On the combinatorics of quadrant marked mesh patterns in 132-avoiding permutations

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Abstract

The study of quadrant marked mesh patterns in 132-avoiding permutations was initiated by Kitaev, Remmel and Tiefenbruck. We refine several results of Kitaev, Remmel and Tiefenbruck by giving new combinatorial interpretations for the coefficients that appear in the generating functions for the distribution of quadrant marked mesh patterns in 132-avoiding permutations. In particular, we study quadrant marked mesh patterns where we specify conditions on exactly one of the four possible quadrants in a quadrant marked mesh pattern. We show that for the first two quadrants, certain of these coefficients are counted by elements of Catalan's triangle and give a new combinatorial interpretation of these coefficients for quadrant four. We also give a new bijection between 132-avoiding permutations and non-decreasing parking functions.

1 Introduction

Mesh patterns were introduced by Brändén and Claesson [1] to provide explicit expansions for certain permutation statistics as (possibly infinite) linear combinations of (classical) permutation patterns. This notion was further studied by Úlfarsson [9] and Kitaev, Remmel and Tiefenbruck in some series of papers refining conditions on permutations and patterns. The present paper focuses on what the trio Kitaev, Remmel and Tiefenbruck call *quadrant marked mesh patterns in* 132*-avoiding permutations* [3]. The goal of this work is to give combinatorial interpretations for permutations avoiding 132 of size n having k values seeing ℓ greater/lower values on their left/right.

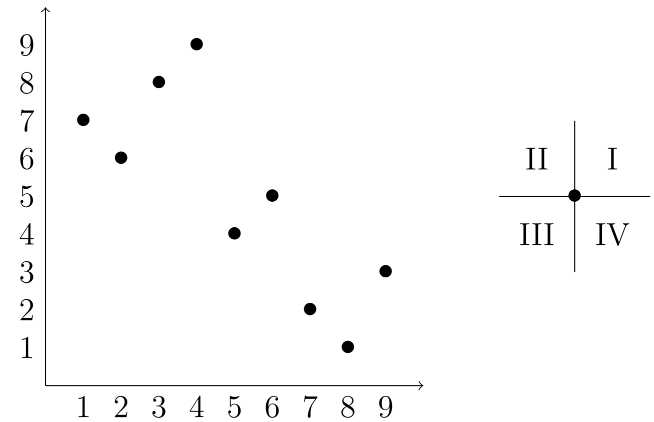

Figure 1: The graph of $\sigma = 768945213$ and the quadrant numeration.

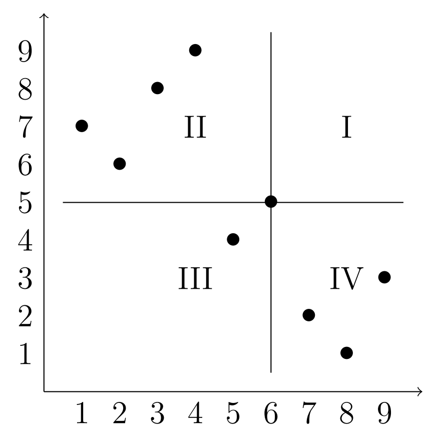

Figure 2: The graph of $\sigma = 768945213$ with coordinate system at position 6.

Let $\sigma = \sigma_1 \ldots \sigma_n$ be a permutation written in one-line notation. We now consider the graph $G(\sigma)$ of σ to be the set of points $\{(i, \sigma_i) : 1 \leqslant i \leqslant n\}$. For example, the graph of the permutation $\sigma = 768945213$ is represented Figure 1.

Now, if we draw a coordinate system centered at the point (i, σ_i), we are interested in the points that lie in the four quadrants I, II, III, IV of that coordinate system as represented on the right of Figure 1. For any a, b, c, d, four nonnegative integers, we say that σ_i matches the simple marked mesh pattern $MMP(a, b, c, d)$ in σ if, in the coordinate system centered at (i, σ_i), $G(\sigma)$ has at least a, b, c and d points in the respective quadrants I, II, III and IV.

For $\sigma = 768945213$, then $\sigma_6 = 5$ matches the simple marked mesh pattern $MMP(0, 3, 1, 1)$, since relative to the coordinate system with origin $(6, 5)$, $G(\sigma)$ has respectively 0, 4, 1 and 3 points in quadrants I, II, III and IV (see Figure 2). If a coordinate in $MMP(a, b, c, d)$ is zero, then there is no condition imposed on the points in the corresponding quadrant. We let $mmp(a, b, c, d)(\sigma)$ denote the number of i such that σ_i matches the marked mesh pattern $MMP(a, b, c, d)$ in σ.

	$\ell=1$					$\ell=2$				$\ell=3$			$\ell=4$		$\ell=5$
$n\backslash k$	**0**	**1**	**2**	**3**	**4**	**0**	**1**	**2**	**3**	**0**	**1**	**2**	**0**	**1**	**0**
1	1					1				1			1		1
2	1	1				2				2			2		2
3	1	2	2			4	1			5			5		5
4	1	3	5	5		8	4	2		13	1		14		14
5	1	4	9	14	14	16	12	9	5	34	6	2	41	1	42

Table 1: Coefficients of the series $Q_{132}^{(\ell,0,0,0)}(t,x)|_{t^n x^k}$ for small values of n, k and ℓ.

We shall study the distribution of quadrant marked mesh patterns on permutations that avoid a fixed pattern. Given a sequence $w = w_1 \dots w_n$ of distinct integers, let $red(w)$ be the permutation obtained by replacing the i-th largest integer that appears in w by i. For example, if $w = 2754$, then $red(w) = 1432$. Given a permutation $\tau = \tau_1 \dots \tau_j$ in the symmetric group $\mathfrak{S}_j$, we say that the pattern τ occurs in a permutation $\sigma \in \mathfrak{S}_n$ if there exist $1 \leqslant i_1 \leqslant \dots \leqslant i_j \leqslant n$ such that $red(\sigma_{i_1} \dots \sigma_{i_j}) = \tau$. We say that a permutation σ avoids the pattern τ if τ does not occur in σ. We will denote by $\mathfrak{S}_n(\tau)$ the set of permutations in $\mathfrak{S}_n$ avoiding τ. This paper presents some combinatorial results concerning the generating functions

$$Q_{132}^{(a,b,c,d)}(t,x) := 1 + \sum_{n \geqslant 1} t^n Q_{n,132}^{(a,b,c,d)}(x)$$

where for any $a, b, c, d \in \mathbb{N}$,

$$Q_{n,132}^{(a,b,c,d)}(x) := \sum_{\sigma \in \mathfrak{S}_n(132)} x^{mmp(a,b,c,d)(\sigma)}.$$

More precisely, we give a combinatorial interpretation of each coefficient of these series when exactly one value among a, b, c, d is non zero. Note that, as explained in [3], $Q_{132}^{(0,0,0,k)}(t,x) = Q_{132}^{(0,k,0,0)}(t,x)$, so that we shall not consider the forth quadrant.

2 Quadrant I: the series $Q_{132}^{(\ell,0,0,0)}(t,x)$

Let ℓ a positive integer, the marked mesh pattern $MMP(\ell,0,0,0)$ splits the 132-avoiding permutations according to the number of values having at least l greater values on their right. Table 1 displays the first values of $Q_{132}^{(\ell,0,0,0)}(t,x)|_{t^n x^k}$:

A lot of different combinatorial families are counted by the Catalan numbers, and here Dyck paths appear to be the objects having a nice behavior with the statistic associated with the pattern $MMP(\ell,0,0,0)$.

The Catalan's triangle $(C_{n,k})_{1 \leqslant k \leqslant n}$ is (A009766 of [7]): There is a nice formula for computing directly any coefficient

$$C_{n,k} = \frac{(n+k)!(n-k+1)}{k!(n+1)!}.$$

$n\backslash k$	1	2	3	4	5	6	7	8
1	1							
2	1	1						
3	1	2	2					
4	1	3	5	5				
5	1	4	9	14	14			
6	1	5	14	28	42	42		
7	1	6	20	48	90	132	132	
8	1	7	27	75	165	297	429	429

Table 2: Catalan's triangle $(C_{n,k})_{1\leqslant k\leqslant n\leqslant 8}$

We recall that a Dyck path with $2n$ steps is a path on the square lattice with steps $(1,1)$ or $(1,-1)$ from $(0,0)$ to $(2n,0)$ that never falls below the x-axis.

Theorem 2.1 *$Q_{132}^{(\ell,0,0,0)}(t,x)|_{t^nx^k}$ counts the number of Dyck paths with $2n$ steps having exactly k steps $(1,-1)$ ending at height greater than or equal to ℓ.*
Proof – This can be seen thanks to a bijection by Krattenthaler [4] between 132-avoiding permutations and Dyck paths. The bijection consists in starting from the empty path and constructing a Dyck path as we read the permutation $\sigma = \sigma_1 \dots \sigma_n$ from left to right. For each value σ_i, we complete the path with some increasing steps followed by a single decreasing step so that the height of the path at this actual point is equal to the number of remaining values σ_j for $j > i$ greater than σ_i (see Figure 3). For the permutation σ in Figure 1, the 7 sees two greater values on its right, and

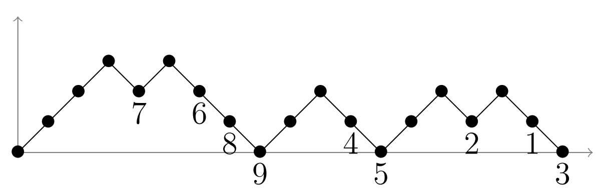

Figure 3: Dyck path obtained with the bijection by Krattenthaler over σ = 768945213.

so does the 6. The 8 sees only the 9 which itself cannot see any greater value as it is the highest. The proof is now immediate since Krattenthaler proved that this construction constitutes a bijection. By construction, $mmp(\ell,0,0,0)$ is equal to the number of points at height greater or equal to ℓ reached by a decreasing step.

As the number of Dick paths of size $2n$ having k decreasing steps that end at of above height 1 are counted by the element $C_{n,k}$ of the Catalan's triangle, we deduce the following corollary.

	$\ell=\mathbf{1}$					$\ell=\mathbf{2}$				$\ell=\mathbf{3}$			$\ell=\mathbf{4}$		$\ell=\mathbf{5}$
$n\backslash k$	**0**	**1**	**2**	**3**	**4**	**0**	**1**	**2**	**3**	**0**	**1**	**2**	**0**	**1**	**0**
1	1					1				1			1		1
2	1	1				2				2			2		2
3	1	2	2			3	2			5			5		5
4	1	3	5	5		4	6	4		9	5		14		14
5	1	4	9	14	14	5	12	15	10	14	18	10	28	14	42

Table 3: Coefficients of the series $Q_{132}^{(0,\ell,0,0)}(t,x)|_{t^n x^k}$ for small values of n, k and ℓ.

Corollary 2.2 *$Q_{132}^{(1,0,0,0)}(t,x)|_{t^n x^k}$ is equal to the value $C_{n,k}$ of the Catalan's triangle.*

Theorem 2.1 gives another proof of Theorem 2 of [3] and the corollary constitutes a refinement of Theorem 5 of [3].

To illustrate Theorem 2.1, here is an example with $\ell = 2$ and $n = 4$. We have $Q_{4,132}^{(2,0,0,0)}(x) = 8 + 4x + 2x^2$ and Figure 4 shows the 14 Dyck paths of length 8 with marked decreasing steps ending at height greater than or equal to 2. We have eight Dyck paths under the horizontal line at height 2, four paths containing a single decreasing step ending at height 2, and two containing 2 such decreasing steps.

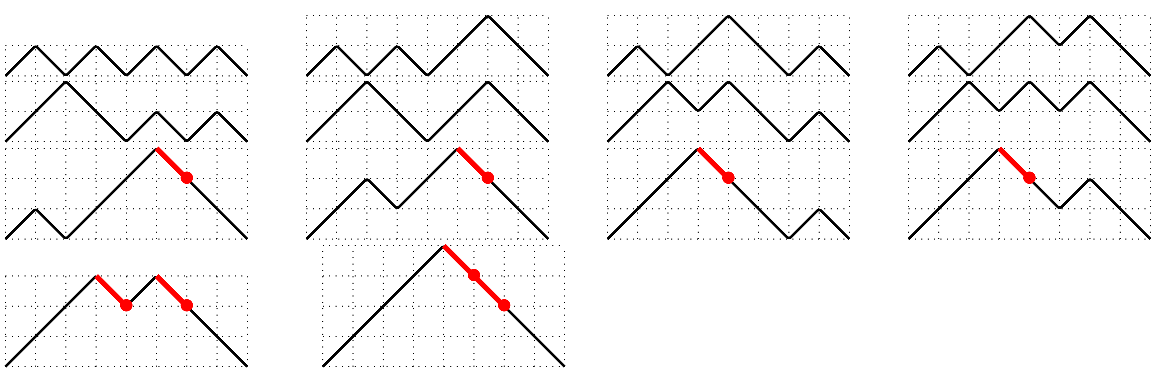

Figure 4: Decreasing steps ending at height greater than or equal to 2 in Dyck paths of length 8.

3 Quadrant II: the series $Q_{132}^{(0,\ell,0,0)}(t,x)$

Quadrant II is about classifying 132-avoiding permutations according to the number of values having ℓ greater values on their left. We will see that this statistic can be read on first values only. Table 3 displays the first values of $Q_{132}^{(0,\ell,0,0)}(t,x)|_{t^n x^k}$:

This statistic appears to be compatible with another kind of combinatorial object, the *nondecreasing parking functions*. Let n be a positive integer, we will call *nondecreasing parking functions* of length n the set of all nondecreasing functions f from $1, 2, \ldots, n$ into $1, 2, \ldots, n$ such that for all $1 \leqslant i \leqslant n : f(i) \leqslant i$. It is well-known that these functions are counted by the Catalan numbers. It is also known that they are counted by the Catalan's triangle (see Table 2) if we refine by the value of $f(n)$ and also counted by Narayana numbers (see Table 7) if we classify such functions by the number of different values of f.

3.1 The general result

Proposition 3.1 *Let σ be a 132-avoiding permutation. The statistic $mmp(0,1,0,0)$ only depends on the first value of σ. More precisely, $mmp(0,1,0,0)(\sigma) = \sigma_1 - 1$.*
Proof – Let $\sigma = \sigma_1 \ldots \sigma_n$ be a 132-avoiding permutation. The $\sigma_1 - 1$ values smaller than σ_1 are to the right of the first position thus see at least one greater value on their left, hence $mmp(0,1,0,0)(\sigma) \geqslant \sigma_1 - 1$. The $n - \sigma_1$ values greater than σ_1 must be ordered increasingly otherwise such two values together with the first value would form a 132 pattern, so all these values cannot see a greater value to their left and $mmp(0,1,0,0)(\sigma) \leqslant \sigma_1 - 1$.

Proposition 3.1 is the key to understanding combinatorics of the coefficients in $Q_{n,132}^{(0,\ell,0,0)}(x)$. This proposition can be rephrased as saying that σ_i matches the pattern $MMP(0,1,0,0)$ in a 132-avoiding permutation $\sigma = \sigma_1 \ldots \sigma_n$ if and only if $\sigma_1 > \sigma_i$. We can then iterate this idea. For example, we can decide by whether σ_i matches $MMP(0,2,0,0)$ in σ if and only if $\sigma_1 > \sigma_i$ and the $i-1$-st element of $red(\sigma_2 \ldots \sigma_n)$ is less than the first element of $red(\sigma_2 \ldots \sigma_n)$.

Theorem 3.2 *For $\ell \geqslant 1$, $Q_{132}^{(0,\ell,0,0)}(t,x)|_{t^n x^k}$ is equal to the number of 132-avoiding permutations in $\mathfrak{S}_n$ such that the* reduction *of its suffix of length $n+1-\ell$ begins with $k+1$.*
Proof – Another interpretation of Proposition 3.1 consists in noticing that if a value σ_i has a greater value on its left, the first value σ_1 of the permutation is also greater than σ_i. By induction on ℓ, we can prove that a value σ_i having ℓ greater values on their left is in fact smaller than the ℓ first values of $\sigma = \sigma_1 \ldots \sigma_n$. The base case $\ell = 1$ is proved by Proposition 3.1. Then if $\ell > 1$ and σ_i matches with $MMP(0,\ell,0,0)$ and $\tau = red(\sigma_2 \ldots \sigma_n) = \tau_1 \ldots \tau_{n-1}$, then we know that $\sigma_1 > \sigma_i$ and τ_{i-1} must match $MMP(0,\ell-1,0,0)$ so that by induction $\tau_{i-1} < \tau_j$ for $j = 1, \ldots, \ell-1$ and, hence, $\sigma_i < \sigma_j$ for $j = 1, \ldots, \ell$. Finally, it follows that if $mmp(0,\ell,0,0)(\sigma) = k$, then there are exactly k values of σ which are less than the first ℓ values of σ which implies that $red(\sigma_l \ldots \sigma_n)$ must start with $k+1$.

For example, with $\ell = 2$ and $n = 4$, we have $Q_{4,132}^{(0,2,0,0)}(x) = 4 + 6x + 4x^2$. Table 4 displays the 14 permutations of length 4 avoiding 132, their suffix of length $4+1-2 = 3$, the *reduction* of the suffix and the statistic $mmp(0,2,0,0)$. The permutation are enumerated by lexicographic order, however the reader can check that four of them contain no value having two greater values on their left, six contain one such value and the last four contain two such values.

σ avoiding 132	suffix of σ of length 3	reduction of the suffix	$mmp(0,2,0,0)(\sigma)$
1234	234	**1**23	0
2134	134	**1**23	0
23$\underline{\mathbf{1}}$4	314	**2**13	1
234$\underline{\mathbf{1}}$	341	**2**31	1
3124	124	**1**23	0
32$\underline{\mathbf{1}}$4	214	**2**13	1
324$\underline{\mathbf{1}}$	241	**2**31	1
34$\underline{\mathbf{12}}$	412	**3**12	2
34$\underline{\mathbf{21}}$	421	**3**21	2
4123	123	**1**23	0
42$\underline{\mathbf{1}}$3	213	**2**13	1
423$\underline{\mathbf{1}}$	231	**2**31	1
43$\underline{\mathbf{12}}$	312	**3**12	2
43$\underline{\mathbf{21}}$	321	**3**21	2

Table 4: Correlation between $mmp(0,2,0,0)$ and first value of the reduction of the suffix of length 3 for permutation of size 4 avoiding 132.

3.2 A new bijection between 132-avoiding permutations and non-decreasing parking functions

Let $\sigma \in \mathfrak{S}_n$ be a 132-avoiding permutation. We set

$$\phi(\sigma) := (mmp(0,n,0,0)+1, mmp(0,n-1,0,0)+1, \ldots mmp(0,1,0,0)+1)$$

Theorem 3.3 *Let n be a positive integer. ϕ is a bijection between 132-avoiding permutations of length n and non-decreasing parking functions of length n.*
Proof – First, we must check that the range of ϕ is contained in the set of non-decreasing parking functions. This is relatively easy to check since a value having k greater values on its left must be at position at least $k+1$ so that $mmp(0,k,0,0)+1 \leqslant n-k+1$. This fact that $\phi(\sigma)$ is non-decreasing following immediately from the fact that for all $\sigma \in S_n$ and for all $i \geqslant 1$, we have $mmp(0,i+1,0,0)(\sigma) \leqslant mmp(0,i,0,0)(\sigma)$.

We can build the inverse bijection using Proposition 3.1. We start with the empty permutation, then insert to its left each value from the non-decreasing parking function by a left to right reading and then add 1 to values greater than or equal to the inserted value (in this way, we keep a 132-avoiding permutation at each step).

Let us apply this algorithm on the parking function $(1, 1, 2, 4, 4, 6, 6, 6, 7)$.

$$\begin{array}{lrcl}
1 \text{ is read} & 1| & \to & 1 \\
1 \text{ is read} & 1|(1+1) & \to & 12 \\
2 \text{ is read} & 2|1(2+1) & \to & 213 \\
4 \text{ is read} & 4|213 & \to & 4213 \\
4 \text{ is read} & 4|(4+1)213 & \to & 45213 \\
6 \text{ is read} & 6|45213 & \to & 645213 \\
6 \text{ is read} & 6|(6+1)45213 & \to & 6745213 \\
6 \text{ is read} & 6|(6+1)(7+1)45213 & \to & 67845213 \\
7 \text{ is read} & 7|6(7+1)(8+1)45213 & \to & 768945213
\end{array}$$

The reader can check that this non-decreasing parking function corresponds to the permutation represented in Figure 1. Proposition 3.1 applied at each step gives the relation between the k^{th} value and $mmp(0, k, 0, 0)(\sigma)$.

Classified according to their last part (which corresponds to $mmp(0, 1, 0, 0)$ on the 132-avoiding permutations), non-decreasing parking function are counted by the Catalan's triangle (see Table 2). We thus have the following corollary which also follows from Corollary 2.2 and the fact that is proved in [3] that $Q_{132}^{(1,0,0,0)}(t, x) = Q_{132}^{(0,1,0,0)}(t, x)$.

Corollary 3.4 *The number of 132-avoiding permutations containing k values having a greater value on their left is the coefficient $C_{n,k}$ of the Catalan's triangle: $Q_{132}^{(0,1,0,0)}(t, x)|_{t^n x^k} = C_{n,k}$.*

3.3 The series $Q_{132}^{(0,k,0,0)}(t, x)$ at $x = 0$

In the article [3], the authors obtain by induction that their function $Q_{132}^{(0,k,0,0)}(t, 0)$ satisfies (Thm. 12 p. 24)

$$Q_{132}^{(0,k,0,0)}(t, 0) = \frac{1 + t \sum_{j=0}^{k-2} C_j t^j (Q_{132}^{(0,k-1-j,0,0)}(t, 0) - 1)}{1 - t}.$$

The first examples are

$$\begin{aligned}
Q_{132}^{(0,1,0,0)}(t, 0) &= \frac{1}{1 - t}, \\
Q_{132}^{(0,2,0,0)}(t, 0) &= \frac{1 - t + t^2}{(1 - t)^2}, \\
Q_{132}^{(0,3,0,0)}(t, 0) &= \frac{1 - 2t + 2t^2 + t^3 - t^4}{(1 - t)^3}, \\
Q_{132}^{(0,4,0,0)}(t, 0) &= \frac{1 - 3t + 4t^2 - t^3 + 3t^4 - 5t^5 + 2t^6}{(1 - t)^4}.
\end{aligned}$$

$n\backslash k$	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2
3	1	3	5	5	5	5	5	5
4	1	4	9	14	14	14	14	14
5	1	5	14	28	42	42	42	42
6	1	6	20	48	90	132	132	132
7	1	7	27	75	165	297	429	429
8	1	8	35	110	275	572	1001	1430

Table 5: Coefficients of $Q_{n,132}^{(0,k,0,0)}(0)$ for small values of n and k

It happens that these series is easy to compute directly. In a further paper [2], Kitaev and Liese explain that the coefficients are related to the Catalan's triangle.

Theorem 3.5 $Q_{n,132}^{(0,k,0,0)}(0)$ *is equal to the sum of the k first values of the n^{th} row of the Catalan's triangle.*

From a combinatorial point of view, it is simpler to present $Q_{n,132}^{(0,k,0,0)}(0)$ as a series in t in columns as in Table 5:

We thus recognize a simple induction :

Theorem 3.6 *We have*

$$Q_{n,132}^{(0,k,0,0)}(0) = \begin{cases} 1 & \text{if } n = 1 \text{ or } k = 1, \\ Q_{n-1,132}^{(0,k,0,0)}(0) + Q_{n,132}^{(0,k-1,0,0)}(0) & \text{if } n \geq k, \\ Q_{n,132}^{(0,k-1,0,0)}(0) & \text{if } n < k. \end{cases}$$

We can prove this property with at least three different approaches: the first one is analytic by noticing that Formula (24) of [3] at $x = 0$ imply our relations, the second method builds a bijection between the related sets, the last one shows that simple combinatorics objects counted by these numbers satisfy the induction.

3.3.1 The analytic proof

In order to simplify the notations, we write $R_n^k = Q_{n,132}^{(0,k,0,0)}(t, 0)$. Here C_n denotes the n^{th} Catalan number.

Formula (24) of [3] at $x = 0$ becomes

$$R_n^k = R_{n-1}^k + \sum_{i=1}^{k-1} C_{i-1} R_{n-i}^{k-i}.$$

If $n < k$, thus $n - 1 < k - 1$ and $n - i < k - i$, hence by induction on $n + k$ and using $R_n^n = C_n$, we get $R_n^k = R_n^{k-1}$.

If $n \geq k$, we assume the statement of the theorem for all pairs (n', k') with $n' + k' < n + k$. We have

$$\begin{aligned}
R_n^k &= R_{n-1}^k + \sum_{i=1}^{k-2} C_{i-1}(R_{n-i}^{k-i-1} + R_{n-i-1}^{k-i}) + C_{k-2}R_{n-k+1}^1 \\
&= R_{n-1}^{k-1} + \sum_{i=1}^{k-2} C_{i-1}R_{n-i}^{k-i-1} + R_{n-2}^k + \sum_{i=1}^{k-1} C_{i-1}R_{n-i-1}^{k-i}) \\
&= R_{n-1}^k + R_n^{k-1}.
\end{aligned}$$

3.3.2 The bijective proof on permutations

Let us denote by $S(n,k)$ the set of permutations avoiding 132 and avoiding the mesh pattern $(0,k,0,0)$. Then,

Proposition 3.7 *The set $S(n,k)$ is composed of permutations of size n avoiding 132 whose position i of 1 satisfy $i \leq k$. In particular, $S(n,k)/S(n,k-1)$ consists in elements where 1 is at the position k. It is equivalent to say that the descents composition of these permutations have their last part equal to $n+1-k$.*

Proof – Let p be a permutation avoiding 132 and let i the position of its 1. If $i > k$, then p does not avoid $(0,k,0,0)$. Otherwise, as p avoids 132, all values after 1 must be in increasing order. That implies that the number of points in quadrant II is maximal for the position i and that it is bounded by $i - 1 < k$. From this remark concerning the order of the elements after 1, we deduce the interpretation in terms of descents composition of p.

The set $S(n,k)$ splits into two blocks: the permutations for which 1 is at position k and the other. The second ones are directly elements of $S(n,k-1)$. Permutations avoiding 132 in which 1 is at position k are in bijection with element of $S(n-1,k)$ using the following process: remove the 1 and reduce the resulting sequence.

For example, one can check that $S(7,3)$ is composed of the 27 following permutations:

$$1234567, 2134567, 2314567, 3124567, 3214567, 3412567, 4123567, 4213567, 4312567,$$
$$4512367, 5123467, 5213467, 5312467, 5412367, 5612347, 6123457, 6213457, 6312457,$$
$$6412357, 6512347, 6712345, 7123456, 7213456, 7312456, 7412356, 7512346, 7612345$$

These 27 permutations are obtained as the union of the 7 elements of $S(7,2)$

$$1234567, 2134567, 3124567, 4123567, 5123467, 6123457, 7123456$$

and the 20 elements of $S(6,3)$

$$123456, 213456, 231456, 312456, 321456, 341256, 412356, 421356, 431256, 451236,$$
$$512346, 521346, 531246, 541236, 561234, 612345, 621345, 631245, 641235, 651234$$

in which we have incremented the values once then added a 1 in third position.

	$\ell=1$					$\ell=2$				$\ell=3$			$\ell=4$		$\ell=5$
$n\backslash k$	**0**	**1**	**2**	**3**	**4**	**0**	**1**	**2**	**3**	**0**	**1**	**2**	**0**	**1**	**0**
1	1					1				1			1		1
2	1	1				2				2			2		2
3	1	3	1			3	2			5			5		5
4	1	6	6	1		5	7	2		9	5		14		14
5	1	10	20	10	1	8	21	11	2	18	19	5	28	14	42

Table 6: Coefficients of the series $Q_{132}^{(0,0,\ell,0)}(t,x)|_{t^n x^k}$ for small values of n,k and ℓ.

3.3.3 Proof using other combinatorics objects

The Catalan's triangle counts for example the number of non-decreasing parking functions of size n and maximum k. Let us denote by $ND(n,k)$ the set of nondecreasing parking functions of size n and maximum at most k. Then $Q_{n,132}^{(0,k,0,0)}(t,0)$ counts the cardinality of $ND(n,k)$.

Thus, it is easy to check the induction formula of Theorem 3.6. That is, either $p \in ND(n,k)$ ends with a value smaller than k in which case it is an element of $ND(n,k-1)$ or it ends with k in which case removing the last of p results in an element of $ND(n-1,k)$. We point out that if $n < k$, p cannot end with k, which justifies why the second case is empty and thus the two cases of the theorem.

4 Quadrant III: the series $Q_{132}^{(0,0,\ell,0)}(t,x)$

These series refine the 132-avoiding permutations according to the number of values having k smaller values on their left. Table 6 displays the first values of the series $Q_{132}^{(0,0,\ell,0)}(t,x)|_{t^n x^k}$:

In their paper [3], the authors compute several terms of the function $Q_{132}^{(0,0,1,0)}(t,x)$. It begins as follows:

$$\begin{aligned}
&Q_{132}^{(0,0,1,0)}(t,x) = 1+t+(1+x)t^2+(1+3x+x^2)t^3+(1+6x+6x^2+x^3)t^4\\
&+(1+10x+20x^2+10x^3+x^4)t^5+(1+15x+50x^2+50x^3+15x^4+x^5)t^6\\
&+(1+21x+105x^2+175x^3+105x^4+21x^5+x^6)t^7\\
&+(1+28x+196x^2+490x^3+490x^4+196x^5+28x^6+x^7)t^8+...
\end{aligned}$$

and we recall that the Narayana triangle (A001263 of [7]) begins as in Table 7. We will denote $N(n,k)$ the Narayana number indexed by n and k.

Theorem 4.1 *We have*

$$\forall\ n \geqslant 1, \forall\ 0 \leqslant k < n, \quad Q_{132}^{(0,0,1,0)}(t,x)|_{t^n x^k} = N(n,k+1) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$$

$n \backslash k$	1	2	3	4	5	6	7	8
1	1							
2	1	1						
3	1	3	1					
4	1	6	6	1				
5	1	10	20	10	1			
6	1	15	50	50	15	1		
7	1	21	105	175	105	21	1	
8	1	28	196	490	490	196	28	1

Table 7: Narayana triangle $N(n,k)$ for $1 \leqslant k \leqslant n \leqslant 8$.

This theorem refines Theorem 11 of [3].

A way to see that is to consider the binary decreasing tree associated with a 132-avoiding permutation. Consider the permutation of Figure 1: $\sigma = 768945213$. We build a binary tree by inserting the greatest value (here 9), then split the permutation into two parts, 768 on the left and 45213 on the right. Values in the first part have to be inserted on the left of this 9 and the other part on the right. We still begin by the greatest remaining values of each sub-part. Therefore, the resulting tree is decreasing as we read paths from top to bottom (see Figure 5).

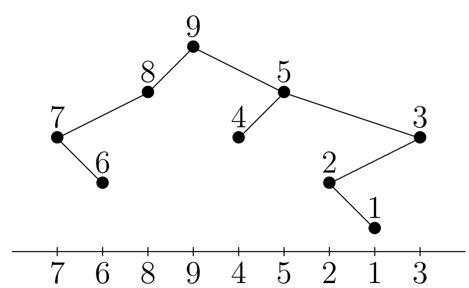

Figure 5: Construction of the binary decreasing tree associated with $\sigma = 768945213$.

This construction is well-known to describe a bijection as we can recover the permutation by gravity (or an infix deep first search reading of the tree). Moreover, if we forget the labels and only keep the binary tree shape, the previous map is still a bijection between 132-avoiding permutations and planar binary trees. The avoidance of pattern 132 ensures that the tree can be uniquely labeled by $1, 2, \ldots, n$ decreasingly. The labels must be set from n, then $n-1, \ldots, 1$ following a prefix depth first search of the tree. As prefix depth first search consists in recursively parsing the root, the left child then the right child, the resulting tree is therefore decreasing and does not contain any 132 pattern.

In the permutation $\sigma = 768945213$, the four values 8, 9, 5 and 3 see at least a point in quadrant III. These values correspond exactly to the nodes of the corresponding tree having a left child.

Proposition 4.2 *Let σ be a 132-avoiding permutation, the number $mmp(0,0,1,0)$ of positions matching $MMP(0,0,1,0)$ is equal to the number of left branches in the binary decreasing tree associated with σ.*
Proof – This fact is obvious due to the construction. A value to the left and smaller than a given value must be inserted on the left and deeper as the tree is decreasing, and on the other side, a node having a left child sees at least one smaller value on its left. As the tree is decreasing and values inserted on the left are left in the permutation, any node having a left child matches the pattern $MMP(0,0,1,0)$.

Proposition 4.3 *The number of binary trees over n nodes containing k left (or right) branches is counted by the Narayana number $N(n,k+1)$.*
Proof – There exist different proofs of this result. One of them consists in checking that both quantities satisfy the same induction. Narayana numbers can be define recursively [5] by

$$\forall n \geqslant 1, 1 \leqslant k \leqslant n : N(n,1) = N(n,n) = 1,$$

$$N(n,k) = N(n-1,k) + \sum_{i=1}^{n-1} \sum_{r=1}^{k-1} N(n-1-i,k-r)N(i,r).$$

We have exactly the same induction when counting binary trees along the number of left branches. Let us consider a binary tree over n nodes having k left branches. If the root of the tree have no left child (case 1 or Figure 6), the right child must be a subtree over $n-1$ nodes that still has k left branches. Otherwise, the root has a left child which is not the empty tree, thus that can contains some left branches (case 2 or Figure 6).

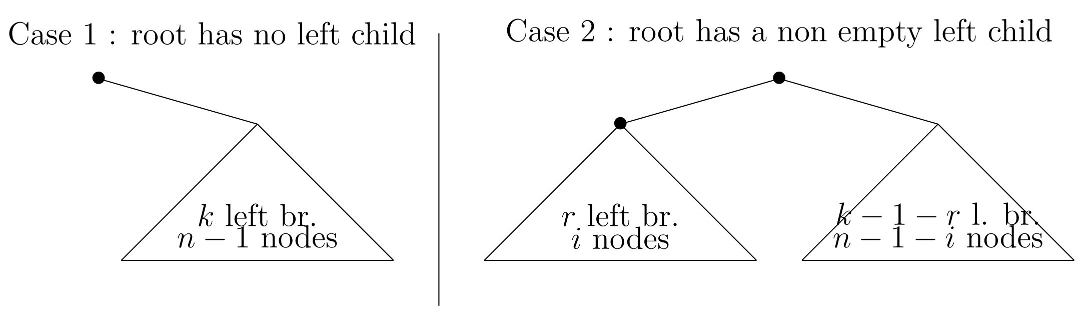

Figure 6: Counting binary trees along number of left branches

In this second case, the right child can be empty. The sum of nodes over both child must be $n-1$ and these must contain together $k-1$ left branches (since the root has already one). Finally, as the two cases are exclusive and cover all possibilities, the number of binary trees over n nodes with k left branches satisfy the induction of $N(n, k+1)$ (modulo the change of indices $k' \leftarrow k+1$). It just remains just to notice that there exists only a single binary tree over n nodes having 0 left branch or $n-1$ left branches. These border conditions are the same than these for Narayana numbers with the change of indices on the second parameter.

Proposition 4.3 and the previous observation prove Theorem 4.1. We now extend this previous statement from pattern $(0,0,1,0)$ to pattern $(0,0,\ell,0)$ with ℓ any positive integer.

Theorem 4.4 $Q_{132}^{(0,0,\ell,0)}|_{t^n x^k}$ *is equal to the number of binary trees over n nodes which contain exactly k left subtrees of at least ℓ nodes.*

Proof – This is a straightforward generalization. The bijection between binary trees and 132-avoiding permutations contains this result, a position matching the pattern $(0,0,\ell,0)$ has a value that sees at least ℓ smaller values on its left. All these smaller and left values are inserted to the left of the corresponding node in the binary decreasing tree. This gives another proof of the second part of Theorem 3 of [3].

For example, with $\ell = 2$ and $n = 4$, we have $Q_{4,132}^{(0,0,\ell,0)}(x) = 5 + 7x + 2x^2$ as one can check on Figure 7 displaying the fourteen binary trees over 4 nodes in which the left sub-trees over at least 2 nodes have been circled.

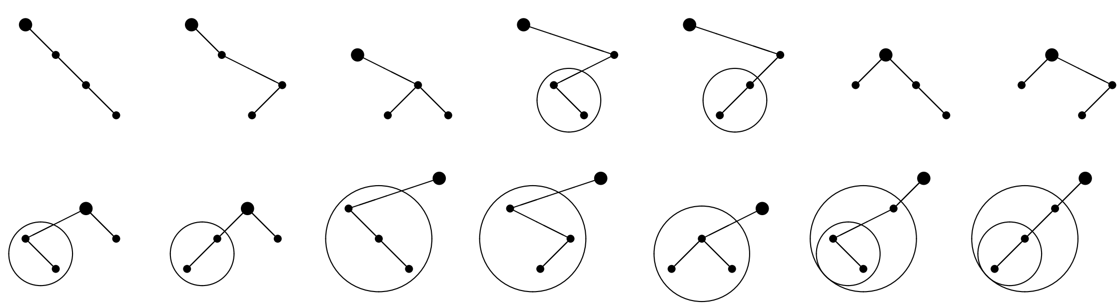

Figure 7: Decreasing trees of 132-avoiding permutations of length 4.

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