Equilibrium pricing under relative performance concerns

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Abstract

We investigate the effects of the social interactions of a finite set of agents on an equilibrium pricing mechanism. A derivative written on non-tradable underlyings is introduced to the market and priced in an equilibrium framework by agents who assess risk using convex dynamic risk measures expressed by Backward Stochastic Differential Equations (BSDE). Each agent is not only exposed to financial and non-financial risk factors, but she also faces performance concerns with respect to the other agents.

Within our proposed model we prove the existence and uniqueness of an equilibrium whose analysis involves systems of fully coupled multi-dimensional quadratic BSDEs. We extend the theory of the representative agent by showing that a non-standard aggregation of risk measures is possible via weighted-dilated infimal convolution. We analyze the impact of the problem's parameters on the pricing mechanism, in particular how the agents' performance concern rates affect prices and risk perceptions. In extreme situations, we find that the concern rates destroy the equilibrium while the risk measures themselves remain stable.

Keywords: Financial innovation, equilibrium pricing, social interactions, performance concerns, representative agent, g-conditional risk measure, multidimensional quadratic BSDE, entropic risk.

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1 Introduction

The importance of relative concerns in human behavior has been emphasized both in economic and sociological studies; making a 1 EUR profit when everyone else made 2 EUR "feels" distinctly different had everyone else lost 2 EUR. A diverse literature handling problems dealing with some form of strategic and/or social interaction in the form of relative performance concerns exists: In both [HRAY10] and [Ped13] the social interaction component appears in the form of peer-based under-performance penalties known as "Minimum Return Guarantees"; the comparison is usually done via tracking a relevant market index, something quite standard in pension fund management. Another type of performance concerns arises in problems where the agents' consumption is taken into account. The utility functions used there exhibit a "keeping up with the Joneses" behavior as introduced in [Due49] and developed by [Abe90], [Abe99] (see further [Gal94], [CK01], [Góm07] and [XZ09]); in other words, the benchmark for the standard of living is the averaged consumption of the population and one computes the individual's consumption preferences in relation to that benchmark. Another type of concern criterion, an internal one, uses the past consumption of the agent as a benchmark for the current consumption; [HERH73] introduced this "habit formation" approach. A more mathematical finance approach, as well as a literature overview, can be found in [ET15] or [FDR11]. These last two papers are the inspiration for this one.

In this paper we study the effects of social interaction between economic agents on a market equilibrium, the efficiency of a securitization mechanism and the global risk. We consider a finite set $\mathbb{A}$ of N agents having access to an incomplete market consisting of an exogenously priced liquidly traded financial asset. The incompleteness stems from a non-tradable external risk factor, such as the amount of rain or the temperature, to which those agents are exposed. In an attempt to reduce the individual and overall market risks, a social planner introduces to the market a derivative written on the external risk source, allowing the agents in $\mathbb{A}$ to reduce their exposures by trading on it. The question of the actual completeness of the resulting market has been addressed in some generality in the literature, and we refer for instance to [Sch15]. Questions about pricing and benefits of such securities written on non-tradable assets have been approached in the literature many times, we refer in particular to [HPDR10] where the new derivative is priced within an equilibrium framework according to supply and demand rules. Equilibrium analysis of incomplete markets is commonly confined to certain cases such as single agent models ([HL93], [GPP09]), multiple agent models where markets are complete in equilibrium ([DH85], [HPDR10], [KLS90]), or models with particular classes of goods ([JRW10]) or preferences ([CFHP10]). An equilibrium in an incomplete market is studied in [KXŽ15].

Although we follow ideas similar to those in [HPDR10], our goal is to understand how such a pricing mechanism and risk assessments are affected when the agents have relative performance concerns with respect to each other. Each agent $a \in \mathbb{A}$ has an endowment H^a over the time period $[0, T]$ depending on both risk factors. Her investment strategy π^a in stock and the newly introduced derivative induces a gains process $(V_t^a(\pi^a))_{t \in [0,T]}$. For a given *performance concern rate* $\lambda^a \in [0, 1]$ the agent seeks to minimize the risk

$$\rho^a \left(H^a + (1 - \lambda^a) V_T^a(\pi^a) + \lambda^a \left(V_T^a(\pi^a) - \frac{1}{N - 1} \sum_{b \in \mathbb{A} \setminus \{a\}} V_T^b(\pi^b) \right) \right), \tag{1.1}$$

where ρ^a is a risk measure (ρ^a is further described below). The first two terms inside ρ^a correspond to the classical situation of an isolated agent a trading optimally in the market to profit from market movements and to hedge the financial risks inherent to H^a. The last term is the *relative performance concern* and corresponds to the difference between her own trading gains and the average trading

gains achieved by her peers. Intuitively, as $\lambda^a \in [0,1]$ increases, the agent is less concerned with the risks associated to her endowment H^a and more concerned with how she fares against the average performance of the other agents in $\mathbb{A}$. For instance, given no endowments, if $\lambda^a = 1/2$ and agent a made 1 EUR from trading while the others all made 2 EUR then she perceives no gain at all.

Each agent $a \in \mathbb{A}$ uses a monetary convex risk measure ρ^a. The theory of monetary, possibly convex, possibly coherent, risk measures was initiated by [ADEH99] and later extended by [FS02] and [FRG02]. One special class of risk measures, the so-called g-*conditional risk measures*, which are closely related to the so-called g-*conditional expectations* (see [Gia06]), are those defined through Backward Stochastic Differential Equations (BSDEs), see [Pen97], [EKR09] and [BE09]. Our use of BSDEs is motivated by two general aspects. The first is that it generically allows to solve stochastic control problems away from the usual Markovian setup where one uses the HJB approach in combination with PDE theory, see e.g. [Tou13]. The second is that optimization can be carried out in closed sets of constraints without the assumption of convexity for which one usually uses duality theory, see [HIM05].

The form of relative performance concern we use and its study using BSDEs can be traced back to [Esp10] and [ET15]. Their setting is quite different from that presented here; the authors show the existence of a Nash equilibrium for a pure-interaction game of optimal investment without idiosyncratic endowments to hedge ($H^a = 0$ for all agents), and where the agents optimize, in a Black–Scholes stock market, the expected utility of the gains they make from trading under individual constraints. These two works are followed by [FDR11], where a general discussion on the existence of equilibrium, with endowments, is given including counter examples to such existence.

Methodology and content of the paper. All agents optimize their respective functional given by (1.1), and since the derivative is priced endogenously via an equilibrium framework, the market price of external risk is also part of the problem's solution. Equilibrium in our game is a set of acceptable investment strategies for the agents and a market price of external risk giving rise to a certain martingale measure.

In the first part of this work, we show the existence of the Nash equilibrium in our problem and how to compute it for general risk measures induced by BSDEs. The analysis is carried out in two steps. The first involves solving the individual optimization problem for each agent given the other agents' actions, the so-called best response problem. The second consists in showing that it is possible to find all best responses simultaneously in such a way that supply and demand for the derivative match, which, in turn, yields the market price of external risk. (We generally think of a market with a zero net supply of the derivative; however, the methodology allows to treat cases where some agents who were allowed to trade in the derivative left the market such that the (active) agents in $\mathbb{A}$ hold together a non-zero position). We then verify that the market price of external risk associated to the best responses satisfies the necessary conditions.

This last step is more complex. Since the agents assess their risks using dynamic risk measures given by BSDEs, the general equilibrium analysis leads to a system of fully coupled non-linear multi-dimensional BSDEs (possibly quadratic). We proceed using the representative agent approach (see [Neg60]) where aggregation of the agents into a single economy and optimal Pareto risk sharing are equivalent to simultaneous individual optimization. From the works of [BEK05] and [BE09], we make use of infimal convolution (short inf-convolution) techniques in order to build a single risk measure that encompasses the risk measures for each agent and through which it is possible to find a single representative economy. We point out that in order to cope with the cross dependence induced by the performance concern rates, standard inf-convolution techniques do not lead to a single representative economy. We use the technique in a non-standard fashion via convenient weighted-dilations of each agent's risk measure ρ^a (see Section 3.2 in [BE09] and Section 4 below);

to the best of our knowledge this type of analysis is new and of independent interest. The closest reference to this is [Rüs13], where some form of weighted inf-convolution appears.

The second part of this work focuses on the case of agents using entropic risk measures, which can be treated more explicitly and allows for an in-depth study of the impact of the concern rates. In identifying the Nash equilibrium, we are led to a system of fully coupled multi-dimensional quadratic BSDEs whose analysis is, in general, quite involved. Based on the works of [Esp10] and [ET15], the authors of [FDR11] give several counter examples to the existence of solutions; nonetheless, positive results do exist although none are very general, see e.g. [Tev08], [CN15], [Fre14], [KP14], [HT14] and [JKL14]. In our case we are able to solve the system.

Findings. Within the case of entropic risk measures, we study in detail a model of two agents a and b with opposite exposures to the external risk factor, so that one has incentives to buy the derivative while the other has incentive to sell. In this model we are able to specify the structure of the equilibrium. Using both analytical methods and numerical computations, when the analytics are not tractable, we explore the behavior of the agents as the model parameters vary. We give particular attention to how the relative performance concern rates, and thereby the strength of the coupling between the agents, deviate from the standard case of non-interacting agents (when $\lambda^a = \lambda^b = 0$).

We find that when either agent's risk tolerance increases, their risk lowers. If any concern rate λ increases, we find that the agents engage in less trading of the derivative. This is because every unit of derivative bought by one is a unit sold by the other and hence the gains of one are the losses of the other. Consequently, if an agent is more concerned about the relative performance, she will tend to trade less volume with the other. Also, as expected, we find that if it is the buyer of derivatives whose concern rate increases, the derivative's price decreases, while it increases in the case of the seller.

Very interestingly, we find that the risk of a single agent increases if the other agents become more concerned with their relative performance but that it *decreases* as this agent becomes more concerned. Consequently, if the agents were to play this game repeatedly and their concern rate were to vary over time, they would both find it more advantageous to become more concerned (or jealous). As they both do so, the trading activity in the derivative decreases, but their activity in the stock increases and explodes - the equilibrium does not exist anymore. Surprisingly, this behavior is *not captured at all by the risk measures*! This non-trivial, and perhaps not desirable, behavior of the system after introduction of the derivative is not without similarities with what is found in the models of [CMV09] and [CML13]. It is a reminder that, when evaluating the benefits of financial innovation, one should not focus of the economy of an individual agent (who sees clear benefits in the form of a risk reduction) but really have a systemic view of the impact of the new instrument.

Organization of the paper. In Section 2, we define the general market, agents, optimization problem and equilibrium that we consider. Sections 3 and 4 are devoted to solving the general optimization problem for a set of agents having arbitrary risk measures. In the former we solve the optimization problem for each agent, given the strategies of all others. In the latter we deal with the aggregation of individual risk measures and identification of the representative agent, and we solves the equilibrium for the whole system. Sections 5 and 6 contain the particular case where the agents use entropic risk measures. In this more tractable setting, Section 5 explores theoretically the influence of various parameters on the global risk while Section 6 focuses on a model with 2 agents with opposite risk profiles, and thoroughly explores the influence of the concern rates, in particular, on the individual behaviors, risks, and the consequences for the whole system. We also present numerical results. Section 7 concludes the study.

2 The model

We consider a finite set $\mathbb{A}$ of N agents, with random endowments H^a, $a \in \mathbb{A}$, to be received at a terminal time $T < \infty$. They trade continuously in the financial market which comprises a *stock* and a newly introduced *structured security* (called derivative), aiming to minimize their risk. For simplicity we assume that money can be lent or borrowed at the risk-free rate zero. Stock prices follow an *exogenous* diffusion process and are not affected by the agents' demand. By contrast, the derivative is traded only by the agents from $\mathbb{A}$ and priced *endogenously* such that demand matches supply.

2.1 The market

Sources of risk, underlyings

Throughout this paper $t \in [0, T]$. In our model, there are two independent sources of randomness, represented by a 2-dimensional standard Brownian motion $W = (W^S, W^R)$ on a standard filtered probability space[1] $(\Omega, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$. The Brownian motion W^R drives the external and non-tradable risk process (R_t), which is thought of as a temperature process or a precipitation index. For analytical convenience we assume that (R_t) follows a Brownian motion with drift being a stochastic process $\mu^R : \Omega \times [0, T] \to \mathbb{R}$ and constant volatility $b > 0$, i.e.,

$$\mathrm{d}R_t = \mu_t^R \mathrm{d}t + b\mathrm{d}W_t^R, \qquad \text{with } R_0 = r_0 \in \mathbb{R}. \tag{2.1}$$

The Brownian motion W^S drives the stock price process (S_t) according to

$$\begin{aligned} \mathrm{d}S_t &= \mu_t^S S_t \, \mathrm{d}t + \sigma_t^S S_t \, \mathrm{d}W_t^S, \\ &= \mu_t^S S_t \, \mathrm{d}t + \langle \sigma_t, \mathrm{d}W_t \rangle \quad \text{with } \sigma_t := (\sigma_t^S S_t, 0) \in \mathbb{R}^2, \text{ and } S_0 = s_0 > 0, \end{aligned} \tag{2.2}$$

for $(\mathcal{F}_t)$-adapted stochastic processes $\mu^S, \sigma^S : \Omega \times [0, T] \to \mathbb{R}$, with $\sigma^S > 0$.

Market price of risk: financial and external

We recall (see e.g. [HM07]) that any linear pricing scheme on the set $L^2(\mathbb{P})$ of square integrable random variables with respect to $\mathbb{P}$ can be identified with a 2-dimensional predictable process θ such that the exponential process $(\mathcal{E}_t^\theta)$ defined by

$$\mathcal{E}_t^\theta := \mathcal{E}\left(-\int_0^\cdot \theta_s \mathrm{d}W_s\right)_t = \exp\left\{-\int_0^t \theta_s \mathrm{d}W_s - \frac{1}{2}\int_0^t \|\theta_s\|^2 \mathrm{d}s\right\}, \quad t \in [0, T], \tag{2.3}$$

is a uniformly integrable martingale. This ensures that the measure $\mathbb{P}^\theta$ defined by having density $\mathcal{E}_T^\theta$ against $\mathbb{P}$ is indeed a probability measure (the *pricing measure*), and the present price of a random terminal payment X is then given by $\mathbb{E}^\theta[X]$, where $\mathbb{E}^\theta$ denotes the expectation with respect to $\mathbb{P}^\theta$. For any such θ, we introduce the $\mathbb{P}^\theta$-Brownian motion

$$W_t^\theta = W_t + \int_0^t \theta_s \, \mathrm{d}s, \qquad t \in [0, T].$$

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[1] $(\mathcal{F}_t)$ is the filtration generated by W and augmented by the $\mathbb{P}$-null sets. We point the reader to Appendix A for a full overview of the notation and stochastic setup.

The first component θ^S of the vector $\theta := (\theta^S, \theta^R)$ is the *market price of financial risk*. Under the assumption that there is no arbitrage, S must be a martingale under $\mathbb{P}^\theta$ and, from the exogenously given dynamics of S, θ^S is necessarily given by $\theta_t^S = \mu_t^S / \sigma_t^S$. The process θ^R on the other hand is currently unknown. It is the *market price of external risk* and will be derived *endogenously* by the market clearing condition and constant net supply.

The agents' endowments and the derivative's payoffs

The agents $a \in \mathbb{A}$ receive at time T the income H^a, which depends on the financial and external risk factors. Specifically,

$$H^a = h^a(S_T, R_T) + \int_0^T \varphi^a(u, S_u, R_u) \mathrm{d}u. \tag{2.4}$$

While the agents are able to trade in the financial market to hedge away some of their financial risk, a basis risk remains originating in the agent's exposure to the non-tradable risk process R.

The derivative also pays a yield at a rate φ^D (received at maturity) and a final amount h^D. The payoff H^D at maturity time T is

$$H^D = h^D(S_T, R_T) + \int_0^T \varphi^D(u, S_u, R_u) \, \mathrm{d}u. \tag{2.5}$$

By trading in the derivative H^D, the agents have now a way to reduce their basis risk. We make the assumption that the functions φ and h, hence the payoffs H^a and H^D, are bounded. We assume the same for the coefficients in the dynamics of S and R.

Assumption 2.1. *The processes μ^R, μ^S, σ^S and $\theta^S = \mu^S / \sigma^S$ are bounded. The random variables H^D and H^a, $a \in \mathbb{A}$, are bounded; the functions $h^\cdot$ and $\varphi^\cdot$ are continuous and bounded.*

Price of the derivative, trading in the market and the agent's strategies

Assuming no arbitrage opportunities, the price process (B_t^θ) of H^D is given by its expected discounted payoff under $\mathbb{P}^\theta$; in other words $B_\cdot^\theta = \mathbb{E}^\theta\left[H^D | \mathcal{F}_\cdot\right]$. Since H^D is bounded, writing the $\mathbb{P}^\theta$-martingale as a stochastic integral against the $\mathbb{P}^\theta$-Brownian motion W^θ (using Martingale representation theorem) yields a 2-dimensional adapted process $\kappa^\theta := (\kappa^S, \kappa^R)$ such that

$$B_t^\theta = \mathbb{E}^\theta[H^D] + \int_0^t \kappa_s^\theta \mathrm{d}W_s^\theta = \mathbb{E}^\theta[H^D] + \int_0^t \kappa_s^\theta \mathrm{d}W_s + \int_0^t \left\langle \kappa_s^\theta, \theta_s \right\rangle \mathrm{d}s. \tag{2.6}$$

We denote by $\pi_t^{a,1}$ and $\pi_t^{a,2}$ the number of units agent $a \in \mathbb{A}$ holds in the stock and the derivative at time $t \in [0, T]$, respectively. Using a self-financing strategy $\pi^a := (\pi^{a,1}, \pi^{a,2}) \in \mathbb{R}^2$, her gains from trading up to time $t \in [0, T]$, under the pricing measure $\mathbb{P}^\theta$ (inducing the prices (B_t^θ) for the derivative), is given by

$$\begin{aligned} V_t^a = V_t(\pi^a) &= \int_0^t \pi_s^{a,1} \, \mathrm{d}S_s + \int_0^t \pi_s^{a,2} \, \mathrm{d}B_s^\theta \\ &= \int_0^t \left\langle \pi_s^{a,1} \sigma_s + \pi_s^{a,2} \kappa_s^\theta, \theta_s \right\rangle \, \mathrm{d}s + \int_0^t (\pi_s^{a,1} \sigma_s + \pi_s^{a,2} \kappa_s^\theta) \, \mathrm{d}W_s. \end{aligned} \tag{2.7}$$

2.2 Preferences, risk minimization and equilibrium

The agents' measure of risk

The agents assess risk using a dynamic convex time-consistent risk measure ρ^a induced by a Backward Stochastic Differential Equation (BSDE). This means that the risk $\rho_t^a(\xi^a)$ which agent a associates at time t with an $\mathcal{F}_T$-measurable random wealth ξ^a is given by Y_t^a, where (Y^a, Z^a) is the solution to the BSDE (with some abuse of notation: $\rho_t^a(\xi^a) := Y_t^a$)

$$-\mathrm{d}Y_t^a = g^a(t, Z_t^a)\mathrm{d}t - Z_t^a\mathrm{d}W_t \quad \text{with terminal condition} \quad Y_T^a = -\xi^a.$$

The driver g^a encodes the agent's risk preferences. Throughout this work we make the standing assumption that the map $g^a : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is a deterministic continuous function, that $z \mapsto g^a(\cdot, z)$ is continuously differentiable, strictly convex and that $\nabla_z g^a = 0$ has at least one solution. The agents' risk measure given by the above BSDE is strongly time consistent, convex and translation invariant (or monetary). We do not give many details on the class of risk measures described by BSDEs, instead, we point the interested reader to [BEK05, Gia06, BE09].

For convenience, we recall the relevant properties of the risk measures that play a role in this work: i) *translation invariance*: for any $m \in \mathbb{R}$ it holds that $\rho_t^a(\xi^a + m) = \rho_t^a(\xi^a) - m$; ii) *time-consistency* of the process $\left(\rho_t^a(\xi^a)\right)$: for any $t, t+s \in [0,T]$ it holds that $\rho_t^a(\xi^a) = \rho_t^a(\rho_{t+s}^a(\xi^a))$; and iii) *convexity*: for $\xi^a, \hat{\xi}^a$ $\mathcal{F}_T$-measurable and $\alpha \in [0,1]$ we have $\rho_t^a\left(\alpha\xi^a + (1-\alpha)\hat{\xi}^a\right) \leq \alpha\rho_t^a(\xi^a) + (1-\alpha)\rho_t^a(\hat{\xi}^a)$.

The individual optimization problem

Agent a's total endowment ξ^a at maturity is given by the sum of her terminal income H^a and the trading gains V_T^a over the time period $[0,T]$. However, the agent compares her trading gains $V_T^a = V_T(\pi^a)$ with the average gains of all other agents. Thus, we define the risky position of each of the N agents $a \in \mathbb{A}$ in the market (compare with (1.1)) at time $t = T$ as

$$\xi^a = \left(H^a + \left(1 - \lambda^a\right)V_T(\pi^a)\right) + \lambda^a\left(V_T(\pi^a) - \frac{1}{N-1}\sum_{b \in \mathbb{A}\setminus\{a\}} V_T(\pi^b)\right)$$

$$= H^a + V_T(\pi^a) - \frac{\lambda^a}{N-1}\sum_{b \in \mathbb{A}\setminus\{a\}} V_T(\pi^b),$$

where $\lambda^a \in [0,1]$ is the *concern rate* (or *jealousy factor*) of agent $a \in \mathbb{A}$.

We make the following assumption on the concern rates $\lambda^{\cdot}$, whose justification will become clear later on in Section 3.3.5.

Assumption 2.2 (Performance concern rates)**.** *We have* $\lambda^a \in [0,1]$ *for each agent[2] and* $\prod_{a \in \mathbb{A}} \lambda^a < 1$.

For notational convenience we introduce the $\left(\mathbb{R}^2\right)^{(N-1)}$-valued vector $\pi^{-a} := (\pi^b)_{b \neq a}$ and

$$\bar{V}_t^{-a} := \widetilde{\lambda}^a \sum_{b \in \mathbb{A}\setminus\{a\}} V_t(\pi^b) = \widetilde{\lambda}^a V_t(\bar{\pi}^{-a}) \quad \text{where} \quad \widetilde{\lambda}^a := \frac{\lambda^a}{N-1} \quad \text{and} \quad \bar{\pi}^{-a} := \sum_{b \in \mathbb{A}\setminus\{a\}} \pi^b. \quad (2.8)$$

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[2] The case of $\lambda^a > 1$ is, as we show in Section 3.3.5, not necessarily intractable, but the analysis of such a situation is beyond the scope of this paper.

The risk associated with the self-financing strategy π^a evolves according to the BSDE

$$-\mathrm{d}Y_t^a = g^a\left(t, Z_t^a\right)\mathrm{d}t - Z_t^a\mathrm{d}W_t \quad\text{and}\quad Y_T^a = -\left(H^a + V_T(\pi^a) - \widetilde{\lambda}^a \sum_{b\in\mathbb{A}\setminus\{a\}} V_T(\pi^b)\right). \tag{2.9}$$

Notice that (Y^a, Z^a) depends on the strategies $\pi = (\pi^a, \pi^{-a})$ and we do not emphasize it in the notation.

Next we introduce a notion of admissibility for our problem. We note that at this point we are not specifying g^a and hence the admissibility here is a general one.

Definition 2.3 (Admissibility)**.** *The $\mathbb{R}^2$-valued strategy process π^a is called admissible with respect to the market price of risk θ if the BSDE (2.9) has a unique solution in $\mathcal{S}^2 \times \mathcal{H}^2$ and if*

$$\mathbb{E}\left[\langle\, V_{\cdot}(\pi^a)\,\rangle_T\right] < \infty, \tag{2.10}$$

where $\langle V_{\cdot}(\pi^a)\rangle$ denotes the quadratic variation of $\left(V_t(\pi^a)\right)_{t\in[0,T]}$. The set of all admissible trading strategies of agent $a \in \mathbb{A}$ with respect to θ is denoted by $\mathcal{A}^{\theta,a} = \mathcal{A}^\theta$.

We point out that in full generality each agent could have her own admissibility set $\mathcal{A}^{\theta,a}$, for example due to different trading constraints, as in [ET15] or [FDR11]. Here we assume that the agents have no trading constraints, aside from their strategies being integrable against the prices.

Endogenous trading and competitive equilibrium

We denote by $n \in \mathbb{R}$ the number of units of derivative present in the market. While each unit of derivative pays H^D at time T, the agents are free to buy and underwrite contracts for any amount of H^D, so that n is not necessarily an integer. We think essentially of the case $n = 0$, where every derivative held by an agent has been underwritten by another agent in $\mathbb{A}$, entailing essentially that agents share their risks with each other (see [BEK05, BE09] or [HM07]). Building upon [HPDR10] allows for a bit more flexibility as $n \neq 0$ is possible[3]. In any case, over $[0, T]$, only the agents in our set $\mathbb{A}$, with trading objectives as described above, are active in the market and so the total number n of derivatives present is constant.

Equilibrium in this setting is defined below; the concept has similarities to that in [CHKP13].

Definition 2.4 (Equilibrium and equilibrium MPR (EMPR))**.** *For a given* Market Price of Risk *(MPR)* $\theta = (\theta^S, \theta^R)$, *we call $\pi^* = (\pi^{*,a})_{a\in\mathbb{A}}$ an equilibrium if $\pi^{*,a} \in \mathcal{A}^\theta$ for all $a \in \mathbb{A}$ and the following two conditions hold:*

1. *For all $a \in \mathbb{A}$ and for all $\pi^a \in \mathcal{A}^\theta$, $Y_0^a(\pi^{*,a}, \pi^{*,-a}) \leq Y_0^a(\pi^a, \pi^{*,-a})$, i.e. individual optimality given the strategies of the other agents; and*

2. *the market clearing condition (or fixed supply condition) for the derivatives holds:*

$$n = \sum_{a\in\mathbb{A}} \pi_0^{*,a,2} = \sum_{a\in\mathbb{A}} \pi_t^{*,a,2} \quad \mathbb{P}\otimes\lambda - a.e. \tag{2.11}$$

We call θ^R an Equilibrium Market Price of external Risk *(EMPeR) if*

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[3]The situation $n \neq 0$ would be possible if, prior to time $t = 0$, another agent $a_0 \notin \mathbb{A}$ was on the derivatives market and then stopped her activity, for instance a_0 might have had as objective to buy $m > 0$ units, according to her own specific criteria, in which case $n = -m < 0$.

- *there exists an equilibrium π^* for θ,*

- *and $\theta = (\theta^S, \theta^R)$ makes $\mathbb{P}^\theta$ a true probability measure (equivalently, $\mathcal{E}^\theta$ from (2.3) is a uniformly integrable martingale); in this situation we name θ the* Equilibrium MPR *(EMPR).*

Otherwise said, π^* defines a Nash equilibrium and satisfies the market clearing condition for H^D. The equilibrium and solution to the problem are the optimal strategies π^* and the EMPR θ.

3 The single agent's optimization and unconstrained equilibrium

In this section and the one following, we study the solvability of the equilibrium problem for a general risk measure induced by a BSDE. Prior to looking if an equilibrium market price of external risk exists, we solve first for the behavior of the individual agent (the so-called best response problem); then, we assert the solvability of the equilibrium as a whole. We solve both results without first imposing the fixed supply-demand constraint. The solution we present is very constructive and it shows the layers of complexity being added to the problem. We close with the constraint problem. Throughout this section assume a given MPR θ.

3.1 Optimal response for one agent

Optimizing residual risk

To solve the optimization problem for agent a, we first observe that, at each time t, the strategy chosen must minimize the residual risk: The additivity of the risk measure implies (with $V_T = (V_T - V_t) + V_t$ and translation invariance)

$$Y_t^a = \rho_t^a \Big(H^a + V_T^a - \widetilde{\lambda}^a \bar{V}_T^{-a} \Big) = \rho_t^a \Big(H^a + (V_T^a - V_t^a) - \widetilde{\lambda}^a (\bar{V}_T^{-a} - \bar{V}_t^{-a}) \Big) - \Big(V_t^a - \widetilde{\lambda}^a \bar{V}_t^{-a} \Big).$$

This suggests applying the following change of variables to (2.9) (recall (2.8)),

$$
\begin{aligned}
\widetilde{Y}_t^a &:= Y_t^a + \big(V_t^a - \widetilde{\lambda}^a \bar{V}_t^{-a} \big), \\
\widetilde{Z}_t^a &:= Z_t^a + \zeta_t, \quad \text{where} \quad \zeta_t = \big(\pi_t^{a,1}\sigma_t + \pi_t^{a,2}\kappa_t^\theta \big) - \widetilde{\lambda}^a \big(\bar{\pi}_t^{-a,1}\sigma_t + \bar{\pi}_t^{-a,2}\kappa_t^\theta \big) \in \mathbb{R}^2.
\end{aligned}
\tag{3.1}
$$

Direct computations imply that $(\widetilde{Y}^a, \widetilde{Z}^a)$ solves the BSDE

$$-d\widetilde{Y}_t^a = \widetilde{g}^a \Big(t, \pi_t^a, \pi_t^{-a}, \widetilde{Z}_t^a \Big) \, dt - \widetilde{Z}_t^a dW_t \quad \text{with terminal condition} \quad \widetilde{Y}_T^a = -H^a, \tag{3.2}$$

where the driver $\widetilde{g}^a \colon \Omega \times [0,T] \times \mathbb{R}^2 \times (\mathbb{R}^2)^{N-1} \times \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$
\begin{aligned}
\widetilde{g}^a(t, \pi_t^a, \pi_t^{-a}, z^a) &:= g^a\big(t, z^a - \zeta_t\big) - \langle \zeta_t, \theta_t \rangle \\
&= g^a \bigg(t, z^a - \Big(\big(\pi_t^{a,1} - \widetilde{\lambda}^a \bar{\pi}_t^{-a,1} \big)\sigma_t + \big(\pi_t^{a,2} - \widetilde{\lambda}^a \bar{\pi}_t^{-a,2} \big)\kappa_t^\theta \Big) \bigg) \\
&\quad - \bigg\langle \big(\pi_t^{a,1} - \widetilde{\lambda}^a \bar{\pi}_t^{-a,1} \big)\sigma_t + \big(\pi_t^{a,2} - \widetilde{\lambda}^a \bar{\pi}_t^{-a,2} \big)\kappa_t^\theta, \theta_t \bigg\rangle.
\end{aligned}
\tag{3.3}
$$

Each individual agent $a \in \mathbb{A}$ seeks to minimize $\widetilde{Y}_0^a$, the solution to (3.2), via her choice of investment strategy $\pi^a \in \mathcal{A}^\theta$, in other words

$$\min_{\pi^a \in \mathcal{A}^\theta} \widetilde{Y}_0^a.$$

We proceed by first identifying the single agent's best response given that all other agents have acted; this is why we leave the dependency on π^{-a} explicit in the definition of $\widetilde{g}^a$ above.

Before we solve the individual optimization we assume that the derivative H^D does indeed complete the market. This, of course, is verified a posteriori and case-by-case depending on the specific underlying model.

Assumption 3.1. *Assume that $\kappa_t^R \neq 0$ and $\sigma_t^S S_t > 0$ for any $t \in [0, T]$, $\mathbb{P}$-a.s. .*

The pointwise minimizer for the single agent's residual risk

In (3.2), the strategy π^a appears only in the driver $\widetilde{g}^a$. The comparison theorem for BSDEs suggests that in order to minimize $\widetilde{Y}_0^a(\pi^a)$ over π^a one needs only to minimize the driver function $\widetilde{g}^a$ over π_t^a, for each fixed t, π_t^{-a} and z^a. We define such pointwise minimizer as

$$\Pi^a(t, \pi^{-a}, z) := \arg \min_{\pi^a \in \mathbb{R}^2} \widetilde{g}^a(t, \pi^a, \pi^{-a}, z), \qquad (t, \pi^{-a}, z) \in [0, T] \times (\mathbb{R}^2)^{N-1} \times \mathbb{R}^2,$$

and $\widetilde{G}^a(t, \pi^{-a}, z^a) := \widetilde{g}^a\big(t, \Pi^a(t, \pi^{-a}, z), \pi^{-a}, z^a\big)$ as the minimized driver.

The pointwise minimization problem is easily solved under the assumptions on g^a and the unique minimizer (due to strict convexity) follows from the *first order conditions (FOC)* for $\widetilde{g}^a$, i.e. $\nabla_{\pi^a} \widetilde{g}^a(t, \pi^a, \pi^{-a}, z^a) = 0$. Recall that $\sigma = (\sigma^S S, 0)$, then from (3.3) we have

$$\begin{aligned}
\partial_{\pi^{a,1}} \widetilde{g}^a(t, \pi^a, \pi^{-a}, z^a) = 0 &\Leftrightarrow -\langle (\nabla_z g^a)(t, z^a - \zeta^a), \sigma \rangle - \langle \sigma, \theta \rangle = 0 \\
&\Leftrightarrow g_{z^1}^a(t, z^a - \zeta^a) = -\theta^S, \qquad (3.4) \\
\partial_{\pi^{a,2}} \widetilde{g}^a(t, \pi^a, \pi^{-a}, z^a) = 0 &\Leftrightarrow -\Big\langle (\nabla_z g^a)(t, z^a - \zeta^a), \kappa^\theta \Big\rangle - \Big\langle \kappa^\theta, \theta \Big\rangle = 0 \\
&\Leftrightarrow -\theta^S \kappa^S + g_{z^2}^a(t, z^a - \zeta^a) \kappa^R = -\kappa^S \theta^S - \kappa^R \theta^R \\
&\Leftrightarrow g_{z^2}^a(t, z^a - \zeta^a) = -\theta^R, \qquad (3.5)
\end{aligned}$$

where we used (3.4) to obtain (3.5) under Assumption 3.1.

We can be more precise about the general structure of Π^a without having specified g^a. Define the $\mathbb{R}^2$-valued map $\mathcal{Z}^a(t, -\theta_t) = \big(\mathcal{Z}^{a,1}(t, -\theta_t), \mathcal{Z}^{a,2}(t, -\theta_t)\big)$ as:

for any (ω, t), $\mathcal{Z}^a(t, -\theta_t)$ is the unique solution (in the z variable) to $\nabla_z g^a(t, z) = -\theta_t$. (3.6)

Then, elementary re-arrangements from (3.4)–(3.5) using that $z^a - \zeta^a = \mathcal{Z}^a(t, -\theta_t)$ holds, yield

$$\begin{aligned}
\Pi^{a,1}(t, \pi_t^{-a}, z^a) - \widetilde{\lambda}^a \bar{\pi}_t^{-a,1} &= \frac{z^{a,1} - \mathcal{Z}^{a,1}(t, -\theta_t)}{\sigma^S S_t} - \frac{z^{a,2} - \mathcal{Z}^{a,2}(t, -\theta_t)}{\kappa_t^R} \frac{\kappa_t^S}{\sigma^S S_t}, \\
\Pi^{a,2}(t, \pi_t^{-a}, z^a) - \widetilde{\lambda}^a \bar{\pi}_t^{-a,2} &= \frac{z^{a,2} - \mathcal{Z}^{a,2}(t, -\theta_t)}{\kappa_t^R}.
\end{aligned} \qquad (3.7)$$

Optimality

Having identified the minimized driver $\widetilde{G}^a$, we let $(\widetilde{Y}^a, \widetilde{Z}^a)$ be the solution to the residual-risk BSDE with terminal condition $-H^a$ and driver $\widetilde{G}^a$, and we define $\pi^{*,a} := \Pi^a(\cdot, \pi_\cdot^{-a}, \widetilde{Z}_\cdot^a)$. We now prove that if the required results from BSDE theory hold, then the above methodology indeed yields the solution to the individual minimization problem.

Theorem 3.2 (Optimality for one agent). *Fix a market price of risk $\theta = (\theta^S, \theta^R)$, an agent $a \in \mathbb{A}$ and a set of admissible strategies π^b for $b \in \mathbb{A} \setminus \{a\}$. If*

- *BSDE (3.2) with the minimized driver $\widetilde{G}^a$ has a unique solution $(\widetilde{Y}^a, \widetilde{Z}^a)$,*

- *and the comparison theorem applies to this BSDE,*

then the minimal risk for agent a is given by $\widetilde{Y}_0^a$ and $\pi^{*,a} = \Pi^a(\cdot, \pi_\cdot^{-a}, \widetilde{Z}_\cdot^a)$ is the optimal strategy.

Proof. The assumptions on the driver suffice to ensure that the map $(t, \pi_t^{-a}, z) \mapsto \Pi^a(t, \pi_t^{-a}, z)$ is well defined. The proof essentially consists in applying the comparison theorem to prove the optimality of $\widetilde{Y}_0^a$ and $\pi^{*,a}$. From the definition of $\widetilde{G}^a$ as a pointwise minimum we naturally have, for any admissible strategy π^a, that $\widetilde{G}^a(t, \pi_t^{-a}, z) = \widetilde{g}^a\left(t, \Pi^a(t, \pi^{-a}, z), \pi_t^{-a}, z\right) \leq \widetilde{g}^a(t, \pi_t^a, \pi_t^{-a}, z)$ for all t and z. By the comparison theorem, we therefore have $\widetilde{Y}_0^a = \widetilde{Y}_0^a(\pi^{*,a}, \pi^{-a}) \leq \widetilde{Y}_0^a(\pi^a, \pi^{-a})$ for any possible $\pi^a \in \mathcal{A}^\theta$ and the statement follows. $\qquad\square$

Remark 3.3. *The above assumptions are easily satisfied in many cases. Consider for instance $g^a(t, z) = \langle \beta, z \rangle + \delta |z|^2$ with $\beta \in \mathbb{R}^2$ and $\delta \geq 0$. This driver is smooth, has quadratic growth in z, and a first derivative with at most linear growth. A well-developed theory is available for BSDEs with bounded terminal conditions and with drivers satisfying this kind of assumptions (in particular: existence and uniqueness of solutions, validity of the comparison theorem). Recalling the definition (3.3) of $\widetilde{g}^a$ and remarking from (3.7) that Π^a is an affine function of z (so in particular it has linear growth), we see that the minimized driver $\widetilde{G}^a$ enjoys the same properties as g^a, and so Theorem 3.2 applies.*

Some results exist for BSDEs when the drivers are less regular, e.g. have a first derivative only in the weak sense. Theorem 3.2 is stated in the above fashion to avoid particularizing the results to special examples of drivers or listing many general properties of g^a. Our focus here is more on the methodology for solving the model than on the BSDE theory.

3.2 The unconstrained Nash equilibrium

Having understood the optimization problem for one agent, finding the Nash equilibrium for a given θ consists in solving (3.7) for the minimizing maps Π^a simultaneously for all agents. For a given t and a family $z^{\mathbb{A}} = (z^a)_{a \in \mathbb{A}} \in (\mathbb{R}^2)^{\mathbb{A}}$, we therefore look for the Nash equilibrium $\Pi^*\left(t, z^{\mathbb{A}}\right) = \left(\Pi^{*,a}\left(t, z^{\mathbb{A}}\right)\right)_{a \in \mathbb{A}}$ of pointwise minimizers. The system (3.7), for the Nash equilibrium maps $\left(\Pi^{*,a}\left(t, z^{\mathbb{A}}\right)\right)_{a \in \mathbb{A}}$, can be rewritten in matrix form as

$$A_N \, \Pi^{*,\cdot,i} = J_N^i, \qquad \text{where} \quad \Pi^{*,\cdot,i} := (\Pi^{*,a,i}, \Pi^{*,b,i}, \cdots, \Pi^{*,N,i}) \in \mathbb{R}^{N \times 1} \text{ and } i \in \{1,2\}, \quad (3.8)$$

and where the a-th entries, $J_N^{a,i}$, of the vectors J_N^1 and J_N^2 are respectively given by

$$J_N^{a,1} := \frac{z^{a,1} - \mathcal{Z}^{a,1}(t, -\theta_t)}{\sigma^S S_t} - \frac{z^{a,2} - \mathcal{Z}^{a,2}(t, -\theta_t)}{\kappa_t^R} \frac{\kappa^S}{\sigma^S S} \qquad \text{and} \qquad J_N^{a,2} := \frac{z^{a,2} - \mathcal{Z}^{a,2}(t, -\theta_t)}{\kappa_t^R},$$

and the $N \times N$-matrix A_N is explicitly given by

$$A_N = \begin{bmatrix} 1 & & -\frac{\lambda^a}{N-1} \\ & \ddots & \\ -\frac{\lambda^N}{N-1} & & 1 \end{bmatrix}, \qquad (3.9)$$

i.e. the j-th line has the entries $-\widetilde{\lambda}^j = -\lambda^j/(N-1)$, everywhere but for the j-th one which is 1.

Proposition 3.4. *Under Assumption 2.2 the matrix A_N is invertible. Moreover, for every market price of external risk θ^R fixed a priori, there exists a Nash equilibrium for the agents.*

Proof. The determinant of the A_N is

$$\det(A_N) = 1 - \sum_{i<j} \widetilde{\lambda}^i \widetilde{\lambda}^j - 2 \sum_{i<j<k} \widetilde{\lambda}^i \widetilde{\lambda}^j \widetilde{\lambda}^k - 3 \sum_{i<j<k<l} \widetilde{\lambda}^i \widetilde{\lambda}^j \widetilde{\lambda}^k \widetilde{\lambda}^l - \ldots - (N-1) \prod_{i=1}^{N} \widetilde{\lambda}^i.$$

If $\lambda^a = 1$ for all $a \in \mathbb{A}$, then $\det(A_N) = 1 - \sum_{k=2}^{N} \frac{k-1}{(N-1)^k} \binom{N}{k} = 0$, so the matrix is not invertible. The determinant is decreasing in each $\widetilde{\lambda}^a$ ($a \in \mathbb{A}$) and therefore also in λ^a. Hence, if $\lambda^a \in [0,1]$ for all $a \in \mathbb{A}$ and if the product $\prod_{a \in \mathbb{A}} \lambda^a < 1$, then at least one factor must be strictly smaller than one, hence the determinant must be positive, i.e. $\det(A_N) > 0$, hence the invertibility.

The Nash equilibrium follows from Theorem 3.2 and the above invertibility result. $\square$

Let us make two remarks: First, if $\lambda^b = 0$ for all $b \in \mathbb{A} \setminus \{a\}$, then A_N is invertible independent of λ^a, i.e. in particular for $\lambda^a = 1$. This shows that the condition $\lambda^a \in [0,1)$ for all $a \in \mathbb{A}$ would be too strict. Second, if we were to allow for $\lambda^a > 1$, then $\prod_a \lambda^a < 1$ is not sufficient for invertibility of A_N, e.g. in the case $N = 3$ take $\lambda^a = \lambda^b = 2$ and $\lambda^c = 0$.

3.3 An example: entropic agents or the entropic risk measure case

We now illustrate the methodology and result of Theorem 3.2 for a particular risk measure, which prepares the ground for the model we study in Sections 5 and 6. We give a sequence of examples, in increasing order of complexity, that show how the structure of the optimal strategies is changing as features are added. As in the above subsections, the examples do not yet take into account the market clearing condition but rather assume that a market price of risk $\theta = (\theta^S, \theta^R)$ is given. Nonetheless they give a flavor for the next section where the equilibrium market price of risk is derived.

We assume that each agent a is assessing her risk using the entropic risk measure ρ^a, that is the one associated with the driver $g^a : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g^a(z) := \frac{1}{2\gamma_a} |z|^2, \quad \text{where} \quad \gamma_a > 0 \quad \text{is agent } a\text{'s } \textit{risk tolerance}, \tag{3.10}$$

and $1/\gamma_a$ is agent a's *risk aversion*. This choice of g^a relates to exponential utilities, and we have (see e.g. [Car09], [FK11] or [HIM05])

$$\rho^a(\xi) = Y_0^a = \gamma_a \ln \mathbb{E}[e^{-\xi/\gamma_a}] = \gamma_a \ln \left(-U_{\gamma_a}(\xi) \right) \quad \text{with} \quad U_{\gamma_a}(\xi) = \mathbb{E}[-e^{-\xi/\gamma_a}],$$

so that, equivalently, the agents are maximizing their expected (exponential) utility.

In what follows, the optimal strategies were computed using the techniques described so far and hence we omit the calculations. They boil down to finding the correct $\mathcal{Z}$ arising from (3.6), then injecting (3.7) in (3.3) and computing the outcome. The computations are straightforward as the LHS of (3.7) appears explicitly in (3.3).

We denote throughout $\pi^{*,a}$ (for $a \in \mathbb{A}$) the optimal Nash equilibrium strategy.

3.3.1 The reference case of a single agent

For comparison, we first give the optimal strategy for a single agent who could trade liquidly in the stock of price S and the derivative of price B with (arbitrary and exogenously-given) market price $\theta = (\theta^S, \theta^R)$. She aims at minimizing her risk, with terminal endowment and trading gains $\xi^a = H^a + V_T^a(\pi^a)$. Here, other agents do not play a role. Since $g_{z^i}^a(z) = z^i/\gamma_a$, it is easily found that $\mathcal{Z}^a(t, -\theta_t) := (-\gamma_a \theta_t^S, -\gamma_a \theta_t^R) = -\gamma_a \theta_t$. Injecting (3.7) in (3.3) yields the minimized driver $\widetilde{G}^a$,

$$\widetilde{G}^a(t, z^a) = \widetilde{g}^a\big(t, \Pi^a(t, \pi^{-a}, z^a), \pi^{-a}, z^a\big) = -\frac{\gamma_a}{2}|\theta_t|^2 - \langle z^a, \theta_t \rangle.$$

The minimized risk is then given by $Y_0^a = \widetilde{Y}_0^a$ where $(\widetilde{Y}^a, \widetilde{Z}^a)$ is the solution to the BSDE with terminal condition $-H^a$ and driver $\widetilde{G}^a$, while the optimal strategy is then given by

$$\pi^{*,a,1} = \frac{\widetilde{Z}^{a,1} + \gamma_a \theta^S}{\sigma^S S} - \frac{\widetilde{Z}^{a,2} + \gamma_a \theta^R}{\kappa^R} \frac{\kappa^S}{\sigma^S S} \qquad \text{and} \qquad \pi^{*,a,2} = \frac{\widetilde{Z}^{a,2} + \gamma_a \theta^R}{\kappa^R}.$$

This result is expected and in line with canonical mathematical finance results. The particular structure of the optimal strategy follows from the fact that the second asset is correlated to the first when $\kappa^S \neq 0$, and the inversion of the volatility matrix for the 2-dimensional price (S, B),

$$\begin{bmatrix} \sigma^S S & \kappa^S \\ 0 & \kappa^R \end{bmatrix}.$$

The market faced by a is complete; the driver for the minimized residual risk $\widetilde{Y}^a$ is affine and

$$\widetilde{Y}_0^a = \mathbb{E}^\theta \left[-H^a - \frac{\gamma_a}{2} \int_0^T |\theta_u|^2 \mathrm{d}u \right] = \mathbb{E}\left[\mathcal{E}_T^\theta \left(-H^a - \frac{\gamma_a}{2} \int_0^T |\theta_u|^2 \mathrm{d}u \right) \right].$$

The minimized risk is linear with respect to H^a – all the risk is hedged away.

3.3.2 The reference case of a single agent that cannot trade in the derivative

It is also instructive, and will be useful later on, to look at the case where this single agent cannot trade in the derivative, and hence faces an incomplete market. We first enforce $\pi^{a,2} = 0$ on (3.3), then we optimize over $\pi^{a,1}$ (as in (3.4)-(3.5)). The minimized driver following the calculations is

$$\widetilde{G}^a(t, z) = -\frac{\gamma_a}{2}(\theta_t^S)^2 - z^1 \theta_t^S + \frac{1}{2\gamma_a}(z^2)^2.$$

The minimized risk is then given by $Y_0^a = \widetilde{Y}_0^a$ where $(\widetilde{Y}^a, \widetilde{Z}^a)$ is the solution to the BSDE with terminal condition $-H^a$ and the above driver $\widetilde{G}^a$, while the optimal strategy is

$$\pi^{*,a,1} = \frac{\widetilde{Z}^{a,1} + \gamma_a \theta^S}{\sigma^S S} \quad \text{and} \quad \pi^{*,a,2} = 0.$$

3.3.3 The case of multiple agents without relative performance concerns

We return to the full set of agents $\mathbb{A}$ and take $\lambda^a = 0$ for all $a \in \mathbb{A}$; this is the setting covered in [HPDR10]. We find the minimized risk-driver for agent a to be

$$\widetilde{G}^a(t, z^a) = -\frac{\gamma_a}{2}|\theta_t|^2 - \langle z^a, \theta_t \rangle.$$

The minimized risk given by $Y_0^a = \widetilde{Y}_0^a$ where $(\widetilde{Y}^a, \widetilde{Z}^a)$ solves the BSDE with terminal condition $-H^a$ and minimized driver $\widetilde{G}^a$, while the optimal strategies are given by

$$\pi^{\lambda=0,a,1} := \frac{\widetilde{Z}^{a,1} + \gamma_a \theta^S}{\sigma^S S} - \frac{\widetilde{Z}^{a,2} + \gamma_a \theta^R}{\kappa^R} \frac{\kappa^S}{\sigma^S S} \quad \text{and} \quad \pi^{\lambda=0,a,2} := \frac{\widetilde{Z}^{a,2} + \gamma_a \theta^R}{\kappa^R} \tag{3.11}$$

Observe that in this case the strategy $\pi^{\lambda=0,a}$ followed by a does not depend *directly* on the strategies of the other agents; its structure is the same as for an agent alone. However, when the price dynamics of the derivative is not fixed but emerges from the equilibrium, the other agents' strategies appear indirectly via θ^R and κ.

3.3.4 The case of multiple agents without relative performance concerns in zero net supply

If one would want to take into account the endogenous trading of the derivative in the particular situation of pure risk trading, where one takes $n = 0$ in (2.11), then the market price of external risk θ^R cannot be fixed arbitrarily. As done above, we must impose $\sum_a \pi^{\lambda=0,a,2} = 0$.

It is not difficult to see, summing the last equation in (3.11) over $a \in \mathbb{A}$, that this requires that $\theta^R = -\sum_a \widetilde{Z}^a / \sum_a \gamma_a$. However the $\widetilde{Z}^a$s are themselves found by solving a system of N BSDEs which involve θ^R. Replacing θ^R by the expression above in the said system of equations leads to a fully coupled system of quadratic BSDEs that is hard to solve in general. We solve this problem with an alternative tool in Section 4.

3.3.5 The general case: multiple agents with performance concerns

In the general case, we obtain the minimal driver injecting again (3.7) in (3.3),

$$\widetilde{G}^a(t, z^a) = -\frac{\gamma_a}{2} |\theta_t|^2 - \langle z^a, \theta_t \rangle. \tag{3.12}$$

The minimized risk is then given by $Y_0^a = \widetilde{Y}_0^a$ where $(\widetilde{Y}^a, \widetilde{Z}^a)$ is the solution to the BSDE with terminal condition $-H^a$ and driver $\widetilde{G}^a$, while the optimal strategies $\pi^* = (\pi^{*,a})_{a \in \mathbb{A}}$ are such that

$$\pi^{*,a,1} - \widetilde{\lambda}^a \sum_{b \neq a} \pi^{*,b,1} = \frac{\widetilde{Z}^{a,1} + \gamma_a \theta^S}{\sigma^S S} - \frac{\widetilde{Z}^{a,2} + \gamma_a \theta^R}{\kappa^R} \frac{\kappa^S}{\sigma^S S} \tag{3.13}$$

$$\pi^{*,a,2} - \widetilde{\lambda}^a \sum_{b \neq a} \pi^{*,b,2} = \frac{\widetilde{Z}^{a,2} + \gamma_a \theta^R}{\kappa^R}. \tag{3.14}$$

The general invertibility of the systems (3.13) and (3.14) given θ is guaranteed by Proposition 3.4.

3.3.6 The general case: multiple agents with performance concerns in zero net supply

If one imposes (2.11) with $n = 0$, implying that $\sum_{b \in \mathbb{A} \setminus \{a\}} \pi^{*,b,2} = -\pi^{*,a,2}$, then the linear system (3.14) for the investment in the derivative simplifies greatly and its solution is explicitly given by

$$\pi^{*,a,2} = \frac{1}{1 + \widetilde{\lambda}^a} \frac{\widetilde{Z}^{a,2} + \gamma_a \theta^R}{\kappa^R} \qquad \text{for all } a. \tag{3.15}$$

Notice how the structure of the optimal investment strategy for the derivative in (3.15) is that of (3.11), scaled down by the factor $\frac{1}{1+\widetilde{\lambda}^a}$.

In Section 6 we study a model with two agents and computations will be done explicitly for the investment in the stock (i.e. the inversion of the system (3.13)).

3.4 Reduction to zero net supply

In this section we give an auxiliary result allowing to simplify the yet to be used condition (2.11). We show how the initial holdings $\pi_{0-}^{a,2} = \pi_0^{a,2} \neq 0$ before/at the beginning of the game can be reduced to the case where $\pi_{0-}^{a,2} = \pi_0^{a,2} = 0$. This allows us to apply (2.11) with $n = 0$, which will prove crucial in later computations. The reduction to $n = 0$ is based on the monotonicity of the risk measures and the following lemma, stated from the point of view of one agent $a \in \mathbb{A}$. To avoid a notational overload, we omit explicit dependencies on π^{-a} in this subsection.

Lemma 3.5. *For a given MPR θ and admissible strategies $\pi^{-a} = (\pi^b)_{b \in \mathbb{A}\backslash\{a\}}$, consider the dynamics of the residual risk BSDE*

$$-\mathrm{d}\widetilde{Y}_t^a(\pi^a) = \widetilde{g}^a\big(t, \pi_t^a, \widetilde{Z}_t^a(\pi^a)\big)\mathrm{d}t - \widetilde{Z}_t^a(\pi^a)\mathrm{d}W_t \tag{3.16}$$

associated with the preferences of agent a using an admissible strategy π^a. Assume further that (3.16) has a unique solution for any given $\mathcal{F}_t$-measurable bounded terminal condition $\widetilde{Y}_T$. Let $\nu \in \mathbb{R}$. Then,

- *if $\pi^a := (\pi^{a,1}, \pi^{a,2})$ minimizes the solution $\widetilde{Y}_0(\pi^a)$ to (3.16) for a terminal condition $-H^a$, then $\widehat{\pi}^a := (\pi^{a,1}, \pi^{a,2} - \nu)$ is optimal for the terminal condition $-(H^a + \nu H^D)$;*

- *if $\pi^a := (\pi^{a,1}, \pi^{a,2})$ minimizes the solution $\widetilde{Y}_0(\pi^a)$ for a terminal condition $-(H^a + \nu H^D)$, then $\widehat{\pi}^a := (\pi^{a,1}, \pi^{a,2} + \nu)$ is optimal for the terminal condition $-H^a$.*

Proof. We prove only the first assertion, as the second is equivalent. Let $t \in [0, T]$. Assume that $\pi^{*,a} \in \mathcal{A}^\theta$ is optimal for (3.16) with $\widetilde{Y}_T^a := -H^a$, i.e. for any $\pi^a \in \mathcal{A}^\theta$ one has $\widetilde{Y}_0^a(\pi^{*,a}) \leq \widetilde{Y}_0^a(\pi^a)$. Define further, for any $\pi^a \in \mathcal{A}^\theta$, the strategies

$$\check{\pi}^a := \pi^a - (0, \nu) = (\pi^{a,1}, \pi^{a,2} - \nu) \qquad \text{and} \qquad \check{\pi}^{*,a} := \pi^{*,a} - (0, \nu).$$

To show that $Y_0^a(\check{\pi}^{*,a}) \leq Y_0^a(\check{\pi}^a)$ for any $\check{\pi}^a$ where Y^a solves (3.16) with $Y_T^a = -(H^a + \nu H^D)$ we first show an identity result between the BSDEs with different terminal conditions. The second step is the optimality.

Step 1: We show that the process $(Y(\check{\pi}^a), Z(\check{\pi}^a)) := (\widetilde{Y}^a(\pi^a) - \nu B^\theta, \widetilde{Z}^a(\pi^a) - \nu \kappa^\theta)$ solves BSDE

$$Y_t(\check{\pi}^a) = -(H^a + \nu H^D) + \int_t^T \widetilde{g}^a\left(s, \check{\pi}_s^a, Z_s(\check{\pi}^a)\right)\mathrm{d}s - \int_t^T Z_s(\check{\pi}^a)\mathrm{d}W_s. \tag{3.17}$$

To this end, we reformulate (2.6) it as a BSDE:

$$B_t^\theta = H^D - \int_t^T \left\langle \kappa_s^\theta, \theta_s \right\rangle \mathrm{d}s - \int_t^T \kappa_s^\theta \mathrm{d}W_s. \tag{3.18}$$

The difference between (3.16) and ν times (3.18) yields

$$\widetilde{Y}_t(\pi^a) - \nu B_t^\theta = -(H^a + \nu H^D) + \int_t^T \left[\widetilde{g}^a(s, \pi_s^a, \widetilde{Z}_s^a(\pi^a)) + \nu \left\langle \kappa_s^\theta, \theta_s \right\rangle\right]\mathrm{d}s - \int_t^T (\widetilde{Z}_s^a(\pi^a) - \nu \kappa_s^\theta)\mathrm{d}W_s$$

$$\Leftrightarrow \quad Y_t(\check{\pi}^a) = -(H^a + \nu H^D) - \int_t^T Z_s(\check{\pi}^a)\mathrm{d}W_s$$

$$+ \int_t^T \left[\widetilde{g}^a\left(s, \check{\pi}^a + (0, \nu), Z_s(\check{\pi}^a) + \nu \kappa_s^\theta\right) + \nu \left\langle \kappa_s^\theta, \theta_s \right\rangle\right]\mathrm{d}s.$$

In view of (3.3), we can manipulate the terms inside driver $\widetilde{g}^a$ above and obtain

$$
\begin{aligned}
\widetilde{g}^a & \left(\cdot, \check{\pi}^a + (0, \nu), Z^a(\check{\pi}^a) + \nu \kappa^\theta \right) + \nu \left\langle \kappa^\theta, \theta \right\rangle \\
& = g^a \left(\cdot, (Z^a(\check{\pi}^a) + \nu \kappa^\theta) - \check{\pi}^{a,1} \sigma - (\check{\pi}^{a,2} + \nu) \kappa^\theta + \widetilde{\lambda}^a \left(\bar{\pi}^{-a,1} \sigma + \bar{\pi}^{-a,2} \kappa^\theta \right) \right) \\
& \qquad - \check{\pi}^{a,1} \left\langle \sigma, \theta \right\rangle - (\check{\pi}^{a,2} + \nu) \left\langle \kappa^\theta, \theta \right\rangle + \widetilde{\lambda}^a \left\langle \bar{\pi}^{-a,1} \sigma + \bar{\pi}^{-a,2} \kappa^\theta, \theta \right\rangle + \nu \left\langle \kappa^\theta, \theta \right\rangle \\
& = \widetilde{g}^a \left(\cdot, \check{\pi}^a, Z^a(\check{\pi}^a) \right).
\end{aligned}
$$

Given the assumed uniqueness of BSDE (3.16) the assertion follows.

Step 2: Given that $(Y(\check{\pi}^a), Z(\check{\pi}^a))$ solve (3.17) and that $\pi^{*,a}$ is the minimizing strategy for $\pi^a \mapsto \widetilde{Y}^a(\pi^a)$, then manipulating $Y(\check{\pi}^a) = \widetilde{Y}^a(\pi^a) - \nu B^\theta$, we have

$$
Y_0(\check{\pi}^a) = \widetilde{Y}_0^a(\pi^a) - \nu B_0^\theta \geq \widetilde{Y}_0^a(\pi^{*,a}) - \nu B_0^\theta = Y_0(\check{\pi}^{*,a}),
$$

and hence $\check{\pi}^{*,a} := \pi^{*,a} - (0, \nu)$ is optimal for BSDE (3.16) with terminal condition $-(H^a + \nu H^D)$. $\square$

This lemma intuitively states that an agent a, owning at time $t = 0$ a portion $\nu^a = \pi_0^{a,2} = \pi_{0-}^{a,2}$ of units of H^D, can be regarded as being in fact endowed with $\widehat{H}^a = H^a + \nu^a H^D$. One then looks only at the relative portfolio $\widehat{\pi}^{a,2} = \pi^{a,2} - \nu^a$, which counts the derivatives bought and sold only from time $t = 0$ onwards: the optimization problem is equivalent. The argument can be extended to all other agents. We note that this reduction is only possible because we do not consider trading constraints in this work, so that the strategies $\pi^{a,2}$ and $\widehat{\pi}^{a,2}$ are equally admissible.

For the rest of this work we assume that each agent receives at $t = T$ a portion[4] n/N of the derivative H^D. By doing so, the market clearing condition in Definition 2.4 transforms into

$$
\sum_{a \in \mathbb{A}} \pi_t^{a,2} = 0 \quad \mathbb{P} \otimes \lambda - a.s.,
$$

and we refer to it as the *zero net supply condition*.

For clarity, we recall that agent $a \in \mathbb{A}$ now assesses her risk by solving the dynamics provided by BSDE (2.9) with terminal condition

$$
Y_T^a = - \left(H^a + \frac{n}{N} H^D + V_T^{a,\theta}(\pi^a) - \widetilde{\lambda}^a \sum_{b \in \mathbb{A} \setminus \{a\}} V_T^{b,\theta}(\pi^b) \right) \tag{3.19}
$$

(instead of that in (2.9)). Moreover, by applying the change of variables (3.1) to BSDE (2.9) with terminal condition (3.19), we reach

$$
-d\widetilde{Y}_t^a = \widetilde{g}^a(t, \pi_t^a, \pi_t^{-a}, \widetilde{Z}_t^a) dt - \widetilde{Z}_t^a dW_t, \qquad \widetilde{Y}_T^a := - \left(H^a + \frac{n}{N} H^D \right), \tag{3.20}
$$

with $\widetilde{g}^a$ given by (3.3) (and $(\widetilde{Y}^a, \widetilde{Z}^a)$ relates to (Y^a, Z^a) via the change of variables (3.1)).

It is straightforward to recompile the results of Section 3.3 under the *zero net supply* condition. It entails no changes is the strategies or drivers, only the terminal condition of the involved BSDEs need to be updated from $-H^a$ to $-(H^a + \frac{n}{N} H^D)$ as in (3.20).

---

[4]Many possibilities for this reduction to zero net supply exist, including endowing one agent with the total amount n of derivatives H^D or endowing each agent with their initial portions of the derivative ν^a. We make the judicious choice of n/N for simplicity.

4 The equilibrium market price of external risk

In the previous section we saw how to compute the Nash equilibrium for a given market price of risk $\theta = (\theta^S, \theta^R)$, without the endogenous trading constraint (market clearing condition). In this section we solve the equilibrium problem, as posed by Definition 2.4, by finding the Equilibrium Market Price of external Risk (EMPeR) θ^R.

The literature contains many general results on equilibrium in complete markets that link competitive equilibria to an optimization problem for a representative agent, and this is the approach we use here. The preferences of the representative agent are usually given by a weighted average of the individual agents' utility functions with the weights depending on the competitive equilibrium to be supported by the representative agent, see [Neg60]. This dependence results in a complex fixed point problem which renders the analysis and computation of equilibria quite cumbersome. The many results on risk sharing under translation invariant preferences, in particular [BEK05], [JST06] and [FK08], suggest that when the preferences are translation invariant, then all the weights are equal. This was an effective strategy in [HPDR10] and it would be such here if $\lambda^a = \lambda \in [0, 1)$ for all $a \in \mathbb{A}$.

In a market without performance concerns, [HPDR10] and [BE09] show that the infimal convolution of risk measures gives rise to a suitable risk measure for the representative agent which, for g-conditional risk measures, corresponds to infimal convolution of the drivers. Due to the performance concerns, we use a weighted-dilated infimal convolution, and in Theorem 4.4 we show that indeed minimizing the risk of our representative agent is equivalent to finding a competitive equilibrium in our market.

4.1 The representative agent

Aggregation of risk and representative agent.

Inspired by the above mentioned results and having in mind [Rüs13] (see Remark 4.5 below) we deal with the added inter-dependency arising from the relative performance component by defining a new risk measure ρ^w. For a set of weights $w = (w^a)_{a \in \mathbb{A}}$ (to be computed later on), we define

$$\rho_{\cdot}^w(X) = \inf \left\{ \sum_{a \in \mathbb{A}} w^a \rho_{\cdot}^a(X^a) \;\middle|\; (X^a) \in (L^\infty)^{\mathbb{A}} : \sum_{a \in \mathbb{A}} w^a X^a = X \right\} \quad \text{for any } X \in L^\infty. \qquad (4.1)$$

In the case of measures defined by BSDEs, [BEK05] shows that the measure ρ^w defined above by inf-convolution of the risk measures $(\rho^a)_{a \in \mathbb{A}}$ can be expressed again as BSDE whose driver is simply the inf-convolution of the BSDE drivers of the risk measures $(\rho^a)_{a \in \mathbb{A}}$.

For the set of weights $w = (w^a)_{a \in \mathbb{A}}$, we define the driver g^w as the inf-convolution of the drivers g^a for $(t, z) \in [0, T] \times \mathbb{R}^2$,

$$g^w(t, z) = \square_w \left((g^a)_{a \in \mathbb{A}} \right)(t, z) = \inf \left\{ \sum_{a \in \mathbb{A}} w^a g^a(t, z^a) \;\middle|\; \sum_{a \in \mathbb{A}} w^a z^a = z \right\}, \qquad (4.2)$$

where the notation $\square \left((g^a)_{a \in \mathbb{A}} \right)$ is that of the standard inf-convolution.

Lemma 4.1. *The map $z \mapsto g^w(z \cdot, z)$ is strictly convex and differentiable. Moreover, $\nabla_z g^w(\cdot, z) = 0$ has a unique solution in z.*

Proof. All properties follow from those of the drivers g^a $a \in \mathbb{A}$ (see Section 2.2) in combination with the properties of the inf-convolution operator. $\qquad \square$

The risk of the random terminal wealth ξ^w, measured through ρ^w, is given by $\rho_0^w(\xi^w) := Y_0^w$ where (Y^w, Z^w) is the solution to the BSDE

$$-dY_t^w = g^w(t, Z_t^w)dt - Z_t^w dW_t, \qquad \text{with terminal condition} \qquad Y_T^w = -\xi^w. \qquad (4.3)$$

The weights $(w^a)_{a \in \mathbb{A}}$ are chosen such that $\sum_a w^a = 1$. In this way, the risk measure ρ^w associated to the BSDE with the above driver is a monetary risk measure. Translation invariance and monotonicity follow from the fact that the driver g^w is independent of y. Convexity follows from the convexity of g^w, which in turns follows from that of the g^as by the envelope theorem.

Remark 4.2. *Notice that* (4.2) *can be rewritten*

$$g^w(t, z) = \inf \left\{ \sum_{a \in \mathbb{A}} w^a g^a \left(t, \frac{z^a}{w^a} \right) \ \bigg| \ \sum_{a \in \mathbb{A}} z^a = z \right\}.$$

In this way, g^w is seen as the (usual) w-weighted infimal convolution of the w^a-dilated drivers g^a, in the terminology from [BE09] *(p.137). For more on dilated risk measures, see Proposition 3.4 in* [BE09].

We compute g^w for the case of entropic agents.

Example 4.3 (Entropic risk measure). *For entropic agents, i.e. with drivers $g^a(z) = \frac{|z|^2}{2\gamma_a}$, one obtains*

$$g^w(z) = \frac{|z|^2}{2\gamma_R}, \quad \text{with} \quad \gamma_R := \sum_{a \in \mathbb{A}} w_a \gamma_a. \qquad (4.4)$$

Having defined the driver of the BSDE for the aggregated risk, we naturally identify a representative agent having a strategy π^w and associated trading gains $V.(\pi^w) = \int_0^{\cdot} \pi_t^{w,1} dS_t + \pi_t^{w,2} dB_t$. Direct computations on (4.1) at time $t = T$ yield that $\rho_T^w(X) = \sum_a w^a \rho^a(X_i) = -\sum_a w^a X_i = X$ and since the zero net supply reduction is effective, from (4.1) and (3.19), the representative agent then assesses the risk of the terminal endowment given by

$$\begin{aligned}
\xi^w &:= \sum_{a \in \mathbb{A}} w^a \xi^a = \sum_{a \in \mathbb{A}} w^a \left(H^a + \frac{n}{N} H^D + V_T^a - \widetilde{\lambda}^a \bar{V}_T^{-a} \right) \\
&= \sum_{a \in \mathbb{A}} w^a (H^a + \frac{n}{N} H^D) + \sum_{a \in \mathbb{A}} w^a \left((1 + \widetilde{\lambda}^a) V_T^a - \widetilde{\lambda}^a \sum_{b \in \mathbb{A}} V_T^b \right) \\
&= \sum_{a \in \mathbb{A}} w^a (H^a + \frac{n}{N} H^D) + \sum_{a \in \mathbb{A}} c^a V_T(\pi^a) \\
&= H^w + V_T(\pi^w),
\end{aligned}$$

where $c^a := w^a(1 + \widetilde{\lambda}^a) - \sum_{b \in \mathbb{A}} w^b \widetilde{\lambda}^b$, $\pi^w = \sum_{a \in \mathbb{A}} c^a \pi^a$ is the representative agent's portfolio, and

$$H^w := \sum_{a \in \mathbb{A}} w^a (H^a + \frac{n}{N} H^D) = \frac{n}{N} H^D + \sum_{a \in \mathbb{A}} w^a H^a \qquad (4.5)$$

is defined as the representative agent's endowment.

There is a certain degree of freedom in the choice of (w^a). We choose them judiciously in order to regain the zero net supply condition for the representative agent, i.e. $\pi^{w,2} = 0$. The weighted sum is a priori not constant or much less equal to zero, but having it equal to zero simplifies the

calculations later on. To this end, we choose the weights $(w^a)_{a \in \mathbb{A}}$ such that $c^a = c$ for any $a \in \mathbb{A}$ for some $c \in (0, +\infty)$, hence the choices

$$w^a := \frac{1}{\Lambda(1 + \widetilde{\lambda}^a)} \qquad \text{for all } a \in \mathbb{A}, \text{ where} \qquad \Lambda := \sum_{a \in \mathbb{A}} \frac{1}{1 + \widetilde{\lambda}^a}. \tag{4.6}$$

Direct verification yields $\sum_a w^a = 1$ and for all $a \in \mathbb{A}$,

$$c^a = c := \frac{1}{\Lambda} - \frac{1}{\Lambda} \sum_{b \in \mathbb{A}} \frac{\widetilde{\lambda}^b}{1 + \widetilde{\lambda}^b} \quad \Rightarrow \quad \pi^{w,2} = \sum_{a \in \mathbb{A}} c^a \pi^{a,2} = c \sum_{a \in \mathbb{A}} \pi^{a,2} = 0,$$

when the zero net supply between the individual agents is satisfied.

Optimization for the representative agent and equilibrium market price of external risk.

We now show that the approach by aggregated risk and representative agent, as motivated above, allows to identify the equilibrium market price of risk as a by-product of minimizing the risk of the representative agent. The latter is given by the solution to BSDE (4.3) with terminal condition $Y_T^w = -\xi^w = -H^w - V_T(\pi^w)$, for any admissible strategy $\pi^w = (\pi^{w,1}, 0)$.

Following Section 3, except that the agent now can only choose $\pi^{w,1}$, we introduce the residual risk

$$\widetilde{Y}_t^w := Y_t^w + V_t^w \qquad \text{and accordingly} \qquad \widetilde{Z}_t^w := Z_t^w + \left(\pi_t^{w,1}\sigma_t + \pi_t^{w,2}\kappa_t\right).$$

The pair $(\widetilde{Y}^w, \widetilde{Z}^w)$ satisfies the BSDE with terminal condition $\widetilde{Y}_T^w = -H^w$ and driver $\widetilde{g}^w$, defined for $(t, \pi^w, z) \in [0, T] \times \mathbb{R}^2 \times \mathbb{R}^2$, by

$$\widetilde{g}^w(t, \pi^w, z) := g^w\left(t, z - \left(\pi^{w,1}\sigma_t + \pi^{w,2}\kappa_t^\theta\right)\right) - \left\langle \pi^{w,1}\sigma_t + \pi^{w,2}\kappa_t^\theta, \theta_t \right\rangle. \tag{4.7}$$

The main result of this section states that minimizing the risk for the representative agent leads to finding the EMPeR θ^R.

Theorem 4.4. *Denote, for $(t, z) \in [0, T] \times \mathbb{R}^2$, by $\Pi^{w,1}(t, z)$ the unique solution $p \in \mathbb{R}$ to*

$$g_{z^1}^w\left(t, z - p\,\sigma_t\right) = -\theta_t^S,$$

and define, for $(t, z) \in [0, T] \times \mathbb{R}^2$, the minimized driver $\widetilde{G}^w$

$$\widetilde{G}^w(t, z) := \widetilde{g}^w\left(t, \left(\Pi^{w,1}(t, z), 0\right), z\right) = g^w\left(t, z - \Pi^{w,1}(t, z)\sigma_t\right) - \Pi^{w,1}(t, z)\,\sigma_t^S S_t\,\theta_t^S.$$

Let $(\widetilde{Y}^w, \widetilde{Z}^w)$ be the unique solution to the BSDE with driver $\widetilde{G}^w$ and terminal condition $-H^w$. Define the strategy $\pi_{\cdot}^{*,w} := \left(\Pi^{w,1}(\cdot, \widetilde{Z}_{\cdot}^w), 0\right)$ and define θ^R by

$$g_{z^2}^w\left(t, \widetilde{Z}_t^w - \pi_t^{*,w,1}\sigma_t\right) = -\theta_t^R.$$

Assume that

- *existence, uniqueness and the comparison theorem hold for BSDEs with driver $\widetilde{G}^w$ (for bounded terminal conditions),*

- $\pi^{*,w}$ is an admissible strategy,

- $\theta^* := (\theta^S, \theta^R)$ makes the density $\mathcal{E}_t^\theta$ defined by (2.3) a uniformly integrable martingale,

- the market is complete, i.e. for all $t \in [0, T]$, we have $\kappa_t^R \neq 0$ (recall (2.6)).

Then $\widetilde{Y}_0^w$ is the minimized risk for the representative agent, $\pi^{*,w}$ is her optimal strategy and θ^* is an EMPR (cf Definition 2.4). Moreover, the minimal aggregated risk $\widetilde{Y}^w$ is linked to the individual risks $(\widetilde{Y}^a)_{a \in \mathbb{A}}$ through the identity $\widetilde{Y}^w = \sum_a w^a \widetilde{Y}^a$ (the same holds for the control components $\widetilde{Z}$).

Proof of Theorem 4.4. Lemma 4.1 states that the map $\Pi^{w,1}$ exists and is well defined. The admissibility of $\pi^{*,w}$ translates to (recall (2.7))

$$\mathbb{E}\left[\left\langle V^{w,\theta}(\pi^{*,w}) \right\rangle_T\right] = \mathbb{E}\left[\int_0^T |\pi_t^{*,w,1} \sigma_t^S S_t|^2 \mathrm{d}t\right] < +\infty$$

and is assumed to be satisfied. Also, the strategy $\pi^{*,w}$ satisfies the first order condition for the representative agent by construction and hence Theorem 3.2 guarantees the optimality of $\pi^{*,w}$.

It remains to prove that θ^* is an EMPR. From Section 3, let $(\pi^a)_{a \in \mathbb{A}}$ be the Nash optimizer for θ^* and let $(\widetilde{Y}^a, \widetilde{Z}^a)$ be the solution to the BSDE (3.20) for the minimized residual risk for agent a. We want to prove that $\sum_a \pi^{a,2} = 0$. For this, we combine the individual strategies into a strategy $\widehat{\pi} := \sum_a c^a \pi^a = c \sum_a \pi^a$ and we show that $\widehat{\pi} = \pi^{*,w}$, the latter being the optimal strategy of the representative agent. The proof is done in several steps. Define $(\widehat{Y}^w, \widehat{Z}^w) = \sum_{a \in \mathbb{A}} w^a (\widetilde{Y}^a, \widetilde{Z}^a)$.

Step 1 - A characterization of g^w. We have the following characterization for the minimum and the minimizer in the definition of g^w, Equation (4.2). For a given z and a family $(z^a)_{a \in \mathbb{A}}$ such that $\sum_a z^a = z$,

$$g^w(t, z) = \sum_{a \in \mathbb{A}} w^a g^a(t, z^a) \quad \Leftrightarrow \quad g_{z^i}^a(t, z^a) = g_{z^i}^b(t, z^b) \quad \text{for all } a, b \in \mathbb{A} \text{ and } i \in \{1, 2\}.$$

As a corollary, since

$$\sum_{a \in \mathbb{A}} w^a \left[\widetilde{Z}_t^a - \left(\left(\pi_t^{a,1} - \widetilde{\lambda}^a \pi_t^{-a,1}\right)\sigma_t + \left(\pi_t^{a,2} - \widetilde{\lambda}^a \pi_t^{-a,2}\right)\kappa_t\right)\right]$$

$$= \sum_{a \in \mathbb{A}} w^a \widetilde{Z}_t^a - \left(\left(\sum_{a \in \mathbb{A}} c^a \pi_t^{a,1}\right)\sigma_t + \left(\sum_{a \in \mathbb{A}} c^a \pi_t^{a,2}\right)\kappa_t\right) = \widehat{Z}_t^w - \left(\widehat{\pi}_t^1 \sigma_t + \widehat{\pi}_t^2 \kappa_t\right),$$

and since the Nash optimality of $(\pi^a)_{a \in \mathbb{A}}$ implies that for all a we have

$$g_{z^1}^a\left(t, \widetilde{Z}_t^a - \left(\left(\pi_t^{a,1} - \widetilde{\lambda}^a \pi_t^{-a,1}\right)\sigma_t + \left(\pi_t^{a,2} - \widetilde{\lambda}^a \pi_t^{-a,2}\right)\kappa_t\right)\right) = -\theta_t^S,$$

$$g_{z^2}^a\left(t, \widetilde{Z}_t^a - \left(\left(\pi_t^{a,1} - \widetilde{\lambda}^a \pi_t^{-a,1}\right)\sigma_t + \left(\pi_t^{a,2} - \widetilde{\lambda}^a \pi_t^{-a,2}\right)\kappa_t\right)\right) = -\theta_t^R,$$

we have

$$g^w\left(t, \widehat{Z}_t^w - \left(\widehat{\pi}_t^1 \sigma_t + \widehat{\pi}_t^2 \kappa_t\right)\right) = \sum_{a \in \mathbb{A}} w^a g^a\left(t, \widetilde{Z}_t^a - \left(\left(\pi_t^{a,1} - \widetilde{\lambda}^a \pi_t^{-a,1}\right)\sigma_t + \left(\pi_t^{a,2} - \widetilde{\lambda}^a \pi_t^{-a,2}\right)\kappa_t\right)\right).$$

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Step 2 - Summation of the individual BSDEs. Summing the individual agents' BSDEs (3.20), with minimized drivers $\widetilde{G}^a(t,z) = \widetilde{g}^a(t, \Pi^a(t, \pi_t^{-a}, z), z)$ and terminal conditions $(H^a + nH^D/N)$, according to the weights w^a, we see, using *Step 1*, that $(\widehat{Y}^w, \widehat{Z}^w)$ solves the BSDE

$$d\widehat{Y}_t^w = -\sum_{a\in\mathbb{A}} w^a \widetilde{g}^a\left(t, \widetilde{Z}_t^a - \left((\pi_t^{a,1} - \widetilde{\lambda}^a \pi_t^{-a,1})\sigma_t + (\pi_t^{a,2} - \widetilde{\lambda}^a \pi_t^{-a,2})\kappa_t\right)\right)dt + \sum_{a\in\mathbb{A}} w^a \widetilde{Z}_t^a \mathrm{d}W_t$$

$$= -\widetilde{g}^w\left(t, \widehat{Z}_t^w - \left(\widehat{\pi}_t^1 \sigma_t + \widehat{\pi}_t^2 \kappa_t\right)\right) + \widehat{Z}_t^w \mathrm{d}W_t,$$

with terminal condition $\sum_a w^a(H^a + nH^D/N) = H^w$. By uniqueness we have $(\widehat{Y}^w, \widehat{Z}^w) = (\widetilde{Y}^w, \widetilde{Z}^w)$.

Step 3 - Optimality of $\widehat{\pi}$. The envelope theorem guarantees that, if a family $(z^a)_{a\in\mathbb{A}}$ such that $\sum_a z^a = z$ is optimal in the definition (4.2) of g^w, that is if $g^w(t,z) = \sum_a w^a g^a(t,z^a)$, then

$$g_{z^i}^w(t,z) = g_{z^i}^a(t,z^a) \qquad \text{for all } a \in \mathbb{A} \text{ and } i \in \{1,2\}.$$

From this follows that

$$g_{z^1}^w\left(t, \widehat{Z}_t^w - \left(\widehat{\pi}_t^1 \sigma_t + \widehat{\pi}_t^2 \kappa_t\right)\right) = g_{z^1}^a\left(t, \widetilde{Z}_t^a - \left((\pi_t^{a,1} - \widetilde{\lambda}^a \pi_t^{-a,1})\sigma_t + (\pi_t^{a,2} - \widetilde{\lambda}^a \pi_t^{-a,2})\kappa_t\right)\right) = -\theta_t^S,$$

$$g_{z^2}^w\left(t, \widehat{Z}_t^w - \left(\widehat{\pi}_t^1 \sigma_t + \widehat{\pi}_t^2 \kappa_t\right)\right) = g_{z^2}^a\left(t, \widetilde{Z}_t^a - \left((\pi_t^{a,1} - \widetilde{\lambda}^a \pi_t^{-a,1})\sigma_t + (\pi_t^{a,2} - \widetilde{\lambda}^a \pi_t^{-a,2})\kappa_t\right)\right) = -\theta_t^R.$$

Step 4 - Conclusion. We have proven with *Step 3* that $\widehat{\pi}$ satisfies the first order condition for the representative agent, given the MPR $\theta = (\theta^S, \theta^R)$. From Theorem 3.2 we infer that $\widehat{\pi}$ is her unique optimal strategy $\pi^{*,w}$. As a by-product, we have also proven the relationship $\sum_{a\in\mathbb{A}} w^a(\widetilde{Y}^a, \widetilde{Z}^a) = (\widetilde{Y}^w, \widetilde{Z}^w)$ between the individual, minimized, residual risk and the aggregated one. $\qquad\square$

Remark 4.5. *In* [Rüs13] *a "weighted minimal convolution" of risk measures is introduced via*

$$\left(\bigwedge \rho_i\right)_\gamma (X) := \inf\left\{\sum_{i=1}^N \gamma_i \rho_i(X_i); \ (X_i) \in A(X)\right\}$$

(see page 271, equation (11.25)) for $\gamma = (\gamma_i) \in \mathbb{R}_{>0}^N$.

Observe that aggregation in our context would not work without the dilation weights $1/w^a$ in the argument of the driver. This can be seen in Step 2 of the proof of Theorem 4.4. The reason is that $\widetilde{G}^a$ is the sum of g^a with the strategies plugged in as arguments and with an additional term with the strategies multiplied by the weights. For the aggregation as a single strategy this adjustment is necessary.

Following on Example 4.3, we illustrate the computations of Theorem 4.4 for the special case of entropic risk measures.

Example 4.6. *With the methodology of Section 3, we find the minimized driver $\widetilde{G}^w$ to be*

$$\widetilde{G}^w(t,z) = -\frac{\gamma_R}{2}(\theta_t^S)^2 - z^1 \theta_t^S + \frac{1}{2\gamma_R}(z^2)^2,$$

with γ_R defined in (4.4). If $(\widetilde{Y}^w, \widetilde{Z}^w)$ is the solution to the BSDE with driver $\widetilde{G}^w$ and terminal condition $-H^w$ as defined by (4.5), then the optimal strategies are

$$\pi^{*,w,1} = \frac{\widetilde{Z}^{w,1} + \gamma_R \theta^S}{\sigma^S S}, \quad \text{and} \quad \pi^{*,w,2} = 0.$$

The EMPeR is a by-product of the FOCs and is given by

$$\theta^R = -\frac{\widetilde{Z}^w}{\gamma_R}. \tag{4.8}$$

4.2 A shortcut to the EMPeR in the case of entropic risk measures

In the previous subsection we gave a general result on how to find the equilibrium market price of risk via the inf-convolution of the risk measures. In the particular case of the entropic risk measure, the general computations are considerably simpler and an easier path allows to reach EMPeR θ^R without the representative agent. Although the BSDE for the representative agent derived above will appear in the following computations, with only these computations one cannot show that the resulting equations lead to the same concept of representative agent. In other words, optimality in the representative agent's economy is equivalent to a Pareto optimal risk distribution among the individual agents – for this we have proved Theorem 4.4. This shorter path consists, as was hinted in Section 3.3.4, in a direct linear combination of the BSDEs (3.2) with the minimized driver $\widetilde{G}^a$ given by (3.12)

Following the computations from Section 3.3.5, we see that the market clearing condition requires

$$0 = \sum_{a\in\mathbb{A}} \pi_t^{*,a,2} = \sum_{a\in\mathbb{A}} \frac{1}{1+\widetilde{\lambda}^a} \frac{\widetilde{Z}_t^{a,2} + \gamma_a \theta_t^R}{\kappa_t^R} \quad\Leftrightarrow\quad \theta_t^R = -\frac{\sum_{a\in\mathbb{A}} \frac{\widetilde{Z}_t^{a,2}}{(1+\widetilde{\lambda}^a)}}{\sum_{a\in\mathbb{A}} \frac{\gamma_a}{(1+\widetilde{\lambda}^a)}} = -\frac{\sum_{a\in\mathbb{A}} w^a \widetilde{Z}_t^{a,2}}{\gamma_R},$$

if we define $\gamma_R = \sum_a w^a \gamma_a$, with $w^a = 1/(\Lambda(1+\widetilde{\lambda}^a))$ and $\Lambda = \sum_a 1/(1+\widetilde{\lambda}^a)$. Notice that here we do not need to normalize the family $w = (w^a)$ so that $\sum_a w^a = 1$, since we are not considering an aggregated risk. Any rescaling Λ' of w would give the same θ^R. We present it in this way for consistency with the general case.

Now, replacing the term θ^R by the above value in the minimized driver given by (3.12), we find that the optimal risk processes for each agent solve the BSDEs with driver given by

$$\widetilde{G}^a(t, \widetilde{Z}_t^{\mathbb{A}}) = -\frac{\gamma_a}{2}(\theta_t^S)^2 - \widetilde{Z}^{a,1}\theta_t^S + \frac{1}{\gamma_R} \widetilde{Z}_t^{a,2}\left(\sum_{a\in\mathbb{A}} w^a \widetilde{Z}_t^{a,2}\right) - \frac{\gamma_a}{2\gamma_R^2}\left(\sum_{a\in\mathbb{A}} w^a \widetilde{Z}_t^{a,2}\right)^2. \tag{4.9}$$

The BSDEs with these drivers form a system of N coupled BSDEs with quadratic growth, which, in general, are difficult to solve, see [ET15], [Esp10], [FDR11] or more recently [Fre14, KP14]. Fortunately, one can take advantage of the structure of (4.9) and find a simpler BSDE for the process $(\widehat{Y}^w, \widehat{Z}^w) = \sum_a w^a(\widetilde{Y}^a, \widetilde{Z}^a)$. It follows easily that $\widehat{Y}_T^w = -\sum_a w^a(H^a + nH^D/N) = H^w$, as in (4.5).

Linearly combining the BSDEs (3.2) with drivers expressed as in (4.9), we find

$$-d\widehat{Y}_t^w = \left[-\frac{\gamma_R}{2}(\theta_t^S)^2 - \widehat{Z}_t^{w,1}\theta_t^S + \frac{1}{2\gamma_R}(\widehat{Z}_t^{w,2})^2\right]dt - \widehat{Z}_t^w dW_t \quad\text{with}\quad \widehat{Y}_T^w = -H^w. \tag{4.10}$$

This is exactly the same BSDE as in Example 4.6. Given that H^w and θ^S are bounded, this BSDE falls in the standard class of quadratic growth BSDE and the existence and uniqueness of $(\widehat{Y}^w, \widehat{Z}^w)$ is easily guaranteed. This allows one to compute θ^R as $-\widehat{Z}^{w,2}/\gamma_R$ and in turn one can finally solve the BSDEs giving the minimized risk processes for each agents, using the driver $\widetilde{G}^a$ as given by (3.12).

Remark 4.7 (No trade-off between risk tolerance and performance concern rate). *Each agent's individual preferences are specified by the parameters γ_a and λ^a, i.e. her risk tolerance and her performance concern respectively. One may ask whether there exists a parametric relation between those parameters such that an agent with (γ_a, λ^a) and another agent with (γ_b, λ^b) would exhibit the same preferences and have the same optimal strategies. Indeed, in most formulas the two parameters appear as coupled. However, one can see that the terminal condition H^w is independent of the risk tolerance parameter γ, hence by changing λ^a and γ^a of any one fixed agent $a \in \mathbb{A}$, one cannot obtain the same outcome.*

5 Further results on the entropic risk measure case

In this section we investigate further the entropic risk measure case. Namely, we introduce further structure that allows to use the theory developed in the previous section and, moreover, to design H^D such that Assumption 3.1 holds true. The ultimate goal of this section is to understand how the concern rates λ affect prices and risks. The first two parts of the section verify that Assumption 3.1 holds and the third one sheds light on the behavior of the aggregated risk and derivative price as the parameters vary.

To ease the analysis we will assume throughout a Black-Scholes market (i.e μ, σ^S, θ^S are constants). Such an assumption is not strictly necessary for the results we obtain here, but we wish to focus on the qualitative analysis and not on obfuscating mathematical techniques. Throughout the rest of this section the next assumption holds.

Assumption 5.1. *Let Assumption 2.1 hold and set $\varphi^D = \varphi^a = 0$ for any $a \in \mathbb{A}$. Let $\mu^R \in C([0,T], \mathbb{R})$, $\theta^S \in (0, \infty)$ and $\mu^S, \theta^S \in \mathbb{R}$. For any $a \in \mathbb{A}$ the functions $h^D, h^a \in C_b^2(\mathbb{R}^2, \mathbb{R})$ are strictly positive and satisfy $(\partial_{x_2} h^D)(x_1, x_2) \neq 0$ for any $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$.*

Remark 5.2 (On notation for the section)**.** *In this section we work mainly with the representative agent BSDE (see Example 4.6 or (4.10)) and the derivative price BSDE (3.18).*

To avoid a notation overload in what the BSDE for the representative agent is concerned, we drop the tilde notation and define (Y^w, Z^w) as the solution to the mentioned BSDE; not to be confused with (4.3) which plays no role here. The solution to the derivative price BSDE is denoted by (B, κ).

5.1 The aggregated risk

The BSDE (4.10) is not difficult to analyze given the existing literature on BSDEs of quadratic growth. Recall that $\theta^S \in \mathcal{S}^\infty$ and $Y_T^w \in L^\infty$ (since it is a weighted sum of bounded random variables). We shortly recall that $\mathbb{D}^{1,2}$ is the space of 1st order Malliavin differentiable processes and D denotes the Malliavin derivative operator, we point the reader to Appendix A.1 for further Malliavin calculus references.

Theorem 5.3. *The BSDE (4.10) has a unique solution $(Y^w, Z^w) \in (\mathcal{S}^\infty \cap \mathbb{D}^{1,2}) \times (\mathcal{H}_{BMO} \cap \mathbb{D}^{1,2})$. Moreover, there exists a strictly negative function $u^w \in C^{0,1}([0,T] \times \mathbb{R}^2, \mathbb{R})$ such that for any $t \in [0,T]$*

$$Y_t^w = u^w(t, S_t, R_t) \qquad and \qquad Z_t^{w,2} = (\partial_{x_2} u^w)(t, S_t, R_t)b, \qquad \mathbb{P}\text{-a.s..} \tag{5.1}$$

i) For any $r, u \in [0,t]$, $t \in [0,T]$ it holds that

$$D_u^{W^R} Y_t^w = D_r^{W^R} Y_t^w \quad \mathbb{P}\text{-a.s.} \qquad and \qquad D_u^{W^R} Z_t^w = D_r^{W^R} Z_t^w \quad \mathbb{P} \otimes \lambda\text{-a.e.} \tag{5.2}$$

and in particular $D_t^{W^R} Y_t = Z_t^w$ $\mathbb{P}$-a.s. for any $t \in [0,T]$.

ii) There exists a constant $C > 0$ for any $t \in [0,T]$ such that $|Z_t^{w,2}| \leq C$, i.e. $Z^{w,2} \in \mathcal{S}^\infty$ and $\partial_{x_2} u^w \in C_b$. Moreover, $\theta^R \in \mathcal{S}^\infty$.

iii) The process $D^{W^R} Z^w$ belongs to $\mathcal{H}_{BMO}$.

Proof. Let $a \in \mathbb{A}$ and $0 \leq u \leq t \leq T$. Existence and uniqueness of the SDE follow from Proposition A.3.

▷ *Proof of i):* By assumption we have $Y_T^w \in L^\infty$ and $\theta^S \in \mathcal{S}^\infty$ which allows to quote Theorem 2.6 in [IDR10] and hence that $(Y^w, Z^w) \in \mathcal{S}^\infty \times \mathcal{H}_{BMO}$. Moreover, given that $Y_T^w < 0$, a standard

comparison principle for quadratic BSDEs (see e.g. Theorem 2.6 in [Kob00]) yields easily that $Y_t^w < 0$ for any $t \in [0, T]$ and hence that $u^w < 0$.

Theorem A.3 ensures that the payoffs H^D and H^a, and hence H^w, are Malliavin differentiable with bounded Malliavin derivatives. Combining this further with $\theta^S \in \mathbb{R}$, the Malliavin differentiability of (4.10) follows from Theorem 2.9 in [IDR10]. Under Assumption 5.1 the results in [IDR10] (or Chapter 4 of [DR11]) along with Theorem 7.6 in [AIdR10] yield the Markov property for Y^w and the parametric differentiability result for the (quadratic) BSDEs.

Since $u^w \in C^{0,1}$ by direct application of the Malliavin differential we have for $0 \le u \le t \le T$

$$
\begin{aligned}
D_u^{W^R} Y_t^w &= D_u^{W^R} \big(u^w(t, S_t, R_t) \big) \\
&= (\partial_{x_2} u^w)(t, S_t, R_t)(D_u^{W^R} R_t) = (\partial_{x_2} u^w)(t, S_t, R_t) b = D_t^{W^R} Y_t^w.
\end{aligned}
$$

It now follows that $D_t^{W^R} Y_t = D_u^{W^R} Y_t = Z_t$ for any $0 \le u \le t \le T$ $\mathbb{P}$-a.s.

▷ *Proof of ii):* Define now the probability measure $\mathbb{Q}$ (equivalent to $\mathbb{P}$) as

$$
\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(-\int_0^T \left\langle (-\theta_s^S, \frac{Z_s^{w,2}}{\gamma_R}), \mathrm{d}W_s \right\rangle \right). \tag{5.3}
$$

The measure $\mathbb{Q}$ is well defined since $\theta^S \in \mathcal{S}^\infty$ and $Z^{w,2} \in \mathcal{H}_{\mathrm{BMO}}$. Then for $0 \le u \le t \le T$ we have (Theorem 2.9 in [IDR10])

$$
\begin{aligned}
D_u^{W^R} Y_t^w &= D_u^{W^R} Y_T^w + \int_t^T [-\theta_s^S D_u^{W^R} Z_s^{w,1} + \frac{1}{\gamma_R} Z_s^{w,2} D_u^{W^R} Z_s^{w,2}] \mathrm{d}s - \int_t^T D_u^{W^R} Z_s^w \mathrm{d}W_s \tag{5.4} \\
&\Rightarrow D_u^{W^R} Y_t^w = \mathbb{E}^{\mathbb{Q}}[D_u^{W^R} Y_T^w | \mathcal{F}_t].
\end{aligned}
$$

The results in Proposition A.3 and the definition of Y_T^w imply that $|D_u^{W^R} Y_t^w| < C$. Path regularity results for BSDEs along with their usual representation formulas (see [IDR10]) yield that $(D_t^{W^S} Y_t) = (Z_t^2) \in \mathcal{S}^\infty$; the boundedness of $\partial_{x_2} u^w$ follows in an obvious way. As a consequence, $\theta^R \in \mathcal{S}^\infty$ since $Z^{w,2} \in \mathcal{S}^\infty$ and (4.8) holds.

▷ *Proof of iii):* Using now the fact that $\theta^S, Z^{w,2} \in \mathcal{S}^\infty$, we apply the results from Theorem 2.6 in [IDR10] to (5.4) and obtain that $D^{W^R} Z^w \in \mathcal{H}_{\mathrm{BMO}}$. The BMO norm of $D^{W^R} Z^w$ depends only on some real constants and T, γ_R, $\sup_u \|D_u^{W^R} Y_T^w\|_{L^\infty}$ and $\|(\theta^S, Z^{w,2})\|_{\mathcal{S}^\infty \times \mathcal{S}^\infty}$ (see Theorem 2.6 in [IDR10]). $\qquad \square$

In the next result we show that the mapping $x_2 \mapsto (\partial_{x_2} u^w)(t, x_1, x_2)$ is Lipschitz. Denote by R and $\widetilde{R}$ the solutions to (2.1) with $R_0 = r_0$ and $R_0 = \widetilde{r}_0$ respectively; denote as well by (Y, Z) and $(\widetilde{Y}, \widetilde{Z})$ the solutions to BSDE (4.10) for the underlying processes R and $\widetilde{R}$ respectively.

Proposition 5.4. *For any $(t, x_1) \in [0, T] \times \mathbb{R}$ the map $\mathbb{R} \ni x_2 \mapsto (\partial_{x_2} u^w)(t, x_1, x_2)$ is Lipschitz continuous uniformly in t and x_1. In particular the process $D^{W^R} Z^w$ is $\mathbb{P}$-a.s. bounded.*

Proof. Let $0 \le u \le t \le T$ and define $\delta DY := D^{W^R} Y^w - D^{W^R} \widetilde{Y}^w$, $\delta DZ^i := D^{W^R} Z^{w,i} - D^{W^R} \widetilde{Z}^{w,i}$ for $i \in \{1, 2\}$ and (intuitively) $\delta DZ := (\delta DZ^1, \delta DZ^2)$. Then, following from (5.4) written under $\mathbb{Q}$ from (5.3), we have

$$
\delta D_u Y_t = \delta D_u Y_T - \int_t^T \delta D_u Z_s \mathrm{d}W_s^{\mathbb{Q}} + \int_t^T \frac{1}{\gamma_R} (Z_s^{w,2} - \widetilde{Z}_s^{w,2}) D_u^{W^R} Z_s^{w,2} \mathrm{d}s.
$$

Define now the process

$$e_t := \exp\left\{\int_0^t \frac{1}{\gamma_R} D_u^{W^R} Z_s^{w,2} \mathrm{d}s\right\}, \quad t \in [0,T] \qquad \text{with} \qquad (e_t) \in \mathcal{H}^p, \quad \forall p \geq 1, \qquad (5.5)$$

where the $\mathcal{H}^p$ integrability of (e_t) follows from Lemma A.1. Observe next that by the results of Theorem 5.3 one has $\delta D_u Y_t = \delta D_t Y_t = Z_t^{w,2} - \widetilde{Z}_t^{w,2}$. Applying Itô's formula to $(e_t \delta D.Y_t)$, using the just mentioned identity and taking $\mathbb{Q}$-conditional expectations it follows at $u = t = 0$ that

$$|(\partial_{x_2} u^w)(0, s_0, r_0) - (\partial_{x_2} u^w)(0, s_0, \widetilde{r}_0)| = \frac{1}{b}|(Z_0^{w,2} - \widetilde{Z}_0^{w,2})| = |\frac{1}{b}\mathbb{E}^{\mathbb{Q}}\left[e_T \delta D_0 Y_T\right]| \leq C|r_0 - \widetilde{r}_0|.$$

The last line is a consequence of Proposition A.3 combined with the fact that $\mathbb{E}^{\mathbb{Q}}[e_T^p]$ ($\forall p \geq 1$) is finite due to the BMO properties of $D^{W^R} Z^{w,2}$, see Lemma A.1. The constant C is independent of $u, r_0, \widetilde{r}_0$ and s_0. Although $D^{W^R} Z^{w,2}$ is a BMO martingale under $\mathbb{P}$, the integrability still carries under $\mathbb{Q}$; this is the same argument as in the final step of the proof of Lemma 3.1 in [IDR10] (see also Lemma 2.2 and Remark 2.7 of the cited work).

The extension of the above result to the whole time interval $[0,T]$ follows via the Markov property of the BSDE solution. This relates to the close link between BSDEs of the Markovian type and certain classes of quasi-linear parabolic PDEs (see e.g. Section 4 in [EKPQ97]).

Finally, the boundedness of $D^{W^R} Z^w$ follows from the Lipschitz property of $x_2 \mapsto (\partial_{x_2} u^w)(\cdot, \cdot, x_2)$ and the boundedness of $D^{W^R} R$, see Proposition A.3, ii). $\qquad \square$

5.2 The EMPR and the derivative's BSDE

The next result shows that Assumption (5.1) implies Assumption 3.1 holds.

Theorem 5.5 (Market completion). *The derivative H^D completes the market, i.e. $\kappa^R \neq 0$ $\mathbb{P}$-a.s. for any $t \in [0,T]$. Moreover, $\kappa^R \in \mathcal{S}^\infty$ and $sgn(\kappa_t^R) = sgn(b\partial_{x_2}h^D)$ for any $t \in [0,T]$.*

Before proving the above result we need an intermediary one. Recall that BSDE (3.18) describes the dynamics of the price process B^θ, that $H^D \in L^\infty$ and $\theta \in \mathcal{S}^\infty \times (\mathcal{H}_{\text{BMO}} \cap \mathbb{D}^{1,2})$ (following from Assumption 5.1 and Theorem 5.3).

Proposition 5.6. *The pair (B, κ) belong to $(\mathcal{S}^\infty \cap \mathbb{D}^{1,2}) \times (\mathcal{H}_{BMO} \cap \mathbb{D}^{1,2})$ and their Malliavin derivatives satisfy for $0 \leq u \leq t \leq T$ the dynamics*

$$D_u^{W^R} B_t^\theta = D_u^{W^R} H^D - \int_t^T \kappa_s^R D_u^{W^R} \theta_s^R + \left\langle \theta_s, D_u^{W^R} \kappa_s^\theta \right\rangle \mathrm{d}s - \int_t^T D_u^{W^R} \kappa_s^\theta \mathrm{d}W_s. \qquad (5.6)$$

The representation $D_t^{W^R} B_t^\theta = \kappa_t^R$ holds $\mathbb{P}$-a.s. for any $0 \leq t \leq T$.

Proof. Let $0 \leq u \leq t \leq T$. Observe that BSDE (3.18) is a BSDE with a linear driver and a bounded terminal condition (throughout $\nu = 1$). The existence and uniqueness of a solution follows from the results of [EKPQ97]. Moreover, the estimation techniques used in [IDR10] yield that $(B, \kappa) \in \mathcal{S}^\infty \times \mathcal{H}_{\text{BMO}}$ (see Theorem 2.6 in [IDR10]). The Malliavin differentiability of (B, κ) follows from Proposition 5.3 in [EKPQ97] and the remark following it since $(\theta^S, \theta^R) \in \mathbb{R} \times (\mathcal{S}^\infty \cap \mathbb{D}^{1,2})$ (see Theorem 5.3). The quoted result and Proposition A.3 yield (5.6) for $D^{W^R} B^\theta$. Moreover, from Theorem 2.9 in [IDR10] we have $\lim_{u \nearrow t} D_u^{W^R} B_t^\theta = \kappa_t^R$ for $0 \leq u \leq t \leq T$ $\mathbb{P} \otimes \lambda$-a.e..

We now prove a finer result on B and κ, namely that $D_t^{W^R} B_t^\theta = \kappa_t^R$ holds $\mathbb{P}$-a.s. for any $0 \leq t \leq T$ instead of just $\mathbb{P} \otimes \lambda$-a.e.. This is done by showing that $(u,t) \mapsto D_u^{W^R} B_t^\theta$ is jointly continuous.

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Remark that the map $t \mapsto D_u^{W^R} B_t^\theta$ for $u \leq t$ is given by (5.6) and hence it is continuous in time ($\forall t \in [u, T]$). Note now that Proposition 5.4 and Proposition A.3 yield that $D^{W^R} Z^{w,2}$ is bounded and $D_u^{W^R} Z_t^{w,2} = D_r^{W^R} Z_t^{w,2} = D_0^{W^R} Z_t^{w,2}$ for any $0 \leq u, r \leq t \leq T$. These properties hold as well for θ^R via the identity $-\gamma_R \theta^R = Z^{w,2}$.

Using the measure $\mathbb{P}^\theta$ (introduced in (2.3)), the fact that $D^{W^R} \theta^S = 0$ and the identity $-\gamma_R \theta^R = Z^{w,2}$ one can rewrite (5.6) as

$$D_u^{W^R} B_t^\theta = D_u^{W^R} H^D + \frac{1}{\gamma_R} \int_t^T \kappa_s^R D_u^{W^R} Z_s^{w,2} \mathrm{d}s - \int_t^T D_u^{W^R} \kappa_s^\theta \mathrm{d}W_s^\theta. \tag{5.7}$$

Writing the same BSDE as above, but for a parameter v (instead of u) we have

$$\begin{aligned} D_v^{W^R} B_t^\theta &= D_v^{W^R} H^D + \frac{1}{\gamma_R} \int_t^T \kappa_s^R D_v^{W^R} Z_s^{w,2} \mathrm{d}s - \int_t^T D_v^{W^R} \kappa_s^\theta \mathrm{d}W_s^\theta \\ &= D_u^{W^R} H^D + \frac{1}{\gamma_R} \int_t^T \kappa_s^R D_u^{W^R} Z_s^{w,2} \mathrm{d}s - \int_t^T D_v^{W^R} \kappa_s^\theta \mathrm{d}W_s^\theta, \end{aligned}$$

where we used the results of Proposition A.3. Since the solution to (5.7) is unique and the BSDE just above has exactly the same parameters as (5.7), we must conclude that for any $t \in [0, T]$ and for $0 \leq u, r \leq t$ it holds $D_u^{W^R} B_t^\theta = D_r^{W^R} B_t^\theta$. From the continuity of $t \mapsto D_\cdot^{W^R} B_t^\theta$ follows now the joint continuity of $(u, t) \mapsto D_u^{W^R} B_t^\theta$ in its time parameters and hence the representation $D_t^{W^R} B_t^\theta = \kappa_t^R$ holds $\mathbb{P}$-a.s. for any $0 \leq t \leq T$. $\qquad \square$

We can now prove Theorem 5.5.

Proof of Theorem 5.5. We proceed in the same way as in the proof of Proposition 5.4. The argument goes as follows: define the process (e_t) just like in (5.5); apply Itô's formula to $(e_t D_\cdot^{W^R} B_t^\theta)$ and write the resulting equation under $\mathbb{P}^\theta$ (just like (5.7)); take $\mathbb{P}^\theta$ conditional expectations. At this point a remaining Lebesgue integral is still in the dynamics:

$$\begin{aligned} D_u^{W^R} B_t^\theta &= (e_t)^{-1} \mathbb{E}^\theta \left[e_T D_u^{W^R} H^D + \frac{1}{\gamma_R} \int_t^T e_s (\kappa_s^R - D_u^{W^R} B_s^\theta) D_u^{W^R} Z_s^{w,2} \mathrm{d}s | \mathcal{F}_t \right] \\ &= (e_t)^{-1} \mathbb{E}^\theta \left[e_T D_u^{W^R} H^D | \mathcal{F}_t \right], \end{aligned}$$

where from the first to the second line we used Proposition 5.6, i.e. that $\kappa_s^R = D_s^{W^R} B_s^\theta = D_u^{W^R} B_s^\theta$ $\mathbb{P}$-a.s. for any $0 \leq u \leq s \leq T$.

Moreover, using Proposition A.3 we conclude that the boundedness of $D_\cdot^{W^R} H^D$ implies that of $(D_\cdot^{W^R} B_t^\theta)$ and hence that of (κ_t^R). Since $b\partial_{x_2} h^D \neq 0$ and since is continuous in its variables, it follows that $\kappa^R \neq 0$ $\mathbb{P}$-a.s. for all $t \in [0, T]$. Finally, $\mathrm{sgn}(\kappa_\cdot^R) = \mathrm{sgn}(b\partial_{x_2} h^D)$. $\qquad \square$

5.3 Parameter Analysis

It is possible to justify at a theoretical level some of the predictable behavior of the processes Y^w, B^θ and θ^R with relation to the problem's parameters: n, γ_R, λ^a and γ_a for $a \in \mathbb{A}$.

Theorem 5.7. *Let θ be the EMPR. The process (Y^w, Z^w) solving BSDE (4.10) is differentiable with relation to λ^a for any $a \in \mathbb{A}$, n and γ_R (see (4.4) and (4.6)).*

Fix agent $a \in \mathbb{A}$. If the differences

$$\gamma_R - \gamma_a \qquad and \qquad \mathbb{E}^\theta \Big[\Big(\sum_{b \in \mathbb{A}} w^b H^b \Big) - H^a \Big] \tag{5.8}$$

are positive (negative respectively) then $\partial_{\widetilde{\lambda}^a} Y_t^w$ is negative (positive respectively) for any $t \in [0, T]$.
  For any $a \in \mathbb{A}$ we have $\mathbb{P}$-a.s that

$$\partial_{\gamma_R} Y_t^w < 0, \quad \partial_{\gamma_a} Y_t^w < 0 \qquad \forall t \in [0, T).$$

Furthermore, $\mathbb{P}$-a.s

$$\partial_n Y_t^w < 0, \quad sgn(\partial_n \theta_t^R) = sgn(b \partial_{x_2} h^D) \ \forall t \in [0, T] \quad and \quad \partial_n B_t^\theta < 0 \quad \forall t \in [0, T).$$

Part of the results are in some way expected. Introducing more derivatives leads to an overall risk reduction and as more derivatives are placed in the market the less the derivative is worth. If γ_R is interpreted as the representative agents' risk tolerance, then as γ_R increases we have a decrease in risk (Y^w decreases) since it represents an increase in the single agents' risk tolerance (i.e. $\gamma_a \nearrow$).

The main message of the above theorem is that the effect of the performance concern of one agent on the aggregate risk depends essentially on how the agent is positioned with respect to the others, both in terms of risk tolerance as well as the personal endowments. If the agent's risk tolerance γ_a is higher than the aggregate risk tolerance γ_R and her endowment position dominates by the aggregate endowment position, then an increase in the agent's concern rate leads to an increase of the aggregate risk.

Before proving the above result we remark that condition (5.8) simplifies under certain conditions; such simplifications are summarized in the below corollary. All results follow by direct manipulation of the involved quantities.

Corollary 5.8. *Let the conditions of Theorem 5.7 hold. If $\gamma_a = \gamma$ for all $a \in \mathbb{A}$, then $\gamma_R - \gamma_a = \gamma \big(\sum_a w^a - 1 \big) = 0$.*
  If $N = 2$, then $w^a + w^b = 1 \Leftrightarrow w^b = 1 - w^a$ and hence

$$\Big(\sum_{c \in \mathbb{A}} w^c H^c \Big) - H^a = -w^b (H^a - H^b) \qquad and \qquad \Big(\sum_{c \in \mathbb{A}} w^c H^c \Big) - H^b = w^a (H^a - H^b).$$

Similarly $\gamma_R - \gamma_a = -w^b (\gamma_a - \gamma_b)$ and $\gamma_R - \gamma_b = w^a (\gamma_a - \gamma_b)$. Moreover, it holds that

$$sgn \Big(\partial_{\lambda^a} Y_t^w \Big) = -sgn \Big(\partial_{\lambda^b} Y_t^w \Big) \qquad \mathbb{P}\text{-a.s. for any } t \in [0, T]. \tag{5.9}$$

Proof of Theorem 5.7. Let $a \in \mathbb{A}$ and $t \in [0, T]$. Theorem 3.1.9 in [DR11] (see also Theorem 2.8 in [IDR10]) ensures the differentiability of BSDE (4.10) with respect to γ_R, γ_a, λ^a and n.

  $\triangleright$ *The derivative of Y^w in γ_R:* Applying ∂_{γ_R} to BSDE (4.10) and writing it under the probability measure $\mathbb{Q}$ defined in (5.3) yields the dynamics

$$\partial_{\gamma_R} Y_t^w = 0 + \int_t^T \left[-\frac{1}{2} (\theta_s^S)^2 - \frac{1}{2\gamma_R^2} \big(Z_s^{w,2} \big)^2 \right] ds - \int_t^T \partial_{\gamma_R} Z_s^w dW_s^{\mathbb{Q}}.$$

Taking $\mathbb{Q}$-conditional expectations and noticing that the Lebesgue integral term is strictly negative for any $t \in [0, T)$, we have then $\partial_{\gamma_R} Y_t^w < 0$ for any $t \in [0, T)$.

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▷ *The derivative of Y^w in γ_a:* This case follows from the previous one as γ_R is defined by (4.4) and the weights $w^{\cdot}$ (see (4.6)) are independent of $\gamma_{\cdot}$.

$$\gamma_R := \sum_{a \in \mathbb{A}} w^a \gamma_a \quad \text{implies} \quad \partial_{\gamma_a} \gamma_R = w^a > 0,$$

and finally $\partial_{\gamma_a} Y^w = \partial_{\gamma_R} Y^w \cdot \partial_{\gamma_a}(\gamma_R)$. The statement follows.

▷ *The derivatives of Y^w in $\widetilde{\lambda}^a$:* We compute only the derivatives with respect to $\widetilde{\lambda}^a$ in order to present simplified calculations as $\widetilde{\lambda}^a := \lambda^a / (N - 1)$. Calculating the involved derivatives leads to

$$\partial_{\widetilde{\lambda}^a} \frac{1}{1 + \widetilde{\lambda}^a} = -(\Lambda w^a)^2, \quad \partial_{\widetilde{\lambda}^a} \frac{1}{\Lambda} = (w^a)^2, \quad \partial_{\widetilde{\lambda}^a} w^a = (w^a)^2 \Lambda (w^a - 1), \quad \partial_{\widetilde{\lambda}^a} w^b = (w^a)^2 \Lambda w^b,$$

$$\partial_{\widetilde{\lambda}^a} \gamma_R = \partial_{\widetilde{\lambda}^a} \sum_{b \in \mathbb{A}} w^b \gamma_b = (w^a)^2 \Lambda (\gamma_R - \gamma_a) \quad \text{and} \quad \partial_{\widetilde{\lambda}^a} H^w = (w^a)^2 \Lambda \Big(\big(\sum_{b \in \mathbb{A}} w^b H^b \big) - H^a \Big).$$

Combining the above results with the BSDE for $\partial_{\widetilde{\lambda}^a} Y^w$ under the $\mathbb{Q}$-measure (just as in the previous two steps) yields

$$\partial_{\widetilde{\lambda}^a} Y_t^w = -(w^a)^2 \Lambda \mathbb{E}^{\mathbb{Q}} \Big[\Big(\big(\sum_{b \in \mathbb{A}} w^b H^b \big) - H^a \Big) + (\gamma_R - \gamma_a) \int_t^T \big[\frac{1}{2}(\theta_s^S)^2 + \frac{1}{2\gamma_R^2} \big(Z_s^{w,2} \big)^2 \big] \mathrm{d}s \Big| \mathcal{F}_t \Big].$$

Since $\mathbb{Q}$ is equivalent to $\mathbb{P}$, the statement follows.

▷ *The derivative of Y^w in n:* Applying ∂_n to BSDE (4.10) and writing it under the probability measure $\mathbb{Q}$ defined in (5.3) yields the dynamics

$$\partial_n Y_t^w = \partial_n Y_T^w - \int_t^T \partial_n Z_s^w \mathrm{d}W_s^{\mathbb{Q}} \quad \Rightarrow \quad \partial_n Y_t^w = \mathbb{E}^{\mathbb{Q}}[\partial_n Y_T^w | \mathcal{F}_t] = -\frac{\mathbb{E}^{\mathbb{Q}}[H^D | \mathcal{F}_t]}{N} \sum_{a \in \mathbb{A}} w^a < 0,$$

where the last sign follows from the definition of Y_T^w and H^D.

▷ *The derivative of θ^R in n:* The analysis of $Z^{w,2}$ and hence of θ^R with respect to n and γ_R follows from the analysis of (5.4). Given representation (4.8), applying ∂_n to BSDE (5.4) and writing it under the probability measure $\mathbb{Q}$ defined in (5.3) yields the dynamics

$$\partial_n D_u^{W^R} Y_t^w = \partial_n D_u^{W^R} Y_T^w - \int_t^T \partial_n D_u^{W^R} Z_s^w \mathrm{d}W_s^{\mathbb{Q}} + \int_t^T [\frac{1}{\gamma_R} D_u^{W^R} Z_s^{w,2} \partial_n Z_s^{w,2}] \mathrm{d}s$$

$$\Leftrightarrow \quad \partial_n Z_t^{w,2} = \partial_n D_u^{W^R} Y_T^w - \int_t^T \partial_n D_u^{W^R} Z_s^w \mathrm{d}W_s^{\mathbb{Q}} + \int_t^T [\frac{1}{\gamma_R} D_u^{W^R} Z_s^{w,2} \partial_n Z_s^{w,2}] \mathrm{d}s$$

$$\Leftrightarrow \quad \partial_n Z_t^{w,2} = (e_t)^{-1} \mathbb{E}^{\mathbb{Q}} \Big[e_T \partial_n D_u^{W^R} Y_T^w | \mathcal{F}_t \Big],$$

where (e_t) is as in (5.5) and the argumentation is similar to that back there.

Notice now that

$$\partial_n D_t^{W^R} Y_T^w = -\frac{1}{N} \Big(\sum_{a \in \mathbb{A}} w^a \Big) D_t^{W^R} H^D = -\frac{1}{N} \Big(\sum_{a \in \mathbb{A}} w^a \Big) b(\partial_{x_2} h^D)(S_T, R_T).$$

Given Assumption 5.1, we are able to conclude that $\operatorname{sgn}(Z_t^{w,2}) = -\operatorname{sgn}(b \partial_{x_2} h^D)$, and hence, from (4.8) that $\operatorname{sgn}(\partial_n \theta_t^R) = \operatorname{sgn}(b \partial_{x_2} h^D)$.

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▷ *The derivative of B^θ in n:* We use justifications similar to those used in Proposition 5.6 and hence we do not give all the details. Recall (3.18), apply the ∂_n-operator to the equation and do the usual change of measure (with $\mathbb{P}^\theta$) to obtain

$$\partial_n B_t^\theta = 0 - \int_t^T \kappa_s^R \partial_n \theta_s^R \mathrm{d}s - \int_t^T \partial_n \kappa_s^\theta \mathrm{d}W_s^\theta \quad \Leftrightarrow \quad \partial_n B_t^\theta = -\mathbb{E}^\theta[\int_t^T \kappa_s^R \partial_n \theta_s^R \mathrm{d}s].$$

By the previous result we have $\mathrm{sgn}(\partial_n \theta_t^R) = \mathrm{sgn}(b \partial_{x_2} h^D)$ and from Theorem 5.5 we have $\mathrm{sgn}(\kappa_t^R) = \mathrm{sgn}(b \partial_{x_2} h^D)$. It easily follows that $\partial_n B_t^\theta < 0$. □

Unfortunately the conditions used above do not allow for similar results on the behavior of, say $\gamma_R \mapsto \theta^R$ or $(\gamma_R, n, \lambda) \mapsto \widetilde{Y}^a$. The conditions required for such results are too restrictive to be of any usefulness. Nonetheless, we will investigate them in Section 6 via numerical simulation.

6 Study of a particular model with two agents

In this section, we investigate in detail a model economy consisting of two agents using entropic risk measures and having opposite exposures to the external non-financial risk. We give particular attention to the impact of the relative performance concern rates on the equilibrium related processes. The model is sufficiently simple to allow extended tractability, when compared with Sections 3, 4 and 5, and nonetheless still sufficiently general as to produce a rich set of results and interpretations. In particular, we are able to explicitly describe the structure of the equilibrium. Using numerical simulations, we are able to explore the dependence of individual quantities (such as the optimal portfolios π^{*a} and minimized risks Y_0^a) on the various parameters, thus complementing the results in Theorem 5.7.

6.1 The particular model and numerical methodology

We consider a stylized market consisting of two agents. We argue that a larger set of N agents with certain exposures to the external risk R can be clustered in two groups: those profiting from the high values of R and those profiting from the low values of R, and we can apply the weighted aggregation technique used in Section 4 to each group. Our two agents can therefore be thought of as representative agents for each group. The external risk process is taken to be the temperature affecting the two agents, who also have access to a stock market.

Temperature and Stock models

We study one period of $T = 1$ month where the temperatures follow an SDE (2.1) with constant coefficients:

$$R_t = r_0 + \mu^R t + b W_t^R,$$

and for the stock we take a standard Black–Scholes model:

$$\frac{dS_t}{S_t} = \mu^S dt + \sigma^S dW_t^S,$$

where the coefficients are $r_0 = 18$, $\mu^R = 2$ and $b = 4$ for the temperature process, and $S_0 = 50$, $\mu^S = -0.2$ and $\sigma^S = 0.25$ (so $\theta^S = \mu^S/\sigma^S = -0.8$) for the stock price process.

Agents' parameters, endowments and the derivative

Define $I(x) := \frac{1}{\pi} \arctan(x) + \frac{1}{2} \in [0,1]$. The agents' endowments, H^a and H^b, are taken to be

$$H^a = 5 + I\Big(2\big(R_T - 24\big)\Big) \cdot 15,$$
$$H^b = 5 + I\Big(2\big(16 - R_T\big)\Big) \cdot \Big(15 + 5\,I(S_T - 40)\Big).$$

Agent a profits from higher temperatures while agent b profits from lower ones. The derivative has a payoff H^D that does not depend on the stock S, and is given by

$$H^D = I(R_T - 20),$$

so that it allows to transfer purely the external risk. All functions satisfy Assumptions 2.1 and 5.1. Given the agents' opposite exposures to R_T and the design of H^D, agent a will act as a seller while agent b will act as the buyer, thus establishing a viable market for the derivative.

We assume throughout that the total supply of derivative is zero, $n = 0$, i.e. every unit of derivative one agent owns is underwritten by the other. The risk tolerance coefficients of the agents are fixed at $\gamma_a = \gamma_b = 1$ unless we are analyzing some behavior with respect to them. Similarly, unless otherwise specified, the concern rates are fixed to be $\lambda^a = \lambda^b = 0.25$ unless we are analyzing some behavior with respect to them.

The numerical procedure

The simulation of the involved processes involves a time discretization and Monte Carlo simulations. The forward processes have explicit solutions and we use them directly. All BSDEs are solved numerically. Regarding their time discretization, we use a standard backward Euler scheme, see [BT04], and we complement the time-discretization procedure with the control variate technique stated in Section 5.4.2 of [LRS14]. The approximation of the conditional expectations in the backward induction steps is done via projection over basis functions, see the Least-Squares Monte Carlo method used in [GT14].

We follow the methodologies in Sections 3 and 4. First, we solve the representative agent's BSDE (4.10). This yields via (4.8) the EMPeR process θ^R. Once this is obtained, we solve the BSDE for the price B^θ of the derivative, Equation (3.18), obtaining (κ^S, κ^R) in the process. Finally, we solve the BSDE (3.20) with driver (3.12) for each agent $a \in \mathbb{A}$ and compute the optimal strategies $\pi^{*,a} = (\pi^{*,a,1}, \pi^{*,a,2})$ via (3.13) and (3.14). We note that in the case of two agents, the system (3.13) is easily inverted.

All plots below are computed using 200.000 simulated paths along a uniform time-discretization grid of 20 time-steps, except the plot of Figure 6.1 which uses 30 time-steps.

6.2 Analysis of the behavior in the model

Figure 6.1 shows a realization of the behavior of the agents over the trading period. One can see that the price of the derivative moves like the temperature, and in particular it is never constant (over a time-interval where the temperature has changed). This means that the derivative does indeed complete the market by providing the agents full exposure to R, or equivalently to W^R – Assumption 3.1 is satisfied. Agent b is always long in the derivative and a always short (the latter following from the former since her position is the opposite of that of b). The fact that both agents only go short in the stock is due to its decreasing trend ($\theta^S < 0$) and the fact that the endowments

depend little on S: it is in mainly an optimal investment in the stock that is observed. However agent b's endowment is higher for lower stock prices, hence she does not go as short in the stock as agent a, to hedge this variability.
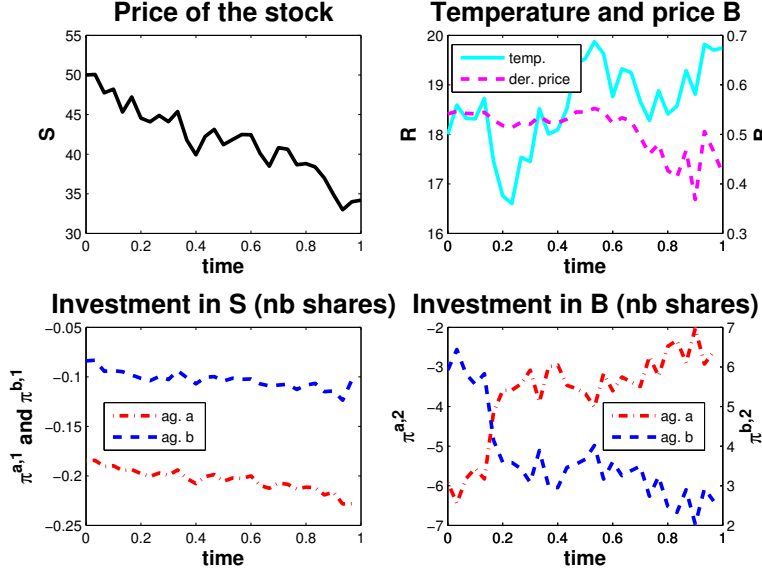

Figure 6.1: Sample paths of the several processes. Stock price on the top left; the temperature and the derivative price on the top right; the investment strategy in the stock on the bottom left and that in the derivative on the bottom right, for each agent. Here $\lambda^a = 0.25$ and $\lambda^b = 0.0$.

Trading activity

The optimal investment strategies for the derivative were seen in Section 3.3.6 and are given by

$$\pi^{*,a,2} = \frac{1}{1+\lambda^a} \frac{\widetilde{Z}^{a,2} + \gamma_a \theta^R}{\kappa^R} \quad \text{and} \quad \pi^{*,b,2} = \frac{1}{1+\lambda^b} \frac{\widetilde{Z}^{b,2} + \gamma_b \theta^R}{\kappa^R}.$$

The optimal investment strategies in the stock follow easily by inverting A_2 from (3.9). This yields

$$\begin{bmatrix} \pi^{*,a,1} \\ \pi^{*,b,1} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1-\lambda^a\lambda^b} & \dfrac{\lambda^a}{1-\lambda^a\lambda^b} \\ \dfrac{\lambda^b}{1-\lambda^a\lambda^b} & \dfrac{1}{1-\lambda^a\lambda^b} \end{bmatrix} \begin{bmatrix} \dfrac{\widetilde{Z}^{a,1} + \gamma_a \theta^S}{\sigma^S S} - \dfrac{\widetilde{Z}^{a,2} + \gamma_a \theta^R}{\kappa^R} \dfrac{\kappa^S}{\sigma^S S} \\ \dfrac{\widetilde{Z}^{b,1} + \gamma_b \theta^S}{\sigma^S S} - \dfrac{\widetilde{Z}^{b,2} + \gamma_b \theta^R}{\kappa^R} \dfrac{\kappa^S}{\sigma^S S} \end{bmatrix}.$$

Remark 6.1 (On the structure of the equilibrium). *The structure of the optimal strategy for the investment in the stock appears clearly in view of the examples treated in Section 3.3. Each agent computes her strategy as a weighted sum of the way both would compute theirs if there was no relative performance concern (compare with Section 3.3.3), using the weights $\left(\frac{1}{1-\lambda^a\lambda^b}, \frac{\lambda^a}{1-\lambda^a\lambda^b}\right)$ for a and $\left(\frac{\lambda^b}{1-\lambda^a\lambda^b}, \frac{1}{1-\lambda^a\lambda^b}\right)$ for b.*

These weights can be understood from Equation (3.13) with $\mathbb{A} = \{a, b\}$: each agent's best response is to invest in the stock according to her natural strategy plus λ^i times the strategy played by the other. Assume now that each agent was initially planning to compute her optimal position using

$$\pi^{(0),i,1} = \frac{\widetilde{Z}^{i,1} + \gamma_i \theta^S}{\sigma^S S} - \frac{\widetilde{Z}^{i,2} + \gamma_i \theta^R}{\kappa^R} \frac{\kappa^S}{\sigma^S S},$$

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and that they are shown, in turn, the strategy that the other is about to play, so that they can update theirs, yielding a sequence of strategies $\pi^{(1),a,1}$, $\pi^{(1),b,1}$, $\pi^{(2),a,1}$, $\pi^{(2),b,1}$, $\pi^{(3),a,1}$, ... for each agent (starting with a's update). Because they both update their strategy according to Equation (3.13), we observe agent a imitating part of agent b, imitating part of agent a, imitating part of agent b, etc. Summing the corresponding series, agent a ends up investing according to $\sum_n (\lambda^a \lambda^b)^n \pi^{(0),a,1} + \lambda^a \sum_n (\lambda^a \lambda^b)^n \pi^{(0),b,1}$, and similarly for b.

The structure of the optimal investment in the derivative is much different, following fundamentally from the endogenous trading condition. If an agent is shown the strategy that the other had decided to follow, she could not unilaterally change her strategy. From this emerges the EMPeR θ^R – see below.

We now look at the behavior of the individual portfolios with respect to the rates of relative performance concern. The intensity of the trading activity at time $t = 0$ on both the stock ($\pi_0^{*,a,1}$) and the derivative ($\pi_0^{*,a,2}$) as maps of the concern rates λ^a, λ^b can be found in Figure 6.2. The positions of agent b are similar in some sense: for the stock, the surface looks very similar; for the derivative, it is the exact opposite (due to the zero net supply condition). For readability we plot only the position of agent a.
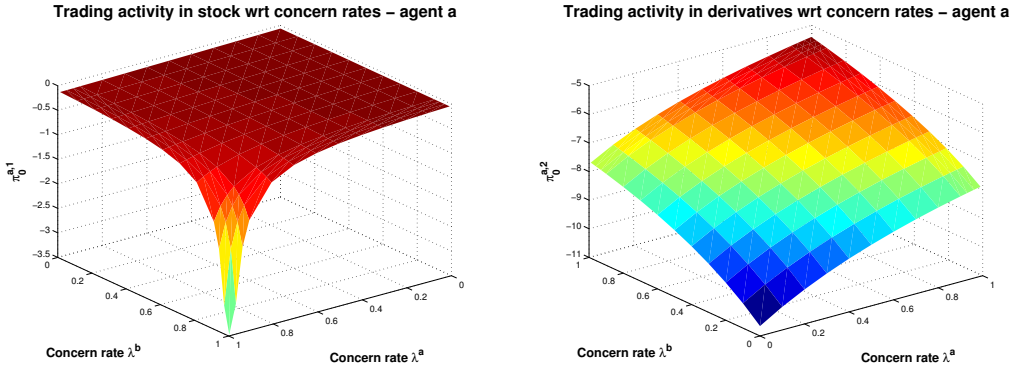

Figure 6.2: Initial number $\pi_0^{a,1}$ and $\pi_0^{a,2}$ of shares of stock (left) and derivative (right) held by agent a, as a function of (λ^a, λ^b). For visualization purposes the axes on the left picture were inverted.

The observed behavior in Figure 6.2 is in line with the intuitive idea that the more the agents are concerned (high λ^i) with their relative performance $V_T^i - V_T^j$, $j \neq i \in \{a, b\}$ (recall (1.1)), the more they will invest in a way that neutralizes this source of risk. This is done by adopting a trading strategy that is as close as possible to that of the other agent.

For the stock, we see from the formulas in Remark 6.1 that when $\lambda^a \lambda^b < 1$, the process of a imitating b imitating a, etc, results in a finite position. But the volume increases with both λ^a and λ^b, and explodes as $(\lambda^a, \lambda^b) \to (1, 1)$. In our example they would both (short-)sell infinitely many shares of the stock. Note that this is possible only because the stock is assumed to be exogenously priced and perfectly liquid. For the derivative, they cannot imitate each other and position themselves in the same direction, as the zero net supply condition implies that the agents must hold exactly opposite positions. Agent b's gains on trading the derivative will be exactly agent a's losses. The only way to reduce the difference in performances for a very concerned agent is to engage less (in volume) in the trading of the derivative. The market clearing condition then forces the other agent to also trade less (in volume). This is seen from the factor $1/(1 + \lambda^i)$ in the formulas in Remark 6.1 and is confirmed in Figure 6.2 (on the right) where agent a, identified as the seller, ends up selling fewer units of the derivative as either concern rate increases. Due to the market clearing condition

between the agents, no explosion is possible.

Price of the derivative

Figure 6.3 shows an opposite dependence of the derivative's price B_0^θ on the concern rates λ^a, λ^b, a behavior not captured by Theorem 5.7. One can make sense of this effect by having in mind Figure 6.2. A higher λ^a implies that agent a wants to trade less and, as she is the seller, this drives the price up. Symmetrically, a higher λ^b implies that agent b wants to trade less and, as she is the buyer, this drives the price down.
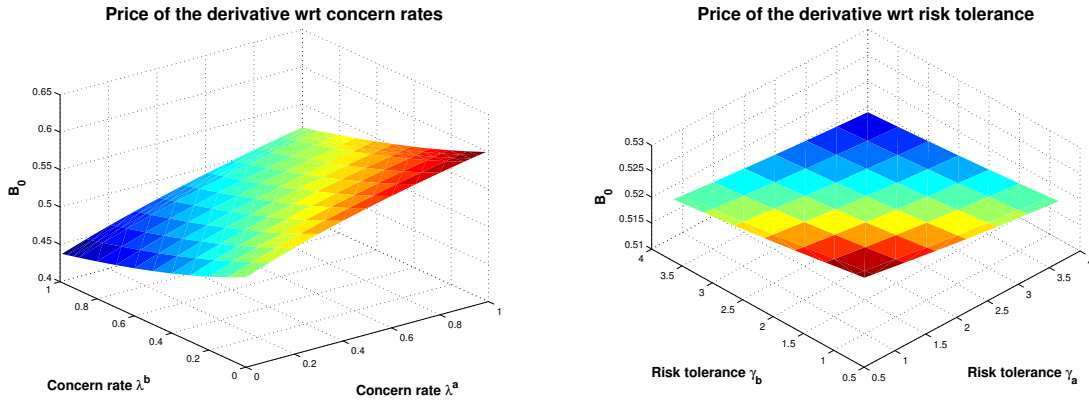
Figure 6.3: Initial price of the derivative, B_0^θ, as map of (λ^a, λ^b) on the left, and as map of (γ_a, γ_b) on the right.

Aggregated risk

Figure 6.4 confirms the analytical results of Theorem 5.7. First note that $\gamma_a = \gamma_b = 1$ and so condition (5.8) simplifies (see Corollary 5.8). As predicted, the increase of the risk tolerances lead to a decrease in the aggregated risk (see Figure 6.4, left picture). The picture on the right shows clearly the cross behavior stated in (5.9).
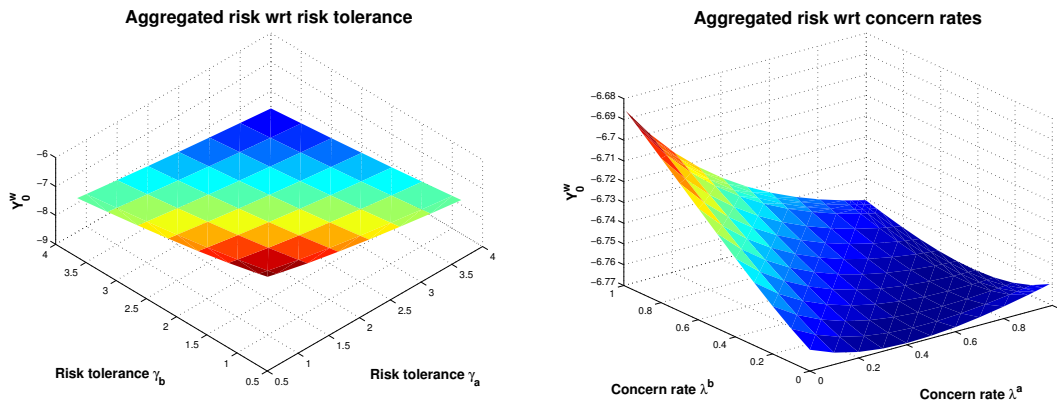

Figure 6.4: Aggregated risk Y_0^w as a function of (γ_a, γ_b) (left) and of (λ^a, λ^b) (right).

Risk of each agent

Theorem 5.7 does not capture the behavior of each agent's risk assessment as a function of the concern rates $\lambda^\cdot$. Figure 6.5 portrays the risk perceptions of each agent as λ^a, λ^b change. Agent
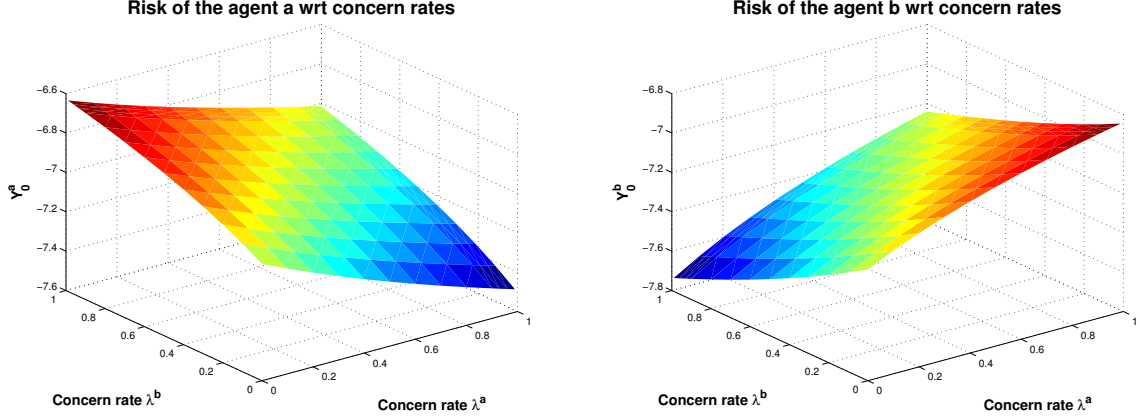

Figure 6.5: Risk Y_0^a (left) and Y_0^b (right) as a function of (λ^a, λ^b).

a's risk Y_0^a increases in λ^b and decreases in λ^a. A possible explanation for the latter behavior (Y_0^a decreases with λ^a) from the perspective of, say a, and having (1.1) or (2.9) in mind is as follows. If a gives more importance to her relative performance concern then she weighs the term $V_T^a - V_T^b$ more than the hedging of the random endowment or the optimization of the personal performance and trades in a way that mimics more of what b does. The net result of this seems to be the ability to neutralize more of the performance risk (as a fluctuation around the mean) and less ability to neutralize the endowment risk. The former apparently carries more weight as Y_0^a does indeed decrease with λ^a.

The explanation of the first behavior (Y_0^a increases with λ^b) seems more direct. As λ^b increases, agent b engages in less trading of the derivative in order to reduce her relative performance concern, and this affects agents a, in particular her ability to hedge H^a.

6.3 Effect of introducing the derivative

We now comment on the effects of introducing the derivative in this model market. Figure 6.6 displays the risks of the representative agent and of agent a with respect to λ^a and λ^b when no derivative is available and when a market for it is available.

We observe in the plot on the right that adding the derivative does not change the aggregated risk. This is clear if one views it as the risk of the representative agent: being alone by construction, the zero net supply condition means that she must keep a zero position in the derivative, and hence does not benefit from its presence (compare the agent of Section 3.3.2 with Example 4.6).

For an individual agent however (left plot), the availability of the derivative always leads to a reduction of risk. We observe that in the absence of the derivative, the risk of agent a does not depend on the concern rates. We can apply the methodology of Sections 3 and 4 to find that the optimal porfolios of the agents in this situation are given by

$$\pi_t^{*,i,1} = \frac{1}{1 - \lambda^a \lambda^b} \frac{\gamma_i \theta^S + \widetilde{Z}_t^{i,1}}{\sigma^S S_t} + \frac{\lambda^i}{1 - \lambda^a \lambda^b} \frac{\gamma_j \theta^S + \widetilde{Z}_t^{j,1}}{\sigma^S S_t} \quad \text{for } j \neq i \in \{a, b\},$$
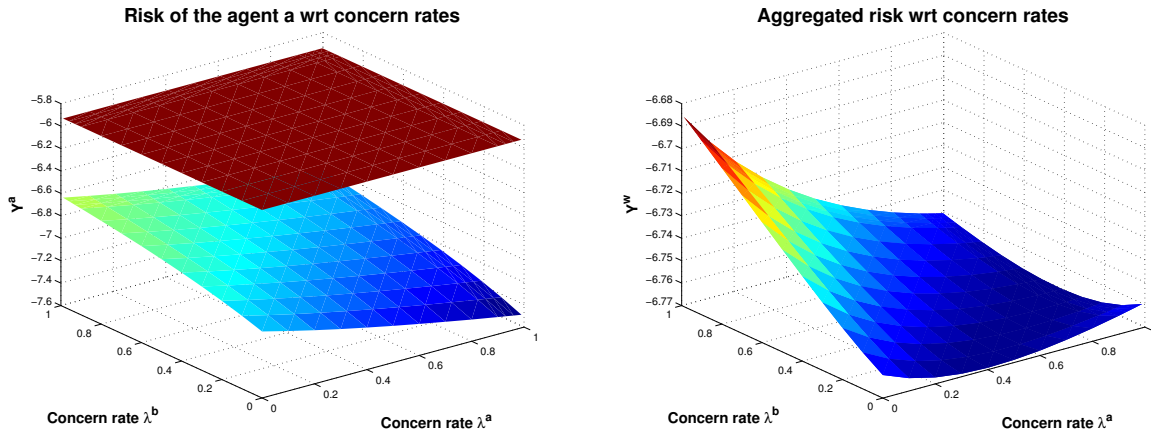
Figure 6.6: Left : risk Y_0^a when the derivative is not available (flat surface) and when it is (tilted surface), as a function of the concern rates (λ^a, λ^b). Right: same plot for the aggregated risk Y_0^w (the two surfaces are equal).

while the minimized risk equation is given by the BSDE

$$d\widetilde{Y}_t^i = -\Big[-\frac{1}{2}\gamma_i\big(\theta^S\big)^2 - \widetilde{Z}_t^{i,1}\theta^S + \frac{1}{2\gamma_i}\big(\widetilde{Z}^{i,2}\big)^2 \Big]dt + \widetilde{Z}_t^i dW_t \quad \text{with} \quad \widetilde{Y}_T^i = -H^i(S_T, R_T).$$

This shows analytically that the value of the problem, Y_0^a, depends on neither λ^a nor λ^b while the optimal strategy does, as was already observed in Proposition 4.1 in [FDR11].

Playing the game repeatedly leads to disaster.

The above study considers a one-period model with (continuous-time) trading until the horizon $T = 1$ month. Imagine now the repetition of this trading period over time. We assume that there are no significant changes to the agents' endowments or the dynamics of the financial and external risks.

At the level of the agents' preferences, with the sole exception of the concern rates, they do not change with time. Specifically, we assume that their risk tolerances, and consequently the entropic risk measures $\rho^{\cdot}$ used to assess their risk in (1.1), are fixed throughout; however, their concern rates $\lambda^{\cdot}$ over their relative performance may vary. This can account for some herding or other behavioral mechanism: after each period, each agent can review the results of everyone's performance, carry this information into the next period and update their concern rate accordingly.

Figure 6.5 sheds some light on the outcome of playing this game repeatedly. Indeed, each agent benefits from a unilateral increase of their concern rate λ while they are worse off with an increase of the other's concern rate. So they have an incentive to increase λ, as the trading periods are repeated, culminating in Assumption 2.2 being violated as $(\lambda^a, \lambda^b) \to (1,1)$.

It is interesting to note that this drifting toward the singularity of the model, $(\lambda^a, \lambda^b) = (1,1)$, is not captured by the risk assessments. Figures 6.4 and 6.5 show that Y_0^w, Y_0^a and Y_0^b remain bounded. At the level of the investment strategies, the trading activity in the derivative slows down but persists. The sharing of the external risk becomes less efficient, because the agents are increasingly concerned about losing out to the other, but does not disappear. However, the investment in the stock explodes (see Figure 6.2).

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We stress that this behavior arises only *after* the derivative is introduced in the market. Indeed, as shown by Figure 6.6, when the derivative is not available and the agents in $\mathbb{A}$ are only concerned with the relative performance of their strategy *over the market*, they have no incentive to having increasingly high concern rates. The particular shape of the surface $(\lambda^a, \lambda^b) \mapsto Y_0^i$, risk decreasing with λ^i but increasing with λ^j, appears only when the derivative is made available. In this situation, the agents are placed in direct interaction (by trading) in addition to the indirect one (social): each agent makes now gains directly *over the other*. The final result is a potential destabilization of the stock market.

7   Conclusion

In this work, we analyzed the effect of a form of social interaction between agents on an equilibrium pricing mechanism. Specifically, we considered the pricing of a (market-completing) derivative introduced to allow market participants to share the risk associated with an external and non-tradable risk factor. The social interaction here takes the form of concerns over relative performance.

From a theoretical point of view, we have shown how to solve the problem for general risk measures and a finite number of agents, when assuming that the derivative completes the market. This involves solving a coupled system of quadratic BSDEs. Due to the heterogeneous rates of concerns of the agents, the risks of the agents cannot be aggregated by the usual infimal convolution technique, so we developed it further and introduced the *weighted-dilated infimal convolution* variant.

We then focused on the particular case of the entropic risk measure and were able to determine sufficient conditions to design a derivative that completes the market. In a market model with two agents representing opposite profiles of exposure to the external risk, we explored the impact of the social interactions on the benefit brought by financial innovation.

We found that the introduction of the derivative always reduces the risk, at the level of individual agents. However, the particular distribution of this risk reduction means that both agents have an incentive to become more concerned with their relative performance. At the global level, while this merely decreases the volume of derivatives exchanged, this leads to an explosion of the volumes traded in the previously-existing financial asset. In practice, the assumption that the agents are small and that the price dynamics of the stock is independent of their actions fails to hold. Thus, although the stock price is fundamentally independent of the external risk, introducing the derivative can lead to unintended consequences on what was a stable stock market.

We stress that this phenomenon is not captured by the risk measures. Therefore one should not only use the performance of the risk measure when evaluating the possible benefits of a new policy (the introduction of the derivative, here). This also stresses the importance of having systemic view: studying the problem from the point of view of an individual investor shows that the availability of the derivative is always beneficial, but at the global level the picture has strong nuances. Strongly undesirable endogenous phenomena can emerge in the dynamics, arising essentially from the interaction between the various agents and their possibility to adapt to the new policy.

A   Stochastic analysis: notation, spaces and base results

Spaces & Notations

We define the following spaces for $p > 1$, $q \geq 1$, $n, m, d, k \in \mathbb{N}$: $C^{0,n}([0,T] \times \mathbb{R}^d, \mathbb{R}^k)$ is the space of continuous functions endowed with the $\|\cdot\|_\infty$-norm that are n-times continuously differentiable in the spatial variable; $C_b^{0,n}$ contains all bounded functions of $C^{0,n}$; the first superscript 0

is dropped for functions independent of time; $L^p(\mathcal{F}_t, \mathbb{R}^d)$, $t \in [0, T]$, is the space of d-dimensional $\mathcal{F}_t$-measurable random variables X with norm $\|X\|_{L^p} = \mathbb{E}[|X|^p]^{1/p} < \infty$; L^∞ refers to the subset of essentially bounded random variables; $\mathcal{S}^p([0,T] \times \mathbb{R}^d)$ is the space of d-dimensional measurable $\mathcal{F}$-adapted processes Y satisfying $\|Y\|_{\mathcal{S}^p} = \mathbb{E}[\sup_{t \in [0,T]} |Y_t|^p]^{1/p} < \infty$; $\mathcal{S}^\infty$ refers to the subset of $\mathcal{S}^p(\mathbb{R}^d)$ of uniformly bounded processes; $\mathcal{H}^p([0,T] \times \mathbb{R}^{n \times d})$ is the space of d-dimensional measurable $\mathcal{F}$-adapted processes Z satisfying $\|Z\|_{\mathcal{H}^p} = \mathbb{E}[\left(\int_0^T |Z_s|^2 \mathrm{d}s\right)^{p/2}]^{1/p} < \infty$; We denote $\mathcal{H}_{BMO}$ as the space of processes $Z \in \mathcal{H}^p$ for any $p \geq 2$ such that for some constant $K_{BMO} > 0$

$$\sup_{\tau \in \mathcal{T}_{[0,T]}} \left\| \mathbb{E}\left[\int_\tau^T |Z_s|^2 \mathrm{d}s \big| \mathcal{F}_\tau \right] \right\|_\infty \leq K_{BMO} < \infty, \tag{A.1}$$

where $\mathcal{T}_{[0,T]}$ is the set of all stopping times $\tau \in [0, T]$. As an easy consequence, if $Z \in \mathcal{H}_{BMO}$, then $\int H \mathrm{d}Z \in \mathcal{H}_{BMO}$ for any bounded adapted process H. Processes in $\mathcal{H}_{BMO}$ have very convenient properties. For more information on BMO spaces and their relation with BSDEs see Subsection 2.3 in [IDR10] or Section 10.1 in [Tou13]; we state, for reference's sake, some of them in the next result.

Lemma A.1. *Let $Z \in \mathcal{H}_{BMO}$ and define $\Phi_. := \int_0^\cdot Z_s \mathrm{d}W_s$. Then we have:*

1) *The stochastic exponential $\mathcal{E}(\Phi_T)$ is uniformly integrable.*

2) *There exists a number $r > 1$ such that $\mathcal{E}(\Phi_T) \in L^r$. This property follows from the* Reverse Hölder inequality. *The maximal r with this property can be expressed explicitly in terms of the BMO norm of $\Phi_.$. There exists as well an upper bound for $\|\mathcal{E}(\Phi_T)\|_{L^r}^r$ depending only on T, r and the BMO norm of $\Phi_.$.*

A.1 Basics of Malliavin's calculus

We briefly introduce the main notation of the stochastic calculus of variations also known as Malliavin's calculus. For more details, we refer the reader to [Nua06], for its application to BSDEs we refer to [Imk08]. Let $\mathcal{S}$ be the space of random variables of the form

$$\xi = F\left(\left(\int_0^T h_s^{1,i} \mathrm{d}W_s^1\right)_{1 \leq i \leq n}, \cdots, \left(\int_0^T h_s^{d,i} \mathrm{d}W_s^d\right)_{1 \leq i \leq n}\right),$$

where $F \in C_b^\infty(\mathbb{R}^{n \times d})$, $h^1, \cdots, h^n \in L^2([0,T]; \mathbb{R}^d)$, $n \in \mathbb{N}$. To simplify notation, assume that all h^j are written as row vectors. For $\xi \in \mathcal{S}$, we define $D = (D^1, \cdots, D^d) : \mathcal{S} \to L^2(\Omega \times [0,T])^d$ by

$$D_\theta^i \xi = \sum_{j=1}^n \frac{\partial F}{\partial x_{i,j}}\left(\int_0^T h_t^1 \mathrm{d}W_t, \ldots, \int_0^T h_t^n \mathrm{d}W_t\right) h_\theta^{i,j}, \quad 0 \leq \theta \leq T, \quad 1 \leq i \leq d,$$

and for $k \in \mathbb{N}$ its k-fold iteration by $D^{(k)} = (D^{i_1} \cdots D^{i_k})_{1 \leq i_1, \cdots, i_k \leq d}$. For $k \in \mathbb{N}$, $p \geq 1$ let $\mathbb{D}^{k,p}$ be the closure of $\mathcal{S}$ with respect to the norm

$$\|\xi\|_{k,p}^p = \mathbb{E}\left[\|\xi\|_{L^p}^p + \sum_{i=1}^k \||D^{(k)]}\xi\|_{(\mathcal{H}^p)^i}^p \right].$$

$D^{(k)}$ is a closed linear operator on the space $\mathbb{D}^{k,p}$. Observe that if $\xi \in \mathbb{D}^{1,2}$ is $\mathcal{F}_t$-measurable, then $D_\theta \xi = 0$ for $\theta \in (t, T]$. Further denote $\mathbb{D}^{k,\infty} = \cap_{p>1} \mathbb{D}^{k,p}$.

We also need Malliavin's calculus for $\mathbb{R}^m$-valued smooth stochastic processes. For $k \in \mathbb{N}, p \geq 1$, denote by $\mathbb{L}^{k,p}(\mathbb{R}^m)$ the set of $\mathbb{R}^m$-valued progressively measurable processes $u = (u^1, \cdots, u^m)$ on $[0,T] \times \Omega$ such that

i) for Lebesgue-a.a. $t \in [0,T]$, $u(t, \cdot) \in (\mathbb{D}^{k,p})^m$;

ii) $[0,T] \times \Omega \ni (t, \omega) \mapsto D^{(k)}u(t,\omega) \in (L^2([0,T]^{1+k}))^{d \times n}$ admits a progressively measurable version;

iii) $\|u\|_{k,p}^p = \|u\|_{\mathcal{H}^p}^p + \sum_{i=1}^{k} \| D^i u \|_{(\mathcal{H}^p)^{1+i}}^p < \infty$.

Note that Jensen's inequality gives[5] for all $p \geq 2$

$$\mathbb{E}\Big[\Big(\int_0^T \int_0^T |D_u X_t|^2 \mathrm{d}u \, \mathrm{d}t\Big)^{\frac{p}{2}}\Big] \leq T^{p/2-1} \int_0^T \|D_u X\|_{\mathcal{H}^p}^p \mathrm{d}u. \tag{A.2}$$

We recall a result from [Imk08] concerning the rule for the Malliavin differentiation of Itô integrals which is of use in applications of Malliavin's calculus to stochastic analysis.

Theorem A.2 (Theorem 2.3.4 in [Imk08])**.** *Let $(X_t)_{t \in [0,T]} \in \mathcal{H}^2$ be an adapted process and define $M_t := \int_0^t X_r \mathrm{d}W_r$ for $t \in [0,T]$. Then, $X \in \mathbb{L}^{1,2}$ if and only if $M_t \in \mathbb{D}^{1,2}$ for any $t \in [0,T]$.*
Moreover, for any $0 \leq s,t \leq T$ we have

$$D_s M_t = X_s \mathbb{1}_{\{s \leq t\}}(s) + \mathbb{1}_{\{s \leq t\}}(s) \int_s^t D_s X_r \mathrm{d}W_r.$$

A.2 Basic Malliavin calculus results for SDEs

With relation to the Brownian motions W^R and W^S, we denote the Malliavin differential operators D^{W^R} and D^{W^S}, see Appendix A.1.

Proposition A.3. *Let Assumption 5.1 hold. Then SDEs (2.1) and (2.2) have a unique solution $R, S \in \mathcal{S}^p$ for any $p \geq 2$ and*

i) *$R, S \in \mathbb{D}^{1,2}$. We have $D_u^{W^S} R_t = D_u^{W^R} S_t = 0$ for any $t, u \in [0,T]$ as well as*

$$D_u^{W^R} R_t = \mathbb{1}_{\{u \leq t\}} b \qquad \text{and} \qquad D_u^{W^S} S_t = \mathbb{1}_{\{u \leq t\}} \sigma^S S_t, \qquad t, u \in [0,T]; \tag{A.3}$$

ii) *For any jointly measurable function $\psi : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that is Lipschitz (in space), it holds that*

$$D_u^{W^R}\big(\psi(t, S_t, R_t)\big) = D_r^{W^R}\big(\psi(t, S_t, R_t)\big) \qquad \forall u, r \in [0,t], \quad t \in [0,T]. \tag{A.4}$$

Furthermore, $\Big(D_0^{W^R}\big(\psi(\cdot, S_\cdot, R_\cdot)\big)\Big) \in \mathcal{S}^\infty$.

iii) *$H^D, H^a \in \mathbb{L}_{1,2} \cap \mathcal{S}^\infty$ for any $a \in \mathbb{A}$ (recall (2.4) and (2.5)). If $\varphi^\zeta = 0$ then there exists $M > 0$ for any $0 \leq r, u \leq T$ and any $\zeta \in \mathbb{A} \cup \{D\}$ such that*

$$D_u^{W^R} H^\zeta = D_r^{W^R} H^\zeta \quad \text{and} \quad 0 < |D_\cdot^{W^R} H^D| \leq M.$$

---

[5]The reason behind this last inequality is that within the BSDE framework the usual tools to obtain a priori estimates yield with much difficulty the LHS while with relative ease the RHS.

iv) Let $\zeta \in \mathbb{A} \cup \{D\}$ and let $r_0 \in \mathbb{R}$. The mapping $r_0 \mapsto (D_u^{W^R} H^\zeta)$ is Lipschitz continuous uniformly in $u \in [0, T]$ for any $s_0 \in (0, +\infty)$.

Proof. Throughout let $\zeta \in \mathbb{A} \cup \{D\}$.

Proof of i) General results on SDEs follow from e.g. Section 2 in [IDR10], standard Malliavin calculus, the fact that S is a Geometric Brownian motion and $\mu^R \in C([0, T], \mathbb{R})$.

The identity $D_u^{W^S} R_t = D_u^{W^R} S_t = 0$ is trivial.

Proof of ii) We prove (A.4): assume ψ to be differentiable, then for $u, r \in [0, t]$

$$D_u^{W^R}\big(\psi(t, S_t, R_t)\big) = (\partial_{x_2}\psi)(t, S_t, R_t)b = D_r^{W^R}\big(\psi(t, S_t, R_t)\big),$$

where we used (A.3). Now a standard approximation by mollification delivers the two results.

Proof of iii) The form of the $\mathcal{F}_T$-measurable payoffs H^D, H^a is quite specific and it is clear that for $0 \le u \le T$ and $\zeta \in \mathbb{A} \cup \{D\}$

$$
\begin{aligned}
D_u^{W^R} H^\zeta &= D_u^{W^R}\big(h^\zeta(S_T, R_T)\big) + \int_0^T \big(D_u^{W^R}\varphi^\zeta(t, S_t, R_t)\big)\mathrm{d}t \\
&= \big\langle (\nabla h^\zeta)(S_T, R_T), (0, \mathbb{1}_{\{u \le T\}}b)\big\rangle + \int_0^T \big\langle (\nabla_x \varphi^\zeta)(t, S_t, R_t), (0, \mathbb{1}_{\{u \le t\}}b)\big\rangle \mathrm{d}t \\
&= b(\partial_{x_2}h^\zeta)(S_T, R_T) + \int_u^T b(\partial_{x_2}\varphi^\zeta)(t, S_t, R_t)\mathrm{d}t.
\end{aligned}
\tag{A.5}
$$

The boundedness of $D^{W^R} H^\zeta$ follows from uniform boundedness of the derivatives of $\varphi^\zeta, h^\zeta \in C_b^2$. We can then conclude that

- if $\partial_{x_2}h^\zeta \ne 0$, $\partial_{x_2}\varphi^\zeta \ne 0$ and $\mathrm{sgn}(\partial_{x_2}h^\zeta) = \mathrm{sgn}(\partial_{x_2}\varphi^\zeta)$, then it follows that $D^{W^R} H^\zeta \ne 0$;

- if $\varphi^\zeta = 0$, then the identity $D_u^{W^R} H^\zeta = D_r^{W^R} H^\zeta$ follows from (A.4).

Proof of iv) We now close with the proof of the last statement. Take $s_0 \in (0, +\infty)$ and let $r_0, \widetilde{r}_0 \in \mathbb{R}$ be two initial conditions for R (see (2.1)) and we denote the corresponding SDE solutions R and $\widetilde{R}$ respectively. We also denote H^ζ and $\widetilde{H}^\zeta$ the random variables depending on R and $\widetilde{R}$ respectively. Notice that due to the linear form of (2.1) it is immediate that $R_t - \widetilde{R}_t = r_0 - \widetilde{r}_0$ for any $t \in [0, T]$.

The properties of $|D_u^{W^R} H^\zeta - D_u^{W^R} \widetilde{H}^\zeta|$ follow from those of $\partial_{x_2}h^\zeta$ and $\partial_{x_2}\varphi^\zeta$ and (A.5). By assumption h^ζ and φ^ζ are twice continuously differentiable (in space) with bounded derivatives, hence, for some $K \ge 0$ (the same computations follows for φ)

$$\big|(\partial_{x_2}h^\zeta)(S_T, R_T) - (\partial_{x_2}h^\zeta)(S_T, \widetilde{R}_T)\big| \le K|R_T - \widetilde{R}_T| = K|r_0 - \widetilde{r}_0|.$$

It follows that for some constant $C \ge 0$ independent of the data u, s_0, r_0 and $\widetilde{r}_0$ one has, as required, $|D_u^{W^R} H^\zeta - D_u^{W^R} \widetilde{H}^\zeta| \le C|r_0 - \widetilde{r}_0|$. $\qquad\qquad\square$

References

[Abe90] A. B. Abel, *Asset prices under habit formation and catching up with the Joneses*, The American Economic Review **80** (1990), no. 2, 38–42.

[Abe99] A. B. Abel, *Risk premia and term premia in general equilibrium*, Journal of Monetary Economics **43** (1999), no. 1, 3 –33.

[ADEH99] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, *Coherent measures of risk*, Math. Finance **9** (1999), no. 3, 203–228. MR1850791 (2002d:91056)

[AIdR10] S. Ankirchner, P. Imkeller, and G. dos Reis, *Pricing and hedging of derivatives based on non-tradable underlyings*, Mathematical Finance **20** (2010April), no. 2, 289 –312. MR2354164 (2008h:60220)

[BE09] P. Barrieu and N. El Karoui, *Pricing, hedging, and designing derivatives with risk measures.*, Indifference pricing: Theory and applications., 2009, pp. 77–146.

[BEK05] P. Barrieu and N. El Karoui, *Inf-convolution of risk measures and optimal risk transfer*, Finance Stoch. **9** (2005), no. 2, 269–298. MR2211128 (2006k:60010)

[BT04] B. Bouchard and N. Touzi, *Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations*, Stochastic Processes and their Applications **111** (2004), no. 2, 175–206.

[Car09] R. Carmona (ed.), *Indifference pricing: theory and applications*, Princeton Univeristy Press, 2009.

[CFHP10] R. Carmona, M. Fehr, J. Hinz, and A. Porchet, *Market design for emission trading schemes*, SIAM Rev. **52** (2010), no. 3, 403–452. MR2680541 (2011j:91226)

[CHKP13] P. Cheridito, U. Horst, M. Kupper, and T. Pirvu, *Equilibrium Pricing in Incomplete Markets under Translation Invariant Preferences*, 2013. Preprint.

[CK01] Y. L. Chan and L. Kogan, *Catching up with the joneses: Heterogeneous preferences and the dynamics of asset prices*, National Bureau of Economic Research, 2001.

[CML13] F. Corsi, S. Marmi, and F. Lillo, *When micro prudence increases macro risk: The destabilizing effects of financial innovation, leverage, and diversification*, Leverage, and Diversification (June 12, 2013) (2013).

[CMV09] F. Caccioli, M. Marsili, and P. Vivo, *Eroding market stability by proliferation of financial instruments*, The European Physical Journal B **71** (2009), no. 4, 467–479.

[CN15] P. Cheridito and K. Nam, *Multidimensional quadratic and subquadratic BSDEs with special structure*, Stochastics An International Journal of Probability and Stochastic Processes **ahead-of-print** (2015), 1–14.

[DH85] D. Duffie and C.-f. Huang, *Implementing Arrow-Debreu equilibria by continuous trading of few long-lived securities*, Econometrica **53** (1985), no. 6, 1337–1356. MR809913 (88b:90021)

[DR11] G. Dos Reis, *Some advances on quadratic BSDE: Theory - Numerics - Applications*, LAP LAMBERT Academic Publishing, 2011.

[Due49] J. Duesenberry, *Income, saving and the theory of consumer behavior*, Harward University Press (1949).

[EKPQ97] N. El Karoui, S. Peng, and M. C. Quenez, *Backward stochastic differential equations in finance*, Math. Finance **7** (1997), no. 1, 1–71. MR1434407

[EKR09] N. El Karoui and C. Ravanelli, *Cash subadditive risk measures and interest rate ambiguity*, Math. Finance **19** (2009), no. 4, 561–590. MR2583520

[Esp10] G.-E. Espinosa, *Stochastic control methods for optimal portfolio investment*, Ph.D. Thesis, 2010.

[ET15] G.-E. Espinosa and N. Touzi, *Optimal Investment under Relative Performance Concerns*, Mathematical Finance **25** (2015), no. 2, 221–257.

[FDR11] C. Frei and G. Dos Reis, *A financial market with interacting investors: does an equilibrium exist?*, Mathematics and financial economics **4** (2011), no. 3, 161–182. MR2796281

[FK08] D. Filipovic and M. Kupper, *Equilibrium prices for monetary utility functions*, International Journal of Theoretical and Applied Finance **11** (2008), no. 03, 325–343.

[FK11] H. Föllmer and T. Knispel, *Entropic risk measures: coherence vs. convexity, model ambiguity, and robust large deviations*, Stoch. Dyn. **11** (2011), no. 2-3, 333–351. MR2836530

[Fre14] C. Frei, *Splitting multidimensional BSDEs and finding local equilibria*, Stochastic Processes and their Applications **124** (2014), no. 8, 2654 –2671.

[FRG02] M. Frittelli and E. Rosazza Gianin, *Putting order in risk measures*, Journal of Banking & Finance **26** (2002), no. 7, 1473–1486.

[FS02] H. Föllmer and A. Schied, *Convex measures of risk and trading constraints*, Finance Stoch. **6** (2002), no. 4, 429–447. MR1932379

[Góm07] J.-P. Gómez, *The impact of keeping up with the Joneses behavior on asset prices and portfolio choice*, Finance Research Letters **4** (2007), no. 2, 95 –103.

[Gal94] J. Gali, *Keeping up with the Joneses: Consumption externalities, portfolio choice, and asset prices*, Journal of Money, Credit and Banking (1994), 1–8.

[Gia06] E. R. Gianin, *Risk measures via g-expectations*, Insurance: Mathematics and Economics **39** (2006), no. 1, 19–34.

[GPP09] N. Garleanu, L. H. Pedersen, and A. M Poteshman, *Demand-based option pricing*, Review of Financial Studies **22** (2009), no. 10, 4259–4299.

[GT14] E. Gobet and P. Turkedjiev, *Linear regression MDP scheme for discrete backward stochastic differential equations under general conditions*, 2014. To appear in Mathematics of Computation.

[HERH73] Jr. Harl E. Ryder and G. M. Heal, *Optimal growth with intertemporally dependent preferences*, The Review of Economic Studies **40** (1973), no. 1, 1–31.

[HIM05] Y. Hu, P. Imkeller, and M. Müller, *Utility maximization in incomplete markets*, Ann. Appl. Probab. **15** (2005), no. 3, 1691–1712. MR2152241

[HL93] H. He and H. Leland, *On equilibrium asset price processes,* Review of Financial Studies **6** (1993), no. 3, 593–617.

[HM07] U. Horst and M. Müller, *On the spanning property of risk bonds priced by equilibrium*, Mathematics of Operations Research **32** (2007), no. 4, 784–807.

[HPDR10] U. Horst, T. Pirvu, and G. Dos Reis, *On securitization, market completion and equilibrium risk transfer*, Mathematics and finance economics **2** (2010), no. 4, 211–252. MR2601853

[HRAY10] R. Hinz, H. Rudolph, P. Antolin, and J. Yermo (eds.), *Evaluating the financial performance of pension funds*, World Bank Publications, 2010.

[HT14] Y. Hu and S. Tang, *Multi-dimensional backward stochastic differential equations of diagonally quadratic generators*, 2014. arXiv:1408.4579.

[IDR10] P. Imkeller and G. Dos Reis, *Path regularity and explicit convergence rate for BSDE with truncated quadratic growth*, Stochastic Process. Appl. **120** (2010), no. 3, 348–379. MR2584898

[Imk08] P. Imkeller, *Malliavin's calculus and applications in stochastic control and finance*, IMPAN Lecture Notes, vol. 1, Polish Academy of Sciences Institute of Mathematics, Warsaw, 2008. MR2494786

[JKL14] A. Jamneshan, M. Kupper, and P. Luo, *Multidimensional quadratic BSDEs with separated generators*, 2014. arXiv:1501.00461.

[JRW10] A Jofre, R. Rockafellar, and R. Wets, *General economic equilibrium with incomplete markets and money*, Preprint (2010).

[JST06] E. Jouini, W. Schachermayer, and N. Touzi, *Law invariant risk measures have the Fatou property*, Advances in mathematical economics. Vol. 9, 2006, pp. 49–71. MR2277714

[KLS90] I. Karatzas, J. P. Lehoczky, and S. E. Shreve, *Existence and uniqueness of multi-agent equilibrium in a stochastic, dynamic consumption/investment model*, Math. Oper. Res. **15** (1990), no. 1, 80–128. MR1038236

[Kob00] M. Kobylanski, *Backward stochastic differential equations and partial differential equations with quadratic growth*, Ann. Probab. **28** (2000), no. 2, 558–602. MR1782267

[KP14] D. Kramkov and S. Pulido, *A system of quadratic BSDEs arising in a price impact model*, 2014. arXiv:1408.0916.

[KXŽ15] C. Kardaras, H. Xing, and G. Žitković, *Incomplete stochastic equilibria with exponential utilities close to pareto optimality*, arXiv preprint arXiv:1505.07224 (2015).

[LRS14] A. Lionnet, G. D. Reis, and L. Szpruch, *Time discretization of FBSDE with polynomial growth drivers and reaction-diffusion PDEs*, To appear in Annals of Applied Probability (2014), available at arXiv:1309.2865.

[Neg60] T. Negishi, *Welfare economics and existence of an equilibrium for a competitive economy*, Metroeconomica **12** (1960), no. 2-3, 92–97.

[Nua06] D. Nualart, *The Malliavin calculus and related topics*, Second, Probability and its Applications (New York), Springer-Verlag, Berlin, 2006. MR2200233

[Ped13] A. Pedraza, *Strategic interactions and portfolio choice in money management: Theory and evidence*, 2013. Woking paper, University of Maryland.

[Pen97] S. Peng, *Backward SDE and related g-expectation*, Backward stochastic differential equations (Paris, 1995–1996), 1997, pp. 141–159. MR1752680

[Rüs13] L. Rüschendorf, *Mathematical Risk analysis. Dependence, Risk Bounds, Optimal Allocations and Portfolios*, Springer Series in Operations Research and Financial Engineering, Springer, 2013.

[Sch15] D. C Schwarz, *Market completion with derivative securities*, arXiv preprint arXiv:1506.00188 (2015).

[Tev08] R. Tevzadze, *Solvability of backward stochastic differential equations with quadratic growth*, Stochastic Process. Appl. **118** (2008), no. 3, 503–515. MR2389055

[Tou13] N. Touzi, *Optimal stochastic control, stochastic target problems, and backward SDE*, Fields Institute Monographs, vol. 29, Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2013. With Chapter 13 by Angès Tourin. MR2976505

[XZ09] C. Xiouros and F. Zapatero, *The representative agent of an economy with external habit-formation and heterogeneous risk-aversion*, 2009. Available at SSRN: http://ssrn.com/abstract=1107495.