A junction condition by specified homogenization and application to traffic lights

Giulio Galise, Cyril Imbert, Régis Monneau

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December 22, 2015

Abstract

Given a coercive Hamiltonian which is quasi-convex with respect to the gradient variable and periodic with respect to time and space at least "far away from the origin", we consider the solution of the Cauchy problem of the corresponding Hamilton-Jacobi equation posed on the real line. Compact perturbations of coercive periodic quasi-convex Hamiltonians enter into this framework for example. We prove that the rescaled solution converges towards the solution of the expected effective Hamilton-Jacobi equation, but whose "flux" at the origin is "limited" in a sense made precise by the authors in [18]. In other words, the homogenization of such a Hamilton-Jacobi equation yields to supplement the expected homogenized Hamilton-Jacobi equation with a junction condition at the single discontinuous point of the effective Hamiltonian. We also illustrate possible applications of such a result by deriving, for a traffic flow problem, the effective flux limiter generated by the presence of a finite number of traffic lights on an ideal road. We also provide meaningful qualitative properties of the effective limiter.

AMS Classification: 35F21, 49L25, 35B27

Keywords: Hamilton-Jacobi equations, quasi-convex Hamiltonians, homogenization, junction condition, flux-limited solution, viscosity solution.

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1 Introduction

1.1 Setting of the general problem

This article is concerned with the study of the limit of the solution $u^\varepsilon(t,x)$ of the following equation

$$u^\varepsilon_t + H\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, u^\varepsilon_x\right) = 0 \quad \text{for } (t,x) \in (0,T)\times\mathbb{R} \tag{1}$$

submitted to the initial condition

$$u^\varepsilon(0,x) = u_0(x) \quad \text{for } x \in \mathbb{R} \tag{2}$$

for a Hamiltonian H satisfying the following assumptions:

(A0) (Continuity) $H\colon \mathbb{R}^3 \to \mathbb{R}$ is continuous.

(A1) (Time periodicity) For all $k \in \mathbb{Z}$ and $(t,x,p) \in \mathbb{R}^3$,

$$H(t+k,x,p) = H(t,x,p).$$

(A2) (Uniform modulus of continuity in time) There exists a modulus of continuity ω such that for all $t,s,x,p \in \mathbb{R}$,

$$H(t,x,p) - H(s,x,p) \le \omega(|t-s|\,(1+\max(H(s,x,p),0))).$$

(A3) (Uniform coercivity)

$$\lim_{|q|\to+\infty} H(t,x,q) = +\infty$$

uniformly with respect to (t,x).

(A4) (Quasi-convexity of H for large x's) There exists some $\rho_0 > 0$ such that for all $x \in \mathbb{R}\setminus(-\rho_0,\rho_0)$, there exists a continuous map $t \mapsto p^0(t,x)$ such that

$$\begin{cases} H(t,x,\cdot) & \text{is non-increasing in} \quad (-\infty, p^0(t,x)), \\ H(t,x,\cdot) & \text{is non-decreasing in} \quad (p^0(t,x), +\infty). \end{cases}$$

(A5) (Left and right Hamiltonians) There exist two Hamiltonians $H_\alpha(t,x,p)$, $\alpha = L,R$, such that

$$\begin{cases} H(t,x+k,p) - H_L(t,x,p) \to 0 & \text{as} \quad \mathbb{Z} \ni k \to -\infty \\ H(t,x+k,p) - H_R(t,x,p) \to 0 & \text{as} \quad \mathbb{Z} \ni k \to +\infty \end{cases}$$

uniformly with respect to $(t,x,p) \in [0,1]^2\times\mathbb{R}$, and for all $k,j \in \mathbb{Z}$, $(t,x,p) \in \mathbb{R}^3$ and $\alpha \in \{L,R\}$,

$$H_\alpha(t+k,x+j,p) = H_\alpha(t,x,p).$$

We have to impose some condition in order to ensure that effective Hamiltonians $\bar{H}_\alpha$ are quasi-convex; indeed, we will see that the effective equation should be solved with *flux-limited solutions* recently introduced by the the second and third authors [18]; such a theory relies on the quasi-convexity of the Hamiltonians.

(B-i) (Quasi-convexity of the left and right Hamiltonians) For each $\alpha = L, R$, H_α does not depend on time and there exists p^0_α (independent on (t,x)) such that

$$\begin{cases} H_\alpha(x,\cdot) & \text{is non-increasing on} \quad (-\infty, p^0_\alpha), \\ H_\alpha(x,\cdot) & \text{is non-decreasing on} \quad (p^0_\alpha, +\infty). \end{cases}$$

(B-ii) (Convexity of the left and right Hamiltonians) For each $\alpha = L, R$, and for all $(t,x) \in \mathbb{R} \times \mathbb{R}$, the map $p \mapsto H_\alpha(t,x,p)$ is convex.

Example 1.1. A simple example of such a Hamiltonian is

$$H(t,x,p) = |p| - f(t,x)$$

with a continuous function f satisfying $f(t+1,x) = f(t,x)$ and $f(t,x) \to 0$ as $|x| \to +\infty$ uniformly with respect to $t \in \mathbb{R}$.

1.2 Main results

Our main result is concerned with the limit of the solution u^ε of (1)-(2). It makes part of the huge literature dealing with homogenization of Hamilton-Jacobi equation, starting with the pioneering work of Lions, Papanicolaou and Varadhan [24]. In particular, we need to use the perturbed test function introduced by Evans [9]. As pointed out to us by the referee, there are few papers dealing with Hamiltonians that depend on time; it implies in particular that so-called correctors also depend on time. The reader is in particular referred to [4, 6] for the large time behaviour and to [10, 13, 11] for homogenization results. This limit satisfies an effective Hamilton-Jacobi equation posed on the real line whose Hamiltonian is discontinuous. More precisely, the effective Hamiltonian equals the one which is expected (see **(A5)**) in $(-\infty; 0)$ and $(0; +\infty)$; in particular, it is discontinuous in the space variable (piecewise constant in fact). In order to get a unique solution, a flux limiter should be identified [18].

Homogenized Hamiltonians and effective flux limiter

The homogenized left and right Hamiltonians are classically determined by the study of some "cell problems".

Proposition 1.2 (Homogenized left and right Hamiltonians)**.** *Assume* **(A0)-(A5)***, and either* **(B-i)** *or* **(B-ii)***. Then for every $p \in \mathbb{R}$, and $\alpha = L, R$, there exists a unique $\lambda \in \mathbb{R}$ such that there exists a bounded solution v^α of*

$$\begin{cases} v^\alpha_t + H_\alpha(t,x,p+v^\alpha_x) = \lambda \quad \text{in} \quad \mathbb{R}\times\mathbb{R}, \\ v^\alpha \ \text{is } \mathbb{Z}^2\text{-periodic.} \end{cases} \tag{3}$$

If $\bar{H}_\alpha(p)$ denotes such a λ, then the map $p \mapsto \bar{H}_\alpha(p)$ is continuous, coercive and quasi-convex.

Remark 1.3. We recall that a function $\bar{H}_\alpha$ is quasi-convex if the sets $\{\bar{H}_\alpha \leq \lambda\}$ are convex for all $\lambda \in \mathbb{R}$. If $\bar{H}_\alpha$ is also coercive, then $\bar{p}_\alpha^0$ denotes in proofs some $p \in \operatorname{argmin} \bar{H}_\alpha$.

The effective flux limiter $\bar{A}$ is the smallest $\lambda \in \mathbb{R}$ for which there exists a solution w of the following global-in-time Hamilton-Jacobi equation

$$\begin{cases} w_t + H(t, x, w_x) = \lambda, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ w \text{ is 1-periodic w.r.t. } t. \end{cases} \tag{4}$$

Theorem 1.4 (Effective flux limiter)**.** *Assume* **(A0)-(A5)** *and either* **(B-i)** *or* **(B-ii)**. *The set*

$$E = \{\lambda \in \mathbb{R} : \exists w \text{ sub-solution of } (4)\}$$

is not empty and bounded from below. Moreover, if $\bar{A}$ denotes the infimum of E, then

$$\bar{A} \geq A_0 := \max_{\alpha = L, R} \left(\min \bar{H}_\alpha\right). \tag{5}$$

Remark 1.5. We will see below (Theorem 4.6) that the infimum is in fact a minimum: there exists a global corrector which, in particular, can be rescaled properly.

We can now define the effective junction condition.

Definition 1.6 (Effective junction condition)**.** The *effective junction function* $F_{\bar{A}}$ is defined by

$$F_{\bar{A}}(p_L, p_R) := \max(\bar{A}, \bar{H}_L^+(p_L), \bar{H}_R^-(p_R))$$

where

$$\bar{H}_\alpha^-(p) = \begin{cases} \bar{H}_\alpha(p) & \text{if } \; p < \bar{p}_\alpha^0, \\ \bar{H}_\alpha(\bar{p}_\alpha^0) & \text{if } \; p \geq \bar{p}_\alpha^0 \end{cases} \quad \text{and} \quad \bar{H}_\alpha^+(p) = \begin{cases} \bar{H}_\alpha(\bar{p}_\alpha^0) & \text{if } \; p \leq \bar{p}_\alpha^0, \\ \bar{H}_\alpha(p) & \text{if } \; p > \bar{p}_\alpha^0 \end{cases}$$

where $\bar{p}_\alpha^0 \in \operatorname{argmin} \bar{H}_\alpha$.

The convergence result

Our main result is the following theorem.

Theorem 1.7 (Junction condition by homogenization)**.** *Assume* **(A0)-(A5)** *and either* **(B-i)** *or* **(B-ii)**. *Assume that the initial datum u_0 is Lipschitz continuous and for $\varepsilon > 0$, let u^ε be the solution of* (1)-(2). *Then u^ε converges locally uniformly to the unique flux-limited solution u^0 of*

$$\begin{cases} u_t^0 + \bar{H}_L(u_x^0) = 0, & t > 0, x < 0, \\ u_t^0 + \bar{H}_R(u_x^0) = 0, & t > 0, x > 0, \\ u_t^0 + F_{\bar{A}}(u_x^0(t, 0^-), u_x^0(t, 0^+)) = 0, & t > 0, x = 0 \end{cases} \tag{6}$$

submitted to the initial condition (2).

Remark 1.8. We recall that the notion of flux-limited solution for (6) is introduced in [18].

This theorem asserts in particular that the slopes of the limit solution at the origin are characterized by the effective flux limiter $\bar{A}$. Its proof relies on the construction of a global "corrector", *i.e.* a solution of (4), which is close to an appropriate V-shaped function after rescaling. This latter condition is necessary so that the slopes at infinity of the corrector fit the expected slopes of the solution of the limit problem at the origin. Here is a precise statement.

Theorem 1.9 (Existence of a global corrector for the junction)**.** *Assume* **(A0)-(A5)** *and either* **(B-i)** *or* **(B-ii)***. There exists a solution w of* (4) *with $\lambda = \bar{A}$ such that, the function*

$$w^\varepsilon(t,x) = \varepsilon w(\varepsilon^{-1}t, \varepsilon^{-1}x)$$

converges locally uniformly (along a subsequence $\varepsilon_n \to 0$) towards a function $W = W(x)$ which satisfies $W(0) = 0$ and

$$\hat{p}_R x 1_{\{x>0\}} + \hat{p}_L x 1_{\{x<0\}} \geq W(x) \geq \bar{p}_R x 1_{\{x>0\}} + \bar{p}_L x 1_{\{x<0\}} \tag{7}$$

where

$$\begin{cases} \bar{p}_R = \min E_R \\ \hat{p}_R = \max E_R \end{cases} \quad \textit{with} \quad E_R := \left\{p \in \mathbb{R}, \quad \bar{H}_R^+(p) = \bar{H}_R(p) = \bar{A}\right\} \tag{8}$$

$$\begin{cases} \bar{p}_L = \max E_L \\ \hat{p}_L = \min E_L \end{cases} \quad \textit{with} \quad E_L := \left\{p \in \mathbb{R}, \quad \bar{H}_L^-(p) = \bar{H}_L(p) = \bar{A}\right\}. \tag{9}$$

The construction of this global corrector is the reason why homogenization is referred to as being "specified". See also Section 1.4 about related results. As a matter of fact, we will prove a stronger result, see Theorem 4.6.

Extension: application to traffic lights

The techniques developed to prove the Theorem 1.7 allow us to deal with a different situation inspired from traffic flow problems. As explained in [19], such problems are related to the study of some Hamilton-Jacobi equations. The problem that we address in Theorem 1.12 below is motivated by its meaningful application to traffic lights. We aim at figuring out how the fraffic flow on an ideal (infinite, straight) road is modified by the presence of a finite number of traffic lights.

We can consider a Hamilton-Jacobi equation whose Hamiltonian does not depend on (t,x) for x outside a (small) interval of the form $N_\varepsilon = (b_1\varepsilon, b_N\varepsilon)$ and is piecewise constant with respect to x in $(b_1\varepsilon, b_N\varepsilon)$. At space discontinuities, junction conditions are imposed with ε-time periodic flux limiters. The limit solution satifies the equation after the "neighbourhood" N_ε disappeared. We will see that the equation keeps memory of what happened there through a flux limiter at the origin $x = 0$.

Let us be more precise now. For $N \geq 1$, (a finite number of) junction points $-\infty = b_0 < b_1 < b_2 < \cdots < b_N < b_{N+1} = +\infty$ and (a finite number of) times $0 = \tau_0 < \tau_1 < \cdots < \tau_K < 1 = \tau_{K+1}$, $K \in \mathbb{N}$ are given. For $N \geq 1$ and $\alpha \in \{0, \dots, N\}$, ℓ_α denotes $b_{\alpha+1} - b_\alpha$. Note that $\ell_\alpha = +\infty$ for $\alpha = 0, N$.

We then consider the solution u^ε of (1) where the Hamiltonian H satifies the following conditions.

(C1) The Hamiltonian is given by

$$H(t,x,p) = \begin{cases} \bar{H}_\alpha(p) & \text{if } b_\alpha < x < b_{\alpha+1} \\ \max(\bar{H}^+_{\alpha-1}(p^-), \bar{H}^-_\alpha(p^+), a_\alpha(t)) & \text{if } x = b_\alpha, \alpha \neq 0. \end{cases}$$

(C2) The Hamiltonians $\bar{H}_\alpha$, for $\alpha = 0, \dots, N$, are continuous, coercive and quasi-convex.

(C3) The flux limiters a_α, for $\alpha = 1, \dots, N$ and $i = 0, \dots, K$, satisfy

$$a_\alpha(s+1) = a_\alpha(s) \quad \text{with} \quad a_\alpha(s) = A^i_\alpha \quad \text{for all} \quad s \in [\tau_i, \tau_{i+1})$$

with $(A^i_\alpha)^{i=0,\dots,K}_{\alpha=1,\dots,N}$ satisfying $A^i_\alpha \geq \max_{\beta=\alpha-1,\alpha} \left(\min \bar{H}_\beta\right)$.

Remark 1.10. The Hamiltonians outside N_ε are denoted by $\bar{H}_\alpha$ instead of H_α in order to emphasize that they do not depend on time and space.

Remark 1.11. In view of the litterature in traffic modeling, the Hamiltonians could be assumed to be convex. But we prefer to stick to the quasi-convex framework since it seems to us that it is the natural one (in view of [18]).

The equation is supplemented with the following initial condition

$$u^\varepsilon(0,x) = U_0^\varepsilon(x) \quad \text{for } x \in \mathbb{R} \tag{10}$$

with

$$U_0^\varepsilon \text{ is equi-Lipschitz continuous and } U_0^\varepsilon \to u_0 \text{ locally uniformly.} \tag{11}$$

Then the following convergence result holds true.

Theorem 1.12 (Time homogenization of traffic lights)**.** *Assume* **(C1)-(C3)** *and* (11). *Let u^ε be the solution of* (1)-(10) *for all $\varepsilon > 0$. Then:*

i) (Homogenization) *There exists some $\bar{A} \in \mathbb{R}$ such that u^ε converges locally uniformly as ε tends to zero towards the unique viscosity solution u^0 of* (6)-(2) *with*

$$\bar{H}_L := \bar{H}_0, \quad \bar{H}_R := \bar{H}_N.$$

ii) (Qualitative properties of $\bar{A}$) *For $\alpha = 1, \dots, N$, $\langle a_\alpha \rangle$ denotes $\int_0^1 a_\alpha(s)\, ds$. The effective limiter $\bar{A}$ satisfies the following properties.*

- *For all α, $\bar{A}$ is non-increasing w.r.t. ℓ_α.*
- *For $N = 1$,*
$$\bar{A} = \langle a_1 \rangle. \tag{12}$$
- *For $N \geq 1$,*
$$\bar{A} \geq \max_{\alpha=1,\dots,N} \quad \langle a_\alpha \rangle. \tag{13}$$
- *For $N \geq 2$, there exists a critical distance $d_0 \geq 0$ such that*
$$\bar{A} = \max_{\alpha=1,\dots,N} \quad \langle a_\alpha \rangle \qquad \text{if} \qquad \min_\alpha \ell_\alpha \geq d_0; \tag{14}$$
this distance d_0 only depends on $\max\limits_{\alpha=1,\dots,N} \|a_\alpha\|_\infty$, $\max\limits_{\alpha=1,\dots,N} \langle a_\alpha \rangle$ and the $\bar{H}_\alpha$'s.
- *We have*
$$\bar{A} \to \langle \bar{a} \rangle \quad \text{as} \quad (\ell_1, \dots, \ell_{N-1}) \to (0, \dots, 0) \tag{15}$$
where $\bar{a}(\tau) = \max_{\alpha=1,\dots,N} a_\alpha(\tau)$.

Remark 1.13. Since the function $a(t)$ is piecewise constant, the way u^ε satisfies (1) has to be made precise. An L^1 theory in time (following for instance the approach of [7, 8]) could probably be developed for such a problem, but we will use here a different, elementary approach. The Cauchy problem is understood as the solution of successive Cauchy problems. This is the reason why we will first prove a global Lipschitz bound on the solution so that there indeed exists such a solution.

Remark 1.14. Note that the result of Theorem 1.4 still holds for equation (1) under Assumptions **(C1)-(C3)**, with the set E defined for sub-solutions which are moreover assumed to be globally Lipschitz (without fixed bound on the Lipschitz constant). The reader can check that the proof is unchanged.

Remark 1.15. It is somewhat easy to get (12) when the Hamiltonians $\bar{H}_\alpha$ are convex by using the optimal control interpretation of the problem. In the more general case of quasi-convex Hamiltonians, the result still holds true but the proof is more involved.

Remark 1.16. We may have $\bar{A} > \max_{\alpha=1,\dots,N} \langle a_\alpha \rangle$. It is possible to deduce it from (15) in the case $N = 2$ by using the traffic light interpretation of the problem. If we have two traffic lights very close to each other (let us say that the distance in between is at most the place for only one car), and if the common period of the traffic lights are exactly in opposite phases (with for instance one minute for the green phase, and one minute for the red phase), then the effect of the two traffic lights together, gives a very low flux which is much lower than the effect of a single traffic light alone (*i.e.* here at most one car every two minutes will go through the two traffic lights).

1.3 Traffic flow interpretation of Theorem 1.12

We mentioned above that there are some connections between our problem and traffic flows.

Inequality (13) has a natural traffic interpretation, saying that the average limitation on the traffic flow created by several traffic lights on a single road is higher or equal to the one created by the traffic light which creates the highest limitation. Moreover this average limitation is smaller if the distances between traffic lights are bigger, as says the monotonicity of $\bar{A}$ with respect to the distances ℓ_α.

Property (14) says that the minimal limitation is reached if the distances between the traffic lights are bigger than a critical distance d_0. The proof of this result is quite involved and is reflected in the fact that the bounds that we have on d_0 are not continuous on the data ($\max_{\alpha=1,\ldots,N} \|a_\alpha\|_\infty$, $\max_{\alpha=1,\ldots,N} \langle a_\alpha \rangle$ and the $\bar{H}_\alpha$'s).

Finally property (15) is very natural from the point of view of traffic, since it corresponds to the case where all the traffic lights would be at the same position.

1.4 Related results

Achdou and Tchou [1] studied a singular perturbation problem which has the same flavor as the one we are looking at in the present paper. More precisely, they consider the simplest network (a so-called junction) embedded in a star-shaped domain. They prove that the value function of an infinite horizon control problem converges, as the star-shaped domain "shrinks" to the junction, to the value function of a control problem posed on the junction. We borrow from them the idea of studying the cell problem on truncated domains with state constraints. We provide a different approach, which is also in some sense more general because it can be applied to problems outside the framework of optimal control theory. Our approach relies in an essential way on the general theory developed in [18].

The general theme of Lions's 2013-2014 lectures at Collège de France [23] is "Elliptic or parabolic equations and specified homogenization". As far as first order Hamilton-Jacobi equations are concerned, the term "specified homogenization" refers to the problem of constructing correctors to cell problems associated with Hamiltonians that are typically the sum of a periodic one H and a compactly supported function f depending only on x, say. Lions exhibits sufficient conditions on f such that the effective Hamilton-Jacobi equation is not perturbed. In terms of flux limiters [18], it corresponds to look for sufficient conditions such that the effective flux limiter $\bar{A}$ given by Theorem 1.4 is (less than or) equal to $A_0 = \min H$.

Barles, Briani and Chasseigne [5, Theorem 6.1] considered the case

$$H(x,p) = \varphi\left(\frac{x}{\varepsilon}\right) H_R(p) + \left(1 - \varphi\left(\frac{x}{\varepsilon}\right)\right) H_L(p)$$

for some continuous increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{s\to-\infty} \varphi(s) = 0 \quad \text{and} \quad \lim_{s\to+\infty} \varphi(s) = 1.$$

They prove that u^ε converges towards a value function denoted by U^-, that they characterize as the solution to a particular optimal control problem. It is proved in [18] that U^- is the solution of (6) with $\bar{H}_\alpha = H_\alpha$ and $\bar{A}$ replaced with $A_I^+ = \max(A_0, A^*)$ with

$$A_0 = \max(\min H_R, \min H_L) \quad \text{and} \quad A^* = \max_{q\in[\min(p_R^0,p_L^0),\max(p_R^0,p_L^0)]} (\min(H_R(q), H_L(q))).$$

In [14], Giga and Hamamuki develop a theory which allows in particular to prove existence and uniqueness for the following Hamilton-Jacobi equation (changing u in $-u$) in $\mathbb{R}^d$,

$$\begin{cases} \partial_t u + |\nabla u| = 0 & \text{for } x \neq 0 \\ \partial_t u + |\nabla u| + c = 0 & \text{at } x = 0. \end{cases}$$

The solutions of [14] are constructed as limits of the following equation

$$\partial_t u^\varepsilon + |\nabla u^\varepsilon| + c(1 - |x|/\varepsilon)^+ = 0.$$

In the monodimensional case ($d = 1$), Theorem 1.7 implies that u^ε converges towards

$$\begin{cases} \partial_t u + |\nabla u| = 0 & \text{for } x \neq 0 \\ \partial_t u + \max(A, |\nabla u|) = 0 & \text{at } x = 0 \end{cases}$$

for some $A \in \mathbb{R}$. In view of Theorem 1.4, it is not difficult to prove that $A = \max(0, c)$. The Hamiltonian $\max(c, |\nabla u|)$ is identified in [14] and is referred to as the *relaxed* one.

It is known that homogenization of Hamilton-Jacobi equations is closely related to the study of the large time behaviour of solutions. In [15], the large time behaviour of Hamilton-Jacobi equations with discontinuous source terms is discussed in two cases: for compactly supported ones and periodic ones. Remark that in our setting, we can adress both and even the sum of a periodic source term and of a compactly supported one. It would be interesting to adress such a problem in the case of traffic lights. In [20], the authors study the large time behaviour of the solutions of a Hamilton-Jacobi equations with an x-periodic Hamiltonian and what can be interpreted as a flux-limiter depending periodically in time.

1.5 Further extensions

It is also possible to adress the time homogenization problem of Theorem 1.12 with any finite number of junctions (with limiter functions $a_\alpha(t)$ piecewise constants – or continuous – and 1-periodic), either separated with distance of order $O(1)$ or with distance of order $O(\varepsilon)$, or mixing both, and even on a complicated network. See also [20] for other connexions between Hamilton-Jacobi equations and traffic light problems and [2] for green waves modelling.

Note that the method presented in this paper can be readily applied (without modifying proofs) to the study of homogeneization on a finite number of branches and not only two branches; the theory developed in [18] should also be used for the limit problem.

Similar questions in higher dimensions with point defects of other co-dimensions will be addressed in future works.

1.6 Organization of the article

Section 2 is devoted to the proof of the convergence result (Theorem 1.7). Section 3 is devoted to the construction of correctors far from the junction point (Proposition 1.2) while the junction case, *i.e.* the proof of Theorem 4.6, is addressed in Section 4. We recall that Theorem 1.9 is a straightforward corollary of this stronger result. The proof of Theorem 4.6 makes use of a comparison principle which is expected but not completely standard. This is the reason why a proof is sketched in Appendix, together with two other ones that are rather standard but included for the reader's convenience.

Notation. A ball centered at x of radius r is denoted by $B_r(x)$. If $\{u^\varepsilon\}_\varepsilon$ is locally bounded, the upper and lower relaxed limits are defined as

$$\begin{cases} \limsup_\varepsilon{}^* u^\varepsilon(X) = \limsup\limits_{Y\to X, \varepsilon\to 0} u^\varepsilon(Y), \\ \liminf_\varepsilon{}_* u^\varepsilon(X) = \liminf\limits_{Y\to X, \varepsilon\to 0} u^\varepsilon(Y). \end{cases}$$

In our proofs, constants may change from line to line.

2 Proof of convergence

This section is devoted to the proof of Theorem 1.7. We first construct barriers.

Lemma 2.1 (Barriers)**.** *There exists a nonnegative constant C such that for any $\varepsilon > 0$*

$$|u^\varepsilon(t,x) - u_0(x)| \le Ct \quad \text{for} \quad (t,x) \in (0,T) \times \mathbb{R}\,. \tag{16}$$

Proof. Let L_0 be the Lipschitz constant of the initial datum u_0. Taking

$$C = \sup_{\substack{(t,x)\in\mathbb{R}\times\mathbb{R} \\ |p|\le L_0}} |H(t,x,p)| < +\infty$$

owing to **(A0)** and **(A5)**, the functions $u^\pm(t,x) = u_0(x) \pm Ct$ are a super- and a sub-solution of (1)-(2) respectively and (16) follows via comparison principle. $\square$

We can now prove the convergence theorem.

Proof of Theorem 1.7. We classically consider the upper and lower relaxed semi-limits

$$\begin{cases} \overline{u} = \limsup_\varepsilon{}^* u^\varepsilon, \\ \underline{u} = \liminf_\varepsilon{}_* u^\varepsilon\,. \end{cases}$$

Notice that these functions are well defined because of Lemma 2.1. In order to prove convergence of u^ε towards u^0, it is sufficient to prove that $\overline{u}$ and $\underline{u}$ are a sub- and a super-solution of (6)-(2) respectively. The initial condition immediately follows from (16). We focus our attention on the sub-solution case since the super-solution one can be handled similarly.

We first check that

$$\overline{u}(t,0) = \limsup_{(s,y)\to(t,0),y>0} \overline{u}(s,y) = \limsup_{(s,y)\to(t,0),y<0} \overline{u}(s,y). \tag{17}$$

This is a consequence of the stability of such a "weak continuity" condition, see [18]. Indeed, it is shown in [18] that classical viscosity solution can be viewed as flux-limited one; in particular, u^ε solves

$$u_t^\varepsilon + H^-\left(\frac{t}{\varepsilon}, \frac{0}{\varepsilon}, u_x^\varepsilon(t,0^+)\right) \vee H^+\left(\frac{t}{\varepsilon}, \frac{0}{\varepsilon}, u_x^\varepsilon(t,0^-))\right) = 0 \quad \text{for } t>0.$$

Since these ε-Hamiltonians are uniformly coercive and u^ε is continuous, we conclude that (17) holds true.

Let φ be a test function such that

$$(\overline{u}-\varphi)(t,x) < (\overline{u}-\varphi)(\overline{t},\overline{x}) = 0 \quad \forall (t,x) \in B_{\overline{r}}(\overline{t},\overline{x}) \setminus \left\{(\overline{t},\overline{x})\right\}. \tag{18}$$

We argue by contradiction by assuming that

$$\varphi_t(\overline{t},\overline{x}) + \bar{H}\left(\bar{x}, \varphi_x(\overline{t},\overline{x})\right) = \theta > 0, \tag{19}$$

where

$$\bar{H}\left(\bar{x}, \varphi_x(\overline{t},\overline{x})\right) := \begin{cases} \bar{H}_R(\varphi_x(\overline{t},\overline{x})) & \text{if} \quad \overline{x}>0, \\ \bar{H}_L(\varphi_x(\overline{t},\overline{x})) & \text{if} \quad \overline{x}<0, \\ F_{\bar{A}}(\varphi_x(\overline{t},0^-), \varphi_x(\overline{t},0^+)) & \text{if} \quad \overline{x}=0. \end{cases}$$

We only treat the case where $\overline{x}=0$ since the case $\overline{x}\neq 0$ is somewhat classical. This latter case is detailed in Appendix for the reader's convenience. Using [18, Proposition 2.8], we may suppose that

$$\varphi(t,x) = \phi(t) + \bar{p}_L x 1_{\{x<0\}} + \bar{p}_R x 1_{\{x>0\}} \tag{20}$$

where ϕ is a C^1 function defined in $(0,+\infty)$. In this case, Eq. (19) becomes

$$\phi'(\overline{t}) + F_{\bar{A}}\left(\bar{p}_L, \bar{p}_R\right) = \phi'(\overline{t}) + \bar{A} = \theta > 0. \tag{21}$$

Let us consider a solution w of the equation

$$w_t + H(t,x,w_x) = \bar{A} \tag{22}$$

provided by Theorem 1.9, which is in particular 1-periodic with respect to time. We recall that the function W is the limit of $w^\varepsilon = \varepsilon w(\cdot/\varepsilon)$ as $\varepsilon \to 0$. We claim that, if $\varepsilon>0$ is small enough, the perturbed test function $\varphi^\varepsilon(t,x) = \phi(t) + w^\varepsilon(t,x)$ [9] is a viscosity super-solution of

$$\varphi_t^\varepsilon + H\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \varphi_x^\varepsilon\right) = \frac{\theta}{2} \quad \text{in} \quad B_r(\overline{t},0)$$

for some sufficiently small $r > 0$. In order to justify this fact, let $\psi(t,x)$ be a test function touching φ^ε from below at $(t_1,x_1) \in B_r(\bar{t},0)$. In this way

$$w\left(\frac{t_1}{\varepsilon},\frac{x_1}{\varepsilon}\right) = \frac{1}{\varepsilon}\left(\psi(t_1,x_1) - \phi(t_1)\right)$$

and

$$w\,(s,y) \geq \frac{1}{\varepsilon}\left(\psi(\varepsilon s,\varepsilon y) - \phi(\varepsilon s)\right)$$

for (s,y) in a neighborhood of $\left(\frac{t_1}{\varepsilon},\frac{x_1}{\varepsilon}\right)$. Hence from (21)-(22)

$$\begin{aligned}\psi_t(t_1,x_1) + H\left(\frac{t_1}{\varepsilon},\frac{x_1}{\varepsilon},\psi_x(t_1,x_1)\right) &\geq \bar{A} + \phi'(t_1)\\ &\geq \bar{A} + \phi'(\bar{t}) - \frac{\theta}{2} \geq \frac{\theta}{2}\end{aligned}$$

provided r is small enough. Hence, the claim is proved.

Combining (7) from Theorem 1.9 with (18) and (20), we can fix $\kappa_r > 0$ and $\varepsilon > 0$ small enough so that

$$u^\varepsilon + \kappa_r \leq \varphi^\varepsilon \quad \text{on} \quad \partial B_r(\bar{t},0).$$

By comparison principle the previous inequality holds in $B_r(\bar{t},0)$. Passing to the limit as $\varepsilon \to 0$ and $(t,x) \to (\bar{t},\bar{x})$, we get the following contradiction

$$\overline{u}(\bar{t},0) + \kappa_r \leq \varphi(\bar{t},0) = \overline{u}(\bar{t},0).$$

The proof of convergence is now complete. □

Remark 2.2. For the super-solution property, φ in (20) should be replaced with

$$\varphi(t,x) = \phi(t) + \hat{p}_L x 1_{\{x<0\}} + \hat{p}_R x 1_{\{x>0\}}.$$

3 Homogenized Hamiltonians

In order to prove Proposition 1.2, we first prove the following lemma. Even if the proof is standard, we give it in full details since we will adapt it when constructing global correctors for the junction.

Lemma 3.1 (Existence of a corrector)**.** *There exists $\lambda \in \mathbb{R}$ and a bounded (discontinuous) viscosity solution of* (3)*.*

Remark 3.2. If H_α does not depend on t, then it is possible to construct a corrector which does not depend on time either. We leave details to the reader.

Proof. For any $\delta > 0$, it is possible to construct a (possibly discontinuous) viscosity solution v^δ of

$$\begin{cases} \delta v^\delta + v^\delta_t + H_\alpha(t,x,p+v^\delta_x) = 0 & \text{in } \mathbb{R}\times\mathbb{R}, \\ v^\delta \text{ is } \mathbb{Z}^2\text{-periodic.} \end{cases}$$

First, the comparison principle implies

$$|\delta v^\delta| \le C_\alpha \tag{23}$$

where

$$C_\alpha = \sup_{(t,x)\in[0,1]^2} |H_\alpha(t,x,p)|.$$

Second, the function

$$m^\delta(x) = \sup_{t\in\mathbb{R}} (v^\delta)^*(t,x)$$

is a sub-solution of

$$H_\alpha(t(x),x,p+m^\delta_x) \le C_\alpha$$

(for some function $t(x)$). Assumptions **(A3)** and **(A5)** imply in particular that there exists $C > 0$ independent of δ such that

$$|m^\delta_x| \le C$$

and

$$v^\delta_t \le C.$$

In particular, the comparison principle implies that for all $t\in\mathbb{R}$ and $x\in\mathbb{R}$ and $h \ge 0$,

$$v^\delta(t+h,x) \le v^\delta(t,x) + Ch.$$

Combining this inequality with the time-periodicity of v^δ yields

$$|v^\delta(t,x) - m^\delta(x)| \le C;$$

in particular,

$$|v^\delta(t,x) - v^\delta(0,0)| \le C. \tag{24}$$

Hence, the half relaxed limits

$$\bar{v} = \limsup_{\delta\to 0}{}^*(v^\delta - v^\delta(0,0)) \quad \text{and} \quad \underline{v} = \liminf_{\delta\to 0}{}_*(v^\delta - v^\delta(0,0))$$

are finite. Moreover, (23) implies that $\delta v^\delta(0,0) \to -\lambda$ (at least along a subsequence). Hence, discontinuous stability of viscosity solutions implies that $\bar{v}$ is a $\mathbb{Z}^2$-periodic sub-solution of (3) and $\underline{v}$ is a $\mathbb{Z}^2$-periodic super-solution of the same equation. Perron's method then allows us to construct a corrector between $\bar{v}$ and $\underline{v} + C$ with $C = \sup(\bar{v} - \underline{v})$. The proof of the lemma is now complete. □

The following lemma is completely standard; the proof is given in Appendix for the reader's convenience.

Lemma 3.3 (Uniqueness of λ)**.** *The real number λ given by Lemma 3.1 is unique. If $\bar{H}_\alpha(p)$ denotes such a real number, the function $\bar{H}_\alpha$ is continuous.*

Lemma 3.4 (Coercivity of $\bar{H}_\alpha$)**.** *The continuous function $\bar{H}_\alpha$ is coercive,*

$$\lim_{|p|\to+\infty} \bar{H}_\alpha(p) = +\infty.$$

Proof. In view of the uniform coercivity in p of H_α with respect to (t,x) (see **(A3)**), for any $R>0$ there exists a positive constant C_R such that

$$|p| \geq C_R \quad \Rightarrow \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}, \quad H_\alpha(t,x,p) \geq R. \tag{25}$$

Let v^α be the discontinuous corrector given by Lemma 3.1 and $(\bar{t},\bar{x})$ be point of supremum of its upper semi-continuous envelope $(v^\alpha)^*$. Then we have

$$H_\alpha(\bar{t},\bar{x},p) \leq \bar{H}_\alpha(p)$$

which implies

$$\bar{H}_\alpha(p) \geq R \quad \text{for} \quad |p| \geq C_R. \tag{26}$$

The proof of the lemma is now complete. □

We first prove the quasi-convexity of $\bar{H}_\alpha$ under assumption **(B-ii)**. We prove in fact more: the effective Hamiltonian is convex in this case.

Lemma 3.5 (Convexity of $\bar{H}_\alpha$ under **(B-ii)**)**.** *Assume* **(A0)-(A5)** *and* **(B-ii)**. *Then the function $\bar{H}_\alpha$ is convex.*

Proof. For p, $q \in \mathbb{R}$, let v_p, v_q be solutions of (3) with $\lambda = \bar{H}_\alpha(p)$ and $\bar{H}_\alpha(q)$ respectively. We also set

$$u_p(t,x) = v_p(t,x) + px - t\bar{H}_\alpha(p)$$

and define similarly u_q.

Step 1: u_p and u_q are locally Lipschitz continuous. In this case, we have almost everywhere:

$$\begin{cases} (u_p)_t + H_\alpha(t,x,(u_p)_x) = 0, \\ (u_q)_t + H_\alpha(t,x,(u_q)_x) = 0. \end{cases}$$

For $\mu \in [0,1]$, let

$$\bar{u} = \mu u_p + (1-\mu)u_q.$$

By convexity, we get almost everywhere

$$\bar{u}_t + H_\alpha(t,x,\bar{u}_x) \leq 0. \tag{27}$$

We claim that the convexity of H_α (in the gradient variable) implies that $\bar{u}$ is a viscosity sub-solution. To see it, we use an argument of [3, Proposition 5.1]. For $P = (t, x)$, we define a mollifier $\rho_\delta(P) = \delta^{-2}\rho(\delta^{-1}P)$ and set

$$\bar{u}_\delta = \bar{u} \star \rho_\delta$$

Then by convexity, we get with $Q = (s, y)$:

$$(\bar{u}_\delta)_t + H_\alpha(P, (\bar{u}_\delta)_x) \le \int dQ \ \{H_\alpha(P, \bar{u}_x(Q)) - H_\alpha(Q, \bar{u}_x(Q)\} \, \rho_\delta(P - Q).$$

The fact that $\bar{u}_x$ is locally bounded and the fact that H_α is continuous imply that the right hand side goes to zero as $\delta \to 0$. We deduce (by stability of viscosity sub-solutions) that (27) holds true in the viscosity sense. Then the comparison principle implies that

$$\mu \bar{H}_\alpha(p) + (1-\mu)\bar{H}_\alpha(q) \ge \bar{H}_\alpha(\mu p + (1-\mu)q). \tag{28}$$

Step 2: u_p and u_q are continuous. We proceed in two (sub)steps.
Step 2.1: the case of a single function u. We first want to show that if $u = u_p$ is continuous and satisfies (27) almost everywhere, then u is a viscosity sub-solution. To this end, we will use the structural assumptions satisfied by the Hamiltonian. The ones that were useful to prove the comparison principle will be also useful to prove the result we want. Indeed, we will revisit the proof of the comparison principle. We also use the fact that

$$u(t, x) - px + t\bar{H}_\alpha(p) \quad \text{is bounded.} \tag{29}$$

For $\nu > 0$, we set

$$u^\nu(t, x) = \sup_{s \in \mathbb{R}} \left(u(s, x) - \frac{(t-s)^2}{2\nu} \right) = u(s_\nu, x) - \frac{(t-s_\nu)^2}{2\nu}.$$

As usual, we get from (29) that

$$|t - s_\nu| \le C\sqrt{\nu} \quad \text{with} \quad C = C(p, T) \tag{30}$$

for $t \in (-T, T)$. In particular $s_\nu \to t$ locally uniformly. If a test function φ touches u^ν from above at some point (t, x), then we have $\varphi_t(t, x) = -\dfrac{t - s_\nu}{\nu}$ and

$$\begin{aligned}
\varphi_t(t, x) + H_\alpha(t, x, \varphi_x(t, x)) &\le H_\alpha(t, x, \varphi_x(t, x)) - H_\alpha(s_\nu, x, \varphi_x(t, x)) \\
&\le \omega(|t - s_\nu| \, (1 + \max(0, H_\alpha(s_\nu, x, \varphi_x(t, x))))) \\
&\le \omega\left(\frac{(t-s_\nu)^2}{\nu} + |t - s_\nu| \right)
\end{aligned} \tag{31}$$

where we have used **(A2)** in the third line. The right hand side goes to zero as ν goes to zero since

$$\frac{(t-s_\nu)^2}{\nu} \to 0 \quad \text{locally uniformly w.r.t. } (t, x)$$

(recall u is continuous). Indeed, this can be checked for (t,x) replaced by (t_ν,x_ν) because for any sequence $(t_\nu,s_\nu,x_\nu)\to(t,t,x)$, we have

$$u(t_\nu,x_\nu)\le u^\nu(t_\nu,x_\nu)=u(s_\nu,x_\nu)-\frac{(t_\nu-s_\nu)^2}{2\nu}$$

where the continuity of u implies the result. For a given $\nu>0$, we see that (30) and (31) imply that

$$|\varphi_t|\,,|\varphi_x|\le C_{\nu,p}.$$

This implies in particular that u^ν is Lipschitz continuous, and then

$$u^\nu_t+H(t,x,u^\nu_x)\le o_\nu(1)\quad\text{a.e.}$$

where $o_\nu(1)$ is locally uniform with respect to (t,x).

Step 2.2: application. Applying Step 2.1, we get for $z=p,q$

$$(u^\nu_z)_t+H(t,x,(u^\nu_z)_x)\le o_\nu(1)\quad\text{a.e.}$$

where $o_\nu(1)$ is locally uniform with respect to (t,x). Step 1 implies that

$$\bar u^\nu:=\mu u^\nu_p+(1-\mu)u^\nu_q$$

is a viscosity sub-solution of

$$(\bar u^\nu)_t+H_\alpha(t,x,(\bar u^\nu)_x)\le o_\nu(1)$$

where $o_\nu(1)$ is locally uniform with respect to (t,x). In the limit $\nu\to0$, we recover (by stability of sub-solutions) that $\bar u$ is a viscosity sub-solution, *i.e.* satisfies (27) in the viscosity sense. This gives then the same conclusion as in Step 1.

Step 3: the general case. To cover the general case, we simply replace u_p by $\tilde u_p$ which is the solution to the Cauchy problem

$$\begin{cases}(\tilde u_p)_t+H_\alpha(t,x,(\tilde u_p)_x)=0, & \text{for}\quad (t,x)\in(0,+\infty)\times\mathbb{R}\\ \tilde u_p(0,x)=px,\end{cases}$$

Then $\tilde u_p$ is continuous and satisfies $|\tilde u_p-u_p|\le C$. Proceeding similarly with $\tilde u_q$ and using Step 2, we deduce the desired inequality (28). The proof is now complete. □

We finally prove the quasi-convexity of $\bar H_\alpha$ under assumption **(B-i)**.

Lemma 3.6 (Quasi-convexity of $\bar H_\alpha$ under **(B-i)**). *Assume* **(A0)-(A5)** *and* **(B-i)**. *Then the function $\bar H_\alpha$ is quasi-convex.*

Proof. We reduce quasi-convexity to convexity by composing with an increasing function γ; notice that such a reduction was already used in optimization and in partial differential equations, see for instance [22, 21].

We first assume that H_α satisfies

$$\begin{cases} H_\alpha \in C^2, \\ D^2_{pp} H_\alpha(x, p^0_\alpha) > 0, \\ D_p H_\alpha(x,p) < 0 \quad \text{for} \quad p \in (-\infty, p^0_\alpha), \\ D_p H_\alpha(x,p) > 0 \quad \text{for} \quad p \in (p^0_\alpha, +\infty), \\ H_\alpha(x,p) \to +\infty \quad \text{as} \quad |p| \to +\infty \quad \text{uniformly w.r.t. } x \in \mathbb{R}. \end{cases} \tag{32}$$

For a function γ such that

$$\gamma \quad \text{is convex,} \quad \gamma \in C^2(\mathbb{R}) \quad \text{and} \quad \gamma' \geq \delta_0 > 0$$

we have

$$D^2_{pp}(\gamma \circ H_\alpha) > 0$$

if and only if

$$(\ln \gamma')'(\lambda) > -\frac{D^2_{pp} H_\alpha(x,p)}{(D_p H_\alpha(x,p))^2} \quad \text{for} \quad p = \pi^\pm_\alpha(x,\lambda) \quad \text{and} \quad \lambda \geq H_\alpha(x,p) \tag{33}$$

where $\pi^\pm_\alpha(x,\lambda)$ is the only real number r such that $\pm r \geq 0$ and $H_\alpha(x,r) = \lambda$. Because $D^2_{pp} H_\alpha(x, p^0_\alpha) > 0$, we see that the right hand side is negative for λ close enough to $H_\alpha(x, p^0_\alpha)$ and it is indeed possible to construct such a function γ.

In view of Remark 3.2, we can construct a solution of $\delta v^\delta + \gamma \circ H_\alpha(x, p + v^\delta_x) = 0$ with $-\delta v^\delta \to \overline{\gamma \circ H_\alpha}(p)$ as $\delta \to 0$, and a solution of

$$\gamma \circ H_\alpha(x, p + v_x) = \overline{\gamma \circ H_\alpha}(p)$$

This shows that

$$\bar{H}_\alpha = \gamma^{-1} \circ \overline{\gamma \circ H_\alpha}.$$

Thanks to Lemmas 3.4 and 3.5, we know that $\overline{\gamma \circ H_\alpha}$ is coercive and convex. Hence $\bar{H}_\alpha$ is quasi-convex.

If now H_α does not satisfies (32), then for all $\varepsilon > 0$, there exists $H^\varepsilon_\alpha \in C^2$ such that

$$\begin{cases} (D^2_{pp} H^\varepsilon_\alpha)(x, p^0_\alpha) > 0 \\ D_p H^\varepsilon_\alpha(x,p) < 0 \quad \text{for} \quad p \in (-\infty, p^0_\alpha), \\ D_p H^\varepsilon_\alpha(x,p) > 0 \quad \text{for} \quad p \in (p^0_\alpha, +\infty), \\ |H^\varepsilon_\alpha - H_\alpha| < \varepsilon. \end{cases}$$

Then we can argue as in the proof of continuity of $\bar{H}_\alpha$ and deduce that

$$\bar{H}_\alpha(p) = \lim_{\varepsilon \to 0} \bar{H}^\varepsilon_\alpha(p).$$

Moreover, the previous case implies that $\bar{H}^\varepsilon_\alpha$ is quasi-convex. Hence, so is $\bar{H}_\alpha$. The proof of the lemma is now complete. □

Proof of Proposition 1.2. Combine Lemmas 3.1, 3.3, 3.4, 3.5 and 3.6. □

4 Truncated cell problems

We consider the following problem: find $\lambda_\rho \in \mathbb{R}$ and w such that

$$\begin{cases} w_t + H(t,x,w_x) = \lambda_\rho, & (t,x) \in \mathbb{R} \times (-\rho,\rho), \\ w_t + H^-(t,x,w_x) = \lambda_\rho, & (t,x) \in \mathbb{R} \times \{-\rho\}, \\ w_t + H^+(t,x,w_x) = \lambda_\rho, & (t,x) \in \mathbb{R} \times \{\rho\}, \\ w \text{ is 1-periodic w.r.t. } t. \end{cases} \tag{34}$$

Even if our approach is different, we borrow here an idea from [1] by truncating the domain and by considering correctors in $[-\rho,\rho]$ with $\rho \to +\infty$.

4.1 A comparison principle

Proposition 4.1 (Comparison principle for a mixed boundary value problem)**.** *Let $\rho_2 > \rho_1 > \rho_0$ and $\lambda \in \mathbb{R}$ and v be a super-solution of the following boundary value problem*

$$\begin{cases} v_t + H(t,x,v_x) \geq \lambda & \text{for } (t,x) \in \mathbb{R} \times (\rho_1,\rho_2), \\ v_t + H^+(t,x,v_x) \geq \lambda & \text{for } (t,x) \in \mathbb{R} \times \{\rho_2\}, \\ v(t,x) \geq U_0(t) & \text{for } (t,x) \in \mathbb{R} \times \{\rho_1\}, \\ v \text{ is 1-periodic w.r.t. } t \end{cases} \tag{35}$$

where U_0 is continuous and for $\varepsilon_0 > 0$ and u be a sub-solution of the following one

$$\begin{cases} u_t + H(t,x,u_x) \leq \lambda - \varepsilon_0 & \text{for } (t,x) \in \mathbb{R} \times (\rho_1,\rho_2), \\ u_t + H^+(t,x,u_x) \leq \lambda - \varepsilon_0 & \text{for } (t,x) \in \mathbb{R} \times \{\rho_2\}, \\ u(t,x) \leq U_0(t) & \text{for } (t,x) \in \mathbb{R} \times \{\rho_1\}, \\ u \text{ is 1-periodic w.r.t. } t. \end{cases} \tag{36}$$

Then $u \leq v$ in $\mathbb{R} \times [\rho_1,\rho_2]$.

Remark 4.2. A similar result holds true if the Dirichlet condition is imposed at $x = \rho_2$ and junction conditions

$$\begin{aligned} v_t + H^-(t,x,v_x) &\geq \lambda \qquad && \text{at } x = \rho_1 \\ u_t + H^-(t,x,u_x) &\leq \lambda - \varepsilon_0 && \text{at } x = \rho_1 \end{aligned}$$

are imposed at $x = \rho_1$.

The proof of Proposition 4.1 is very similar to (in fact simpler than) the proof of the comparison principle for Hamilton-Jacobi equations on networks contained in [18]. The main difference lies in the fact that in our case, u and v are global in time and the space domain is bounded. A sketch of the proof is provided in Appendix shedding some light on the main differences. Here the parameter $\varepsilon_0 > 0$ in (36) is used in place of the standard correction term $-\eta/(T-t)$ for a Cauchy problem.

4.2 Correctors on truncated domains

Proposition 4.3 (Existence and properties of a corrector on a truncated domain)**.** *There exists a unique $\lambda_\rho \in \mathbb{R}$ such that there exists a solution $w^\rho = w$ of* (34)*. Moreover, there exists a constant $C > 0$ independent of $\rho \in (\rho_0, +\infty)$ and a function $m^\rho \colon [-\rho, \rho] \to \mathbb{R}$ such that*

$$\begin{cases} |\lambda_\rho| \le C, & \\ |m^\rho(x) - m^\rho(y)| \le C\,|x-y| & \textit{for } x, y \in [-\rho, \rho], \\ |w^\rho(t,x) - m^\rho(x)| \le C & \textit{for } (t,x) \in \mathbb{R} \times [-\rho, \rho]. \end{cases} \tag{37}$$

Proof. In order to construct a corrector on the truncated domain, we proceed classically by considering

$$\begin{cases} \delta w^\delta + w^\delta_t + H(t,x,w^\delta_x) = 0, & (t,x) \in \mathbb{R} \times (-\rho, \rho)\,, \\ \delta w^\delta + w^\delta_t + H^-(t,x,w^\delta_x) = 0, & (t,x) \in \mathbb{R} \times \{-\rho\}\,, \\ \delta w^\delta + w^\delta_t + H^+(t,x,w^\delta_x) = 0, & (t,x) \in \mathbb{R} \times \{\rho\}\,, \\ w^\delta \text{ is 1-periodic w.r.t. } t. & \end{cases} \tag{38}$$

A discontinuous viscosity solution of (38) is constructed by Perron's method (in the class of 1-periodic functions with respect to time) since $\pm\delta^{-1}C$ are trivial super-/sub-solutions if C is chosen as follows

$$C = \sup_{t \in \mathbb{R},\ x \in \mathbb{R}} |H(t,x,0)|.$$

In particular, the solution w^δ satisfies by construction

$$|w^\delta| \le \frac{C}{\delta}. \tag{39}$$

We next consider

$$m^\delta(x) = \sup_{t \in \mathbb{R}} (w^\delta)^*(t,x).$$

We remark that the supremum is reached since w^δ is periodic with respect to time; we also remark that m^δ is a viscosity sub-solution of

$$H(t(x), x, m^\delta_x) \le C, \quad x \in (-\rho, \rho)$$

(for some function $t(x)$). In view of **(A3)**, we conclude that m^δ is globally Lipschitz continuous and

$$|m^\delta_x| \le C \tag{40}$$

for some constant C which still only depends on H. Assumption **(A3)** also implies that,

$$w^\delta_t \le C$$

(with C only depending on H). In particular, the comparison principle implies that for all $t \in \mathbb{R}$, $x \in (-\rho, \rho)$ and $h \ge 0$,

$$w^\delta(t+h, x) \le w^\delta(t,x) + Ch.$$

Combining this information with the periodicity of w^δ with respect to t, we conclude that for $t \in \mathbb{R}$ and $x \in (-\rho, \rho)$,

$$|w^\delta(t,x) - m^\delta(x)| \le C.$$

In particular,

$$|w^\delta(t,x) - w^\delta(0,0)| \le C.$$

We then consider

$$\overline{w} = \limsup_\delta {}^*(w^\delta - w^\delta(0,0)) \quad \text{and} \quad \underline{w} = \liminf_\delta {}_*(w^\delta - w^\delta(0,0)).$$

We next remark that (39) and (40) imply that there exists $\delta_n \to 0$ such that

$$\begin{aligned} m^{\delta_n} - m^{\delta_n}(0) &\to m^\rho \quad \text{as } n \to +\infty \\ \delta_n w^{\delta_n}(0,0) &\to -\lambda_\rho \quad \text{as } n \to +\infty \end{aligned}$$

(the first convergence being locally uniform). In particular, λ, $\overline{w}$, $\underline{w}$ and m^ρ satisfies

$$\begin{aligned} |\lambda_\rho| &\le C \\ |\overline{w} - m^\rho| &\le C \\ |\underline{w} - m^\rho| &\le C \\ |m^\rho_x| &\le C. \end{aligned}$$

Discontinuous stability of viscosity solutions of Hamilton-Jacobi equations imply that $\overline{w} - 2C$ and $\underline{w}$ are respectively a sub-solution and a super-solution of (34) and

$$\overline{w} - 2C \le \underline{w}.$$

Perron's method is used once again in order to construct a solution w^ρ of (34) which is 1-periodic with respect to time. In view of the previous estimates, λ_ρ, m^ρ and w^ρ satisfy (37). Proving the uniqueness of λ_ρ is classical so we skip it. The proof of the proposition is now complete. □

Proposition 4.4 (First definition of the effective flux limiter)**.** *The map $\rho \mapsto \lambda_\rho$ is non-decreasing and bounded in $(0, +\infty)$. In particular,*

$$\bar{A} = \lim_{\rho \to +\infty} \lambda_\rho$$

exists and $\bar{A} \ge \lambda_\rho$ for all $\rho > 0$.

Proof. For $\rho' > \rho > 0$, we see that the restriction of $w^{\rho'}$ to $[-\rho, \rho]$ is a sub-solution, as a consequence of [18, Proposition 2.19]. The boundedness of the map follows from Proposition 4.3. The proof is thus complete. □

We next prove that we can control w^ρ from below under appropriate assumptions on $\bar{A}$.

Proposition 4.5 (Control of slopes on a truncated domain)**.** *Assume first that $\bar{A} > \min \bar{H}_R$. Then for all $\delta > 0$, there exists $\rho_\delta > 0$ and $C_\delta > 0$ (independent on ρ) such that for $x \geq \rho_\delta$ and $h \geq 0$,*

$$w^\rho(t, x+h) - w^\rho(t, x) \geq (\bar{p}_R - \delta)h - C_\delta. \tag{41}$$

If now we assume that $\bar{A} > \min \bar{H}_L$, then for $x \leq -\rho_\delta$ and $h \geq 0$,

$$w^\rho(t, x-h) - w^\rho(t, x) \geq (-\bar{p}_L - \delta)h - C_\delta \tag{42}$$

for some $\rho_\delta > 0$ and $C_\delta > 0$ as above.

Proof. We only prove (41) since the proof of (42) follows along the same lines. Let $\delta > 0$. In view of **(A5)**, we know that there exists ρ_δ such that

$$|H(t, x, p) - H_R(t, x, p)| \leq \delta \quad \text{for} \quad x \geq \rho_\delta. \tag{43}$$

Assume that $\bar{A} > \min \bar{H}_R$. Then Proposition 1.2 implies that we can pick p_R^δ such that

$$\bar{H}_R(p_R^\delta) = \bar{H}_R^+(p_R^\delta) = \lambda_\rho - 2\delta$$

for $\rho \geq \rho_0$ and $\delta \leq \delta_0$, by choosing ρ_0 large enough and δ_0 small enough.

We now fix $\rho \geq \rho_\delta$ and $x_0 \in [\rho_\delta, \rho]$. In view of Proposition 1.2 applied to $p = p_R^\delta$, we know that there exists a corrector v_R solving (3) with $\alpha = R$. Since it is $\mathbb{Z}^2$-periodic, it is bounded and $w_R = p_R^\delta x + v_R(t, x)$ solves

$$(w_R)_t + H_R(t, x, (w_R)_x) = \lambda_\rho - 2\delta, \quad (t, x) \in \mathbb{R} \times \mathbb{R}.$$

In particular, the restriction of w_R to $[\rho_\delta, \rho]$ satisfies (see [18, Proposition 2.19]),

$$\begin{cases} (w_R)_t + H_R(t, x, (w_R)_x) \leq \lambda_\rho - 2\delta & \text{for } (t, x) \in \mathbb{R} \times (\rho_\delta, \rho), \\ (w_R)_t + H_R^+(t, x, (w_R)_x) \leq \lambda_\rho - 2\delta & \text{for } (t, x) \in \mathbb{R} \times \{\rho\}\,. \end{cases}$$

In view of (43), this implies

$$\begin{cases} (w_R)_t + H(t, x, (w_R)_x) \leq \lambda_\rho - \delta & \text{for } (t, x) \in \mathbb{R} \times (\rho_\delta, \rho), \\ (w_R)_t + H^+(t, x, (w_R)_x) \leq \lambda_\rho - \delta & \text{for } (t, x) \in \mathbb{R} \times \{\rho\}\,. \end{cases}$$

Now we remark that $v = w^\rho - w^\rho(0, x_0)$ and $u = w_R - w_R(0, x_0) - 2C - 2\|v_R\|_\infty$ satisfies

$$v(t, x_0) \geq -2C \geq u(t, x_0)$$

where C is given by (37). Thanks to the comparison principle from Proposition 4.1, we thus get for $x \in [x_0, \rho]$,

$$w^\rho(t, x) - w^\rho(t, x_0) \geq p_R^\delta(x - x_0) - C_\delta$$

where C_δ is a large constant which does not depend on ρ. In particular, we get (41), reducing δ if necessary. $\square$

4.3 Construction of global correctors

We now state and prove a result which implies Theorem 1.9 stated in the introduction.

Theorem 4.6 (Existence of a global corrector for the junction)**.** *Assume* **(A0)-(A5)** *and either* **(B-i)** *or* **(B-ii)**.

i) (General properties) *There exists a solution w of* (4) *with $\lambda = \bar{A}$ such that for all $(t,x) \in \mathbb{R}^2$,*
$$|w(t,x) - m(x)| \leq C \tag{44}$$
for some globally Lipschitz continuous function m, and
$$\bar{A} \geq A_0.$$

ii) (Bound from below at infinity) *If $\bar{A} > \max_{\alpha=L,R}\left(\min \bar{H}_\alpha\right)$, then there exists $\delta_0 > 0$ such that for every $\delta \in (0,\delta_0)$, there exists $\rho_\delta > \rho_0$ such that w satisfies*
$$\begin{cases} w(t,x+h) - w(t,x) \geq (\bar{p}_R - \delta)h - C_\delta & \text{for } x \geq \rho_\delta \quad \text{and} \quad h \geq 0,\\ w(t,x-h) - w(t,x) \geq (-\bar{p}_L - \delta)h - C_\delta & \text{for } x \leq -\rho_\delta \quad \text{and} \quad h \geq 0. \end{cases} \tag{45}$$
The first line of (45) also holds if we have only $\bar{A} > \min \bar{H}_R$, while the second line of (45) also holds if we have only $\bar{A} > \min \bar{H}_L$.

iii) (Rescaling w) *For $\varepsilon > 0$, we set*
$$w^\varepsilon(t,x) = \varepsilon w(\varepsilon^{-1}t, \varepsilon^{-1}x).$$
Then (along a subsequence $\varepsilon_n \to 0$), we have that w^ε converges locally uniformly towards a function $W = W(x)$ which satisfies
$$\begin{cases} |W(x) - W(y)| \leq C\,|x-y| & \text{for all} \quad x,y \in \mathbb{R},\\ \bar{H}_R(W_x) = \bar{A} \quad \text{and} \quad \hat{p}_R \geq W_x \geq \bar{p}_R & \text{for } x \in (0,+\infty),\\ \bar{H}_L(W_x) = \bar{A} \quad \text{and} \quad \hat{p}_L \leq W_x \leq \bar{p}_L & \text{for } x \in (-\infty,0). \end{cases} \tag{46}$$
In particular, we have $W(0) = 0$ and
$$\hat{p}_R x 1_{\{x>0\}} + \hat{p}_L x 1_{\{x<0\}} \geq W(x) \geq \bar{p}_R x 1_{\{x>0\}} + \bar{p}_L x 1_{\{x<0\}}. \tag{47}$$

Proof. We consider (up to some subsequence)
$$\overline{w} = \limsup_{\rho\to+\infty}{}^*(w^\rho - w^\rho(0,0)), \quad \underline{w} = \liminf_{\rho\to+\infty}{}_*(w^\rho - w^\rho(0,0)) \quad \text{and} \quad m = \lim_{\rho\to+\infty}(m^\rho - m^\rho(0)).$$
We derive from (37) that $\underline{w}$ and $\overline{w}$ are finite and
$$m - C \leq \underline{w} \leq \overline{w} \leq m + C.$$

Moreover, discontinuous stability of viscosity solutions imply that $\overline{w} - 2C$ and $\underline{w}$ are respectively a sub-solution and a super-solution of (4) with $\lambda = \bar{A}$ (recall Proposition 4.4). Hence, a discontinuous viscosity solution w of (4) can be constructed by Perron's method (in the class of functions that are 1-periodic with respect to time).

Using again (37), w and m satisfy (44). We also get (45) from Proposition 4.5 (use (37) and pass to the limit with m instead of w if necessary).

We now study $w^\varepsilon(t,x) = \varepsilon w(\varepsilon^{-1}t, \varepsilon^{-1}x)$. Remark that (37) implies in particular that

$$w^\varepsilon(t,x) = \varepsilon m(\varepsilon^{-1}x) + O(\varepsilon).$$

In particular, we can find a sequence $\varepsilon_n \to 0$ such that

$$w^{\varepsilon_n}(t,x) \to W(x) \quad \text{locally uniformly as } n \to +\infty,$$

with $W(0) = 0$. Arguing as in the proof of convergence away from the junction point (see the case $\bar{x} \neq 0$ in Appendix), we deduce that W satisfies

$$\begin{aligned} \bar{H}_R(W_x) &= \bar{A} \text{ for } x > 0, \\ \bar{H}_L(W_x) &= \bar{A} \text{ for } x < 0. \end{aligned}$$

We also deduce from (45) that for all $\delta > 0$ and $x > 0$,

$$W_x \geq \bar{p}_R - \delta$$

in the case where $\bar{A} > \min \bar{H}_R$. Assume now that $\bar{A} = \min \bar{H}_R$. This implies that

$$\bar{p}_R \leq W_x \leq \hat{p}_R$$

and, in all cases, we thus get (47) for $x > 0$.

Similarly, we can prove for $x < 0$ that

$$\hat{p}_L \leq W_x \leq \bar{p}_L$$

and the proof of (46) of is achieved. This implies (47). The proof of Theorem 4.6 is now complete. $\square$

4.4 Proof of Theorem 1.4

Proof of Theorem 1.4. Let $\bar{A}$ denote the limit of A_ρ (see Proposition 4.4). We want to prove that $\bar{A} = \inf E$ where we recall that

$$E = \{\lambda \in \mathbb{R} : \exists w \text{ sub-solution of (4)}\}.$$

In view of (4), sub-solutions are assumed to be periodic with respect to time; we will see that they also automatically satisfy some growth conditions at infinity, see (48) below.

We argue by contradiction by assuming that there exist $\lambda < \bar{A}$ and a sub-solution w_λ of (4). The function

$$m_\lambda(x) = \sup_{t\in\mathbb{R}} (w_\lambda)^*(t,x)$$

satisfies

$$H(t(x), x, (m_\lambda)_x) \le C$$

(for some function $t(x)$). Assumption **(A3)** implies that m_λ is globally Lipschitz continuous. Moreover, since w_λ is 1-periodic w.r.t. time and $(w_\lambda)_t \le C$, then

$$|w_\lambda(t,x) - m_\lambda(x)| \le C.$$

Hence

$$w_\lambda^\varepsilon(t,x) = \varepsilon w_\lambda(\varepsilon^{-1}t, \varepsilon^{-1}x)$$

has a limit W^λ which satisfies

$$\bar{H}_R(W_x^\lambda) \le \lambda \quad \text{for } x > 0.$$

In particular, for $x > 0$,

$$W_x^\lambda \le \hat{p}_R^\lambda := \max\{p \in \mathbb{R} : \bar{H}_R(p) = \lambda\} < \bar{p}_R$$

where $\bar{p}_R$ is defined in (8). Similarly,

$$W_x^\lambda \ge \hat{p}_L^\lambda := \min\{p \in \mathbb{R} : \bar{H}_L(p) = \lambda\} > \bar{p}_L$$

with $\bar{p}_L$ defined in (9). Those two inequalities imply in particular that for all $\delta > 0$, there exists $\tilde{C}_\delta$ such that

$$w_\lambda(t,x) \le \begin{cases} (\hat{p}_R^\lambda + \delta)x + \tilde{C}_\delta & \text{for } x > 0, \\ (\hat{p}_L^\lambda + \delta)x + \tilde{C}_\delta & \text{for } x < 0. \end{cases} \tag{48}$$

In particular,

$$w_\lambda < w \text{ for } |x| \ge R$$

if δ is small enough and R is large enough. In particular,

$$w_\lambda < w + C_R \text{ for } x \in \mathbb{R}.$$

Remark finally that $u(t,x) = w(t,x) + C_R - \bar{A}t$ is a solution and $u_\lambda(t,x) = w_\lambda(t,x) - \lambda t$ is a sub-solution of (1) with $\varepsilon = 1$ and $u_\lambda(0,x) \le u(0,x)$. Hence the comparison principle implies that

$$w_\lambda(t,x) - \lambda t \le w(t,x) - \bar{A}t + C_R.$$

Dividing by t and letting t go to $+\infty$, we get the following contradiction

$$\bar{A} \le \lambda.$$

The proof is now complete. $\square$

5 Proof of Theorem 1.12

This section is devoted to the proof of Theorem 1.12. As pointed out in Remark 1.13 above, the notion of solutions for (1) has to be first made precise because the Hamiltonian is discontinuous with respect to time.

Notion of solutions for (1)**.** For $\varepsilon = 1$, a function u is a *solution* of (1) if it is globally Lipschitz continuous (in space and time) and if it solves successively the Cauchy problems on time intervals $[\tau_i + k, \tau_{i+1} + k)$ for $i = 0, \ldots, K$ and $k \in \mathbb{N}$.

Because of this definition (approach), we have to show that if the initial datum u_0 is globally Lipschitz continuous, then the solution to the successive Cauchy problems is also globally Lipschitz continuous (which of course insures its uniqueness from the classical comparison principle). See Lemma 5.1 below.

Proof of Theorem 1.12 i). In view of the proof of Theorem 1.7, the reader can check that it is enough to get a global Lipschitz bound on the solution u^ε and to construct a global corrector in this new framework. The proof of these two facts is postponed, see Lemmas 5.1 and 5.2 following this proof. Notice that half-relaxed limits are not necessary anymore and that the reasoning can be completed by considering locally converging subsequences of $\{u^\varepsilon\}_\varepsilon$. Notice also that the perturbed test function method of Evans [9] still works. As usual, if the viscosity sub-solution inequality is not satisfied at the limit, this implies that the perturbed test function is a super-solution except at times $\varepsilon\,(\mathbb{Z} + \{\tau_0, \ldots, \tau_K\})$. Still a localized comparison principle in each slice of times for each Cauchy problem is sufficient to conclude. □

Lemma 5.1 (Global Lipschitz bound)**.** *The function u^ε is equi-Lipschitz continuous with respect to time and space.*

Proof. Remark that it is enough to get the result for $\varepsilon = 1$ since $u(t, x) = \varepsilon^{-1} u^\varepsilon(\varepsilon t, \varepsilon x)$ satisfies the equation with $\varepsilon = 1$ and the initial condition

$$u_0^\varepsilon(x) = \frac{1}{\varepsilon} U_0^\varepsilon(\varepsilon x)$$

is equi-Lipschitz continuous. For the sake of clarity, we drop the ε superscript in u_0^ε and simply write u_0.

We first derive bounds on the time interval $[\tau_0, \tau_1) = [0, \tau_1)$. In order to do so, we assume that the initial data satisfies $|(u_0)_x| \le L$. Then as usual, there is a constant $C > 0$ such that

$$u^\pm(t, x) = u_0(x) \pm Ct$$

are super-/sub-solutions of (1)-(10) with H given by **(C1)** with for instance

$$C := \max\left(\max_{\alpha=1,\ldots,N} \|a_\alpha\|_\infty,\ \max_{\alpha=0,\ldots,N} \left(\max_{|p|\le L} |\bar{H}_\alpha(p)| \right) \right). \tag{49}$$

Let u be the standard (continuous) viscosity solution of (1) on the time interval $(0, \tau_1)$ with initial data given by u_0 (recall that $\varepsilon = 1$). Then for any $h > 0$ small enough, we have $-Ch \leq u(h, x) - u(0, x) \leq Ch$. The comparison principle implies for $t \in (0, \tau_1 - h)$

$$-Ch \leq u(t + h, x) - u(t, x) \leq Ch$$

which shows the Lipschitz bound in time, on the time interval $[0, \tau_1)$:

$$|u_t| \leq C. \tag{50}$$

From the Hamilton-Jacobi equation, we now deduce the following Lipschitz bound in space on the time interval $(0, \tau_1)$:

$$|\bar{H}_\alpha(u_x(t, \cdot))|_{L^\infty(b_\alpha, b_{\alpha+1})} \quad \leq C \quad \text{for} \quad \alpha = 0, \ldots, N. \tag{51}$$

We can now derive bounds on the time interval $[\tau_1, \tau_2)$ as follows. We deduce first that (51) still holds true at time $t = \tau_1$. Combined with our definition (49) of the constant C, we also deduce that

$$v^\pm(t, x) = u(\tau_1, x) \pm C(t - \tau_1)$$

are sub/super-solutions of (6) for $t \in (\tau_1, \tau_2)$ where H is given by **(C1)**. Reasoning as above, we get bounds (50) and (51) on the time interval $[\tau_1, \tau_2)$.

Such a reasoning can be used iteratively to get the Lipschitz bounds (50) and (51) for $t \in [0, +\infty)$. The proof of the lemma is now complete. □

Lemma 5.2. *The conclusion of Theorem 4.6 still holds true in this new framework.*

Proof. The proof proceeds in several steps.

Step 1. Construction of a time periodic corrector w^ρ on $[-\rho, \rho]$. We first construct a Lipschitz corrector on a truncated domain. In order to do so, we proceed in several steps.

Step 1.1. First Cauchy problem on $(0, +\infty)$. The method presented in the proof of Proposition 4.3, using a term δw^δ has the inconvenience that it would not clearly provide a Lipschitz solution. In order to stick to our notion of globally Lipschitz solutions, we simply solve the Cauchy problem for $\rho > \rho_0 := \max_{\alpha=1,\ldots,N} |b_\alpha|$:

$$\begin{cases} w_t^\rho + H(t, x, w_x^\rho) = 0 & \text{on} \quad (0, +\infty) \times (-\rho, \rho)\,, \\ w_t^\rho + \bar{H}_N^-(w_x^\rho) = 0 & \text{on} \quad (0, +\infty) \times \{-\rho\}\,, \\ w_t^\rho + \bar{H}_0^+(w_x^\rho) = 0 & \text{on} \quad (0, +\infty) \times \{\rho\}\,, \\ w^\rho(0, x) = 0 & \text{for} \quad x \in [-\rho, \rho]\,. \end{cases} \tag{52}$$

As in the proof of the previous lemma, we get global Lipschitz bounds with a constant C (independent on $\rho > 0$ and independent on the distances $\ell_\alpha = b_{\alpha+1} - b_\alpha$):

$$|w_t^\rho|, \quad |\bar{H}_\alpha(w_x^\rho(t, \cdot))|_{L^\infty((b_\alpha, b_{\alpha+1}) \cap (-\rho, \rho))} \quad \leq C, \quad \text{for} \quad \alpha = 0, \ldots, N. \tag{53}$$

Arguing as in [12] for instance, we deduce that there exists a real number λ_ρ with

$$|\lambda_\rho| \le C$$

and a constant C_0 (that depends on ρ) such that we have

$$|w^\rho(t,x) + \lambda_\rho t| \le C_0. \tag{54}$$

Details are given in Appendix for the reader's convenience.

STEP 1.2. GETTING GLOBAL SUB AND SUPER-SOLUTIONS. Let us now define the following function (up to some subsequence $k_n \to +\infty$):

$$w^\rho_\infty(t,x) = \lim_{k_n \to +\infty} \left(w^\rho(t+k_n, x) + \lambda_\rho k_n \right)$$

which still satisfies (53) and (54). Then we also define the two functions

$$\overline{w}^\rho_\infty(t,x) = \inf_{k\in\mathbb{Z}} \left(w^\rho_\infty(t+k,x) + k\lambda_\rho \right), \quad \underline{w}^\rho_\infty(t,x) = \sup_{k\in\mathbb{Z}} \left(w^\rho_\infty(t+k,x) + k\lambda_\rho \right).$$

They still satisfy (53) and (54) and are respectively a super- and a sub-solution of the problem in $\mathbb{R}\times[-\rho,\rho]$. They satisfy moreover that $\overline{w}^\rho_\infty(t,x)+\lambda_\rho t$ and $\underline{w}^\rho_\infty(t,x)+\lambda_\rho t$ are 1-periodic in time, which implies the following bounds

$$|\overline{w}^\rho_\infty(t,x) - \overline{w}^\rho_\infty(0,x) + \lambda_\rho t| \le C, \quad |\underline{w}^\rho_\infty(t,x) - \underline{w}^\rho_\infty(0,x) + \lambda_\rho t| \le C.$$

STEP 1.3: A NEW CAUCHY PROBLEM ON $(0,+\infty)$ AND CONSTRUCTION OF A TIME PERIODIC SOLUTION. We note that $\overline{w}^\rho_\infty + 2C_0 \ge \underline{w}^\rho_\infty$, and we now solve the Cauchy problem with new initial data $\underline{w}^\rho_\infty(0,x)$ instead of the zero initial data and call $\tilde{w}^\rho$ the solution of this new Cauchy problem. From the comparison principle, we get

$$\underline{w}^\rho_\infty \le \tilde{w}^\rho \le \overline{w}^\rho_\infty + 2C_0.$$

In particular,

$$\tilde{w}^\rho(1,x) \ge \underline{w}^\rho_\infty(1,x) \ge \tilde{w}^\rho(0,x) - \lambda_\rho.$$

This implies, by comparison, that

$$\tilde{w}^\rho(k+1,x) \ge \tilde{w}^\rho(k,x) - \lambda_\rho. \tag{55}$$

Moreover $\tilde{w}^\rho$ still satisfies (53) (indeed with the same constant because, by construction, this is also the case for $\underline{w}^\rho_\infty$). We now define (up to some subsequence $k_n \to +\infty$):

$$\tilde{w}^\rho_\infty(t,x) = \lim_{k_n \to +\infty} \left(\tilde{w}^\rho(t+k_n,x) + \lambda_\rho k_n \right)$$

which, because of (55) and the fact that $\tilde{w}^\rho(t,x)+\lambda_\rho t$ is bounded, satisfies

$$\tilde{w}^\rho_\infty(k+1,x) + \lambda = \tilde{w}^\rho_\infty(k,x)$$

and then $\tilde{w}^\rho_\infty(t,x)+\lambda_\rho t$ is 1-periodic in time. Moreover $\tilde{w}^\rho_\infty$ is still a solution of the Cauchy problem and satisfies (53). We define

$$w^\rho := \tilde{w}^\rho_\infty + \lambda_\rho t$$

which satisfies (37) and then provides the analogue of the function given in Proposition 4.3.

Step 2. Contruction of w on $\mathbb{R}$. The result of Theorem 4.6 still holds true for

$$w = \lim_{\rho\to+\infty} (w^\rho - w^\rho(0,0))$$

which is globally Lipschitz continuous in space and time and satisfies (53) with $\rho = +\infty$, and

$$\bar{A} = \lim_{\rho\to+\infty} \lambda_\rho. \quad \square$$

Proof of (12) *from Theorem 1.12.* We recall that $\bar{H}_L = \bar{H}_0$ and $\bar{H}_R = \bar{H}_1$ and set $a = a_1$ and (up to translation) $b_1 = 0$.

Step 1: The convex case: identification of $\bar{A}$.

STEP 1.1: A CONVEX SUBCASE. We first work in the particular case where both $\bar{H}_\alpha$ for $\alpha = L, R$ are convex and given by the Legendre-Fenchel transform of convex Lagrangians L_α which satisfy for some compact interval I_α:

$$L_\alpha(p) = \begin{cases} \text{finite} & \text{if} \quad q \in I_\alpha, \\ +\infty & \text{if} \quad q \notin I_\alpha. \end{cases} \tag{56}$$

Then it is known (see for instance the section on optimal control in [18]) that the solution of (1) on the time interval $[0, \varepsilon\tau_1)$, is given by

$$u^\varepsilon(t,x) = \inf_{y\in\mathbb{R}} \left(\inf_{X\in S_{0,y;t,x}} \left\{ u^\varepsilon(0, X(0)) + \int_0^t L_\varepsilon(s, X(s), \dot{X}(s))\, ds \right\} \right) \tag{57}$$

with

$$L_\varepsilon(s,x,p) = \begin{cases} \bar{H}_L^*(p) & \text{if} \quad x < 0, \\ \bar{H}_R^*(p) & \text{if} \quad x > 0, \\ \min(-a(\varepsilon^{-1}s), \min\limits_{\alpha=L,R} L_\alpha(0)) & \text{if} \quad x = 0, \end{cases}$$

and for $s < t$, the following set of trajectories:

$$S_{s,y;t,x} = \{X \in \mathrm{Lip}((s,t);\mathbb{R}), \quad X(s) = y, \quad X(t) = x\}.$$

Combining this formula with the other one on the time interval $[\varepsilon\tau_1, \varepsilon\tau_2)$, and iterating on all necessary intervals, we get that (57) is a representation formula of the solution u^ε of (1) for all $t > 0$. We also know (see the section on optimal control in [18]), that the optimal trajectories from $(0, y)$ to (t_0, x_0) intersect the axis $x = 0$ at most on a time interval $[t_1^\varepsilon, t_2^\varepsilon]$ with $0 \le t_1^\varepsilon \le t_2^\varepsilon \le t_0$. If this interval is not empty, then we have $t_i^\varepsilon \to t_i^0$ for $i = 1, 2$ and we can easily pass to the limit in (57). In general, u^ε converges to u^0 given by the formula

$$u^0(t,x) = \inf_{y\in\mathbb{R}} \left(\inf_{X\in S_{0,y;t,x}} \left\{ u^0(0, X(0)) + \int_x^t L_0(s, X(s), \dot{X}(s))\, ds \right\} \right)$$

with

$$L_0(s,x,p) = \begin{cases} \bar{H}_L^*(p) & \text{if} \quad x<0, \\ \bar{H}_R^*(p) & \text{if} \quad x>0, \\ \min(-\langle a\rangle, \min\limits_{\alpha=L,R} L_\alpha(0)) & \text{if} \quad x=0, \end{cases}$$

and from [18] we see that u^0 is the unique solution of (6)-(2) with $\bar{A} = \langle a\rangle$.

Step 1.2: The general convex case. The general case of convex Hamiltonians is recovered, because for Lipschitz continuous initial data u_0, we know that the solution is globally Lipschitz continuous. Therefore, we can always modify the Hamiltonians $\bar{H}_\alpha$ outside some compact intervals such that the modified Hamiltonians satisfy (56).

Step 2: General quasi-convex Hamiltonians: identification of $\bar{A}$.

Step 2.1: Sub-Solution inequality. From Theorem 2.10 in [18], we know that $w(t,0)$, as a function of time only, satisfies in the viscosity sense

$$w_t(t,0) + a(t) \le \bar{A} \quad \text{for all} \quad t \notin \bigcup_{i=1,\dots,K+1} \tau_i + \mathbb{Z}.$$

Using the 1-periodicity in time of w, we see that the integration in time on one period implies:

$$\langle a\rangle \le \bar{A}. \tag{58}$$

Step 2.2: Super-solution inequality. Recall that $\bar{A} \ge \langle a\rangle \ge A_0 := \max\limits_{\alpha=L,R} \min(\bar{H}_\alpha)$. If $\bar{A} = A_0$, then obviously, we get $\bar{A} = \langle a\rangle$. Hence, it remains to treat the case $\bar{A} > A_0$.

Step 2.3: Construction of a super-solution for $x \neq 0$. Recall that $\bar{p}_R$ and $\bar{p}_L$ are defined in (8) and (9) and the minimum of $\bar{H}_\alpha$ is reached for $\bar{p}_\alpha^0$, $\alpha = R, L$. Since $\bar{A} > A_0$, there exists some $\delta > 0$ such that

$$\bar{p}_L + 2\delta < \bar{p}_L^0 \qquad \text{and} \quad \bar{p}_R^0 < \bar{p}_R - 2\delta. \tag{59}$$

If w denotes a global corrector given by Lemma 5.2 (or Theorem 4.6), let us define

$$\underline{w}_R(t,x) = \inf_{h\ge 0} \big(w(t,x+h) - \bar{p}_R^0 h\big) \quad \text{for} \quad x \ge 0,$$

and similarly

$$\underline{w}_L(t,x) = \inf_{h\ge 0} \big(w(t,x-h) + \bar{p}_L^0 h\big) \quad \text{for} \quad x \le 0.$$

From (45) with $\rho_\delta = 0$, we deduce that we have for some $\bar{h} \ge 0$

$$w(t,x) \ge \underline{w}_R(t,x) = w(t,x+\bar{h}) - \bar{p}_R^0 \bar{h} \ge w(t,x) + (\bar{p}_R - \delta - \bar{p}_R^0)\bar{h} - C_\delta.$$

From (59), this implies

$$0 \le \bar{h} \le C_\delta/\delta \tag{60}$$

and using the fact that w is globally Lipschitz continuous, we deduce that for $\alpha = R$:

$$w \geq \underline{w}_\alpha \geq w - C_1. \tag{61}$$

Moreover, by constrution (as an infimum of (globally Lipschitz continuous) super-solutions), $\underline{w}_R$ is a (globally Lipschitz continuous) super-solution of the problem in $\mathbb{R} \times (0, +\infty)$. We also have for $x = y + z$ with $z \geq 0$:

$$\begin{aligned}
\underline{w}_R(t,x) - \underline{w}_R(t,y) &= w(t, x + \bar{h}) - \bar{p}_R^0 \bar{h} - \underline{w}_R(t,y) \\
&\geq w(t, x + \bar{h}) - \bar{p}_R^0 \bar{h} - \big(w(t, y + \bar{h} + z) - \bar{p}_R^0(\bar{h} + z)\big) \\
&\geq \bar{p}_R^0 z = \bar{p}_R^0 (x - y)
\end{aligned}$$

which shows that

$$(\underline{w}_R)_x \geq \bar{p}_R^0. \tag{62}$$

Similarly (and we can also use a symmetry argument to see it), we get that $\underline{w}_L$ is a (globally Lipschitz continuous) super-solution in $\mathbb{R} \times (-\infty, 0)$, it satisfies (61) with $\alpha = L$ and

$$(\underline{w}_L)_x \leq \bar{p}_L^0. \tag{63}$$

We now define

$$\underline{w}(t,x) = \begin{cases} \underline{w}_R(t,x) & \text{if} \quad x > 0, \\ \underline{w}_L(t,x) & \text{if} \quad x < 0, \\ \min(\underline{w}_L(t,0), \underline{w}_R(t,0)) & \text{if} \quad x = 0 \end{cases} \tag{64}$$

which by constrution is lower semi-continuous and satisfies (61), and is a super-solution for $x \neq 0$.

Step 2.4: Checking the super-solution property at $x = 0$. Let φ be a test function touching $\underline{w}$ from below at $(t_0, 0)$ with $t_0 \notin \bigcup_{i=1,\ldots,K+1} \tau_i + \mathbb{Z}$. We want to check that

$$\varphi_t(t_0, 0) + F_{a(t_0)}(\varphi_x(t_0, 0^-), \varphi_x(t_0, 0^+)) \geq \bar{A}. \tag{65}$$

We may assume that

$$\underline{w}(t_0, 0) = \underline{w}_R(t_0, 0)$$

since the case $\underline{w}(t_0, 0) = \underline{w}_L(t_0, 0)$ is completely similar. Let $\bar{h} \geq 0$ be such that

$$\underline{w}_R(t_0, 0) = w(t_0, 0 + \bar{h}) - \bar{p}_R^0 \bar{h}.$$

We distinguish two cases. Assume first that $\bar{h} > 0$. Then we have for all $h \geq 0$

$$\varphi(t, 0) \leq w(t, 0 + h) - \bar{p}_R^0 h$$

with equality for $(t, h) = (t_0, \bar{h})$. This implies the viscosity inequality

$$\varphi_t(t_0, 0) + \bar{H}_R(\bar{p}_R^0) \geq \bar{A}$$

which implies (65), because $F_{a(t_0)}(\varphi_x(t_0,0^-),\varphi_x(t_0,0^+)) \geq a(t_0) \geq A_0 \geq \min \bar{H}_R = \bar{H}_R(\bar{p}_R^0)$.

Assume now that $\bar{h} = 0$. Then we have $\varphi \leq \underline{w} \leq w$ with equality at $(t_0, 0)$. This implies immediately (65).

STEP 2.5: CONCLUSION. We deduce that $\underline{w}$ is a super-solution on $\mathbb{R} \times \mathbb{R}$. Now let us consider a C^1 function $\psi(t)$ such that

$$\psi(t) \leq \underline{w}(t, 0)$$

with equality at $t = t_0$. Because of (62) and (63), we see that

$$\varphi(t,x) = \psi(t) + \bar{p}_L^0 x 1_{\{x<0\}} + \bar{p}_R^0 x 1_{\{x>0\}}$$

satisfies

$$\varphi \leq \underline{w}$$

with equality at $(t_0, 0)$. This implies (65), and at almost every point t_0 where the Lipschitz continuous function $\underline{w}(t,0)$ is differentiable, we have

$$\underline{w}_t(t_0, 0) + a(t_0) \geq \bar{A}.$$

Because w is 1-periodic in time, we get after an integration on one period,

$$\langle a \rangle \geq \bar{A}. \tag{66}$$

Together with (58), we deduce that $\langle a \rangle = \bar{A}$, which is the desired result, for $N = 1$. □

Proof of (13) *in Theorem 1.12.* . We simply remark, using the sub-solution viscosity inequality at each junction condition, that for $\alpha = 1, \ldots, N$,

$$\bar{A} \geq \langle a_\alpha \rangle$$

which is the desired result. This achieves the proof of (12) and (13). □

Proof of the monotonicity of $\bar{A}$ in Theorem 1.12. Let $N \geq 2$, and for $i = c, d$, let us assume some given $b_1^i < \cdots < b_N^i$. and let us call w^i a global corrector given by Lemma 5.2 (or Theorem 4.6) with $\lambda = \bar{A}^i$ and $H = H^i$ with $i = c, d$ respectively.
We call $\ell_\alpha^i = b_{\alpha+1}^i - b_\alpha^i > 0$ and assume that

$$0 < \ell_{\alpha_0}^d - \ell_{\alpha_0}^c =: \delta_{\alpha_0} \quad \text{for some} \quad \alpha_0 \in \{1, \ldots, N-1\}$$

and

$$\ell_\alpha^d = \ell_\alpha^c \quad \text{for all} \quad \alpha \in \{1, \ldots, N-1\} \setminus \{\alpha_0\}.$$

Calling $\bar{p}_{\alpha_0}^0$ a point of global minimum of $\bar{H}_{\alpha_0}$, we define

$$\tilde{w}^d(t,x) = \begin{cases} w^c(t, x - b_{\alpha_0}^d + b_{\alpha_0}^c) & \text{if} \quad x \leq b_{\alpha_0}^d + \ell_{\alpha_0}^c/2 =: x_-, \\ w^c(t, x_- - b_{\alpha_0}^d + b_{\alpha_0}^c) + \bar{p}_{\alpha_0}^0(x - x_-) & \text{if} \quad x_- \leq x \leq x_+, \\ w^c(t, x - b_{\alpha_0+1}^d + b_{\alpha_0+1}^c) + \bar{p}_{\alpha_0}^0(x_+ - x_-) & \text{if} \quad x \geq b_{\alpha_0+1}^d - \ell_{\alpha_0}^c/2 =: x_+. \end{cases}$$

Recall that w^i for $i = c, d$, are globally Lipschitz continuous in space and time. This shows that $\tilde{w}^d$ is also Lipschitz continuous in space and time by construction, because it is continuous at $x = x_-, x_+$. Moreover $\tilde{w}^d$ is 1-periodic in time. We now want to check that $\tilde{w}^d$ is a sub-solution of the equation satisfied by w^d with $\bar{A}^c$ on the right hand side instead of $\bar{A}^d$. We only have to check it for all times $\bar{t} \notin \{\tau_0, \dots, \tau_K\}$ and $\bar{x} \in [x_-, x_+]$, *i.e.* we have to show that

$$\tilde{w}^d_t(\bar{t}, \bar{x}) + \bar{H}_{\alpha_0}(\tilde{w}^d_x(\bar{t}, \bar{x})) \leq \bar{A}^c \quad \text{for all} \quad \bar{x} \in [x_-, x_+]. \tag{67}$$

Assume that φ is a test function touching $\tilde{w}^d$ from above at such a point $(\bar{t}, \bar{x})$ with $\bar{x} \in [x_-, x_+]$. Then this implies in particular that $\psi(t,x) = \varphi(t,x) - \bar{p}^0_{\alpha_0}(x - x_-)$ touches $\tilde{w}^d(\cdot, x_-) = w^c(\cdot, x_0)$ from above at time $\bar{t}$ with $x_0 = b^c_{\alpha_0} + \ell^c_{\alpha_0}/2$. Recall that w^c is solution of

$$w^c_t + \bar{H}_{\alpha_0}(w^c_x) = \bar{A}^c \quad \text{on} \quad (b^c_{\alpha_0}, b^c_{\alpha_0+1}).$$

From the characterization of sub-solutions (see Theorem 2.10 in [18]), we then deduce that

$$\psi_t(\bar{t}) + \bar{H}_{\alpha_0}(\bar{p}^0_{\alpha_0}) \leq \bar{A}^c.$$

If $\bar{x} \in (x_-, x_+)$, then we have $\varphi_x(\bar{t}, \bar{x}) = \bar{p}^0_{\alpha^0}$. This means in particular

$$\varphi_t + \bar{H}_{\alpha_0}(\varphi_x) \leq \bar{A}^c \quad \text{at} \quad (\bar{t}, \bar{x}) \quad \text{if} \quad \bar{x} \in (x_-, x_+). \tag{68}$$

Using now (68), and still from Theorem 2.10 in [18], we deduce that we have in the viscosity sense

$$\tilde{w}^d_t(\bar{t}, \bar{x}) + \max\left(\bar{H}^-_{\alpha_0}(\tilde{w}^d_x(\bar{t}, \bar{x}^+)), \bar{H}^+_{\alpha_0}(\tilde{w}^d_x(\bar{t}, \bar{x}^-))\right) \leq \bar{A}^c \quad \text{for} \quad \bar{x} = x_\pm. \tag{69}$$

Therefore (68) and (69) imply (67).

Let us now call H^d the Hamiltonian in assumption **(C1)** constructed with the points $\{b^d_\alpha\}_{\alpha=1,\dots,N}$. Then we have

$$\tilde{w}^d_t + H^d(t, x, \tilde{w}^d_x) \leq \bar{A}^c \quad \text{for all} \quad t \notin \{\tau_0, \dots \tau_K\}.$$

Note that the proof of Theorem 1.4 is unchanged for the present problem, and then Theorem 1.4 still holds true. This shows that

$$\bar{A}^d \leq \bar{A}^c \tag{70}$$

which shows the expected monotonicity. The proof is now complete. □

Remark 5.3. Note that, in the previous proof, it would also be possible to compare the sub-solution given by the restriction of $\tilde{w}^d$ on some interval $[-\rho, \rho]$ with $\rho > 0$ large enough (see [18, Proposition 2.19]), with the approximation $w^{d,\rho}$ of w^d on $[-\rho, \rho]$ with $\bar{A}^d \geq \bar{A}^d_\rho \to \bar{A}^d$ as $\rho \to +\infty$. The comparison for large times would imply $\bar{A}^d_\rho \leq \bar{A}^c$. As $\rho \to +\infty$, this would give the same conclusion (70).

Proof of (14) *in Theorem 1.12.* Let w be a global corrector associated to $\bar{A}$.

Recall that

$$\bar{A} \geq \bar{A}_0 := \max_{\alpha=1,\dots,N} \langle a_\alpha \rangle \geq A_0 := \max_{\alpha=1,\dots,N} A_0^\alpha \quad \text{with} \quad A_0^\alpha = \max_{\beta=\alpha-1,\alpha} (\min \bar{H}_\beta). \tag{71}$$

Our goal is to prove that $\bar{A} = \bar{A}_0$ when all the distances ℓ_α are large enough, *i.e.* (14). Let us assume that

$$\bar{A} > \bar{A}_0.$$

Step 1: Considering another corrector with the same $\langle \widehat{a}_\alpha \rangle = \bar{A}_0$. Let $\mu_\alpha \geq 0$ such that

$$\widehat{a}_\alpha = \mu_\alpha + a_\alpha \quad \text{with} \quad \langle \widehat{a}_\alpha \rangle = \bar{A}_0 \quad \text{for all} \quad \alpha = 1, \dots, N.$$

Let us call $\widehat{w}$ the corresponding corrector with associated constant $\widehat{A}$. Then Theorem 1.4 (still valid here) implies that

$$\widehat{A} \geq \bar{A} > \bar{A}_0.$$

We also split the set $\{1, \dots, N\}$ into two disjoint sets

$$I_0 = \left\{ \alpha \in \{1, \dots, N\}, \quad \bar{A}_0 = A_0^\alpha \right\}$$

and

$$I_1 = \left\{ \alpha \in \{1, \dots, N\}, \quad \bar{A}_0 > A_0^\alpha \right\}.$$

Note that by (71), if $\alpha \in I_0$, then $\langle a_\alpha \rangle = A_0^\alpha$, and then by **(C3)**, we have $a_\alpha(t) = const = A_0^\alpha$ for all time $t \in \mathbb{R}$. For later use, we then claim that $\widehat{w}$ satisfies

$$\widehat{w}_t(t,x) + \max(\bar{H}_\alpha^-(\widehat{w}_x(t,x^+)), \bar{H}_{\alpha-1}^+(\widehat{w}_x(t,x^-))) = \hat{A} \quad \text{for all} \quad (t,x) \in \mathbb{R} \times \{b_\alpha\} \tag{72}$$

and not only for $t \in \mathbb{R} \backslash (\mathbb{Z} + \{\tau_0, \dots, \tau_K\})$. Let us show it for sub-solutions (the proof being similar for super-solutions). Let φ be a test function touching $\widehat{w}$ from above at some point $(\bar{t}, \bar{x}) = (j + \tau_k, b_\alpha)$ for some $j \in \mathbb{Z}$, $k \in \{0, \dots, K\}$. Assume also that the contact between φ and $\widehat{w}$ only holds at that point $(\bar{t}, \bar{x})$. The proof is a variant of a standard argument. For $\eta > 0$, let us consider the test function

$$\varphi_\eta(t,x) = \varphi(t,x) + \frac{\eta}{\bar{t} - t} \quad \text{for} \quad t \in (-\infty, \bar{t}).$$

Then for $r > 0$ fixed, we have

$$\inf_{(t,x) \in \overline{B_r(\bar{t},\bar{x})},\ t<\bar{t}} (\varphi_\eta - \widehat{w})(t,x) = (\varphi_\eta - \widehat{w})(t_\eta, x_\eta)$$

with

$$\begin{cases} P_\eta = (t_\eta, x_\eta) \to (\bar{t}, \bar{x}) = \bar{P} & \text{as} \quad \eta \to 0, \\ \varphi_t(\bar{P}) \leq \limsup_{\eta \to 0} (\varphi_\eta)_t(P_\eta). \end{cases}$$

This implies that $\widehat{w}$ is a relaxed viscosity sub-solution at $(\bar{t}, \bar{x})$ in the sense of Definition 2.2 in [18]. By [18, Proposition 2.6], we deduce that $\widehat{w}$ is also a standard (*i.e.* not relaxed) viscosity sub-solution at $(\bar{t}, \bar{x})$. Finally we get (72).

STEP 2: DEFINING A SPACE SUPER-SOLUTION. Let us define the function

$$M(x) = \inf_{t \in \mathbb{R}} \widehat{w}(t, x).$$

Because $\widehat{w}$ is globally Lispschitz continuous, we deduce that M is also globally Lipschitz continuous. Moreover we have the following viscosity super-solution inequality

$$\bar{H}_\alpha(M_x(x)) \geq \widehat{A} > \bar{A}_0 \quad \text{for all} \quad x \in (b_\alpha, b_{\alpha+1}), \quad \text{for all} \quad \alpha = 0, \dots, N.$$

Let us call for $\alpha = 0, \dots, N$:

$$\bar{p}_{\alpha,R} = \min E_{\alpha,R} \quad \text{with} \quad E_{\alpha,R} = \left\{ p \in \mathbb{R}, \quad \bar{H}_\alpha^+(p) = \bar{H}_\alpha(p) = \bar{A}_0 \right\},$$

$$\bar{p}_{\alpha,L} = \max E_{\alpha,L} \quad \text{with} \quad E_{\alpha,L} = \left\{ p \in \mathbb{R}, \quad \bar{H}_\alpha^-(p) = \bar{H}_\alpha(p) = \bar{A}_0 \right\}.$$

Let us now consider $\alpha = 0, \dots, N$ and two points $x_- < x_+$ with $x_\pm \in (b_\alpha, b_{\alpha+1})$. Let us assume that there is a test function $\varphi^\pm$ touching M from below at $x_\pm$. Then we have

$$\bar{H}_\alpha(\varphi_x^\pm(x_\pm)) \geq \widehat{A} > \bar{A}_0$$

with

$$\varphi_x^\pm(x_\pm) \geq \bar{p}_{\alpha,R} \quad \text{or} \quad \varphi_x^\pm(x_\pm) \leq \bar{p}_{\alpha,L}.$$

Moreover, if $\bar{A}_0 > \min \bar{H}_\alpha$, then we have

$$\bar{p}_{\alpha,L} < \bar{p}_\alpha^0 < \bar{p}_{\alpha,R}$$

for any $\bar{p}_\alpha^0$ which is a point of global minimum of $\bar{H}_\alpha$.

STEP 3: A PROPERTY OF THE SPACE SUPER-SOLUTION. We now claim that the following case is impossible:

$$p^- := \varphi_x^-(x_-) < \varphi_x^+(x_+) =: p^+ \quad \text{and} \quad \inf_{[p^-, p^+]} \bar{H}_\alpha < \widehat{A}.$$

If it is the case, then let $\bar{p} \in (p^-, p^+)$ such that $\bar{H}_\alpha(\bar{p}) < \widehat{A}$. Therefore the geometry of the graph of the function M implies that

$$\inf_{x \in [x_-, x_+]} (M(x) - x\bar{p}) = M(\bar{x}) - \bar{x}\bar{p} \quad \text{for some} \quad \bar{x} \in (x_-, x_+)$$

and then we have the viscosity super-solution inequality at $\bar{x}$:

$$\bar{H}_\alpha(\bar{p}) \geq \widehat{A}$$

which leads to a contradiction. Therefore (in all cases $\bar{A}_0 > \min \bar{H}_\alpha$ or $\bar{A}_0 = \min \bar{H}_\alpha$), it is possible to check that there is a point $\bar{x}_\alpha \in [b_\alpha, b_{\alpha+1}]$ such that the Lipschitz continuous function M satisfies in the viscosity sense

$$\begin{cases} M_x \geq \bar{p}_{\alpha,R} & \text{in} \quad (b_\alpha, \bar{x}_\alpha), \\ -M_x \geq -\bar{p}_{\alpha,L} & \text{in} \quad (\bar{x}_\alpha, b_{\alpha+1}). \end{cases}$$

Moreover from Theorem 4.6 ii) (see Lemma 5.2), we deduce from $\widehat{A} > \max(\min H_N, \min H_0)$ that

$$\bar{x}_N = +\infty \quad \text{and} \quad \bar{x}_0 = -\infty.$$

In particular, we deduce that there exists at least one $\alpha_0 \in \{1, \dots, N\}$ such that

$$\bar{x}_{\alpha_0} - b_{\alpha_0} \geq \ell_{\alpha_0}/2 \quad \text{and} \quad b_{\alpha_0} - \bar{x}_{\alpha_0 - 1} \geq \ell_{\alpha_0 - 1}/2. \tag{73}$$

Step 4: The case $\alpha_0 \in I_0$. In this case, we see that there exists a time $\bar{t}$ such that the test function

$$\varphi(t,x) = \begin{cases} \bar{p}_{\alpha_0,R}(x - b_{\alpha_0}) & \text{for} \quad x \geq b_{\alpha_0}, \\ \bar{p}_{\alpha_0-1,L}(x - b_{\alpha_0}) & \text{for} \quad x \leq b_{\alpha_0} \end{cases}$$

is a test function touching (up to some additive constant) $\widehat{w}$ from below at $(\bar{t}, b_{\alpha_0})$. By (72), this implies

$$\bar{A}_0 = \max(\bar{H}_{\alpha_0}(\bar{p}_{\alpha_0,R}), \bar{H}_{\alpha_0-1}(\bar{p}_{\alpha_0-1,L})) \geq \widehat{A} \geq \bar{A}.$$

Contradiction.

Step 5: Consequences on $\widehat{w}$. From the fact that $\widehat{w}$ is 1-periodic in time and C-Lipschitz continuous in time (with a constant C depending only on $\max_{\alpha=1,\dots,N} \|\widehat{a}_\alpha\|_\infty$ and the $\bar{H}_\alpha$'s, see (49)), we deduce that we have

$$\begin{cases} \widehat{w}(t, x+h) - \widehat{w}(t,x) \geq \bar{p}_{\alpha,R}h - 2C & \text{for} \quad x, x+h \in (b_\alpha, \bar{x}_\alpha), \\ \widehat{w}(t, x-h) - \widehat{w}(t,x) \geq -\bar{p}_{\alpha,R}h - 2C & \text{for} \quad x, x+h \in (\bar{x}_\alpha, b_{\alpha+1}). \end{cases} \tag{74}$$

Step 6: The case $\alpha_0 \in I_1$: definition of a space-time super-solution. Proceeding similarly to Step 3 of the proof of (12), we define

$$\underline{\widehat{w}}_{\alpha_0,R}(t,x) = \inf_{\frac{\ell_{\alpha_0}}{4} \geq h \geq 0} \left(\widehat{w}(t, x+h) - \bar{p}^0_{\alpha_0} h \right) \quad \text{for} \quad b_{\alpha_0} \leq x \leq b_{\alpha_0} + \frac{\ell_{\alpha_0}}{4}$$

and

$$\underline{\widehat{w}}_{(\alpha_0-1),L}(t,x) = \inf_{\frac{\ell_{(\alpha_0-1)}}{4} \geq h \geq 0} \left(\widehat{w}(t, x-h) + \bar{p}^0_{\alpha_0-1} h \right) \quad \text{for} \quad b_{\alpha_0} - \frac{\ell_{\alpha_0-1}}{4} \leq x \leq b_{\alpha_0}.$$

From (74), we deduce that we have for some $\bar{h} \in [0, \frac{\ell_{\alpha_0}}{4}]$

$$\widehat{w}(t,x) \geq \underline{\widehat{w}}_{\alpha_0,R}(t,x) = \widehat{w}(t, x+\bar{h}) - \bar{p}^0_{\alpha_0}\bar{h} \geq \widehat{w}(t,x) + (\bar{p}_{\alpha_0,R} - \bar{p}^0_{\alpha_0})\bar{h} - 2C$$

which implies

$$0 \le \bar{h} \le \frac{2C}{\bar{p}_{\alpha_0,R} - \bar{p}^0_{\alpha_0}}.$$

As in Step 3 of the proof of (12), if

$$\frac{\ell_{\alpha_0}}{4} > \frac{2C}{\bar{p}_{\alpha_0,R} - \bar{p}^0_{\alpha_0}} \tag{75}$$

this implies that $\underline{\widehat{w}}_{\alpha_0,R}$ is a super-solution for $x \in (b_{\alpha_0}, b_{\alpha_0} + \frac{\ell_{\alpha_0}}{4})$. Similarly, if

$$\frac{\ell_{\alpha_0-1}}{4} > \frac{2C}{\bar{p}^0_{\alpha_0-1} - \bar{p}_{\alpha_0-1,L}} \tag{76}$$

then $\underline{\widehat{w}}_{\alpha_0-1,L}$ is a super-solution for $x \in (b_{\alpha_0} - \frac{\ell_{\alpha_0-1}}{4}, b_{\alpha_0})$. We now define

$$\underline{\widehat{w}}(t,x) = \begin{cases} \underline{\widehat{w}}_{\alpha_0,R}(t,x) & \text{if} \quad x \in (b_{\alpha_0}, b_{\alpha_0} + \frac{\ell_{\alpha_0}}{4}), \\ \underline{\widehat{w}}_{\alpha_0-1,L}(t,x) & \text{if} \quad x \in (b_{\alpha_0} - \frac{\ell_{\alpha_0-1}}{4}, b_{\alpha_0}), \\ \min(\underline{\widehat{w}}_{\alpha_0-1,L}(t,b_{\alpha_0}),\ \underline{\widehat{w}}_{\alpha_0,R}(t,b_{\alpha_0})) & \text{if} \quad x = b_{\alpha_0}. \end{cases}$$

Then as in Steps 4 and 5 of the proof (12), we deduce that $\underline{\widehat{w}}$ is a super-solution up to the junction point $x = b_{\alpha_0}$ and that

$$\bar{A}_0 = \langle \widehat{a}_{\alpha_0} \rangle \ge \widehat{A} \ge \bar{A}.$$

Contradiction.

Step 7: Conclusion. If (75) and (76) hold true for any $\alpha_0 \in I_1$, then we deduce that $\bar{A} \le \bar{A}_0$, which implies $\bar{A} = \bar{A}_0$. This ends the proof of (14) in Theorem 1.12. □

Proof of (15) *in Theorem 1.12.* Let us consider

$$\bar{a}(t) = \max_{\alpha=1,\dots,N} a_\alpha(t),$$

and $(w, \bar{\bar{A}})$ a solution (given by Theorem 4.6 (see also Lemma 5.2)) of

$$\begin{cases} w_t + \bar{H}_0(w_x) = \bar{\bar{A}} & \text{if} \quad x < 0, \\ w_t + \bar{H}_N(w_x) = \bar{\bar{A}} & \text{if} \quad x > 0, \\ w_t(t,0) + \max(\bar{a}(t), \bar{H}^-_N(w_x(t,0^+)), \bar{H}^+_0(w_x(t,0^-))) = \bar{\bar{A}} & \text{if} \quad x = 0, \\ w \quad \text{is 1-periodic with respect to } t. \end{cases}$$

From Theorem 1.12, we also know that

$$\bar{\bar{A}} = \langle \bar{a} \rangle.$$

For $N \geq 2$, we set $\ell = (\ell_1, \dots, \ell_{N-1}) \in (0, +\infty)^{N-1}$ and consider $b_0 = -\infty < b_1 < \cdots < b_N < b_{N+1} = +\infty$ with

$$\ell_\alpha = b_{\alpha+1} - b_\alpha \quad \text{for} \quad \alpha = 1, \dots N-1.$$

We now call $(w^\ell, \bar{A}^\ell)$ a global corrector given by Theorem 4.6 (see also Lemma 5.2). The remaining of the proof is divided into several steps.

Step 1: Bound from above on $\bar{A}^\ell$. We define

$$\tilde{w}(t,x) = \begin{cases} w(t, x - b_1) & \text{if} \quad x \leq b_1, \\ \\ w(t,0) + \bar{p}^0_\alpha (x - b_\alpha) + \displaystyle\sum_{\beta = 1, \dots, \alpha - 1} \bar{p}^0_\beta (b_{\beta+1} - b_\beta) & \text{if} \quad \left\{ \begin{array}{l} b_\alpha \leq x \leq b_{\alpha+1}, \\ \alpha \in \{1, \dots, N-1\}, \end{array} \right. \\ \\ w(t, x - b_N) + \displaystyle\sum_{\beta = 1, \dots, N-1} \bar{p}^0_\beta (b_{\beta+1} - b_\beta) & \text{if} \quad x \geq b_N. \end{cases}$$

Proceeding as in Step 1 of the proof of Theorem 1.12 ii), it is then easy to check that $\tilde{w}$ is a sub-solution of the equation satisfied by w^ℓ with $\bar{\bar{A}}$ on the right hand side instead of $\bar{A}^\ell$. Then Theorem 1.4 implies that

$$\bar{A}^\ell \leq \bar{\bar{A}} = \langle \bar{a} \rangle. \tag{77}$$

Step 2: Bound from below on $\bar{A}^\ell$. From Theorem 2.10 in [18], we deduce that we have in the viscosity sense (in time only)

$$w^\ell_t (t, b_\alpha) + a_\alpha(t) \leq \bar{A}^\ell \quad \text{for all} \quad t \notin \cup_{k=0}^K \{\tau_k + \mathbb{Z}\}.$$

Let us call

$$\underline{A} = \liminf_{\ell \to 0} \bar{A}^\ell.$$

We also know that w^ℓ is 1-periodic and globally Lipschitz continuous with a constant which is independent on ℓ. Therefore there exists a 1-periodic and Lipschitz continuous function $g = g(t)$ such that

$$w^\ell(t, b_\alpha) \to g(t) \quad \text{for all} \quad \alpha = 1, \dots, N, \quad \text{as} \quad \ell \to 0.$$

The stability of viscosity solutions implies in the viscosity sense

$$g'(t) + a_\alpha(t) \leq \underline{A}, \quad \text{for all} \quad \alpha = 1, \dots, N, \quad \text{for all} \quad t \notin \cup_{k=0}^K \{\tau_k + \mathbb{Z}\}.$$

Because g is Lipschitz continuous, this inequality also holds for almost every $t \in \mathbb{R}$. This implies

$$g'(t) + \bar{a}(t) \leq \underline{A} \quad \text{for a.e.} \quad t \in \mathbb{R}.$$

An integration on one period gives

$$\langle \bar{a} \rangle \leq \underline{A}. \tag{78}$$

Step 3: Conclusion. Combining (77) with (78) finally yields that $\bar{A}^\ell \to \langle \bar{a} \rangle$ as $\ell \to 0$. The proof of (15) in Theorem 1.12 is now complete. □

Acknowledgements. The authors thank the referees for their valuable comments. The authors thank Y. Achdou, K. Han and N. Tchou for stimulating discussions. The authors thank N. Seguin for interesting discussions on green waves. C. I. thanks Giga for the interesting discussions they had together and for drawing his attention towards papers such as [15]. R. M. thanks G. Costeseque for his comments on traffic lights modeling and his simulations which inspired certain complementary results. The second and third authors are partially supported by ANR-12-BS01-0008-01 HJnet project.

A Proofs of some technical results

A.1 The case $\bar{x} \neq 0$ in the proof of convergence

The case $\bar{x} \neq 0$ in the proof of Theorem 1.7. We only deal with the subcase $\bar{x} > 0$ since the subcase $\bar{x} < 0$ is treated in the same way. Reducing $\overline{r}$ if necessary, we may assume that $B_{\overline{r}}(\overline{t}, \overline{x})$ is compactly embedded in the set $\{(t,x) \in (0,+\infty) \times (0,+\infty) : x > 0\}$: there exists a positive constant $c_{\overline{r}}$ such that

$$(t,x) \in B_{\overline{r}}(\overline{t}, \overline{x}) \quad \Rightarrow \quad x > c_{\overline{r}}\,. \tag{79}$$

Let $p = \varphi_x(\overline{t}, \overline{x})$ and let $v^R = v^R(t,x)$ be a solution of the cell problem

$$v^R_t + H_R\left(t, x, p + v^R_x\right) = \bar{H}_R(p) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}\,. \tag{80}$$

We claim that if $\varepsilon > 0$ is small enough, the perturbed test function [9]

$$\varphi^\varepsilon(t,x) = \varphi(t,x) + \varepsilon v^R\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$$

satisfies, in the viscosity sense, the inequality

$$\varphi^\varepsilon_t + H\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \varphi^\varepsilon_x\right) \geq \frac{\theta}{2} \quad \text{in} \quad B_r(\overline{t}, \overline{x}) \tag{81}$$

for sufficiently small $r > 0$. To see this let ψ be a test function touching φ^ε from below at $(t_1, x_1) \in B_r(\overline{t}, \overline{x}) \subseteq B_{\overline{r}}(\overline{t}, \overline{x})$. In this way the function

$$\eta(s,y) = \frac{1}{\varepsilon}\left(\psi(\varepsilon s, \varepsilon y) - \varphi(\varepsilon s, \varepsilon y)\right)$$

touches v^R from below at $(s_1, y_1) = \left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}\right)$ and (80) yields

$$\psi_t(t_1,x_1) - \varphi_t(t_1,x_1) + H_R\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}, p + \psi_x(t_1,x_1) - \varphi_x(t_1,x_1)\right) \geq \bar{H}_R(p). \tag{82}$$

Since (79) implies that $\frac{x}{\varepsilon} \to +\infty$, as $\varepsilon \to 0$, uniformly with respect to $(t,x) \in B_{\overline{r}}(\overline{t}, \overline{x})$, we can find, owing to **(A5)**, an $\varepsilon_0 > 0$ independent of ψ and (t_1, x_1) such that the inequality

$$H\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}, \psi_x(t_1,x_1)\right) \geq H_R\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}, \psi_x(t_1,x_1)\right) - \frac{\theta}{4} \tag{83}$$

holds true for $\varepsilon < \varepsilon_0$. Combining (19)-(82)-(83) and using the continuity of φ_x and φ_t we have

$$\begin{aligned}
&\psi_t(t_1,x_1) + H\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}, \psi_x(t_1,x_1)\right) \\
&\geq \psi_t(t_1,x_1) + H_R\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}, p + \psi_x(t_1,x_1) - \varphi_x(t_1,x_1)\right) + \\
&\quad H_R\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}, \psi_x(t_1,x_1)\right) - H_R\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}, \varphi_x(\bar{t},\bar{x}) + \psi_x(t_1,x_1) - \varphi_x(t_1,x_1)\right) - \frac{\theta}{4} \\
&\geq \frac{\theta}{2}
\end{aligned}$$

if r is sufficiently close to 0. The claim (81) is proved.

Since φ is strictly above $\overline{u}$, if ε and r are small enough

$$u^\varepsilon + \kappa_r \leq \varphi^\varepsilon \quad \text{on} \quad \partial B_r(\overline{t}, \overline{x})$$

for a suitable positive constant κ_r. By comparison principle we deduce

$$u^\varepsilon + \kappa_r \leq \varphi^\varepsilon \quad \text{in} \quad B_r(\overline{t}, \overline{x})$$

and passing to the limit as $\varepsilon \to 0$ and $(t,x) \to (\bar{t}, \bar{x})$ on both sides of the previous inequality, we produce the contradiction

$$\overline{u}(\overline{t}, \overline{x}) < \overline{u}(\overline{t}, \overline{x}) + \kappa_r \leq \varphi(\overline{t}, \overline{x}) = \overline{u}(\overline{t}, \overline{x}) \,. \qquad \square$$

A.2 Proof of Lemma 3.3

Proof of Lemma 3.3. We first adress uniqueness. Let us assume that we have two solutions (v^i, λ^i) for $i = 1, 2$ of (3). Let

$$u^i(t,x) = v^i(t,x) + px - \lambda^i t$$

Then u^i solves

$$u^i_t + H_\alpha(t, x, u^i_x) = 0$$

with

$$u^1(0,x) \leq u^2(0,x) + C$$

The comparison principle implies

$$u^1 \leq u^2 + C \quad \text{for all} \quad t > 0$$

and then $\lambda^1 \geq \lambda^2$. Similarly we get the reverse inequality and then $\lambda^1 = \lambda^2$.

We now turn to the continuity of the map $p \mapsto \bar{H}_\alpha(p)$. It follows from the stability of viscosity sub- and super-solutions, from the fact that the constant C in (24) is bounded for bounded p's and from the comparison principle. This achieves the proof of the lemma. $\square$

A.3 Sketch of the proof of Proposition 4.1

Sketch of the proof of Proposition 4.1. Consider

$$M_\nu = \sup_{x\in[\rho_1,\rho_2],s,t\in\mathbb{R}} \left\{ u(t,x) - v(s,x) - \frac{(t-s)^2}{2\nu} \right\}.$$

We want to prove that

$$M = \lim_{\nu\to 0} M_\nu \le 0.$$

We argue by contradiction by assuming that $M > 0$. The supremum defining M_ν is reached, let s_ν, t_ν, x_ν denote a maximizer. Choose ν small enough so that $M_\nu \ge \frac{M}{2} > 0$. We classically get,

$$|t_\nu - s_\nu| \le C\sqrt{\nu}.$$

If there exists $\nu_n \to 0$ such that $x_{\nu_n} = \rho_1$ for all $n \in \mathbb{N}$, then

$$\frac{M}{2} \le M_{\nu_n} \le U_0(t_{\nu_n}) - U_0(s_{\nu_n}) \le \omega_0(t_{\nu_n} - s_{\nu_n}) \le \omega_0(C\sqrt{\nu_n})$$

where ω_0 denotes the modulus of continuity of U_0. The contradiction $M \le 0$ is obtained by letting n go to $+\infty$.

Hence, we can assume that for ν small enough, $x_\nu > \rho_1$. Reasoning as in [18, Theorem 7.8], we can easily reduce to the case where $H(t_\nu, x_\nu, \cdot)$ reaches its minimum for $p = p_0 = 0$. We can also consider the vertex test function G^γ associated with the single Hamiltonian H (using notation of [18], it corresponds to the case $N = 1$) and the free parameter γ. If $x_\nu < \rho_2$, then $G^\gamma(x,y)$ reduces to the standard test function $\frac{(x-y)^2}{2}$.

We next consider

$$M_{\nu,\varepsilon} = \sup_{\substack{x,y\in[\rho_1,\rho_2]\cap\overline{B_r(x_\nu)} \\ s,t\in\mathbb{R}}} \left\{ u(t,x) - v(s,y) - \frac{(t-s)^2}{2\nu} - \varepsilon G^\gamma(\varepsilon^{-1}x, \varepsilon^{-1}y) - \varphi^\nu(t,s,x) \right\}$$

where $r = r_\nu$ is chosen so that $\rho_1 \notin \overline{B_r(x_\nu)}$ and the localization function

$$\varphi^\nu(t,s,x) = \frac{1}{2}((t-t_\nu)^2 + (s-s_\nu)^2 + (x-x_\nu)^2).$$

The supremum defining $M_{\nu,\varepsilon}$ is reached and if (t,s,x,y) denotes a maximizer, then

$$(t,s,x,y) \to (t_\nu, s_\nu, x_\nu, x_\nu) \quad \text{as } (\varepsilon,\gamma) \to 0.$$

In particular, $x, y \in B_r(x_\nu)$ for ε and γ small enough. The remaining of the proof is completely analogous (in fact much simpler). $\square$

A.4 Construction of λ_ρ in the proof of Lemma 5.2

In order to get λ_ρ, it is enough to apply the following lemma.

Lemma A.1. *Let u be the solution of a Hamilton-Jacobi equation of evolution-type submitted to the initial condition: $u(0,x) = 0$ and posed on a compact set K. Assume that*

- *the comparison principle holds true;*
- *u is L-globally Lipschitz continuous in time and space;*
- *$u(k + \cdot, \cdot) + C$ is a solution for all $k \in \mathbb{N}$ and $C \in \mathbb{R}$.*

There then exists $\lambda \in \mathbb{R}$ such that

$$|u(t,x) - \lambda t| \leq C_0$$

and

$$|\lambda| \leq L$$

where $C_0 = L(2 + 3\rho)$ if ρ denotes the diameter of K.

Proof. Define

$$\lambda^+(T) = \sup_{\tau \geq 0} \frac{u(\tau + T, 0) - u(\tau, 0)}{T} \quad \text{and} \quad \lambda^-(T) = \inf_{\tau \geq 0} \frac{u(\tau + T, 0) - u(\tau, 0)}{T}.$$

Remark that $T \mapsto \pm T\lambda^\pm(T)$ is sub-additive. Remark that the fact that u is L-Lipschitz continuous with respect to time implies that $\lambda^\pm(T)$ are both finite:

$$|\lambda^\pm(T)| \leq L.$$

the ergodic theorem implies that $\lambda^\pm(T)$ converges towards $\lambda^\pm$ and

$$\lambda^+ = \inf_{T>0} \lambda^+(T) \quad \text{and} \quad \lambda^- = \sup_{T>0} \lambda^-(T).$$

If moreover

$$|\lambda^+(T) - \lambda^-(T)| \leq \frac{C}{T}, \tag{84}$$

then the proof of the lemma is complete. Indeed, (84) implies in particular that $\lambda^+ = \lambda^-$ and

$$-\frac{C}{T} \leq \lambda^-(T) - \lambda \leq \lambda^+(T) - \lambda \leq \frac{C}{T}.$$

This implies that

$$|u(t,0) - \lambda t| \leq C.$$

Finally, we get

$$|u(t,x) - \lambda t| \leq C + L\rho.$$

It remains to prove (84). There exists $k \in \mathbb{Z}$ and $\beta \in [0, 1)$ such that $\tau^+ = k + \tau^- + \beta$. Moreover,

$$u(\tau^+, x) \le u(\tau^- + \beta, x) + u(\tau^+, 0) - u(\tau^- + \beta, 0) + 2L\rho$$

where $\rho = \text{diam } K$. Now remark that $u(\tau^- + \beta + t, x) + D$ is a solution in $[\tau^+, +\infty)$ for all constant D. Hence, we get by comparison that for all $t > 0$ and $x \in K$,

$$u(\tau^+ + t, x) \le u(\tau^- + \beta + t, x) + u(\tau^+, 0) - u(\tau^- + \beta, 0) + 2L\rho.$$

In particular,

$$\begin{aligned} u(\tau^+ + T, 0) - u(\tau^+, 0) &\le u(\tau^- + \beta + T, 0) - u(\tau^- + \beta, 0) + 2L\rho \\ &\le u(\tau^- + T, 0) - u(\tau^-, 0) + 2L(1 + \rho). \end{aligned}$$

Finally, we get (after letting $\varepsilon \to 0$),

$$\lambda^+(T) \le \lambda^-(T) + \frac{2L(1+\rho)}{T}.$$

Similarly, we can get

$$\lambda^+(T) \ge \lambda^-(T) - \frac{2L(1+\rho)}{T}.$$

This implies (84) with $C = 2L(1 + \rho)$. The proof of the lemma is now complete. $\square$

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