where we used the oddness of the Biot and Savart kernel K for the second equality.

Remark 2.8 *When the vorticities a_i of the particles are different, p_∞ is no longer a solution of the stationary Fokker-Planck equation. Indeed, $L^* p_\infty(z) = \frac{p_\infty(z)}{2\nu} \sum_{i \neq j} a_j K(z^i - z^j).z^i = \frac{p_\infty(z)}{2\nu} \sum_{i \neq j} (a_j - a_i) \frac{(z^i - z^j)^\perp.(z^i + z^j)}{8\pi|z^i - z^j|^2}$. Supposing for instance that $a_1 < a_2$, and fixing $z^2, \ldots, z^n \in \mathbb{R}^2$ all distinct with $z^2 \neq 0$, one deduces that $L^* p_\infty(z)$ goes to $+\infty$ when $z^1 = z^2 + \lambda z^{2\perp}$ with $\lambda \to 0^+$.*

The following computations about the exponential decay of the relative entropy are inspired from the proofs of Lemma 3.2 [5] and Proposition 8 [7]. Since $\nabla_{z^i}.K(z^i - z^j) = 0$, one can replace the expression $K(z^i - z^j)\nabla_{z^i} p_t$ by $\nabla_{z^i}.\left(K(z^i - z^j)p_t \right)$. Thus, by the Fokker-Planck equation satisfied by p_t, Stokes' formula and remarking that $\int_{\mathbb{R}^{2n}} \nabla_z.(z p_t) \ln(p_t) = -\int_{\mathbb{R}^{2n}} z.\nabla_z p_t = \int_{\mathbb{R}^{2n}} 2n p_t$, one obtains

$$\partial_t \int_{\mathbb{R}^{2n}} p_t \ln\left(\frac{p_t}{p_\infty}\right) = \int_{\mathbb{R}^{2n}} \partial_t p_t \ln\left(\frac{p_t}{p_\infty}\right)$$

$$= \int_{\mathbb{R}^{2n}} \sum_{i=1}^{n} \nabla_{z^i}.\left(\nu \nabla_{z^i} p_t + \frac{z^i}{2} p_t - a \sum_{\substack{j=1 \\ j \neq i}}^{n} K(z^i - z^j)p_t \right) \times (\ln(p_t) - \ln(p_\infty))$$

$$= -\int_{\mathbb{R}^{2n}} \nu \frac{|\nabla_z p_t|^2}{p_t} - n p_t - a \sum_{j \neq i} K(z^i - z^j).\nabla_{z^i} p_t$$

$$+ \int_{\mathbb{R}^{2n}} \left(-a \sum_{j \neq i} K(z^i - z^j).\nabla_{z^i} \ln(p_\infty) + \frac{z}{2}.\nabla_z \ln(p_\infty) - \nu \Delta_z \ln(p_\infty) \right) p_t.$$

Next, dividing the stationary Fokker-Planck equation by p_∞, one obtains

$$\frac{z}{2}.\nabla_z \ln(p_\infty) - a \sum_{j \neq i} K(z^i - z^j).\nabla_{z^i} \ln(p_\infty) = -\nu \left(\Delta_z \ln(p_\infty) + |\nabla_z \ln(p_\infty)|^2 \right) - n. \quad (2.8)$$

Using this and $\int_{\mathbb{R}^{2n}} \sum_{j \neq i} K(z^i - z^j).\nabla_{z^i} p_t = \sum_{i=1}^{n} \int_{\mathbb{R}^{2n}} \nabla_{z^i}.\left(\sum_{\substack{j=1 \\ j \neq i}}^{n} K(z^i - z^j) p_t \right) = 0$ yields

$$\partial_t \int_{\mathbb{R}^{2n}} p_t \ln\left(\frac{p_t}{p_\infty}\right) = -\nu \int_{\mathbb{R}^{2n}} \frac{|\nabla_z p_t|^2}{p_t} + 2\Delta_z \ln(p_\infty) p_t + |\nabla_z \ln(p_\infty)|^2 p_t$$

$$= -\nu \int_{\mathbb{R}^{2n}} \left| \nabla_z \ln\left(\frac{p_t}{p_\infty}\right) \right|^2 p_t \quad (2.9)$$

$$= -4\nu \int_{\mathbb{R}^{2n}} \left| \nabla_z \sqrt{\frac{p_t}{p_\infty}} \right|^2 p_\infty.$$

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The Logarithmic Sobolev Inequality satisfied by the invariant density p_∞ (see [6]) ensures that

$$\int_{\mathbb{R}^{2n}} p_t \ln\left(\frac{p_t}{p_\infty}\right) \leq 4\nu \int_{\mathbb{R}^{2n}} \left|\nabla_z \sqrt{\frac{p_t}{p_\infty}}\right|^2 p_\infty,$$

and one easily concludes by comparison with the ODE $\alpha'(t) = -\alpha(t)$.

Proof of Proposition 2.4. Since the regularized kernel K_ε also satisfies $K_\varepsilon(z^i - z^j) \cdot (z^i - z^j) = 0$, the arguments before Remark 2.8 permit to check that

$$\int_{\mathbb{R}^{2n}} L^\varepsilon f(z) p_\infty(z) dz = 0$$

for each compactly supported smooth function $f : \mathbb{R}^{2n} \to \mathbb{R}$, where L^ε is the generator of the process $Z^\varepsilon = (Z_t^{\varepsilon,1}, \ldots, Z_t^{\varepsilon,n})_{t \geq 0}$ defined by

$$Z_t^{\varepsilon,i} = X_0^i + \sqrt{2\nu} B_t^i + \int_0^t \left(\sum_{\substack{j=1 \\ j \neq i}}^n a_j K_\varepsilon(Z_s^{\varepsilon,i} - Z_s^{\varepsilon,j}) - \frac{Z_s^{\varepsilon,i}}{2}\right) ds, \ 1 \leq i \leq n.$$

This implies by the criterion of Echeverria [1] that the density p_∞ is invariant for the process Z^ε. In particular, for each function f as before, one has

$$\mathbb{E}_\infty(f(Z_t^\varepsilon)) = \int_{\mathbb{R}^{2n}} f(z) p_\infty(z) dz,$$

where $\mathbb{E}_\infty$ denotes the expectation when the initial condition $(X_0^1, \ldots, X_0^n)$ has the density p_∞. Since by the proof of Lemma 2.3, for each $t > 0$, $Z_t^\varepsilon \to Z_t$ in law as $\varepsilon \to 0$ under any initial distribution of $Z_0^\varepsilon = Z_0$, we deduce that

$$\mathbb{E}_\infty(f(Z_t)) = \int_{\mathbb{R}^{2n}} f(z) p_\infty(z) dz$$

for all such f and all $t > 0$ which entails the desired result. Since the invariant density p_∞ of Z^ε has a locally integrable gradient, assumptions $H1)$, $H2)_{p_\infty}$, $H3)_{p_\infty}$ in [3] are satisfied. For any initial condition $(X_0^1, \ldots, X_0^n)$, Z_t^ε admits a positive density by the boundedness of K_ε and Girsanov theorem. Let p_0 be a density on $\mathbb{R}^{2n}$ and p_t^ε denote the density of Z_t^ε when $(X_0^1, \ldots, X_0^n)$ has the density p_0. Let us first suppose that $\frac{p_0}{p_\infty}$ has polynomial growth. By Remark 2.5 [3], the densities p_t^ε then satisfy assumption $H3)_{p^\varepsilon}$. By Theorem 2.4 and Corollary 2.6 in [3] we obtain for all $t \geq 0$,

$$\partial_t \int_{\mathbb{R}^{2n}} p_t^\varepsilon \ln\left(\frac{p_t^\varepsilon}{p_\infty}\right) \leq -4\nu \int_{\mathbb{R}^{2n}} \left|\nabla_z \sqrt{\frac{p_t^\varepsilon}{p_\infty}}\right|^2 p_\infty.$$

The Logarithmic Sobolev Inequality ensures that

$$\forall t \geq 0, \ \int_{\mathbb{R}^{2n}} p_t^\varepsilon \ln\left(\frac{p_t^\varepsilon}{p_\infty}\right) \leq e^{-t} \int_{\mathbb{R}^{2n}} p_0 \ln\left(\frac{p_0}{p_\infty}\right) \tag{2.10}$$

for all such p_0. In order to extend the above inequality to a general initial density p_0, recall that the relative entropy $H(\cdot|p_\infty)$ defined on probability measures on $\mathbb{R}^{2n}$ by

$$m \mapsto H(m|p_\infty) = \begin{cases} \int_{\mathbb{R}^{2n}} \ln\left(\frac{p}{p_\infty}(z)\right) p(z) dz & \text{if } m(dz) = p(z) dz \\ +\infty \text{ otherwise,} \end{cases}$$

is lower semi-continuous with respect to the weak convergence.

For $k \in \mathbb{N}^*$, let $\alpha_k = \int_{\mathbb{R}^{2n}} (p_0(z) \wedge k p_\infty(z)) dz$, $\tilde{p}_0^k = (p_0 \wedge k p_\infty)/\alpha_k$, and $p_0^k = \alpha_k \tilde{p}_0^k + (1 - \alpha_k) p_\infty$. By convexity of $m \mapsto H(m|p_\infty)$,

$$
\begin{aligned}
H(p_0^k|p_\infty) \leq & \alpha_k H(\tilde{p}_0^k|p_\infty) + (1 - \alpha_k) \times 0 \\
= & - \alpha_k \ln(\alpha_k) + \int_{\mathbb{R}^{2n}} p_0(z) \ln \left(\frac{p_0}{p_\infty}(z) \right) 1_{\{p_0(z) \leq p_\infty(z)\}} dz \\
& + \int_{\mathbb{R}^{2n}} (p_0(z) \wedge k p_\infty(z)) \ln \left(\frac{p_0}{p_\infty}(z) \wedge k \right) 1_{\{p_0(z) > p_\infty(z)\}} dz
\end{aligned}
$$

where, by a slight abuse of notations, for a probability density p on $\mathbb{R}^{2n}$, $H(p|p_\infty)$ denotes the relative entropy of $p(z)dz$ w.r.t. $p_\infty(z)dz$. As $k \to \infty$, the third term in the r.h.s. converges to $\int_{\mathbb{R}^{2n}} p_0(z) \ln \left(\frac{p_0}{p_\infty}(z) \right) 1_{\{p_0(z) > p_\infty(z)\}} dz$ by monotone convergence and $\alpha_k \to 1$ by Lebesgue's theorem. With the lower semi-continuity of the relative entropy, this ensures that $H(p_0^k|p_\infty) \to H(p_0|p_\infty)$.

Denoting by $p_t^{k,\varepsilon}$ the density of Z_t^ε when $(X_0^1, \ldots, X_0^n)$ admits the density p_0^k, one has by the semi-group property

$$
\int_{\mathbb{R}^{2n}} |p_t^\varepsilon(z) - p_t^{k,\varepsilon}(z)| dz \leq \int_{\mathbb{R}^{2n}} |p_0(z) - p_0^k(z)| dz = 2(1 - \alpha_k)
$$

so that $H(p_t^\varepsilon|p_\infty) \leq \liminf_{k \to \infty} H(p_t^{k,\varepsilon}|p_\infty)$. Hence (2.10) holds for general p_0. Finally, using the convergence in law $Z_t^\varepsilon \to Z_t$ as $\varepsilon \to 0$ following from the proof of Lemma 2.3 and once again the lower semicontinuity of the relative entropy, we conclude that the required inequality holds for (Z_t) .

∎

3   Two particles case : $n = 2$

This section is devoted to the simple case where only two vortices interact :

$$
\begin{cases}
X_t^1 = X_0^1 + \sqrt{2\nu} W_t^1 + \int_0^t a_2 K(X_s^1 - X_s^2) ds \\
X_t^2 = X_0^2 + \sqrt{2\nu} W_t^2 + \int_0^t a_1 K(X_s^2 - X_s^1) ds
\end{cases}.
\tag{3.1}
$$

Existence and uniqueness for this system follows from [19] when $a_1 a_2 > 0$ and from [14] for general vorticities. After the change of variables $Z_t^i = e^{-t/2} X_{e^t - 1}^i$ for $i \in \{1, 2\}$, one obtains the following dynamics :

$$
\begin{cases}
Z_t^1 = X_0^1 + \sqrt{2\nu} B_t^1 + \int_0^t a_2 K(Z_s^1 - Z_s^2) ds - \frac{1}{2} \int_0^t Z_s^1 ds \\
Z_t^2 = X_0^2 + \sqrt{2\nu} B_t^2 + \int_0^t a_1 K(Z_s^2 - Z_s^1) ds - \frac{1}{2} \int_0^t Z_s^2 ds
\end{cases}.
\tag{3.2}
$$

The difference $Z_t = Z_t^1 - Z_t^2$ solves

$$
Z_t = X_0^1 - X_0^2 + 2\sqrt{\nu} B_t + a \int_0^t K(Z_s) ds - \frac{1}{2} \int_0^t Z_s ds
\tag{3.3}
$$

where $a \stackrel{\text{def}}{=} a_1 + a_2$ and $B_t \stackrel{\text{def}}{=} \frac{B_t^1 - B_t^2}{\sqrt{2}}$ is a two-dimensional Brownian motion independent from the two dimensional Brownian motion $B_t^\perp \stackrel{\text{def}}{=} \frac{B_t^1 + B_t^2}{\sqrt{2}}$. Remarking that

$$
\begin{cases}
Z_t^1 = e^{-\frac{t}{2}} X_0^1 + \sqrt{\nu} \int_0^t e^{\frac{s-t}{2}} dB_s + \sqrt{\nu} \int_0^t e^{\frac{s-t}{2}} dB_s^\perp + a_2 \int_0^t e^{\frac{s-t}{2}} K(Z_s) ds \\
Z_t^2 = e^{-\frac{t}{2}} X_0^2 - \sqrt{\nu} \int_0^t e^{\frac{s-t}{2}} dB_s + \sqrt{\nu} \int_0^t e^{\frac{s-t}{2}} dB_s^\perp - a_1 \int_0^t e^{\frac{s-t}{2}} K(Z_s) ds
\end{cases},
\tag{3.4}
$$

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we see that to understand the long-time behaviour of (Z_t^1, Z_t^2), it is enough to study the long-time behaviour of the triplet $(\int_0^t e^{\frac{s-t}{2}} dB_s, \int_0^t e^{\frac{s-t}{2}} K(Z_s) ds, \int_0^t e^{\frac{s-t}{2}} dB_s^\perp)$. The last coordinate is independent from the two first and converges in law to a two-dimensional standard normal random variable. So we only need to study the long-time behaviour of the couple $(\int_0^t e^{\frac{s-t}{2}} dB_s, \int_0^t e^{\frac{s-t}{2}} K(Z_s) ds)$. In fact,

$$\forall t \geq 0, \ Z_t = (X_0^1 - X_0^2) e^{-\frac{t}{2}} + 2\sqrt{\nu} \int_0^t e^{\frac{s-t}{2}} dB_s + a \int_0^t e^{\frac{s-t}{2}} K(Z_s) ds \tag{3.5}$$

and we are going to study the limit behaviour of

$$\mu_t \stackrel{\text{def}}{=} \text{Law}\left(Z_t, \int_0^t e^{\frac{s-t}{2}} dB_s, \int_0^t e^{\frac{s-t}{2}} K(Z_s) ds \right).$$

Let $q_\infty : z \in \mathbb{R}^2 \mapsto \frac{1}{8\nu\pi} e^{-\frac{|z|^2}{8\nu}}$ be the density of the law $\mathcal{N}_2(0, 4\nu I_2)$ and $\bar{Z}_0$ denote a $\mathbb{R}^2$-valued random variable distributed according to this law independent from the Brownian motions B^1 and B^2. The stochastic differential equation

$$\bar{Z}_t = \bar{Z}_0 + 2\sqrt{\nu} B_t - a \int_0^t K(\bar{Z}_s) ds - \frac{1}{2} \int_0^t \bar{Z}_s ds \tag{3.6}$$

admits a unique solution. Indeed, for all $\varepsilon > 0$, K is globally Lipschitz continuous and bounded on $\{z \in \mathbb{R}^2 : |z| \geq \varepsilon\}$ and $\bar{R}_t \stackrel{\text{def}}{=} |\bar{Z}_t|^2$ solves the Cox-Ingersoll-Ross stochastic differential equation

$$d\bar{R}_t = 4\sqrt{\nu \bar{R}_t} d\beta_t + (8\nu - \bar{R}_t) dt$$

with $\beta_t = \int_0^t \frac{\bar{Z}_s . dB_s}{\sqrt{\bar{R}_s}}$ a standard Brownian motion and does not vanish since $8\nu = \frac{(4\sqrt{\nu})^2}{2}$ (see Proposition 6.2.3 [8]). By an easy adaptation of the proof of the first statement in Proposition 2.4, one checks that the density q_∞ is stationary for both the SDEs satisfied by Z_t and $\bar{Z}_t$. As a consequence, for all $s \geq 0$, $\bar{Z}_s$ admits the density q_∞ and $\mathbb{E}(|\bar{Z}_s|) = \mathbb{E}(|\bar{Z}_0|) < +\infty$, $\mathbb{E}(|K(\bar{Z}_s)|) = \frac{1}{2\pi} \mathbb{E}(|\bar{Z}_0|^{-1}) < +\infty$. One deduces that the random vector

$$\left(\bar{Z}_0, \int_0^\infty e^{-\frac{s}{2}} \left(\bar{Z}_s / \sqrt{4\nu} \, ds - dB_s \right), \int_0^\infty e^{-\frac{s}{2}} K(\bar{Z}_s) ds \right)$$

is well defined. Let μ_∞ denote its distribution.

Proposition 3.1 *If $X_0^1 - X_0^2$ admits a density q_0 with respect to the Lebesgue measure on $\mathbb{R}^2$ such that $\int_{\mathbb{R}^2} q_0 \ln\left(\frac{q_0}{q_\infty}\right) < +\infty$, then for all $t \geq 0$, the density q_t of Z_t is such that*

$$\int_{\mathbb{R}^2} q_t \ln\left(\frac{q_t}{q_\infty}\right) \leq e^{-t} \int_{\mathbb{R}^2} q_0 \ln\left(\frac{q_0}{q_\infty}\right). \tag{3.7}$$

Moreover, for all $\alpha \in [1, 2)$, $W_\alpha(\mu_t, \mu_\infty)$ converges to 0 exponentially fast as $t \to +\infty$. If, in addition, $\mathbb{E}(|X_0^1 - X_0^2|^\rho) < +\infty$ for some $\rho > 2$, then this exponential convergence still holds for all $\alpha \in [1, \rho)$.

Remark 3.2 *By the transport inequality satisfied by the Gaussian law (see [20]), $W_2(q_0, q_\infty) \leq 2\sqrt{\nu \int_{\mathbb{R}^2} q_0 \ln\left(\frac{q_0}{q_\infty}\right)}$ (where by a slight abuse of notations q_∞ and q_0 stand for the measures $q_\infty(z) dz$ and $q_0(z) dz$). Hence, the finiteness of the relative entropy $\int_{\mathbb{R}^2} q_0 \ln\left(\frac{q_0}{q_\infty}\right)$ implies the finiteness of $\mathbb{E}(|X_0^1 - X_0^2|^2)$.*

This proposition, the proof of which is postponed, is the main step in the derivation of the long-time behaviour of (Z_t^1, Z_t^2).

Theorem 3.3 *If $X_0^1 - X_0^2$ admits a density q_0 such that $\int_{\mathbb{R}^2} q_0 \ln\left(\frac{q_0}{q_\infty}\right) < +\infty$, then the 2×2 matrix (Z_t^1, Z_t^2) (with first column equal to Z_t^1 and second column equal to Z_t^2) converges in law to*

$$\frac{\bar{Z}_0}{2}(1, -1) + \int_0^{+\infty} e^{-\frac{s}{2}} \left(\sqrt{\nu} dB_s^\perp + \frac{(a_2 - a_1)}{2} K(\bar{Z}_s) ds \right) (1, 1).$$

Let $\mu_t^{1,2}$ and $\mu_\infty^{1,2}$ respectively denote the law of (Z_t^1, Z_t^2) and of the above limit. For $\alpha \geq 1$ such that $\mathbb{E}(|X_0^1|^\alpha + |X_0^2|^\alpha) < +\infty$, $W_\alpha(\mu_t^{1,2}, \mu_\infty^{1,2})$ converges to 0 exponentially fast when $\alpha < 2$ and under the additional condition $\mathbb{E}(|X_0^1 - X_0^2|^\rho) < +\infty$ for some $\rho > \alpha$ when $\alpha \geq 2$.
Last, unless $a_2 = a_1$, the limiting distribution $\mu_\infty^{1,2}$ is not Gaussian.

Remark 3.4 • *The linear combination $a_1 Z_t^1 + a_2 Z_t^2$ is an Ornstein-Uhlenbeck process and converges in law to some Gaussian limit. The difference $Z_t = Z_t^1 - Z_t^2$, which is linearly independent of $a_1 Z_t^1 + a_2 Z_t^2$ as soon as $a_1 + a_2 \neq 0$, also converges to some Gaussian limit. Nethertheless, unless $a_1 = a_2$, the limit distribution of (Z_t^1, Z_t^2) is not Gaussian.*

• *When $a_1 = a_2$, (Z_t^1, Z_t^2) converge in distribution to $\frac{\bar{Z}_0}{2}(1,1) + \int_0^{+\infty} e^{-\frac{s}{2}} \sqrt{\nu} dB_s^\perp (1, -1)$ with $\frac{\bar{Z}_0}{2}$ and $\int_0^{+\infty} e^{-\frac{s}{2}} \sqrt{\nu} dB_s^\perp$ both distributed according to $\mathcal{N}_2(0, \nu I_2)$ and independent. One thus recovers the limit distribution obtained in Proposition 2.4.*

Let us first check that the Theorem follows from Proposition 3.1 before proving this Proposition.
Proof of Theorem 3.3. Let $\alpha \geq 1$. The random vector $\int_0^t e^{\frac{s-t}{2}} dB_s^\perp$ distributed according to $\mathcal{N}_2(0, (1 - e^{-t})I_2)$ converges in law to $G \sim \mathcal{N}_2(0, I_2)$. Moreover, $W_\alpha(\mathcal{N}_2(0, (1 - e^{-t})I_2), \mathcal{N}_2(0, I_2)) \leq e^{-\frac{t}{2}} \left(\mathbb{E}(|G|^\alpha) \right)^{1/\alpha}$ converges exponentially fast to 0. Since for $x \in \mathbb{R}^6$ and $y \in \mathbb{R}^2$, $(|x|^2 + |y|^2)^{\alpha/2} \leq 2^{(\frac{\alpha}{2}-1)^+}(|x|^\alpha + |y|^\alpha)$, one gets

$$W_\alpha\left(\mu_t \otimes \mathcal{N}_2(0, (1 - e^{-t})I_2), \mu_\infty \otimes \mathcal{N}_2(0, I_2) \right) \leq 2^{(\frac{1}{2}-\frac{1}{\alpha})^+} \left(W_\alpha(\mu_t, \mu_\infty) + W_\alpha(\mathcal{N}_2(0, (1 - e^{-t})I_2), \mathcal{N}_2(0, I_2)) \right)$$

since, for ρ_1 and ρ_2 optimal couplings respectively between μ_t and μ_∞ and between $\mathcal{N}_2(0, (1 - e^{-t})I_2)$ and $\mathcal{N}_2(0, I_2)$, the probability measure $\rho(dxdy, d\tilde{x}d\tilde{y}) = \rho_1(dx, d\tilde{x})\rho_2(dy, d\tilde{y})$ is a coupling between $\mu_t \otimes \mathcal{N}_2(0, (1 - e^{-t})I_2)$ and $\mu_\infty \otimes \mathcal{N}_2(0, I_2)$. Hence, by Proposition 3.1, the Wasserstein distance between

$$\mu_t \otimes \mathcal{N}_2(0, (1 - e^{-t})I_2) = \mathcal{L}aw\left(Z_t, \int_0^t e^{\frac{s-t}{2}} dB_s, \int_0^t e^{\frac{s-t}{2}} K(Z_s) ds, \int_0^t e^{\frac{s-t}{2}} dB_s^\perp \right)$$

and

$$\mu_\infty \otimes \mathcal{N}_2(0, I_2) = \mathcal{L}aw\left(\bar{Z}_0, \int_0^\infty e^{-\frac{s}{2}}(\bar{Z}_s / \sqrt{4\nu}\, ds - dB_s), \int_0^\infty e^{-\frac{s}{2}} K(\bar{Z}_s) ds, \int_0^\infty e^{-\frac{s}{2}} dB_s^\perp \right)$$

converges exponentially fast to 0 when $\alpha < 2$ and under the condition $\mathbb{E}(|X_0^1 - X_0^2|^\rho) < +\infty$ for some $\rho > \alpha$ when $\alpha \geq 2$. The first statement then follows from (3.4) and Slutsky's theorem : (Z_t^1, Z_t^2) writes as the sum of $e^{-\frac{t}{2}}(X_0^1, X_0^2)$ which converges a.s. to zero and the vector

$$\left(\sqrt{\nu} \int_0^t e^{\frac{s-t}{2}}(dB_s + dB_s^\perp) + a_2 \int_0^t e^{\frac{s-t}{2}} K(Z_s) ds, \sqrt{\nu} \int_0^t e^{\frac{s-t}{2}}(dB_s^\perp - dB_s) - a_1 \int_0^t e^{\frac{s-t}{2}} K(Z_s) ds \right)$$

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with law $\tilde{\mu}_t^{1,2}$ which converges weakly to

$$\int_0^{+\infty} e^{-\frac{s}{2}}\left(\sqrt{\nu}dB_s^\perp + \frac{(a_2-a_1)}{2}K(\bar{Z}_s)ds\right)(1,1) + \int_0^{+\infty} e^{-\frac{s}{2}}\left(-\sqrt{\nu}dB_s + \frac{1}{2}\bar{Z}_sds + \frac{a}{2}K(\bar{Z}_s)ds\right)(1,-1),$$

where $e^{-\frac{s}{2}}\left(-\sqrt{\nu}dB_s + \frac{1}{2}\bar{Z}_sds + \frac{a}{2}K(\bar{Z}_s)ds\right) = -\frac{1}{2}d(e^{-\frac{s}{2}}\bar{Z}_s)$ by (3.6).

The second statement is a consequence of the triangle inequality

$$W_\alpha(\mu_t^{1,2}, \mu_\infty^{1,2}) \leq W_\alpha(\mu_t^{1,2}, \tilde{\mu}_t^{1,2}) + W_\alpha(\tilde{\mu}_t^{1,2}, \mu_\infty^{1,2})$$

where by (3.4) and the above definition of $\tilde{\mu}_t^{1,2}$, the first term of the right-hand-side is not greater than $e^{-\frac{t}{2}}\left(\mathbb{E}(|(X_0^1, X_0^2)|^\alpha)\right)^{1/\alpha}$. Since $\tilde{\mu}_t^{1,2}$ (resp. $\tilde{\mu}_\infty^{1,2}$) is the image of $\mu_t \otimes \mathcal{N}_2(0, (1-e^{-t})I_2)$ (resp.

$\mu_\infty \otimes \mathcal{N}_2(0, I_2)$) by the mapping $x \in \mathbb{R}^8 \mapsto \begin{pmatrix} 0 & 0 & \sqrt{\nu} & 0 & a_2 & 0 & \sqrt{\nu} & 0 \\ 0 & 0 & 0 & \sqrt{\nu} & 0 & a_2 & 0 & \sqrt{\nu} \\ 0 & 0 & -\sqrt{\nu} & 0 & -a_1 & 0 & \sqrt{\nu} & 0 \\ 0 & 0 & 0 & -\sqrt{\nu} & 0 & -a_1 & 0 & \sqrt{\nu} \end{pmatrix} x,$

the second term is not greater than $W_\alpha\left(\mu_t \otimes \mathcal{N}_2(0, (1-e^{-t})I_2), \mu_\infty \otimes \mathcal{N}_2(0, I_2)\right)$ multiplied by the operator norm of the above matrix.

Let us now suppose that $a_1 \neq a_2$ and that $\mu_\infty^{1,2}$ is Gaussian and obtain some contradiction. Then, as can be seen using the characteristic function, the limit $(\bar{Z}_0, \int_0^{+\infty} e^{-\frac{s}{2}}K(\bar{Z}_s)ds)$ of $(Z_t, \int_0^t e^{\frac{s-t}{2}}K(Z_s)ds)$ is Gaussian by independence of $(\bar{Z}_t)_{t\geq 0}$ and $\int_0^\infty e^{-\frac{s}{2}}dB_s^\perp$.

Let $\zeta_t = 2\sqrt{\nu}\int_0^t e^{\frac{s-t}{2}}dB_s \sim \mathcal{N}_2(0, 4\nu(1-e^{-t})I_2)$ which solves $d\zeta_t = 2\sqrt{\nu}dB_t - \frac{\zeta_t}{2}dt$. Since by (3.5),

$$\zeta_t = Z_t - e^{-\frac{t}{2}}Z_0 - a\int_0^t e^{\frac{s-t}{2}}K(Z_s)ds,$$

(Z_t, ζ_t) converges in law to some Gaussian random vector (Z_∞, ζ_∞) with Z_∞ and ζ_∞ both distributed according to $\mathcal{N}_2(0, 4\nu I_2)$. Let β and γ be the respective correlations of $\zeta_{1,\infty}$ with $Z_{1,\infty}$ and $Z_{2,\infty}$ and

$$L = 2\nu((\partial_{z_1} + \partial_{\zeta_1})(\partial_{z_1} + \partial_{\zeta_1}) + (\partial_{z_2} + \partial_{\zeta_2})(\partial_{z_2} + \partial_{\zeta_2})) + (aK(z) - \frac{z}{2}).\nabla_z - \frac{\zeta}{2}.\nabla_\zeta$$

where $z = (z_1, z_2)$ and $\zeta = (\zeta_1, \zeta_2)$ be the infinitesimal generator of the process (Z_t, ζ_t). When both $f : \mathbb{R}^4 \to \mathbb{R}$ and Lf are continuous and bounded functions, then $\mathbb{E}(f(Z_t, \zeta_t))$ and $\mathbb{E}(Lf(Z_t, \zeta_t))$ converge respectively to $\mathbb{E}(f(Z_\infty, \zeta_\infty))$ and $\mathbb{E}(Lf(Z_\infty, \zeta_\infty))$ as $t \to \infty$. If moreover ∇f is bounded, taking expectations in Itô's formula, one obtains

$$\frac{d}{dt}\mathbb{E}(f(Z_t, \zeta_t)) = \mathbb{E}(Lf(Z_t, \zeta_t)),$$

so that the previous convergences ensure that $\mathbb{E}(Lf(Z_\infty, \zeta_\infty)) = 0$. Assume for a while that this centering property may be extended to the choices $f(z, \zeta)$ respectively equal to $\zeta_1 z_1$, $\zeta_1 z_2$ and $\zeta_1^2 z_1^2$, for which Lf is equal to $4\nu - z_1\zeta_1 - a\frac{z_2\zeta_1}{2\pi|z|^2}$, $-\zeta_1 z_2 + a\frac{\zeta_1 z_1}{2\pi|z|^2}$ and $4\nu(\zeta_1^2 + z_1^2 + 4\zeta_1 z_1) - 2z_1^2\zeta_1^2 - \frac{2az_1 z_2\zeta_1^2}{2\pi|z|^2}$ and $\mathbb{E}(Lf(Z_\infty, \zeta_\infty))$ to $4\nu(1-\beta) - \frac{a\gamma}{4\pi}$, $-4\nu\gamma + \frac{a\beta}{4\pi}$ and $(4\nu)^2(2 + 4\beta) - 2(4\nu)^2(2\beta^2 + 1) - \frac{2\nu}{\pi}a\beta\gamma$. One obtains the system

$$\begin{cases} 4\nu(1-\beta) - \frac{a\gamma}{4\pi} = 0 \\ -4\nu\gamma + \frac{a\beta}{4\pi} = 0 \\ \beta\left(16\nu(1-\beta) - \frac{a\gamma}{2\pi}\right) = 0 \end{cases} \Leftrightarrow \begin{cases} a\beta\gamma = 0 \\ -4\nu\gamma + \frac{a\beta}{4\pi} = 0 \\ 4\nu(1-\beta) - \frac{a\gamma}{4\pi} = 0 \end{cases} \qquad (3.8)$$

which has no solution (β, γ) when $a \neq 0$. This provides the required contradiction in the case $a \neq 0$, once we are able to extend the centering property to the three above choices of f. To

deal with the singularity of the terms like $\frac{z_2\zeta_1}{2\pi|z|^2}$, for $\varepsilon \in (0,1)$, we introduce a non-decreasing odd C^2 cutoff function $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that

$$\varphi_\varepsilon(x) = \begin{cases} 0 \text{ if } |x| \leq \frac{\varepsilon}{2} \\ x \text{ if } |x| \in [\varepsilon, \frac{1}{\varepsilon}] \\ \frac{3x}{2\varepsilon|x|} \text{ if } x \geq \frac{2}{\varepsilon} \end{cases} , \quad \|\varphi_\varepsilon'\|_\infty \leq 2 \text{ and } \sup_{\varepsilon < 1, |x| \geq \varepsilon} |\varphi_\varepsilon''(x)| + \varepsilon \sup_{\varepsilon < 1, |x| \leq \varepsilon} |\varphi_\varepsilon''(x)| < \infty.$$

The centering property $\mathbb{E}(Lf_\varepsilon(Z_\infty, \zeta_\infty)) = 0$ holds for f_ε respectively equal to $\zeta_1\varphi_\varepsilon(z_1)$, $\zeta_1\varphi_\varepsilon(z_2)$ and $\zeta_1^2\varphi_\varepsilon^2(z_1)$ by replacing the boundedness by the uniform integrability derived from the bound

$$\forall n \in \mathbb{N}, \sup_{t \geq 0} \mathbb{E}(|\zeta_t|^n) = \mathbb{E}(|\zeta_\infty|^n) < +\infty.$$

For $f(z, \zeta) = \zeta_1 z_1$ and $f_\varepsilon(z, \zeta) = \zeta_1\varphi_\varepsilon(z_1)$, one has

$$L(f_\varepsilon - f)(Z_\infty, \zeta_\infty) = 2\nu\zeta_{1,\infty}\varphi_\varepsilon''(Z_{1,\infty}) + (1 - \varphi_\varepsilon'(Z_{1,\infty})) \times \left(-4\nu + \frac{Z_{1,\infty}\zeta_{1,\infty}}{2} + a\frac{Z_{2,\infty}\zeta_{1,\infty}}{2\pi|Z_\infty|^2} \right).$$

The expectation of the second term in the right-hand-side tends to 0 as $\varepsilon \to 0$ by Lebesgue's theorem. On the other hand, using the controls on φ_ε'' for the inequality, one obtains

$$|\mathbb{E}(\zeta_{1,\infty}\varphi_\varepsilon''(Z_{1,\infty}))| = |\mathbb{E}(\mathbb{E}(\zeta_{1,\infty}\varphi_\varepsilon''(Z_{1,\infty}))|Z_{1,\infty})| = |\beta\mathbb{E}(Z_{1,\infty}\varphi_\varepsilon''(Z_{1,\infty}))|$$
$$\leq C \left(\mathbb{E}\left(\frac{|Z_{1,\infty}|}{\varepsilon} 1_{\{\frac{\varepsilon}{2} \leq |Z_{1,\infty}| \leq \varepsilon\}} \right) + \mathbb{E}\left(|Z_{1,\infty}| 1_{\{|Z_{1,\infty}| \geq \frac{1}{\varepsilon}\}} \right) \right) \overset{\varepsilon \to 0}{\longrightarrow} 0,$$

so that $\mathbb{E}(Lf(Z_\infty, \zeta_\infty)) = 0$. By similar arguments, the centering is still true for f equal to $\zeta_1 z_2$ and $\zeta_1^2 z_1^2$

When $a = 0$, $(\beta, \gamma) = (1, 0)$ solves the system (3.8), which is not surprising since $\forall t$, $Z_t = \zeta_t$. We then work with $\xi_t = \int_0^t e^{\frac{s-t}{2}} K(Z_s)ds$. As $t \to \infty$, (Z_t, ξ_t) converges in distribution to some Gaussian random vector (Z_∞, ξ_∞) with Z_∞ distributed according to $\mathcal{N}_2(0, 4\nu I_2)$ and ξ_∞ centered since $\int_0^\infty e^{-\frac{s}{2}} K(\bar{Z}_s)ds$ is centered. The infinitesimal generator of (Z_t, ξ_t) is $L = 2\nu\Delta_z - \frac{z}{2}.\nabla_z + (K(z) - \frac{\xi}{2}).\nabla_\xi$. For $f(z, \xi)$ respectively equal to $-\xi_1 z_2$ and $\xi_1^2 z_2^2$, Lf is equal to $\xi_1 z_2 + \frac{z_2^2}{2\pi|z|^2}$ and $4\nu\xi_1^2 - 2\xi_1^2 z_2^2 - \frac{\xi_1 z_2^3}{\pi|z|^2}$ and the equality $\mathbb{E}(Lf(Z_\infty, \xi_\infty)) = 0$ yields

$$\begin{cases} \text{Cov}(Z_{2,\infty}, \xi_{1,\infty}) = -\frac{1}{4\pi} \\ 4\nu\text{Var}(\xi_{1,\infty}) - 2\left(4\nu\text{Var}(\xi_{1,\infty}) + 2\text{Cov}^2(Z_{2,\infty}, \xi_{1,\infty}) \right) - \frac{3}{4\pi}\text{Cov}(Z_{2,\infty}, \xi_{1,\infty}) = 0 \end{cases}.$$

These equalities imply $\text{Var}(\xi_{1,\infty}) = -\frac{1}{64\nu\pi^2}$ which is the desired contradiction. To justify the equality $\mathbb{E}(Lf(Z_\infty, \xi_\infty)) = 0$ for the above choices of f one first construct approximations $f_{\varepsilon,\eta}$ for $\varepsilon, \eta \in (0,1)$ by replacing the factors z_1, z_2, ξ_1, ξ_2 respectively by $\varphi_\varepsilon(z_1)$, $\varphi_\varepsilon(z_2)$, $\varphi_\eta(\xi_1)$ and $\varphi_\eta(\xi_2)$. For $\varepsilon \in (0,1)$, f_ε is obtained similarly by only replacing the factors z_1 and z_2 in f. Then $\mathbb{E}(Lf_{\varepsilon,\eta}(Z_\infty, \xi_\infty)) = 0$. Since no second order derivative of φ_η appears in $Lf_{\varepsilon,\eta}$, one may apply Lebesgue's theorem to take the limit $\eta \to 0$ and obtain $\mathbb{E}(Lf_\varepsilon(Z_\infty, \xi_\infty)) = 0$. Then one concludes as previously by taking the limit $\varepsilon \to 0$.

■

Proof of Proposition 3.1. The first statement is obtained by an easy adaptation of the proof of Proposition 2.4. Let us deal with the second statement. Let $t > 0$. Csiszar-Kullback

inequality writes $\|q_{t/2}-q_\infty\|_1 \leq \sqrt{2\int_{\mathbb{R}^2} q_{t/2}\ln\left(\frac{q_{t/2}}{q_\infty}\right)}$. By (3.7), the left-hand-side is smaller than $Ce^{-\frac{t}{4}}$ with C not depending on t. Let $\rho_{t/2} = \frac{q_{t/2}\wedge q_\infty}{q_{t/2}}$ and $(U,\zeta_{t/2})$ be a couple of independent random variables independent from (B^1,B^2,X_0^1,X_0^2) with U uniformly distributed on $[0,1]$ and $\zeta_{t/2}$ distributed according to the density $\frac{(q_\infty-q_{t/2})^+}{\int_{\mathbb{R}^2}(q_\infty-q_{t/2})^+}$ (when $\int_{\mathbb{R}^2}(q_\infty-q_{t/2})^+ = 0$, $q_{t/2} = q_\infty$ and $\zeta_{t/2}$ is not needed in what follows). The random variable $Z_{t/2}^{t/2} = 1_{\{U\leq\rho_{t/2}(Z_{t/2})\}}Z_{t/2} + 1_{\{U>\rho_{t/2}(Z_{t/2})\}}\zeta_{t/2}$ admits the density q_∞ and is such that $\mathbb{P}(Z_{t/2} \neq Z_{t/2}^{t/2}) = \frac{1}{2}\|q_{t/2} - q_\infty\|_1$. Let $(Z_s^{t/2})_{s\geq t/2}$ be the unique solution of the SDE

$$Z_s^{t/2} = Z_{t/2}^{t/2} + 2\sqrt{\nu}(B_s - B_{t/2}) + \int_{t/2}^s \left(aK(Z_r^{t/2}) - \frac{1}{2}Z_r^{t/2}\right) dr$$

and $\tilde{\mu}_t$ denote the law of $(Z_t^{t/2}, \int_{t/2}^t e^{\frac{s-t}{2}} dB_s, \int_{t/2}^t e^{\frac{s-t}{2}} K(Z_s^{t/2})ds)$. By trajectorial uniqueness, $(Z_s)_{s\geq t/2}$ and $(Z_s^{t/2})_{s\geq t/2}$ coincide on $\{U \leq \rho_{t/2}(Z_{t/2})\}$. Let $A_{t/2} = \{U > \rho_{t/2}(Z_{t/2})\}$. By the triangle inequality, for $\alpha \geq 1$,

$$W_\alpha(\mu_t,\mu_\infty) \leq W_\alpha(\mu_t,\tilde{\mu}_t) + W_\alpha(\tilde{\mu}_t,\mu_\infty). \tag{3.9}$$

For $\gamma > \alpha$, one has, using Hölder's inequality for the second step,

$$W_\alpha^\alpha(\mu_t,\tilde{\mu}_t) \leq \mathbb{E}\Bigg(\Bigg(\Big(1_{A_{t/2}}|Z_t - Z_t^{t/2}|^2 + \Big|\int_0^{t/2} e^{\frac{s-t}{2}} dB_s\Big|^2$$
$$+ \Big|\int_0^{t/2} e^{\frac{s-t}{2}} K(Z_s)ds + 1_{A_{t/2}}\int_{t/2}^t e^{\frac{s-t}{2}}(K(Z_s) - K(Z_s^{t/2}))ds\Big|^2\Big)^{\alpha/2}\Bigg)$$

$$\leq 3^{(\frac{\alpha}{2}-1)^+}\Bigg[2^{\alpha-1}\mathbb{P}(A_{t/2})^{\frac{\gamma-\alpha}{\gamma}}(\mathbb{E}^{\frac{\alpha}{\gamma}}|Z_t|^\gamma + \mathbb{E}^{\frac{\alpha}{\gamma}}|Z_t^{t/2}|^\gamma) + \mathbb{E}\Big|\int_0^{t/2} e^{\frac{s-t}{2}} dB_s\Big|^\alpha$$

$$+ 3^{\alpha-1}\Bigg\{\mathbb{E}\Big|\int_0^{t/2} e^{\frac{s-t}{2}} K(Z_s)ds\Big|^\alpha + \mathbb{P}(A_{t/2})^{\frac{\gamma-\alpha}{\gamma}}\Big(\mathbb{E}^{\frac{\alpha}{\gamma}}\Big|\int_{t/2}^t e^{\frac{s-t}{2}} K(Z_s)ds\Big|^\gamma + \mathbb{E}^{\frac{\alpha}{\gamma}}\Big|\int_{t/2}^t e^{\frac{s-t}{2}} K(Z_s^{t/2})ds\Big|^\gamma\Big)\Bigg\}\Bigg]$$

The term $\mathbb{E}\Big|\int_0^{t/2} e^{\frac{s-t}{2}} dB_s\Big|^\alpha$ is equal to $(e^{-t/2} - e^{-t})^{\alpha/2}\mathbb{E}|G|^\alpha$ where $G \sim \mathcal{N}_2(0,I_2)$ and converges exponentially fast to 0. So does $\mathbb{P}(A_{t/2})^{\frac{\gamma-\alpha}{\gamma}} = \left(\frac{1}{2}\|q_{t/2} - q_\infty\|_1\right)^{\frac{\gamma-\alpha}{\gamma}}$ according to the beginning of the proof. According to Lemma 3.5 below and since $\forall z \in \mathbb{R}^2$, $|K(z)| \leq \frac{2\pi}{|z|}$, the terms involving the Biot and Savart kernel K also converge to 0 exponentially fast. The expectation $\mathbb{E}|Z_t^{t/2}|^\gamma$ does not depend on t and is finite. When $\int_{\mathbb{R}^2} q_0 \ln\left(\frac{q_0}{q_\infty}\right) < +\infty$, choosing $\gamma = 2$, we deduce from Remark 3.2 and Lemma 3.6 below that $W_\alpha(\mu_t,\tilde{\mu}_t)$ converges to 0 exponentially fast for $\alpha \in [1,2)$. When moreover $\mathbb{E}|X_0^1 - X_0^2|^\rho < +\infty$ for some $\rho > 2$, choosing $\gamma = \rho$, we obtain that this exponential convergence holds for $\alpha \in [1,\rho)$.

By Lemma 3.7 below, which is based on a time-reversal argument, $\tilde{\mu}_t$ is the law of $(\bar{Z}_0, \int_0^{t/2} e^{-\frac{s}{2}}(\bar{Z}_s/\sqrt{4\nu}\,ds - dB_s), \int_0^{t/2} e^{-\frac{s}{2}} K(\bar{Z}_s)ds)$. Since μ_∞ is the law of $(\bar{Z}_0, \int_0^\infty e^{-\frac{s}{2}}(\bar{Z}_s/\sqrt{4\nu}\,ds - dB_s), \int_0^\infty e^{-\frac{s}{2}} K(\bar{Z}_s)ds)$, one deduces that

$$W_\alpha^\alpha(\tilde{\mu}_t,\mu_\infty) \leq 2^{(\frac{\alpha}{2}-1)^+}\left(\frac{1}{2\nu^{\alpha/2}}\mathbb{E}\Big|\int_{t/2}^{+\infty} e^{-\frac{s}{2}}\bar{Z}_s ds\Big|^\alpha + 2^{\alpha-1}\mathbb{E}\Big|\int_{t/2}^{+\infty} e^{-\frac{s}{2}} dB_s\Big|^\alpha + \mathbb{E}\Big|\int_{t/2}^{+\infty} e^{-\frac{s}{2}} K(\bar{Z}_s)\Big|^\alpha\right).$$

The term $\mathbb{E}\left|\int_{t/2}^{+\infty} e^{-\frac{s}{2}} dB_s\right|^\alpha$ is equal to $e^{-\frac{\alpha t}{4}}\mathbb{E}|G|^\alpha$ where $G \sim \mathcal{N}_2(0, I_2)$ and converges exponentially fast to 0. Moreover, by Hölder's inequality

$$\mathbb{E}\left|\int_{t/2}^{+\infty} e^{-\frac{s}{2}} \bar{Z}_s ds\right|^\alpha \leq 2^{\alpha-1} e^{-\frac{(\alpha-1)t}{4}} \int_{t/2}^{+\infty} e^{-\frac{s}{2}} \mathbb{E}|\bar{Z}_s|^\alpha ds = 2^\alpha e^{-\frac{\alpha t}{4}} \mathbb{E}|\bar{Z}_0|^\alpha$$

with $\mathbb{E}|\bar{Z}_0|^\alpha < +\infty$. The third term of the right-hand-side is equal to $e^{-\frac{\alpha t}{4}}\mathbb{E}\left|\int_0^{+\infty} e^{-\frac{s}{2}} K(\bar{Z}_s) ds\right|^\alpha$ where the expectation is finite according to Lemma 3.5 below.

∎

Lemma 3.5 *Let $(\zeta_t)_{t\geq 0}$ solve the SDE*

$$d\zeta_t = 2\sqrt{\nu} dB_t + cK(\zeta_t) dt - \frac{\zeta_t}{2} dt$$

for some real constant c. Then,

$$\forall n \in \mathbb{N}^*, \ \exists C < +\infty, \ \forall \zeta_0, \ \mathbb{E}\left(\left(\int_0^{+\infty} \frac{e^{-\frac{s}{2}}}{|\zeta_s|} ds\right)^n\right) \leq C \ and \ \forall t \geq 0, \ \mathbb{E}\left(\left(\int_0^t \frac{ds}{|\zeta_s|}\right)^n\right) \leq C(1+t)^n.$$

Proof. The process $R_t \overset{\text{def}}{=} |\zeta_t|^2$ solves the Cox-Ingersoll-Ross stochastic differential equation

$$dR_t = 4\sqrt{\nu R_t} d\beta_t + (8\nu - R_t) dt$$

with $d\beta_t = \frac{\zeta_t \cdot dB_t}{\sqrt{R_t}}$ and does not vanish since $8\nu = \frac{(4\sqrt{\nu})^2}{2}$. By the comparison principle satisfied by this stochastic differential equation (see Theorem 3.7 p394 [18]), for $t > s$,

$$\mathbb{E}\left(\frac{1}{\sqrt{R_t}}\Big| R_s\right) \leq \int_0^{+\infty} \frac{1}{\sqrt{r}} p(t-s, 0, r) dr$$

where for $t > 0$, $p(t, 0, r) = \frac{1}{8\nu(1-e^{-t})} e^{-\frac{r}{8\nu(1-e^{-t})}}$ denotes the transition density from the state 0. One deduces that $\mathbb{E}\left(\frac{1}{\sqrt{R_t}}\Big| R_s\right) \leq \sqrt{\frac{\pi}{8\nu(1-e^{s-t})}}$. By successive conditionings, one deduces that

$$\forall n \in \mathbb{N}^*, \ \exists C_n > 0, \ \forall 0 = t_0 < t_1 < \ldots < t_n, \ \mathbb{E}\left(\frac{1}{\prod_{k=1}^n |\zeta_{t_k}|}\right) \leq \frac{C_n}{\prod_{k=1}^n \sqrt{1 - e^{t_{k-1}-t_k}}}.$$

As a consequence

$$\mathbb{E}\left(\left(\int_0^t \frac{ds}{|\zeta_s|}\right)^n\right) = n! \int_0^t \int_{t_1}^t \ldots \int_{t_{n-1}}^t \mathbb{E}\left(\frac{1}{\prod_{k=1}^n |\zeta_{t_k}|}\right) dt_n dt_{n-1} \ldots dt_1$$

$$\leq C \int_0^t \int_{t_1}^t \ldots \int_{t_{n-1}}^t \frac{dt_n dt_{n-1} \ldots dt_1}{\prod_{k=1}^n \sqrt{1 - e^{t_{k-1}-t_k}}},$$

Since $\forall s \in [0, t]$, $\int_s^t \frac{1}{\sqrt{1-e^{s-r}}} dr \leq \int_0^t \frac{1}{\sqrt{1-e^{-r}}} dr \leq C(1+t)$, one concludes that $\mathbb{E}\left(\left(\int_0^t \frac{ds}{|\zeta_s|}\right)^n\right) \leq C(1+t)^n$. The finiteness of $\mathbb{E}\left(\left(\int_0^{+\infty} \frac{e^{-\frac{s}{2}}}{|\zeta_s|} ds\right)^n\right)$ is obtained by a similar argument remarking that $\forall s \geq 0$, $\int_s^{+\infty} \frac{e^{-\frac{r}{2}}}{\sqrt{1-e^{s-r}}} dr \leq \int_0^{+\infty} \frac{e^{-\frac{r}{2}}}{\sqrt{1-e^{-r}}} dr < +\infty$.

∎

Lemma 3.6 *If for some $\gamma \geq 2$, $\mathbb{E}|X_0^1 - X_0^2|^\gamma < +\infty$, then $\sup_{t \geq 0} \mathbb{E}|Z_t|^\gamma < +\infty$.*

Proof . By Itô's formula, for $\delta \geq 2$,

$$d|Z_t|^\delta = 2\sqrt{\nu}\delta|Z_t|^{\delta-2}Z_t.dB_t + \left(2\nu\delta^2|Z_t|^{\delta-2} - \frac{\delta}{2}|Z_t|^\delta\right)dt.$$

For the choice $\delta = 2$, since $\mathbb{E}|Z_0|^2 < +\infty$, by a standard localization argument we obtain that $d\mathbb{E}|Z_t|^2 = (8\nu - \mathbb{E}|Z_t|^2)dt$ and deduce that $\sup_{t \geq 0} \mathbb{E}|Z_t|^2 < +\infty$. By induction, we then check that for all $k \in \{1, \ldots, \lfloor\frac{\gamma}{2}\rfloor\}$, $d\mathbb{E}|Z_t|^{2k} = (8\nu k^2\mathbb{E}|Z_t|^{2(k-1)} - k\mathbb{E}|Z_t|^{2k})dt$ so that

$$\mathbb{E}|Z_t|^{2k} \leq e^{-kt}\mathbb{E}|Z_0|^{2k} + 8\nu k^2 \int_0^t e^{k(s-t)}\mathbb{E}|Z_s|^{2(k-1)}ds \text{ and } \sup_{t \geq 0}\mathbb{E}|Z_t|^{2k} < +\infty.$$

With the choice $\delta = \gamma$, we then conclude that $d\mathbb{E}|Z_t|^\gamma = (2\nu\gamma^2\mathbb{E}|Z_t|^{\gamma-2} - \frac{\gamma}{2}\mathbb{E}|Z_t|^\gamma)dt$ and that $\sup_{t \geq 0}\mathbb{E}|Z_t|^\gamma < +\infty$.

∎

Lemma 3.7 *For $s \in [0, t/2]$, let $\mathcal{F}_s^t = \sigma(Z_t^{t/2}, (B_r - B_t)_{r \in [t-s,t]})$. Then $(\hat{B}_s^t \overset{\text{def}}{=} B_{t-s} - B_t + \frac{1}{2\sqrt{\nu}}\int_{t-s}^t Z_r^{t/2}dr)_{s \in [0,t/2]}$ is a $\mathcal{F}_s^t$-Brownian motion and $(\hat{Z}_s^t \overset{\text{def}}{=} Z_{t-s}^{t/2})_{s \in [0,t/2]}$ solves*

$$d\hat{Z}_s^t = 2\sqrt{\nu}d\hat{B}_s^t - aK(\hat{Z}_s^t)ds - \frac{1}{2}\hat{Z}_s^t ds.$$

Moreover, $(Z_t^{t/2}, \int_{t/2}^t e^{\frac{s-t}{2}}dB_s, \int_{t/2}^t e^{\frac{s-t}{2}}K(Z_s^{t/2})ds)$ and $(\bar{Z}_0, \int_0^{t/2} e^{-\frac{s}{2}}(\bar{Z}_s/\sqrt{4\nu}\,ds - dB_s), \int_0^{t/2} e^{-\frac{s}{2}}K(\bar{Z}_s)ds)$ have the same distribution.

Proof . One has

$$\left(Z_t^{t/2}, \int_{t/2}^t e^{\frac{s-t}{2}}dB_s, \int_{t/2}^t e^{\frac{s-t}{2}}K(Z_s^{t/2})ds\right) = \left(\hat{Z}_0^t, \int_0^{t/2} e^{-\frac{s}{2}}(\hat{Z}_s^t/\sqrt{4\nu}\,ds - d\hat{B}_s^t), \int_0^{t/2} e^{-\frac{s}{2}}K(\hat{Z}_s^t)ds\right).$$

Moreover, the trajectorial uniqueness for the stochastic differential equation (3.6) implies that $(\bar{Z}_s, B_s)_{s \in [0,t/2]}$ and $(\hat{Z}_s^t, \hat{B}_s^t)_{s \in [0,t/2]}$ have the same distribution. Therefore the second statement is a consequence of the first one.

Without the singularity of the Biot and Savart kernel at the origin, the first statement would be a consequence of [17], Theorem 2.2 and Corollary 2.4 (see also [13] for more general results concerning the time-reversal of diffusion processes). To deal with this singularity, we use the smooth approximations K_ε of this kernel defined at the end of the introduction. For $\varepsilon > 0$, let $(Z_{\varepsilon,s}^{t/2})_{s \geq t/2}$ be the unique solution of the SDE

$$Z_{\varepsilon,s}^{t/2} = Z_{t/2}^{t/2} + 2\sqrt{\nu}(B_s - B_{t/2}) + \int_{t/2}^s \left(aK_\varepsilon(Z_{\varepsilon,r}^{t/2}) - \frac{1}{2}Z_{\varepsilon,r}^{t/2}\right)dr.$$

By an adaptation of the proof of the first statement in Proposition 2.4, for all $s \geq t/2$, $Z_{\varepsilon,s}^{t/2}$ is distributed according to the density q_∞. By Theorem 2.2 [17] which deals with time-reversal of

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Brownian motions, one deduces that for $g : \mathbb{R}^2 \to \mathbb{R}$ continuous and bounded and $0 \leq r \leq s \leq t/2$,

$$\mathbb{E}\left(\left(B_{t-s} - B_{t-r} + \frac{1}{2\sqrt{\nu}}\int_{t-s}^{t-r} Z_{\varepsilon,u}^{t/2}du\right)g(Z_{\varepsilon,t-r}^{t/2})\right) = 0.$$

Hence

$$\left|\mathbb{E}\left((\hat{B}_s^t - \hat{B}_r^t)g(Z_{t-r}^{t/2})\right)\right| \leq \frac{1}{2\sqrt{\nu}}\left|\mathbb{E}\left(g(Z_{\varepsilon,t-r}^{t/2})\int_{t-s}^{t-r}(Z_u^{t/2} - Z_{\varepsilon,u}^{t/2})du\right)\right|$$

$$+ \left|\mathbb{E}\left(\left(B_{t-s} - B_{t-r} + \frac{1}{2\sqrt{\nu}}\int_{t-s}^{t-r} Z_u^{t/2}du\right)(g(Z_{t-r}^{t/2}) - g(Z_{\varepsilon,t-r}^{t/2}))\right)\right|. \quad (3.10)$$

Since $Z_u^{t/2}$ does not vanish, $\tau_\varepsilon \overset{\text{def}}{=} \inf\{u \geq \frac{t}{2} : |Z_u^{t/2}| \leq \varepsilon\}$ goes to infinity when ε goes to 0. The processes $(Z_u^{t/2})_{u\in[t/2,t]}$ and $(Z_{\varepsilon,u}^{t/2})_{u\in[t/2,t]}$ coincide on $\tau_\varepsilon \geq t$. By Lebesgue's theorem, the second term of the right-hand-side of (3.10) converges to 0 as $\varepsilon \to 0$. So does the first term since

$$\left|\mathbb{E}\left(g(Z_{\varepsilon,t-r}^{t/2})\int_{t-s}^{t-r}(Z_u^{t/2} - Z_{\varepsilon,u}^{t/2})du\right)\right| \leq \|g\|_\infty \int_{t-s}^{t-r}\mathbb{E}\left(1_{\{\tau_\varepsilon < t\}}\left|Z_u^{t/2} - Z_{\varepsilon,u}^{t/2}\right|\right)du$$

$$\leq \|g\|_\infty\sqrt{\mathbb{P}(\tau_\varepsilon < t)}\int_{t-s}^{t-r}\sqrt{2\mathbb{E}\left(|Z_u^{t/2}|^2 + |Z_{\varepsilon,u}^{t/2}|^2\right)}du$$

$$= 2\|g\|_\infty(s-r)\sqrt{\mathbb{P}(\tau_\varepsilon < t)\mathbb{E}(|\bar{Z}_0|^2)}.$$

Therefore $\mathbb{E}(\hat{B}_s^t - \hat{B}_r^t|Z_{t-r}^{t/2}) = 0$. Since $\mathcal{F}_r^t = \sigma(Z_{t-r}^{t/2}, (B_u - B_t)_{u\in[t-r,t]})$ and $\hat{B}_s^t - \hat{B}_r^t = B_{t-s} - B_{t-r} + \frac{1}{2\sqrt{\nu}}\int_{t-s}^{t-r}Z_u^{t/2}du$, one deduces that $\mathbb{E}(\hat{B}_s^t - \hat{B}_r^t|\mathcal{F}_r^t) = 0$. Taking into account the quadratic variation, one concludes that $\hat{B}_s$ is a $\mathcal{F}_s^t$-Brownian motion. The second statement follows easily.

■

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