Gog and GOGAm pentagons

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GOG AND GOGAM PENTAGONS

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Abstract. We consider the problem of finding a bijection between the sets of alternating sign matrices and of totally symmetric self complementary plane partitions, which can be reformulated using Gog and Magog triangles. In a previous work we introduced GOGAm triangles, which are images of Magog triangles by the Schützenberger involution. In this paper we introduce Gog and GOGAm pentagons. We conjecture that they are equienumerated. We provide some numerical evidence as well as an explicit bijection in the case when they have one or two diagonals. We also consider some interesting statistics on Gog and Magog triangles.

1. Introduction

Finding a bijection between the set of alternating sign matrices and the set of totally symmetric self complementary plane partitions is a well known open problem in combinatorics. One can reformulate this problem using so-called Gog and Magog triangles, which are particular species of Gelfand-Tsetlin triangles. For example below are a Gog and a Magog triangle of size 4 (precise definitions are recalled below):

$$\begin{array}{ccccccc} 1 & & 2 & & 3 & & 4 \\ & 1 & & 3 & & 4 & \\ & & 2 & & 3 & & \\ & & & 3 & & & \end{array} \qquad \begin{array}{ccccccc} 1 & & 1 & & 2 & & 4 \\ & 1 & & 2 & & 2 & \\ & & 1 & & 2 & & \\ & & & 1 & & & \end{array}$$

In particular, Gog triangles are in simple bijection with alternating sign matrices of the same size, while Magog triangles are in bijection with totally symmetric self complementary plane partitions. In [9], Mills, Robbins and Rumsey introduced trapezoids in this problem by considering k diagonals on the right of a triangle of size n, as below, where we show a Magog trapezoid:

$$\begin{array}{ccccccccc} * & & * & & * & & 2 & & 3 \\ & * & & * & & 1 & & 3 & \\ & & * & & 1 & & 2 & & \\ & & & 1 & & 2 & & & \\ & & & & 1 & & & & \end{array}$$

They conjectured that Gog and Magog trapezoids of the same shape are equienumerated. Some related conjectures can be found in [8]. Zeilberger [12] proved the conjecture of Mills, Robbins and Rumsey, but no explicit bijection is known, except for $k = 1$ (which is a relatively easy problem) and for $k = 2$, this bijection being the main result of [4]. In this last paper a new class of triangles and trapezoids was introduced, called GOGAm triangles (or trapezoids),

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which are in bijection with the Magog triangles by the Schützenberger involution acting on Gelfand-Tsetlin triangles. In this paper we introduce Gog and GOGAm pentagons, obtained by taking the intersection of some leftmost and rightmost diagonals and some bottom rows of Gog and GOGAm triangles, as below

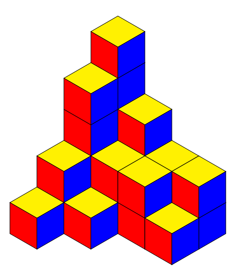
$$\begin{array}{cccccccc}
 & & 4 & & 5 & & 6 & \\
 & 3 & & 5 & & 6 & & 7 \\
2 & & 3 & & 5 & & 6 & \\
 & 3 & & 5 & & 6 & & \\
 & & 4 & & 5 & & & \\
 & & & 4 & & & &
\end{array}$$

We conjecture that the Gog and GOGAm pentagons of the same shape are equienumerated. Actually we formulate a refined conjecture where the bottom entries of the pentagons coincide. This generalizes the conjecture of Mills Robbins and Rumsey. We give some numerical evidence for this conjecture and give a bijective proof for several shapes of pentagons, for example for pentagons composed of one or two diagonals. We will also consider a new statistic (which we call the β-statistic) on the different species of triangles, which is of independent interest.

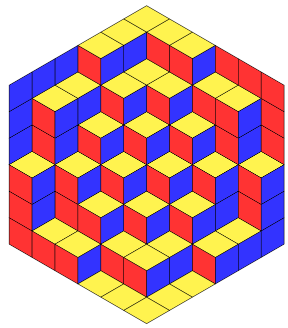
This paper is organized as follows: in section 2 we recall basic definitions, then in section 3 we introduce Gog and Magog triangles and trapezoids, as well as GOGAm triangles and some new statistic on them. In section 4 we introduce the main objects of the paper, which are pentagons, and we formulate the main conjecture for which we give some numerical evidence. Finally in section 5 we give some partial results towards the conjecture.

2. Basic definitions

2.1. **Totally Symmetric Self-Complementary Plane Partitions and Alternating Sign Matrices.** A *Plane Partition* is a stack of cubes in a corner.



Choosing a large cube that contains a plane partition, one can also encode the partition as a lozenge tiling of an hexagon. A *Totally Symmetric Self-Complementary Plane Partition* (TSSCPP) of size n is a plane partition, inside a cube of side $2n$, such that the lozenge tiling has all the symmetries of the hexagon and the partition is the same as its complement inside the cube, as in the picture below, where $n = 3$.



An *Alternating Sign Matrix* (ASM) is a square matrix with entries in $\{-1, 0, +1\}$ such that, on each line and on each column, the non zero entries alternate in sign, the sum of each line and each column being equal to 1. Below is an alternating sign matrix of size 5.

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

It is well known [1] that the number of TSSCPP of size n is

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = 1, 2, 7, 42, 429, \dots \tag{2.1}$$

As first proved by Zeilberger [12], this is also the number of ASM of size n (more about this story is in [5]), however no explicit bijection between these classes of objects has ever been constructed, and finding one is a major open problem in combinatorics. In this paper we use Gog and Magog triangles, defined below, in order to investigate this question.

2.2. **Gelfand-Tsetlin triangles.**

2.2.1.

Definition 2.1. *A Gelfand-Tsetlin triangle of size n is a triangular array $X = (X_{i,j})_{n \geqslant i \geqslant j \geqslant 1}$ of positive integers*

$$\begin{array}{ccccccccc} X_{n,1} & & X_{n,2} & & \dots & & X_{n,n-1} & & X_{n,n} \\ & X_{n-1,1} & & X_{n-1,2} & & \dots & & X_{n-1,n-1} & \\ & & \dots & & \dots & & \dots & & \\ & & & X_{2,1} & & X_{2,2} & & & \\ & & & & X_{1,1} & & & & \end{array}$$

such that

$$X_{i+1,j} \leqslant X_{i,j} \leqslant X_{i+1,j+1} \quad \textit{for } n-1 \geqslant i \geqslant j \geqslant 1.$$

For example here is a Gelfand-Tsetlin triangle of size 5:

$$\begin{array}{ccccccccc} 2 & & 2 & & 7 & & 11 & & 15 \\ & 2 & & 6 & & 8 & & 13 & \\ & & 6 & & 6 & & 9 & & \\ & & & 6 & & 7 & & & \\ & & & & 7 & & & & \end{array}$$

2.2.2. *Schützenberger involution.* Gelfand-Tsetlin triangles label bases of irreducible representations of general linear groups. As such, they are in simple bijection with semi-standard Young tableaux (SSYT). It follows that the Schützenberger involution, which is defined on SSYTs, can be transported to Gelfand-Tsetlin triangles. We recall the definition of this involution here. Since we will not use SSYTs in this paper we give only a brief sketch and we insist on the definition of the involution directly on Gelfand-Tsetlin triangles. Let A be a finite totally ordered alphabet. The Robinson-Schensted correspondance is a bijection between words and pairs of Young tableaux of the same shape, more precisely, it assigns to every word w of length n on A a pair $(P(w), Q(w))$ formed of a semi-standard Young tableau P of size n, and a standard tableau Q of the same shape. Let $a \mapsto \bar{a}$ be the order reversing involution on A, then the Schützenberger involution on words maps a word $w = a_1 \dots a_n$ to $S(w) = \bar{a}_n \dots \bar{a}_1$. It turns out that for two words w_1 and w_2 with $P(w_1) = P(w_2)$ one has $P(S(w_1)) = P(S(w_2))$, therefore S descends to an involution $S : P \mapsto S(P)$ on semi-standard Young tableaux, moreover this involution preserves the shapes of the tableaux. It is much less easy to describe the map S directly on Gelfand-Tsetlin triangles, however this has been done by Berenstein and Kirillov [3]. First define involutions s_k, for $k \leqslant n-1$, acting on the set of Gelfand-Tsetlin triangles of size n. If $X = (x_{i,j})_{n\geqslant i\geqslant j\geqslant 1}$ is such a triangle the action of s_k on X is given by $s_k X = (\tilde{X}_{i,j})_{n\geqslant i\geqslant j\geqslant 1}$ with

$$\begin{array}{rcl} \tilde{X}_{i,j} & = & X_{i,j}, \qquad \text{if } i \neq k \\ \tilde{X}_{k,j} & = & \max(X_{k+1,j}, X_{k-1,j-1}) + \min(X_{k+1,j+1}, X_{k-1,j}) - X_{i,j} \end{array}$$

It is understood that $\max(a, b) = \max(b, a) = a$ and $\min(a, b) = \min(b, a) = a$ if the entry b of the triangle is not defined. The geometric meaning of the transformation of an entry is the following: on row k, any entry $X_{k,j}$ is surrounded by four (or less if it is on the boundary) numbers, increasing from left to right.

$$\begin{array}{ccc} X_{k+1,j} & & X_{k+1,j+1} \\ & X_{k,j} & \\ X_{k-1,j-1} & & X_{k-1,j} \end{array}$$

These four numbers determine a smallest interval containing $X_{k,j}$, namely

$$[\max(X_{k+1,j}, X_{k-1,j-1}), \min(X_{k+1,j+1}, X_{k-1,j})]$$

and the transformation maps $X_{k,j}$ to its mirror image with respect to the center of this interval.

Define $\omega_j = s_j s_{j-1} \dots s_2 s_1$.

Definition 2.2. *The Schützenberger involution, acting on Gelfand-Tsetlin triangles of size n, is given by the formula*

$$S = \omega_1 \omega_2 \dots \omega_{n-1}.$$

It is a non trivial result that S is an involution and coincides with the Schützenberger involution when transported to SSYTs, see [3]; beware that the s_k *do not* satisfy the braid relations. As an example, below is a Gelfand-Tsetlin triangle X of size 4, together with its images by the successive maps $s_1, s_2s_1, s_3s_2s_1, s_1s_3s_2s_1, s_2s_1s_3s_2s_1$ and $S = s_1s_2s_1s_3s_2s_1$.

$$X = \begin{array}{ccccccc} 1 & & 3 & & 3 & & 5 \\ & 1 & & 3 & & 4 & \\ & & 2 & & 4 & & \\ & & & 3 & & & \end{array} \qquad s_1(X) = \begin{array}{ccccccc} 1 & & 3 & & 3 & & 5 \\ & 1 & & 3 & & 4 & \\ & & 2 & & 4 & & \\ & & & 3 & & & \end{array}$$

$$s_2s_1(X) = \begin{array}{ccccccc} 1 & & 3 & & 3 & & 5 \\ & 1 & & 3 & & 4 & \\ & & 2 & & 3 & & \\ & & & 3 & & & \end{array} \qquad s_3s_2s_1(X) = \begin{array}{ccccccc} 1 & & 3 & & 3 & & 5 \\ & 2 & & 3 & & 4 & \\ & & 2 & & 3 & & \\ & & & 3 & & & \end{array}$$

$$s_1s_3s_2s_1(X) = \begin{array}{ccccccc} 1 & & 3 & & 3 & & 5 \\ & 2 & & 3 & & 4 & \\ & & 2 & & 3 & & \\ & & & 2 & & & \end{array} \qquad s_2s_1s_3s_2s_1(X) = \begin{array}{ccccccc} 1 & & 3 & & 3 & & 5 \\ & 2 & & 3 & & 4 & \\ & & 2 & & 4 & & \\ & & & 2 & & & \end{array}$$

$$S(X) = \begin{array}{ccccccc} 1 & & 3 & & 3 & & 5 \\ & 2 & & 3 & & 4 & \\ & & 2 & & 4 & & \\ & & & 4 & & & \end{array}$$

3. Gog, Magog and GOGAm triangles

3.1. **Gog triangles.**

Definition 3.1. *A Gog triangle of size n is a Gelfand-Tsetlin triangle such that*

(i) $$X_{n,j} = j, \qquad 1 \leqslant j \leqslant n$$

and

(ii) $$X_{i,j} < X_{i,j+1}, \qquad j < i \leqslant n-1$$
in other words, such that its rows are strictly increasing.

Here is an example:

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & 4 & & 5 \\ & 1 & & 3 & & 4 & & 5 & \\ & & 1 & & 4 & & 5 & & \\ & & & 2 & & 4 & & & \\ & & & & 3 & & & & \end{array}$$

There is a simple bijection between Gog triangles and alternating sign matrices (see e.g. [5]). If $(M_{ij})_{1\leqslant i,j\leqslant n}$ is an ASM of size n, then the matrix $\bar{M}_{ij} = \sum_{k=i}^{n} M_{kj}$ has exactly $i-1$ entries 0 and $n-i+1$ entries 1 on row i. Let $(X_{ij})_{j=1,\dots,i}$ be the columns (in increasing order) with a 1 entry of $\bar{M}$ on row $n-i+1$. The triangle $X = (X_{ij})_{n\geqslant i\geqslant j\geqslant 1}$ is the Gog triangle corresponding to M. For example, the following matrix M is the alternating sign matrix of size 5 corresponding to the above Gog triangle. We also show the matrix $\bar{M}$.

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \bar{M} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

3.2. **Magog triangles.**

Definition 3.2. *A Magog triangle of size n is a Gelfand-Tsetlin triangle such that $X_{jj} \leqslant j$ for all $1 \leqslant j \leqslant n$.*

There is a well known bijection between Magog triangles and TSSCPP of the same size (see e.g. [5]). The problem of finding an explicit bijection between ASM and TSSCPP can therefore be reduced to that of finding an explicit bijection between Gog and Magog triangles.

3.3. **Statistics.**

We will define several statistics on the sets of Gelfand-Tsetlin triangles.

3.3.1. *The α statistics.*

For a triangle X of size n we define

$$\alpha_{Gog}(X) = X_{1,1}, \text{ the bottom entry of the triangle}$$

$$\alpha_{Magog}(X) = \textstyle\sum_1^n X_{n,i} - \sum_1^{n-1} X_{n-1,i}.$$

It is a well known property of the Schützenberger involution that, for all Gelfand-Tsetlin triangles, $\alpha_{Gog}(X) = \alpha_{Magog}(S(X))$ and, of course, $\alpha_{Gog}(S(X)) = \alpha_{Magog}(X)$.

3.3.2. *The β statistics.*

For Gelfand-Tsetlin triangle X, of size n, we define

$$\beta_{Gog}(X) = \text{ the number of indices } k \text{ such that } X_{k,k} = n.$$

The definition of β_{Magog} is more involved. Let k be the largest integer such that $X_{k,k} = k$. If there is no such k, then $\beta_{Magog}(X) = 0$. If k exists then we define a sequence of pairs $(i_l, j_l)_{1 \leqslant l \leqslant k}$ by $(i_1, j_1) = (k, k)$ and, if $k > 1$,

$(i_{l+1}, j_{l+1}) = (i_l, j_l - 1)$ if $X_{i_l, j_l - 1} = X_{i_l - 1, j_l - 1}$,

$(i_{l+1}, j_{l+1}) = (i_l - 1, j_l - 1)$ if $X_{i_l, j_l - 1} < X_{i_l - 1, j_l - 1}$

Since j decreases by 1 at each step, one has $j_l = k - l + 1$ and the sequence ends at step k, when $j_k = 1$. We define

$$\beta_{Magog}(X) = i_k.$$

Each step of the above algorithm can be described as follows: suppose that the current entry with index (i_l, j_l) is a and consider the two neighbouring entries in the triangle as below:

$$\begin{array}{ccc} b & & a \\ & c & \end{array}$$

If $b < c$ then the next entry is c as below

$$\begin{array}{ccc} b & & a \\ & c & \end{array}$$

whereas is $b = c$ the next entry is the upper entry b

$$\begin{array}{ccc} b & & a \\ & b & \end{array}$$

As an illustration, the following triangle has $k = 5$ and $\beta_{Magog}(X) = 3$. The sequence of indices obtained by applying the above algorithm is $(5,5), (4,4), (4,3), (3,2), (3,1)$. The corresponding entries are $5, 3, 3, 2, 1$, they are highlighted on the picture.

$$\begin{array}{ccccccccccc}
1 & & 1 & & 2 & & 2 & & 4 & & 5 \\
& 1 & & 1 & & 2 & & 3 & & 5 & \\
& & 1 & & 1 & & 3 & & 3 & & \\
& & & 1 & & 2 & & 3 & & & \\
& & & & 1 & & 2 & & & & \\
& & & & & 1 & & & & &
\end{array}$$

3.3.3. *A conjecture.* It is easy to see that, on Gog (resp. Magog) triangles of size n, the statistics α_{Gog} and β_{Gog} (resp. α_{Magog} and β_{Magog}) take values in $[1, n]$. If we identify Gog triangles with alternating sign matrices, then the two statistics $\alpha_{Gog}, \beta_{Gog}$ correspond to the position of the 1 in, respectively, the bottom row and the rightmost column. It is known that the α statistic is equienumerated on Gog and Magog triangles, i.e. for all k the number of Gog triangles with $\alpha_{Gog}(X) = k$ is equal to the number of Magog triangles with $\alpha_{Magog}(X) = k$.

Conjecture 3.3. *For any n the two statistics α, β are equienumerated on, respectively, Gog and Magog triangles.*

We have verified this conjecture numerically for triangles of sizes up to $n = 14$. For example, the distribution of the two statistics on triangles of size 8 is given in the following table:

	8	7	6	5	4	3	2	1
8	218348	0	0	0	0	0	0	0
7	0	218348	210912	184886	137566	80782	33462	7436
6	0	210912	377208	444548	390104	253214	111956	26026
5	0	184886	444548	620256	604890	421486	196014	47320
4	0	137566	390104	604890	635180	467740	227136	56784
3	0	80782	253214	421486	467740	360880	182546	47320
2	0	33462	111956	196014	227136	182546	96252	26026
1	0	7436	26026	47320	56784	47320	26026	7436

The distribution is symmetric in α and β, as is clear from the interpretation in terms of Alternating Sign Matrices.

3.4. **Trapezoids.** Trapezoids where introduced by Mills, Robbins and Rumsey [9], they are obtained by taking the k rightmost SW-NE diagonals in a triangle. For example, below is a Magog trapezoid with $n = 5$ and $k = 2$.

$$\begin{array}{cccccc}
& & & 2 & & 3 \\
& & 2 & & 3 & \\
& 1 & & 2 & & \\
1 & & 2 & & & \\
& 1 & & & &
\end{array}$$

A Magog trapezoid can be completed into a Magog triangle by putting a triangle of 1's to its left in order to obtain a Magog triangle as follows

$$\begin{array}{ccccccccc}
1 & & 1 & & 1 & & \mathbf{2} & & \mathbf{3} \\
 & 1 & & 1 & & \mathbf{2} & & \mathbf{3} & \\
 & & 1 & & \mathbf{1} & & \mathbf{2} & & \\
 & & & \mathbf{1} & & \mathbf{2} & & & \\
 & & & & \mathbf{1} & & & &
\end{array}$$

Similarly a Gog trapezoid can be completed by putting to its left the smallest Gog triangle, which is made of constant NW-SE diagonals with respective entries $1, 2, 3, \ldots$

$$\begin{array}{ccccccccc}
1 & & 2 & & 3 & & \mathbf{4} & & \mathbf{5} \\
 & 1 & & 2 & & \mathbf{4} & & \mathbf{5} & \\
 & & 1 & & \mathbf{2} & & \mathbf{4} & & \\
 & & & \mathbf{2} & & \mathbf{3} & & & \\
 & & & & \mathbf{2} & & & &
\end{array}$$

In this way Gog (or Magog) trapezoids can be seen as forming a subset of Gog (or Magog) triangles. This leads us to a refined version of the conjecture.

Conjecture 3.4. *For any n, k the two statistics α, β are equienumerated on, respectively, Gog and Magog trapezoids of shape (n, k).*

Again, we have checked this conjecture numerically for all values of $n, k \leqslant 14$.

3.5. **GOGAm triangles.**

Definition 3.5. *A GOGAm triangle of size n is a Gelfand-Tsetlin triangle whose image by the Schützenberger involution is a Magog triangle (of size n).*

Of course, the problem of finding an explicit bijection between Gog and Magog triangles can be reduced to that of finding an explicit bijection between Gog and GOGAm triangles. It is shown in [4] that the GOGAm triangles are the Gelfand-Tsetlin triangles $X = (X_{i,j})_{n \geqslant i \geqslant j \geqslant 1}$ such that $X_{nn} \leqslant n$ and, for all $1 \leqslant k \leqslant n-1$ and, for all $n = j_0 > j_1 > j_2 \ldots > j_{n-k} \geqslant 1$, one has

$$\left(\sum_{i=0}^{n-k-1} X_{j_i+i, j_i} - X_{j_{i+1}+i, j_{i+1}} \right) + X_{j_{n-k}+n-k, j_{n-k}} \leqslant k. \tag{3.1}$$

It is easier to describe this last condition with a picture. Consider a Gelfand-Tsetlin triangle as below and the highlighted entries a, b, c, d, e, f, g: the first one is the upper right corner, the second one is below on the right most SW-NE diagonal, the next one is just one step to the NW, the next is below on the second rightmost SW-NE diagonal, etc. The quantity to evaluate in equation (3.1) is $a - b + c - d + e - f + g$, the sum of the local maxima minus the sum of the local minima of the zigzag path thus created. In order that the triangle be a GOGAm triangle, this quantity has to be at most k where k is the index of the SW-NE diagonal of the last entry of the path (numbering them from the left), which is 5 in our example.

$$
\begin{array}{ccccccccccccccc}
* & & * & & * & & * & & * & & * & & * & & a \\
& * & & * & & * & & * & & * & & c & & * & \\
& & * & & * & & * & & e & & * & & b & & \\
& & & * & & * & & * & & d & & * & & & \\
& & & & g & & * & & * & & * & & & & \\
& & & & & f & & * & & * & & & & & \\
& & & & & & * & & * & & & & & & \\
& & & & & & & * & & & & & & &
\end{array}
$$

One can define the α_{GOGAm} and β_{GOGAm} statistics by

$$\alpha_{GOGAm}(X) = \alpha_{Magog}(S(X)), \qquad \beta_{GOGAm}(X) = \beta_{Magog}(S(X)).$$

It follows from well-known properties of the Schützenberger involution, recalled in section 3.3.1, that

$$\alpha_{GOGAm}(X) = \alpha_{Gog}(X) = X_{1,1}.$$

As with Gog and Magog we can define GOGAm trapezoids by taking diagonals on the right, and we can embed them into GOGAm triangles by adding a triangle of 1's on the upper left. This is compatible with the embedding of Magog trapezoids into Magog triangles: if a Magog triangle is obtained from a Magog trapezoid as in

$$
\begin{array}{ccccccccc}
1 & & 1 & & 1 & & \mathbf{2} & & \mathbf{3} \\
& 1 & & 1 & & \mathbf{2} & & \mathbf{3} & \\
& & 1 & & \mathbf{1} & & \mathbf{2} & & \\
& & & \mathbf{1} & & \mathbf{2} & & & \\
& & & & \mathbf{1} & & & &
\end{array}
$$

then the triangle of 1's on the upper left of the Magog trapezoid remains invariant when applying the Schützenberger involution, therefore we obtain a GOGAm triangle which comes from a GOGAm trapezoid.

4. Pentagons

We now introduce the main object of our paper.

Definition 4.1. *For positive integers k, l, m, n, satisfying*

$$\max(k, l, m) \leqslant n \qquad \max(k, l) \leqslant m \leqslant k + l - 1$$

a (k, l, m, n) Gog (resp. GOGAm) pentagon is an array of positive integers $X = (X_{i,j})$ with $\min(m, k + j - 1) \geqslant i \geqslant j \geqslant 1; l \geqslant j$, extracted from a Gog (resp. GOGAm) triangle of size n.

In other words, a (n, k, l, m) pentagon is obtained by keeping the intersection of the k right most SW-NE diagonals, the l leftmost SE-NW diagonals, and the m bottom lines of a triangle of size n. For example, the following is a $(4, 5, 6, 7)$ Gog pentagon.

$$\begin{array}{cccccccc}
 & & \mathbf{4} & & \mathbf{5} & & \mathbf{6} & \\
 & \mathbf{3} & & \mathbf{5} & & \mathbf{6} & & \mathbf{7} \\
\mathbf{2} & & \mathbf{3} & & \mathbf{5} & & \mathbf{6} & \\
 & \mathbf{3} & & \mathbf{5} & & \mathbf{6} & & \\
 & & \mathbf{4} & & \mathbf{5} & & & \\
 & & & \mathbf{4} & & & &
\end{array}$$

$k = 4$ $\qquad l = 5$ $\qquad m = 6$

It can be extracted from the following Gog triangle of size 7:

$$\begin{array}{ccccccccccccc}
1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 \\
 & 1 & & 2 & & \mathbf{4} & & \mathbf{5} & & \mathbf{6} & & 7 & \\
 & & 1 & & \mathbf{3} & & \mathbf{5} & & \mathbf{6} & & \mathbf{7} & & \\
 & & & \mathbf{2} & & \mathbf{3} & & \mathbf{5} & & \mathbf{6} & & & \\
 & & & & \mathbf{3} & & \mathbf{5} & & \mathbf{6} & & & & \\
 & & & & & \mathbf{4} & & \mathbf{5} & & & & & \\
 & & & & & & \mathbf{4} & & & & & &
\end{array}$$

Note that, when $l = m = n$, a Gog or GOGAm pentagon is actually a trapezoid.

4.1. **Completion of GOGAm pentagons.** Given a (k, l, m, n) pentagon, the smallest Gelfand-Tsetlin triangle of size n containing it is obtained by completing each SW-NE diagonal of X with the maximal value of this diagonal in order to obtain a trapezoid, then adding a triangle of 1's to the upper left of this trapezoid, see below a $(4, 4, 5, 7)$ GOGAm pentagon and the smallest Gelfand-Tsetlin triangle containing it:

$$\begin{array}{ccccccccccccc}
1 & & 1 & & 1 & & 1 & & 2 & & 3 & & 5 \\
 & 1 & & 1 & & 1 & & 2 & & 3 & & 5 & \\
 & & 1 & & \mathbf{1} & & \mathbf{2} & & \mathbf{3} & & 5 & & \\
 & & & \mathbf{1} & & \mathbf{1} & & \mathbf{2} & & \mathbf{5} & & & \\
 & & & & \mathbf{1} & & \mathbf{2} & & \mathbf{3} & & & & \\
 & & & & & \mathbf{1} & & \mathbf{3} & & & & & \\
 & & & & & & \mathbf{2} & & & & & &
\end{array}$$

The following remark will be useful in order to check whether a pentagon is a GOGAm pentagon.

Proposition 4.2. *Let X be a (k, l, m, n) GOGAm pentagon then the smallest Gelfand-Tsetlin triangle of size n containing X is a GOGAm triangle.*

Proof. The proof relies on the following lemma.

Lemma 4.3. *Let X be a GOGAm triangle of size n.*

(1) *If one replaces all entries of X in some upper left triangle by 1's, the resulting triangle is still a GOGAm triangle.*
(2) *Let $n \geqslant m \geqslant k \geqslant 1$. Assume that X is constant on each partial SW-NE diagonal of the form $(X_{i+l,k+l}; n - i \geqslant l \geqslant 0)$ for $i \geqslant m + 1$, then the triangle obtained from X by replacing the entries $(X_{m+l,k+l}; n - m \geqslant l \geqslant 1)$ by $X_{m,k}$ is a GOGAm triangle.*

Proof. It is easily seen that the above replacements give a Gelfand-Tsetlin triangle. The claim now follow by inspection of the formula (3.1), which shows that, upon making the above replacements, the quantity on the left cannot increase. □

End of proof of Proposition 4.2. Let X be a (k, l, m, n) GOGAm pentagon and let Y be a GOGAm triangle of size n containing X. One can replace the entries of the $n-k$ leftmost SW-NE diagonals of Y to the left by 1's and get a GOGAm triangle. Then one can replace successively the SW-NE partial diagonals as in the Lemma above. The resulting GOGAm triangle is the smallest Gelfand-Tsetlin triangle containing X. □

4.2. **The main conjecture.**

Conjecture 4.4. *For all k, l, m, n, x the numbers of (k, l, m, n) Gog and GOGAm pentagons with bottom entry equal to x are equal.*

4.3. **Numerical verifications.** We have verified the conjecture 4.4 for all values of the parameters n, k, l, m, x at most 7. We have also investigated larger values of the parameter n for pentagons with k or $l = 3$. As a small sample of the numerical values obtained, we have verified that the numbers of Gog and GOGAm pentagons of shape $(5, 4, 7, 7)$ are given, according to the value of the bottom entry x, by

x	1	2	3	4	5	6	7
	4862	17488	32839	40903	35699	20920	6578

while the numbers of Gog and GOGAm pentagons of shape $(5, 3, 6, 9)$ are given by

x	1	2	3	4	5	6	7	8	9
	22402	98264	239686	419076	570857	620188	528820	330255	120763

5. Some results

5.1. **Trapezoids.**

Theorem 5.1 (Zeilberger [12]). *For all $k \leqslant n$, the (n, k) Gog and Magog trapezoids are equienumerated.*

Composing by the Schützenberger involution yields that, for all $k \leqslant n$, the (n, k) Gog and GOGAm trapezoids are equienumerated. In [4] a bijective proof of this last fact is given for $(n, 1)$ and $(n, 2)$ trapezoids. Moreover our bijection preserves the statistic α (the value of the bottom entry). As far as we know, apart from this case it is not known whether the statistic α is equienumerated on Gog and Magog trapezoids with $k < n$. It seems also that our bijection has the virtue of preserving the statistic β. We have verified this on a number of examples, but we do not have a proof of this fact. Also it is possible to embed $(2, l, m, n)$ pentagons into triangles of size n so that our bijection restrict to a bijection between $(2, l, m, n)$ Gog and GOGAm pentagons (the embedding consists simply in completing the pentagons in the minimal way). Details of this claim are not difficult to check but rather cumbersome to write down so we omit them here.

Remark 5.2. *Some related conjectures have been formulated by Krattenthaler [8]*

5.2. **Inversions and standard procedure.** We will now describe a bijective approach to the problem which yields some partial results. For this we need a few preliminaries.

5.2.1. *Inversions.*

Definition 5.3. *An inversion in a Gelfand-Tsetlin triangle X is a pair (i,j) such that $X_{i,j} = X_{i+1,j}$.*

For example the following Gog triangle contains three inversions, $(2,2)$, $(3,1)$, $(4,1)$, the respective equalities being depicted on the picture:

$$\begin{array}{ccccccccc}
1 & & 2 & & 3 & & 4 & & 5 \\
 & 1 & & 3 & & 4 & & 5 & \\
 & & 1 & & 4 & & 5 & & \\
 & & & 2 & & 4 & & & \\
 & & & & 3 & & & &
\end{array}$$

The name *inversion* comes from the fact that, for a Gog triangle corresponding to an alternating sign matrix which is a permutation σ, its inversions are in one to one correspondance with the inversions of the permutation σ.

Definition 5.4. *Let $X = (X_{i,j})_{n \geqslant i \geqslant j \geqslant 1}$ be a Gog triangle and let (k,l) be an inversion of X, we say that this inversion covers the entry (i,j) if $i = k+p$ and $j = l+p$ for some p with $1 \leqslant p \leqslant n-k$.*

The entries (i,j) covered by an inversion are depicted with " $+$ " on the following picture.

$$\begin{array}{ccccccccc}
\bullet & & \bullet & & \bullet & & + & & \bullet \\
 & \bullet & & \bullet & & + & & \bullet & \\
 & & \bullet & & \bullet & & \bullet & & \\
 & & & \bullet & & \bullet & & & \\
 & & & & \bullet & & & &
\end{array}$$

5.2.2. *Standard procedure.* The basic idea for our bijection is that, for any inversion in the Gog triangle, we should subtract 1 from the entries covered by this inversion, scanning the inversions along the successive NW-SE diagonals, starting from the rightmost diagonal, and scanning each diagonals from NW to SE. We call this the *standard procedure*. If the successive triangles obtained after each of these steps are Gelfand-Tsetlin triangles, then we say that the initial triangle is *admissible*.

Below is an admissible Gog triangle with three inversions.

$$\begin{array}{ccccccc}
1 & & 2 & & 3 & & 4 \\
 & 1 & & 3 & & 4 & \\
 & & 1 & & 3 & & \\
 & & & 2 & & &
\end{array}$$

Applying the standard procedure in three steps we obtain respectively

$$\begin{array}{ccccccc}
1 & & 2 & & 3 & & 3 \\
 & 1 & & 3 & & 3 & \\
 & & 1 & & 3 & & \\
 & & & 2 & & &
\end{array}
\qquad
\begin{array}{ccccccc}
1 & & 1 & & 3 & & 3 \\
 & 1 & & 3 & & 3 & \\
 & & 1 & & 3 & & \\
 & & & 2 & & &
\end{array}
\qquad
\begin{array}{ccccccc}
1 & & 1 & & 2 & & 3 \\
 & 1 & & 2 & & 3 & \\
 & & 1 & & 3 & & \\
 & & & 2 & & &
\end{array}$$

One can check that the result is a GOGAm triangle. More generally one has:

Proposition 5.5. *Let X be an admissible Gog triangle of size n, then the triangle obtained by applying the standard procedure to X is a GOGAm triangle of size n.*

Proof. Let us denote by Y the triangle obtained from X by the standard procedure. One has $Y_{i,j} = X_{i,j} - c_{i,j}$ where $c_{i,j} \geqslant 0$ is the number of inversions which are covered by (i,j). Note that this number weakly increases along SW-NE diagonals: $c_{i+1,j+1} \geqslant c_{i,j}$.

We have to prove that for all $n = j_0 > j_1 > j_2 \ldots > j_{n-k} \geqslant 1$, one has

$$\left(\sum_{i=0}^{n-k-1} Y_{j_i+i,j_i} - Y_{j_{i+1}+i,j_{i+1}}\right) + Y_{j_{n-k}+n-k,j_{n-k}} \leqslant k$$

Let us rewrite the sum on the left hand side as

$$\begin{aligned} S &= X_{n,n} - c_{n,n} - \sum_{i=1}^{n-k}(X_{j_i+i-1,j_i} - c_{j_i+i-1,j_i} - X_{j_i+i,j_i} + c_{j_i+i,j_i}) \\ &= X_{n,n} - \left[\sum_{i=1}^{n-k}(X_{j_i+i-1,j_i} - X_{j_i+i,j_i}) + (c_{j_{i-1}+i-1,j_{i-1}} - c_{j_i+i-1,j_i})\right] - c_{j_{n-k}+n-k,j_{n-k}}. \end{aligned}$$

One has $X_{n,n} = n$, furthermore, for each i one has either

$$X_{j_i+i-1,j_i} - X_{j_i+i,j_i} \geqslant 1$$

or

$$X_{j_i+i-1,j_i} = X_{j_i+i,j_i}$$

in which case $(j_i + i - 1, j_i)$ is an inversion, therefore

$$c_{j_{i-1}+i-1,j_{i-1}} - c_{j_i+i-1,j_i} \geqslant 1.$$

It follows that for each term in the sum

$$(X_{j_i+i-1,j_i} - X_{j_i+i,j_i}) + (c_{j_{i-1}+i-1,j_{i-1}} - c_{j_i+i-1,j_i}) \geqslant 1$$

therefore

$$S \leqslant n - (n-k) - c_{j_{n-k}+n-k,j_{n-k}} \leqslant k.$$

□

For admissible triangles the standard procedure is invertible, meaning that the triangle can recovered uniquely from its associated GOGAm triangle. Indeed it suffices for that to scan the inversions of the GOGAm triangle in the reverse order and to add one to the entries covered by each inversion, in order to recover the original admissible triangle. We will call admissible the GOGAm triangles for which the inverse of the standard procedure works. Thus the previous proposition asserts that the standard procedure is a bijection between admissible Gog and GOGAm triangles.

The following property is not difficult to check, we leave it as an exercise to the reader.

Proposition 5.6. *The Gog triangles corresponding to permutation matrices, in the correspondance between alternating sign matrices and Gog triangles, are all admissible.*

This yields a bijection between Gog triangles corresponding to permutations matrices and a subset of the GOGAm triangles (or of the Magog triangles by applying Schützenberger involution). A bijection between permutation matrices and a subset of the Magog triangles has also been proposed by J. Striker [11], however it is not the same as the one obtained here.

5.2.3. *Triangles of size* $2, 3$ *and* 4. All Gog triangles of size 2 or 3 are admissible and the standard procedure then gives a bijection between Gog and GOGAm triangles of this size. Moreover, this bijection preserves the statistics α and β, more precisely, if ω denotes this bijection then, for any Gog triangle X, one has $\alpha_{Gog}(X) = \alpha_{GOGAm}(\omega(X))$ and $\beta_{Gog}(X) = \beta_{GOGAm}(\omega(X))$. For $n = 3$ the bijection is described in the following table, where we have also shown the corresponding Magog triangles, as well as the value of the two statistics α and β.

Gog	GOGAm	Magog	(α, β)
$\begin{array}{ccccc}1&&2&&3\\&1&&2&\\&&1&&\end{array}$	$\begin{array}{ccccc}1&&1&&1\\&1&&1&\\&&1&&\end{array}$	$\begin{array}{ccccc}1&&1&&1\\&1&&1&\\&&1&&\end{array}$	$(1, 1)$
$\begin{array}{ccccc}1&&2&&3\\&1&&2&\\&&2&&\end{array}$	$\begin{array}{ccccc}1&&1&&2\\&1&&2&\\&&2&&\end{array}$	$\begin{array}{ccccc}1&&1&&2\\&1&&1&\\&&1&&\end{array}$	$(2, 1)$
$\begin{array}{ccccc}1&&2&&3\\&1&&3&\\&&1&&\end{array}$	$\begin{array}{ccccc}1&&1&&2\\&1&&2&\\&&1&&\end{array}$	$\begin{array}{ccccc}1&&1&&2\\&1&&2&\\&&1&&\end{array}$	$(1, 2)$
$\begin{array}{ccccc}1&&2&&3\\&1&&3&\\&&2&&\end{array}$	$\begin{array}{ccccc}1&&1&&3\\&1&&3&\\&&2&&\end{array}$	$\begin{array}{ccccc}1&&1&&3\\&1&&2&\\&&1&&\end{array}$	$(2, 2)$
$\begin{array}{ccccc}1&&2&&3\\&1&&3&\\&&3&&\end{array}$	$\begin{array}{ccccc}1&&1&&3\\&1&&3&\\&&3&&\end{array}$	$\begin{array}{ccccc}1&&1&&3\\&1&&1&\\&&1&&\end{array}$	$(3, 3)$
$\begin{array}{ccccc}1&&2&&3\\&2&&3&\\&&2&&\end{array}$	$\begin{array}{ccccc}1&&2&&2\\&2&&2&\\&&2&&\end{array}$	$\begin{array}{ccccc}1&&2&&2\\&1&&2&\\&&1&&\end{array}$	$(2, 2)$
$\begin{array}{ccccc}1&&2&&3\\&2&&3&\\&&3&&\end{array}$	$\begin{array}{ccccc}1&&2&&3\\&2&&3&\\&&3&&\end{array}$	$\begin{array}{ccccc}1&&2&&3\\&1&&2&\\&&1&&\end{array}$	$(3, 3)$

There are 42 Gog triangles of size 4, among which 41 are admissible. The only non admissible Gog triangle is

$$\begin{array}{ccccccc}1&&2&&3&&4\\&1&&3&&4&\\&&2&&3&&\\&&&2&&&\end{array}$$

One thus obtain a bijection between Gog and GOGAm triangles of size 4, in which this last triangle is mapped to the only non admissible GOGAm triangle which is

$$\begin{array}{ccccccc}1&&1&&2&&2\\&1&&2&&2&\\&&1&&2&&\\&&&2&&&\end{array}$$

One can check that, again, the two statistics α and β are preserved by this bijection.

5.3. **The** $(3, 3, 3, n)$ **pentagons.** The $(3, 3, 3, n)$ Gog or GOGAm pentagons are Gelfand-Tsetlin triangles composed of positive integers, of the form:

$$\begin{array}{ccccc} a & & d & & f \\ & b & & e & \\ & & c & & \end{array}$$

In order that such a Gelfand-Tsetlin triangle be a $(3,3,3,n)$ Gog pentagon, it is necessary and sufficient that:

$$a < d < f; \quad b < e, \quad f \leqslant n.$$

For GOGAm pentagons the conditions are:

$$f \leqslant n; \quad f - e + d \leqslant n - 1; \quad f - c + b \leqslant n - 1; \quad f - e + d - b + a \leqslant n - 2.$$

We will now describe the bijection from Gog to GOGAm by decomposing the set of Gog pentagons according to the inversion pattern. In the left column we put the different Gog pentagons, on the right the corresponding GOGAm pentagons. There are three possible places for inversions, hence $2^3 = 8$ cases to consider. In 7 of these cases, the triangle is admissible, and we apply the standard procedure. There is only one case where this procedure does not apply.

Verifying that this table gives a bijection between Gog and GOGAm pentagons of this form is straightforward, but tedious, so we leave this task to the interested reader.

Gog	GOGAm	Gog	GOGAm
a d f b e c	a d f b e c	a d f b d b	a d-1 f-2 a d-1 b
a d f b d c	a d f-1 b d c	a d f a d c	a d-1 f-1 a d c
a d f a e c	a d-1 f a e c	a d f a e a	a d-1 f-1 a e-1 a
a d f b e b	a d f-1 b e-1 b	a d f a d a	a d-1 f-2 a d-1 a

5.4. **Pentagons with $l = 1$.** The sets of $(k,1,m,n)$ Gog and GOGAm pentagons actually both coincide with the set of nondecreasing sequences $X_{n,1} \leqslant \ldots \leqslant X_{1,1}$ satisfying

$$X_{j,1} \leqslant n - j + 1$$

(note that these sets are counted by Catalan numbers). Therefore the identity map provides a trivial bijection between these two sets. Clearly the α statistic is preserved.

5.5. **Pentagons with $l = 2$.** Using the completion of a pentagon, one sees that the family of inequalities (3.1) simplifies in the case of $(n,2,n,n)$ GOGAm pentagons and they reduce to

$$X_{i,2} \leqslant n - i + 2 \tag{5.1}$$

$$X_{i,2} - X_{i-1,1} + X_{i,1} \leqslant n - i + 1 \tag{5.2}$$

Remark that, since $-X_{i-1,1}+X_{i,1} \leqslant 0$, the inequality (5.2) follows from (5.1) unless $X_{i-1,1} = X_{i,1}$. We will now describe a bijection between Gog and GOGAm $(n, 2, n, n)$ pentagons. One can embed the $(k, 2, m, n)$ pentagons inside $(n, 2, n, n)$ pentagons by completing them in the minimal way. One can check that the bijection that we will produce maps $(k, 2, m, n)$ Gog pentagons to $(k, 2, m, n)$ GOGAm pentagons and gives a bijection between these subsets. We leave this verification to the interested reader.

5.5.1. *From Gog to GOGAm.* Let X be a $(n, 2, n, n)$ Gog pentagon. We shall construct a $(n, 2, n, n)$ GOGAm pentagon Y by scanning the inversions in the leftmost NW-SE diagonal of X, starting from NW. Let us denote by $n > i_1 > \ldots > i_k \geqslant 1$ these inversions, so that $X_{i,1} = X_{i+1,1}$ if and only if $i \in \{i_1, \ldots, i_k\}$. We also put $i_0 = n$. We will construct a sequence of pentagons of the same shape $Y^{(0)} = X, Y^{(1)}, Y^{(2)}, \ldots, Y^{(k)} = Y$.

Let us assume that we have constructed the trapezoids up to $Y^{(l)}$, that $Y^{(l)} \leqslant X$, that $Y^{(l)}_{ij} = X_{ij}$ for $i \leqslant i_l$, and that inequalities (5.1) and (5.2) are satisfied by $Y^{(l)}$ for $i \geqslant i_l + 1$. This is the case for $l = 0$.

Let m be the largest integer such that $Y^{(l)}_{m,2} = Y^{(l)}_{i_{l+1}+1,2}$. We put

$$\begin{array}{lcll}
Y^{(l+1)}_{i,1} & = & Y^{(l)}_{i,1} & \text{for} \quad n \geqslant i \geqslant m \text{ and } i_{l+1} \geqslant i \\
Y^{(l+1)}_{i,1} & = & Y^{(l)}_{i+1,1} & \text{for} \quad m-1 \geqslant i > i_{l+1} \\
Y^{(l+1)}_{i,2} & = & Y^{(l)}_{i,2} & \text{for} \quad n \geqslant i \geqslant m+1 \text{ and } i_{l+1} \geqslant i \\
Y^{(l+1)}_{i,2} & = & Y^{(l)}_{i,2} - 1 & \text{for} \quad m \geqslant i \geqslant i_{l+1} + 1.
\end{array}$$

From the definition of m, and the fact that X is a $(n, 2, n, n)$ Gog pentagon, we see that this new triangle is a Gelfand-Tsetlin triangle, that $Y^{(l+1)} \leqslant X$, and that $Y^{(l+1)}_{ij} = X_{ij}$ for $i \leqslant i_{l+1}$. Let us now check that $Y^{(l+1)}$ satisfies the inequalities (5.1) and (5.2) for $i \geqslant i_{l+1} + 1$. The first series of inequalities, for $i \geqslant i_{l+1} + 1$, follow from the fact that $Y^{(l)} \leqslant X$. For the second series, they are satisfied for $i \geqslant m + 1$ since this is the case for $Y^{(l)}$. For $m \geqslant i \geqslant i_{l+1} + 1$, observe that

$$Y^{(l+1)}_{i,2} - Y^{(l+1)}_{i+1,1} + Y^{(l+1)}_{i+1,1} \leqslant Y^{(l+1)}_{i,2} = Y^{(l+1)}_{m,2} = Y^{(k)}_{m,2} - 1 \leqslant n - m + 1$$

by (5.1) for $Y(l)$, from which (5.2) follows.

This proves that $Y^{(l+1)}$ again satisfies the induction hypothesis. Finally $Y = Y^{(k)}$ is a $(n, 2, n, n)$ GOGAm pentagon: indeed inequalities (5.1) follow again from $Y^{(l+1)} \leqslant X$, and (5.2) for $i \leqslant i_k$ follow from the fact that there are no inversions in this range. It follows that the above algorithm provides a map from $(n, 2, n, n)$ Gog pentagons to $(n, 2, n, n)$ GOGAm pentagons. Also, in this map, the α statistic is preserved.

5.5.2. *Inverse map.* We now describe the inverse map, from GOGAm to Gog pentagons.

We start from an $(n, 2)$ GOGAm pentagon Y, and construct a sequence

$$Y = Y^{(k)}, Y^{(k-1)}, Y^{(k-2)}, \ldots, Y^{(0)} = X$$

of intermediate Gelfand-Tsetlin trapezoids.

Let $n - 1 \geqslant \iota_1 > \iota_2 \ldots > \iota_k \geqslant 1$ be the inversions of the leftmost diagonal of Y, and let $\iota_{k+1} = 0$. Assume that $Y^{(l)}$ has been constructed and that $Y^{(l)}_{ij} = Y_{ij}$ for $i - j \geqslant \iota_{l+1}$. This is the case for $l = k$.

Let p be the smallest integer such thatinversion $Y^{(l)}_{\iota_l+1,2} = Y^{(l)}_{p,2}$. We put

$$\begin{array}{rclll} Y^{(l-1)}_{i,1} & = & Y^{(l)}_{i,1} & \text{for} & n \geqslant i \geqslant \iota_l + 1 \text{ and } p \geqslant i \\ Y^{(l-1)}_{i,1} & = & Y^{(l)}_{i-1,1} & \text{for} & \iota_l \geqslant i \geqslant p \\ Y^{(l-1)}_{i,2} & = & Y^{(l)}_{i,2} & \text{for} & n \geqslant i \geqslant \iota_l + 2 \text{ and } p-1 \geqslant i \\ Y^{(l-1)}_{i,2} & = & X^{(l)}_{i,2} - 1 & \text{for} & \iota_l + 1 \geqslant i \geqslant p. \end{array}$$

It is immediate to check that if X is an $(n, 2, n, n)$ Gog pentagon, and Y is its image by the first algorithm then the above algorithm applied to Y yields X back, actually the sequence $Y^{(l)}$ is the same. Therefore in order to prove the bijection we only need to show that if Y is a GOGAm pentagon then the algorithm is well defined and X is a Gog pentagon. This is a bit cumbersome, but not difficult and very similar to the opposite case, so we leave this task to the reader.

5.6. **An example.** In this section we work out an example of the algorithm from the Gog pentagon X to the GOGAm pentagon Y by showing the successive pentagons $Y^{(k)}$. At each step we indicate the inversion, as well as the entry covered by this inversion, and the values of the parameters i_l, p. The algorithm also runs backwards to yield the GOGAm→Gog bijection.

$$\begin{array}{cccccccc} 1 & & 2 & & & & & \\ & 1 & & 2 & & & & \\ & & 1 & & 4 & & & \\ & & & 2 & & 4 & & \\ & & & & 3 & & 4 & \\ & & & & & 3 & & 4 \\ & & & & & & 3 & \end{array}$$

$X = Y^{(0)}$

$i_1 = 6$

$m = 7$

$$\begin{array}{cccccccc} 1 & & 1 & & & & & \\ & 1 & & 2 & & & & \\ & & 1 & & 4 & & & \\ & & & 2 & & 4 & & \\ & & & & 3 & & 4 & \\ & & & & & 3 & & 4 \\ & & & & & & 3 & \end{array}$$

$Y^{(1)}$

$i_2 = 5$

$m = 6$

$$\begin{array}{cccccccc} 1 & & 1 & & & & & \\ & 1 & & 1 & & & & \\ & & 1 & & 4 & & & \\ & & & 2 & & 4 & & \\ & & & & 3 & & 4 & \\ & & & & & 3 & & 4 \\ & & & & & & 3 & \end{array}$$

$Y^{(2)}$

$i_3 = 2$

$m = 5$

$$\begin{array}{cccccccc} 1 & & 1 & & & & & \\ & 1 & & 1 & & & & \\ & & 1 & & 3 & & & \\ & & & 1 & & 3 & & \\ & & & & 2 & & 3 & \\ & & & & & 3 & & 4 \\ & & & & & & 3 & \end{array} \qquad \begin{array}{l} Y^{(3)} \\ \\ i_4 = 1 \\ m = 2 \end{array}$$

$$\begin{array}{cccccccc} 1 & & 1 & & & & & \\ & 1 & & 1 & & & & \\ & & 1 & & 3 & & & \\ & & & 1 & & 3 & & \\ & & & & 2 & & 3 & \\ & & & & & 3 & & 3 \\ & & & & & & 3 & \end{array} \qquad Y^{(4)} = Y$$

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