Operads, quasiorders, and regular languages

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SAMUELE GIRAUDO, JEAN-GABRIEL LUQUE, LUDOVIC MIGNOT, AND FLORENT NICART

Abstract. We generalize the construction of multi-tildes in the aim to provide double multi-tilde operators for regular languages. We show that the underlying algebraic structure involves the action of some operads. An operad is an algebraic structure that mimics the composition of the functions. The involved operads are described in terms of combinatorial objects. These operads are obtained from more primitive objects, namely precompositions, whose algebraic counter-parts are investigated. One of these operads acts faithfully on languages in the sense that two different operators act in two different ways.

Contents

Introduction 2
1. Some combinatorial operators in language theory 4
1.1. Nonsymmetric operads 4
1.2. Multi-tildes and related operads 5
2. Breaking operads 6
2.1. Some monoids of infinite matrices 6
2.2. Precompositions 8
2.3. From precompositions to operads 10
2.4. Quotients of precompositions 12
3. Multi-tildes and precompositions 14
3.1. The operad $\mathcal{T}$ revisited 14
3.2. The operad RAS revisited 15
3.3. The operad POSet revisited 16
4. The operad of double multi-tildes 16
4.1. $\mathcal{DT}$ and antireflexive relations 17
4.2. An operad on quasiorders 19
4.3. Infinite matrices again 20
5. Action on languages 21
5.1. Action of ARef 21
5.2. Operadic expressions for regular languages 24
5.3. Action of QOSet 28
5.4. Back to (simple) multi-tildes 30
Conclusion and perspectives 31
References 31

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Introduction

Following the Chomsky-Schützenberger hierarchy [Cho56], regular languages are defined to be the formal languages that are generated by Type-3 grammars (also called regular grammars). These particular languages have been studied from several years since they have many applications in several areas such as pattern matching, compilation, verification, and bioinformatics. Their generalization as rational series links them to various algebraic or combinatorial topics like enumeration (manipulation of generating functions), rational approximation (for instance Pade approximation), representation theory (modules viewed as automata), and combinatorial optimization ($(\max, +)$-automata).

One of the main specificity of regular languages is that they can be represented by various tools: regular grammars, automata, regular expressions, *etc.* Whilst they can be represented by both automata and regular expressions [Kle56], these two tools are not equivalent. Indeed, Ehrenfeucht and Zeiger [EZ76] showed a one parameter family of automata whose shortest equivalent regular expressions have a width exponentially growing with the numbers of states. Note that it is possible to compute an automaton from a regular expression E such that the number of its states is a linear function of the alphabet width (*i.e.*, the number of occurrences of alphabet symbols) of E [Ant96, CZ01, Glu61, MY60].

In the aim to increase expressiveness of regular expressions for a bounded length, Caron *et al.* [CCM11] introduced the so-called multi-tilde operators and applied these to represent finite languages. Investigating the equivalence of two multi-tilde expressions, they define a natural notion of composition which endows the set of multi-tilde operators with a structure of operad. This structure has been investigated in [LMN13].

Originating from the algebraic topology [May72, BV72], operad theory has been developed as a field of abstract algebra concerned by prototypical algebras that model classical properties such as commutativity and associativity [LV10]. Generally defined in terms of categories, this notion can be naturally applied to computer science. Indeed, an operad is just a set of operations, each one having exactly one output and a fixed finite number of inputs, endowed with the composition operation. An operad can then model the compositions of functions occurring during the execution of a program. In terms of theoretical computer science, this can be represented by trees with branching rules. The whole point of the operads in the context of the computer science is that this allows to use different tools and concepts from algebra (such as morphisms, quotients, substructures, generating sets).

In order to illustrate this point of view, let us recall the main results of our previous paper [LMN13]. In this paper, we first showed that the set of multi-tilde operators has a structure of operad. We used the concept of morphism in the aim to choose the operad allowing us to describe in the simplest way a given operation or a property. For instance, the original definition of the action of the multi-tildes on languages is rather complicated. But, *via* an intermediate operad based on set of boolean vectors, the action was described in a more natural way. In the same way, the equivalence problem is clearer when asked in a operad based on antisymmetric and reflexive relations which is isomorphic to the operad of multi-tildes: two operators are equivalent if and only if they have the same transitive closure. The transitive closure being compatible with the composition, we defined an operad based on partial ordered sets as a quotient of the previous operad and we showed that this representation is optimal in the sense that two different operators act in two different ways on languages. This not only helps to clarify constructions but also to ask new questions. For instance, how many different ways do n-ary multi-tildes act on languages? Precisely, the answer is the number of posets on $\{1, \ldots, n+1\}$ that are compatible with the natural order on integers.

The goal of this paper is to generalize this construction to regular languages. We investigate several operads (based on double multi-tildes, antireflexive relations or quasiorders) allowing to represent a regular language as an n-ary operator acting on a n-tuple of symbols $(\alpha_1, \ldots, \alpha_n)$ where the α_i are symbols or $\emptyset$. These operators generalize the multi-tildes and the investigated properties involve their underlying operads. Such a generalization induces the definition of new parametrized operators that allow to increase the number of regular languages denoted by an expression with a fixed alphabetical width *i.e.*, the number of occurrences of its symbols. One of the main properties of such a family of operators is that the expressions can be easily translated in terms of ε-automata. This paper is the first step of this process: multi-tildes were shown to be able to replace the operators of sum and catenation; this work shows that they may replace the Kleene star too. The notion of precomposition is the keystone of this modus operandi; using functors and category theory, the next step is to link the notion of operad and the conversions between expressions and automata.

This paper is organized as follows. First we recall in Section 1 several notions concerning operad theory and multi-tilde operations. In Section 2, we observe that many of the operads involved in [LMN13] and in this paper have some common properties. More precisely, they can be described completely by means of *shifting* operations. This leads to the definition of the category of precompositions together with a functor to the category of operads. We also define and investigate the notion of quotients of precompositions. These structures serve as model for the operads defined in the sequel. To illustrate how to use these tools, we revisit in Section 3 the operads defined in [LMN13] and describe them in terms of precompositions. In Section 4, we define the double multi-tilde operad $\mathcal{DT}$ as the graded tensor square of the multi-tilde operad. We construct also an isomorphic operad ARef based on antireflexive relations and a quotient based on quasiorders QOSet. In Section 5, we describe the action of the operads on the languages. In particular, we show that any regular language can be written as $\mathbf{p}_n(\alpha_1, \ldots, \alpha_n)$ where the α_i are letters or $\emptyset$ and $\mathbf{p}_n$ is an n-ary operation belonging to ARef, $\mathcal{DT}$, or QOSet. Finally, we prove that the action of QOSet on regular languages is faithful, that is two different operators act in two different ways.

The operad studied in this paper fit into the following diagram

$$\begin{tikzcd}
& \mathrm{ARef} \simeq \mathbb{H}(\mathrm{ARAS}, \mathrm{ARAS}) \simeq \mathcal{DT} \arrow[dl, two heads] & \\
\mathrm{QOSet} & & \mathrm{ARAS} \simeq \mathrm{RAS} \simeq \mathcal{T} \arrow[ul, tail] \arrow[dl, two heads] \\
& \mathrm{POSet} \arrow[ul, tail] &
\end{tikzcd} \tag{0.1}$$

where arrows $\rightarrowtail$ (resp. $\twoheadrightarrow$) are injective (resp. surjective) morphisms of operads.

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1. Some combinatorial operators in language theory

We recall here some basic notions about the theory of operads and set our notations for the sequel of the paper. In particular, we recall what are operads, free operads, and modules over an operad. We conclude this section by presenting the operad of multi-tildes introduced in [LMN13].

1.1. **Nonsymmetric operads.**

Since we shall consider in this paper only nonsymmetric operads, we shall call these simply *operads*. Operads are algebraic graded structures which mimic the composition of n-ary operators. Let us recall the main definitions and properties. Let $\mathfrak{P} = \bigsqcup_{n\geqslant 1} \mathfrak{P}_n$ be a graded set ($\bigsqcup$ means that the sets are disjoint); the elements of $\mathfrak{P}_n$ are called *n-ary operators*. The set $\mathfrak{P}$ is endowed with maps

$$\circ_i : \mathfrak{P}_n \times \mathfrak{P}_m \to \mathfrak{P}_{n+m-1}, \tag{1.1}$$

where $1 \leqslant i \leqslant n$, called *partial compositions* and satisfying for all $\mathbf{p}_1 \in \mathfrak{P}_n$, $\mathbf{p}_2 \in \mathfrak{P}_m$, and $\mathbf{p}_3 \in \mathfrak{P}_p$ the two following rules.

(1) *Associativity 1*: if $1 \leqslant i < j \leqslant n$, then

$$(\mathbf{p}_1 \circ_i \mathbf{p}_2) \circ_{j+m-1} \mathbf{p}_3 = (\mathbf{p}_1 \circ_j \mathbf{p}_3) \circ_i \mathbf{p}_2. \tag{1.2}$$

(2) *Associativity 2*: if $j \leqslant m$, then

$$(\mathbf{p}_1 \circ_i \mathbf{p}_2) \circ_{i+j-1} \mathbf{p}_3 = \mathbf{p}_1 \circ_i (\mathbf{p}_2 \circ_j \mathbf{p}_3). \tag{1.3}$$

Moreover, in an operad, there is a special element $\mathbf{1} \in \mathfrak{P}_1$ called *unit* and satisfying for all $\mathbf{p} \in \mathfrak{P}_n$ the following rule.

(3) *Unitality relation*: if $1 \leqslant i \leqslant n$, then

$$\mathbf{p} \circ_i \mathbf{1} = \mathbf{p} = \mathbf{1} \circ_1 \mathbf{p}. \tag{1.4}$$

The reader could refer to [LV10, MSS02] for a complete description of the structures.

Consider two operads $(\mathfrak{P}, \circ, \mathbf{1})$ and $(\mathfrak{P}', \circ', \mathbf{1}')$. A *morphism of operads* is a graded map $\phi : \mathfrak{P} \to \mathfrak{P}'$ such that $\phi(\mathbf{1}) = \mathbf{1}'$ and

$$\phi(\mathbf{p}_1 \circ_i \mathbf{p}_2) = \phi(\mathbf{p}_1) \circ'_i \phi(\mathbf{p}_2) \tag{1.5}$$

for all $\mathbf{p}_1 \in \mathfrak{P}_n$, $\mathbf{p}_2 \in \mathfrak{P}_m$ and $1 \leqslant i \leqslant n$. If $\mathfrak{Q} \subset \mathfrak{P}$, the *suboperad* of $\mathfrak{P}$ generated by $\mathfrak{Q}$ is the smallest subset of $\mathfrak{P}$ containing $\mathfrak{Q}$ and $\mathbf{1}$ which is stable by composition.

Let $\mathfrak{G} = \bigsqcup_{n\geqslant 1} \mathfrak{G}_n$ be a graded set. The set $\mathrm{Free}(\mathfrak{G})_n$ is the set of planar rooted trees with n leaves and where any internal node with n children is labeled on $\mathfrak{G}_n$. The *free operad* on $\mathfrak{G}$ is obtained by endowing the set $\mathrm{Free}(\mathfrak{G}) = \bigsqcup_{n\geqslant 1} \mathrm{Free}(\mathfrak{G})_n$ with the partial compositions $\circ_i$ where $\mathbf{p}_1 \circ_i \mathbf{p}_2$ is the tree obtained by grafting the ith leaf of $\mathbf{p}_1$ on the root of $\mathbf{p}_2$. Observe that $\mathrm{Free}(\mathfrak{G})$ contains a copy of $\mathfrak{G}$ which is the set of the trees with only one internal node (the root) labeled on $\mathfrak{G}$; for simplicity we will identify it with $\mathfrak{G}$. Moreover, $\mathrm{Free}(\mathfrak{G})$ is clearly generated by $\mathfrak{G}$. The universality means that for any map $\varphi : \mathfrak{G} \to \mathfrak{P}$ there exists a unique operad morphism $\phi : \mathrm{Free}(\mathfrak{G}) \to \mathfrak{P}$ such that $\phi(\mathbf{g}) = \varphi(\mathbf{g})$ for each $\mathbf{g} \in \mathfrak{G}$.

A graded equivalence relation $\equiv$ on $\mathfrak{P}$ is an *operad congruence* if for all $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}'_1, \mathbf{p}'_2 \in \mathfrak{P}$, $\mathbf{p}_1 \equiv \mathbf{p}'_1$, and $\mathbf{p}_2 \equiv \mathbf{p}'_2$ imply $\mathbf{p}_1 \circ_i \mathbf{p}_2 \equiv \mathbf{p}'_1 \circ_i \mathbf{p}'_2$. The set $\mathfrak{P}/_{\equiv}$ is then naturally endowed with a structure of operad, called *quotient operad*. Note that if $\phi : \mathfrak{P} \to \mathfrak{P}'$ is a surjective morphism of operads then the equivalence defined by $\mathbf{p}_1 \equiv \mathbf{p}'_1$ if and only if $\phi(\mathbf{p}_1) = \phi(\mathbf{p}_1)$ is an operad congruence.

The *Hadamard product* $\mathbb{H}(\mathfrak{P}, \mathfrak{P}')$ of two operads $(\mathfrak{P}, \circ)$ and $(\mathfrak{P}', \circ')$ is the operad defined as follows. The elements of arity n of $\mathbb{H}(\mathfrak{P}, \mathfrak{P}')$ are pairs $(\mathbf{p}, \mathbf{p}')$ where $\mathbf{p} \in \mathfrak{P}_n$ and $\mathbf{p}' \in \mathfrak{P}'_n$ and its partial compositions are defined by

$$(\mathbf{p}_1, \mathbf{p}'_1) \circ_i (\mathbf{p}_2, \mathbf{p}'_2) := (\mathbf{p}_1 \circ_i \mathbf{p}_2, \mathbf{p}'_1 \circ'_i \mathbf{p}'_2), \tag{1.6}$$

for all $(\mathbf{p}_1, \mathbf{p}'_1) \in \mathbb{H}(\mathfrak{P}, \mathfrak{P}')_n$, $(\mathbf{p}_2, \mathbf{p}'_2) \in \mathbb{H}(\mathfrak{P}, \mathfrak{P}')_m$, and $1 \leqslant i \leqslant n$.

Consider a set $\mathbf{S}$ together with a left action of an operad $\mathfrak{P}$. That is, each $\mathbf{p} \in \mathfrak{P}_n$ defines a map

$$\mathbf{p} : \mathbf{S}^n \to \mathbf{S}. \tag{1.7}$$

We say that $\mathbf{S}$ is a $\mathfrak{P}$-*module* if the action of $\mathfrak{P}$ is compatible with the composition in the following sense. For any $\mathbf{p}_1 \in \mathfrak{P}_n$, $\mathbf{p}_2 \in \mathfrak{P}_m$, $1 \leqslant i \leqslant n$, and $s_1, \ldots, s_{n+m-1} \in \mathbf{S}$, one has

$$\mathbf{p}_1(s_1, \ldots, s_{i-1}, \mathbf{p}_2(s_i, \ldots, s_{i+m-1}), s_{i+m}, \ldots, s_{n+m-1}) = (\mathbf{p}_1 \circ_i \mathbf{p}_2)(s_1, \ldots, s_{n+m-1}). \tag{1.8}$$

Furthermore, if for each $n \geqslant 1$ and $\mathbf{p} \neq \mathbf{p}' \in \mathfrak{P}_n$ there exist $s_1, \ldots, s_n \in \mathbf{S}$ such that $\mathbf{p}(s_1, \ldots, s_n) \neq \mathbf{p}'(s_1, \ldots, s_n)$, we say that the module $\mathbf{S}$ is *faithful*.

1.2. **Multi-tildes and related operads.** In [LMN13], we have defined several operads. Let us recall briefly the main constructions. First we defined the *operad of multi-tildes* $\mathcal{T} = \bigsqcup_n \mathcal{T}_n$. A multi-tilde of $\mathcal{T}_n$ is a subset of $\{(x, y) : 1 \leqslant x \leqslant y \leqslant n\} \subset \mathbb{N}^2$. Note that $\bigsqcup_n$ means that sets belonging in two different graded components $\mathcal{T}_n$ and $\mathcal{T}_m$ are considered as different operators. Let $i \geqslant 0$ and $n \geqslant 1$, for any pair (x, y) of positive integers, we define

$$\overset{i,n}{\circ\!\!\rightarrow}(x, y) := \begin{cases} (x, y) & \text{if } y \leqslant i - 1, \\ (x, y + n - 1) & \text{if } x \leqslant i \leqslant y, \\ (x + n - 1, x + n - 1) & \text{otherwise.} \end{cases} \tag{1.9}$$

The actions of the operators are extended to sets E of pairs of positive integers by

$$\overset{i,n}{\circ\!\!\rightarrow}(E) := \{\overset{i,n}{\circ\!\!\rightarrow}(x, y) : (x, y) \in E\}. \tag{1.10}$$

We have shown the following result:

Theorem 1 ([LMN13])**.** *The set $\mathcal{T}$ endowed with the partial compositions*

$$\circ_i : \left\{ \begin{array}{ccc} \mathcal{T}_n \times \mathcal{T}_m & \to & \mathcal{T}_{n+m-1} \\ T_1 \circ_i T_2 & \mapsto & \overset{i,m}{\circ\!\!\rightarrow}(T_1) \cup \overset{0,i}{\circ\!\!\rightarrow}(T_2), \end{array} \right. \tag{1.11}$$

is an operad.

We have also defined the operators

$$\overset{i,n}{\diamond\!\!\rightarrow}(x) := \begin{cases} x & \text{if } x \leqslant i, \\ x + n - 1 & \text{otherwise,} \end{cases} \tag{1.12}$$

and have extended these respectively to pairs and sets of pairs by

$$\overset{i,n}{\diamond\!\!\rightarrow}(x, y) = (\overset{i,n}{\diamond\!\!\rightarrow}(x), \overset{i,n}{\diamond\!\!\rightarrow}(y)) \tag{1.13}$$

and

$$\overset{i,n}{\diamond\!\!\rightarrow}(E) = \{\overset{i,n}{\diamond\!\!\rightarrow}(x, y) : (x, y) \in E\}, \tag{1.14}$$

where x and y are positive integers and E is a set of pairs of positive integers. The operad $(\mathcal{T}, \circ)$ is isomorphic to another operad $(\mathrm{RAS}, \diamond)$ whose underlying set is the set $\mathrm{RAS} = \bigsqcup_n \mathrm{RAS}_n$ where RAS_n denotes the set of Reflexive and Antisymmetric Subrelations of the natural order $\leqslant$ on $\{1, \dots, n+1\}$. The partial compositions of RAS are defined by

$$R_1 \diamond_i R_2 := \overset{i,m}{\diamond\!\!\rightarrow}(R_1) \cup \overset{0,i}{\diamond\!\!\rightarrow}(R_2), \tag{1.15}$$

if $R_1 \in \mathrm{RAS}_n$ and $R_2 \in \mathrm{RAS}_m$. The isomorphism of operads $\phi : \mathcal{T} \to \mathrm{RAS}$ satisfies, for all $T \in \mathcal{T}_n$,

$$\phi(T) = \{(x, y+1) : (x, y) \in T\} \cup \{(x, x) : x \in \{1, \dots, n+1\}\}. \tag{1.16}$$

See [LMN13] for more details.

2. Breaking operads

The objective of this section is to introduce new algebraic objects, namely the *precompositions*. The precompositions are a kind of representation of a certain monoid denoted by $\square$ which can be described in terms of infinite matrices. We present here a functor from the category of precompositions to the category of operads. We shall use this functor in the sequel to reconstruct some already known operads and to construct new ones.

2.1. Some monoids of infinite matrices.

We consider the set $\overline{M}_\infty$ of infinite matrices with a finite number of non-zero diagonals whose entries belong to the boolean semiring $\mathbb{B}$. A typical element $(a_{ij})_{i,j \in \mathbb{Z}}$ of $\overline{M}_\infty$ is a finite linear combination of elements $D_k^\lambda := \sum_{i \in \mathbb{Z}} \lambda_i E_{i+k,i}$ where $\lambda = (\lambda_i)_{i \in \mathbb{Z}}$ and $E_{k,\ell} = (\delta_{i,k}\delta_{j,\ell})_{i,j \in \mathbb{Z}}$ and $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise is the Kronecker symbol *i.e.*, the matrix $E_{k,\ell}$ has 1 at the cell (k, ℓ) and 0 elsewhere.

Observing that

$$D_k^\lambda D_{k'}^{\lambda'} = \left(\sum_{i \in \mathbb{Z}} \lambda_i E_{i+k,i}\right)\left(\sum_{i \in \mathbb{Z}} \lambda'_i E_{i+k',i}\right) = \sum_{i \in \mathbb{Z}} \lambda_{i+k'} \lambda'_i E_{i+k+k',i} = D_{k+k'}^{\lambda \bullet_{k'} \lambda'}, \tag{2.1}$$

with $\lambda \bullet_{k'} \lambda' = (\lambda_{i+k'} \lambda'_i)_{i \in \mathbb{Z}}$, we deduce that $\overline{M}_\infty$ is stable by product. So

Proposition 1. *The space $\overline{M}_\infty$ is an algebra.*

Remark 1. *Notice that when the entries belong to $\mathbb{C}$ instead of $\mathbb{B}$, the algebraic structure of $\overline{M}_\infty$ is very rich and has many connexions with the study of infinite Lie algebras (see* e.g. *[Kac94]).*

Here, for our purpose, we consider only the structure of monoid; the unit of $\overline{M}_\infty$ is $Id := D_1^{(\dots,1,1,1,\dots)} = \sum_{i \in \mathbb{Z}} E_{i,i}$. In particular, we define the submonoid $\overline{P}_\infty$ generated by the matrices

$$\mathfrak{m}_{i,n} = \sum_{j \leqslant i} E_{j,j} + \sum_{i < j} E_{j+n-1,j} \tag{2.2}$$

for each $i \in \mathbb{Z}$ and each $n > 0$. With these notations we have

$$\mathfrak{m}_{i,1} = Id, \tag{2.3}$$

for any i.

Let $]-\infty, i]$ be the vector such that $]-\infty, i]_j = 1$ if $j \leqslant i$ and 0 otherwise, and $]i, \infty[= (1 -]-\infty, i]_j)_{j \in \mathbb{Z}}$. With these notations one has

$$\mathfrak{m}_{i,n} = D_1^{]-\infty,i]} + D_n^{]i,\infty[}. \tag{2.4}$$

Lemma 1. *The two following identities hold:*

- *if* $i \leqslant j$ *then*

$$\mathfrak{m}_{i,n}\mathfrak{m}_{j,n'} = \mathfrak{m}_{j+n-1,n'}\mathfrak{m}_{i,n}, \tag{2.5}$$

- *if* $0 \leqslant j < n'$ *then*

$$\mathfrak{m}_{i+j,n}\mathfrak{m}_{i,n'} = \mathfrak{m}_{i,n+n'-1}. \tag{2.6}$$

Proof. From (2.4), one has

$$\begin{aligned} \mathfrak{m}_{i,n}\mathfrak{m}_{j,n'} &= \left(D_0^{]-\infty,i[} + D_{-1}^{]i,\infty[}\right)\left(D_0^{]-\infty,j]} + D_{n'-1}^{]j,\infty[}\right) \\ &= D_0^{]-\infty,i]}D_0^{]-\infty,j]} + D_0^{]-\infty,i]}D_{n'-1}^{]j,\infty[} + D_{n-1}^{[i,\infty]}D_0^{]-\infty,j]} + D_{n-1}^{[i,\infty]}D_{n'-1}^{]j,\infty[}. \end{aligned}$$

But $i \leqslant j$ implies $]-\infty,i]\bullet_0]-\infty,j]=]-\infty,i]$, $]-\infty,i]\bullet_{n'-1}]j,\infty[=[\dots,0,0,0,\dots]$, $]i,\infty[\bullet_0]-\infty,j]=]-\infty,j+n-1]\bullet_{n-1}]i,\infty[$ and $]i,\infty[\bullet_{n'-1}]j,\infty[=]j,\infty[$. Hence, (2.1) implies

$$\mathfrak{m}_{i,n}\mathfrak{m}_{j,n'} = D_0^{]-\infty,i]} + D_{n-1}^{]-\infty,j+n-1]\bullet_{n-1}]i,\infty[} + D_{n+n'-2}^{]j,\infty[} = \mathfrak{m}_{j+n-1,n'}\mathfrak{m}_{i,n}.$$

This proves formula (2.5).

Now, from (2.4), we obtain

$$\begin{aligned} \mathfrak{m}_{i+j,n}\mathfrak{m}_{i,n'} &= (D_0^{]-\infty,i+j]} + D_{n-1}^{]i+j,\infty[})(D_0^{]-\infty,i]} + D_{n'-1}^{]i,\infty[}) \\ &= D_0^{]-\infty,i+j]}D_0^{]-\infty,i]} + D_0^{]-\infty,i+j]}D_{n'-1}^{]i,\infty[} + D_{n-1}^{]i+j,\infty[}D_0^{]-\infty,i]} + D_{n-1}^{]i+j,\infty[}D_{n'-1}^{]i,\infty[} \end{aligned}$$

But, since $0 \leqslant j < n'$, one has $]-\infty,i+j]\bullet_0]-\infty,i]=]-\infty,i]$, $]-\infty,i+j]\bullet_{n'-1}]i,\infty[=]i+j,\infty[\bullet_0]-\infty,i]=[\dots,0,0,0,\dots,]$ and $]i+j,\infty[\bullet_{n'-1}]i,\infty[=]i,\infty[$. Hence, (2.1) allows us to recover (2.6):

$$\mathfrak{m}_{i+j,n}\mathfrak{m}_{i,n'} = D_0^{]-\infty,i]} + D_{n+n'-2}^{]i,\infty[} = \mathfrak{m}_{i,n+n'-1}.$$

□

Proposition 2. *(Presentation of* $\overline{P}_\infty$*)*

The monoid $\overline{P}_\infty$ *is isomorphic to the monoid* $\overline{\square}$ *generated by the symbols* $\{\overset{i,n}{\square\!\!\Rightarrow} : i \in \mathbb{Z}, n \geqslant 1\}$ *and the relations*

$$\overset{i,1}{\square\!\!\Rightarrow} = \mathbf{1}_{\overline{\square}} \quad \textit{for any } i \in \mathbb{Z}. \tag{2.7}$$

$$\overset{i,n}{\square\!\!\Rightarrow}\overset{j,n'}{\square\!\!\Rightarrow} = \overset{j+n-1,n'}{\square\!\!\Rightarrow}\overset{i,n}{\square\!\!\Rightarrow} \quad \textit{if } i \leqslant j, \tag{2.8}$$

$$\overset{i+j,n}{\square\!\!\Rightarrow}\overset{i,n'}{\square\!\!\Rightarrow} = \overset{i,n+n'-1}{\square\!\!\Rightarrow} \quad \textit{if } 0 \leqslant j < n'. \tag{2.9}$$

Proof. First let us prove that the map φ sending $\overset{i,n}{\square\!\!\Rightarrow}$ to $\mathfrak{m}_{i,n}$ can be extended as a morphism of monoids from $\overline{\square}$ to $\overline{P}_\infty$. It suffices to show that $\varphi(\overset{i,1}{\square\!\!\Rightarrow}) = Id$ for any $i \in \mathbb{Z}$, $\varphi(\overset{i,n}{\square\!\!\Rightarrow})\varphi(\overset{j,n'}{\square\!\!\Rightarrow}) = \varphi(\overset{j+n-1,n'}{\square\!\!\Rightarrow})\varphi(\overset{i,n}{\square\!\!\Rightarrow})$ if $i \leqslant j$ and $\varphi(\overset{i+j,n}{\square\!\!\Rightarrow})\varphi(\overset{i,n'}{\square\!\!\Rightarrow}) = \varphi(\overset{i,n+n'-1}{\square\!\!\Rightarrow})$ when $0 \leqslant j < n'$. These equalities are respectively the consequences of (2.3), (2.5), and (2.6). Hence, φ is extended to an into morphism of monoids, called also φ. It remains to prove that φ is into.

Using (2.8) and (2.9), any element of $\overline{\square}$ can be written as

$$\overset{i_0,n_0}{\square\!\!\Rightarrow}\cdots\overset{i_\ell,n_\ell}{\square\!\!\Rightarrow}$$

for some $i_0 < \cdots < n_\ell$ and $n_0, \dots, n_\ell > 1$. Therefore, since φ is a morphism, any element of $\overline{P}_\infty$ can be written into the form $\mathfrak{m}_{i_0,n_0} \cdots \mathfrak{m}_{i_\ell,n_\ell}$. Furthermore, we observe that

$$\begin{aligned}\mathfrak{m}_{i_0,n_0} \cdots \mathfrak{m}_{i_\ell,n_\ell} \quad = \quad & \sum_{j \leqslant i_0} E_{j,j} + \sum_{i_0 < j \leqslant i_1} E_{j+n_0-1,j} + \sum_{i_1 < j \leqslant i_2} E_{j+n_0+n_1-1,j} + \cdots \\ & + \sum_{i_{\ell-1} < j \leqslant i_\ell} E_{j+n_0+\cdots+n_{\ell-1}-\ell,j} + \sum_{i_\ell < j} E_{j+n_0+\cdots+n_\ell-\ell-1,j}.\end{aligned}$$

As a consequence, it is easy to check that the factorization is unique . Hence, since the image of $\overset{i_0,n_0}{\square\!\!\Rightarrow} \cdots \overset{i_\ell,n_\ell}{\square\!\!\Rightarrow}$ by φ is $\mathfrak{m}_{i_0,n_0} \cdots \mathfrak{m}_{i_\ell,n_\ell}$, each element of $\overline{P}_\infty$ admits a unique preimage and so φ is injective. It follows that the φ is an isomorphism. $\square$

Now, let us consider the algebra M_∞ of the matrices $(a_{ij})_{i,j\in\mathbb{N}\setminus\{0\}}$ with a finite number of non-zero diagonals. Let $\pi : \overline{M}_\infty \longrightarrow M_\infty$ be the projection sending $(a_{ij})_{i,j\in\mathbb{Z}}$ to $(a_{ij})_{i,j\in\mathbb{N}\setminus\{0\}}$ and $\mathfrak{m}_{i,n} = \pi(\mathfrak{n}_{i,n}) = \sum_{0<j\leqslant i} E_{j,j} + \sum_{j>i,j>0} E_{j+n-1,j}$. Remarking that $\pi(\mathfrak{m}_{i,n} + \mathfrak{m}_{j,n'}) = \mathfrak{n}_{i,n} + \mathfrak{n}_{j,n'}$, the restriction of π to $\overline{P}_\infty$ is a morphism of monoid from $\overline{P}_\infty$ to the submonoid P_∞ of M_∞ generated by the matrices $\mathfrak{n}_{i,n}$. We notice also that $\mathfrak{n}_{-i,n} = \sum_{j\geqslant 1} E_{j+n-1,j} = \mathfrak{n}_{0,n}$ for any $i > 0$. Furthermore, we have

Proposition 3. *The monoid P_∞ is isomorphic to the quotient $\square$ of the monoid $\overline{\square}$ by the relations $\overset{-i,n}{\square\!\!\Rightarrow} = \overset{0,n}{\square\!\!\Rightarrow}$ for any $i > 0$. That is the monoid defined by generators $\{\overset{i,n}{\square\!\!\Rightarrow} : i \in \mathbb{Z}, n \geqslant 1\}$ and relations:*

$$\overset{i,n}{\square\!\!\Rightarrow} = \overset{0,n}{\square\!\!\Rightarrow} \qquad \text{for any } i < 0, \tag{2.10}$$

$$\overset{i,1}{\square\!\!\Rightarrow} = \overset{0,1}{\square\!\!\Rightarrow} = \mathbf{1}_{\square} \qquad \text{for any } i, \tag{2.11}$$

$$\overset{i,n}{\square\!\!\Rightarrow}\overset{j,m}{\square\!\!\Rightarrow} = \overset{j+n-1,m}{\square\!\!\Rightarrow}\overset{i,n}{\square\!\!\Rightarrow} \qquad \text{if } i \leqslant j \text{ or } i,j \leqslant 0, \tag{2.12}$$

$$\overset{i+j,n}{\square\!\!\Rightarrow}\overset{i,m}{\square\!\!\Rightarrow} = \overset{i,n+m-1}{\square\!\!\Rightarrow} \qquad \text{if } 0 \leqslant j < m. \tag{2.13}$$

Remark 2. *We have explained the construction when the entries are taken in $\mathbb{B}$. One can make a similar construction for any semiring $\mathbb{K}$ and obtain monoids $\overline{P}_\infty(\mathbb{K})$ and $P_\infty(\mathbb{K})$. In all the case, the monoid $\overline{P}_\infty(\mathbb{K})$ is isomorphic to $\overline{P}_\infty$ and the monoid $P_\infty(\mathbb{K})$ is isomorphic to P_∞.*

2.2. **Precompositions.** Let $(\mathcal{S}, \oplus)$ be a commutative monoid endowed with a filtration $\mathcal{S} = \bigcup_{n\geqslant 1} \mathcal{S}_n$ with

$$\mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{S}_n \subset \cdots \tag{2.14}$$

and such that each $\mathcal{S}_n$ is a submonoid is $\mathcal{S}$. We will denote by $\mathbf{0}_\mathcal{S}$ the unit of $\mathcal{S}$.

A *precomposition* is a monoid morphism $\circ : \square \to \mathrm{Hom}(\mathcal{S}, \mathcal{S})$ satisfying

$$\circ(\overset{i,n}{\square\!\!\Rightarrow}) : \mathcal{S}_m \to \mathcal{S}_{n+m-1}, \tag{2.15}$$

$$\circ(\overset{i,n}{\square\!\!\Rightarrow})|_{\mathcal{S}_m} = Id_{\mathcal{S}_m} \text{ if } i \geqslant m+1, \tag{2.16}$$

where $|_{\mathcal{S}_m}$ denotes the restriction to $\mathcal{S}_m$. For simplicity, we denote by $\overset{i,n}{\circ\!\!\to}$ the map $\circ(\overset{i,n}{\square\!\!\Rightarrow})$. Observe that the maps $\overset{i,k}{\circ\!\!\to}$ have the following intuitive meaning. If s is an element of $\mathcal{S}_m$, $\overset{i,n}{\circ\!\!\to}(s)$ is an element of $\mathcal{S}_{n+m-1}$ obtained by inserting in s a gap of length $n-1$ at position i. Axioms (2.11), (2.12), and (2.13) can be understood in the light of this interpretation.

Example 1. We consider the set $\mathcal{V}$ of infinite vectors with only a finite numbers of non zero entries. This set, endowed with the sums is a monoid. If $\mathcal{V}_n$ denotes the set of the vectors v such that $m > n$ implies $v_m = 0$, each $\mathcal{V}_n$ is a submonoid of $\mathcal{V}$. We define a precomposition by setting $\overset{i,n}{\bigcirc\!\!\rightarrow} v = \mathfrak{n}_{i,n} v$. For instance,

$$\mathfrak{n}_{2,3} = E_{1,1} + E_{2,2} + \sum_{j\geqslant 3} E_{j+2,j} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & \cdots \\ & \vdots & & & \ddots & & \vdots & \end{bmatrix}.$$

Hence,

$$\mathfrak{n}_{2,3} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ \vdots \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \\ 0 \\ v_3 \\ v_4 \\ \vdots \end{bmatrix}$$

Notice that $\mathfrak{n}_{i,n} v$ for $i \leqslant 0$ is obtained by shifting down the entries of v of $n-1$ cells and by replacing the first $n-1$ entries by zero. For instance:

$$\mathfrak{n}_{0,3} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \end{bmatrix}$$

Let $\circ : \square \to \mathrm{Hom}(\mathcal{S}, \mathcal{S})$ and $\triangleright : \square \to \mathrm{Hom}(\mathcal{S}', \mathcal{S}')$ be two precompositions. A map $\phi : \mathcal{S} \to \mathcal{S}'$ is a *precomposition morphism* from $\circ$ to $\triangleright$ if it is a monoid morphism and satisfies

$$\phi : \mathcal{S}_n \to \mathcal{S}'_n, \tag{2.17}$$

$$\overset{i,n}{\triangleright\!\!\rightarrow}(\phi(x)) = \phi(\overset{i,n}{\bigcirc\!\!\rightarrow}(x)). \tag{2.18}$$

We denote by $\mathrm{Hom}(\circ, \triangleright)$ the set of precomposition morphisms from $\circ$ to $\triangleright$.

It is easy to check that the class PreComp of precompositions endowed with the arrows $\mathrm{Hom}(\circ, \triangleright)$ for each $\circ, \triangleright \in \mathrm{PreComp}$ is a category.

Let $\circ$ be a precomposition, We define on $\mathcal{S}$ the binary operators $\circ_k^{(n)}$ by

$$s \circ_i^{(n)} s' = \overset{i,n}{\bigcirc\!\!\rightarrow} s \oplus \overset{0,i}{\bigcirc\!\!\rightarrow} s'.$$

We recall that $\oplus$ denotes the binary operation of the monoid $\mathcal{S}$.

Lemma 2. *We have*

- *For each $n, m \geqslant 1$, $1 \leqslant i < j \leqslant n$, $s, s'' \in \mathcal{S}$ and $s' \in \mathcal{S}_m$, we have*

$$\left(s \circ_i^{(m)} s'\right) \circ_{j+m-1}^{(n)} s'' = \left(s \circ_j^{(n)} s''\right) \circ_i^{(m)} s'. \tag{2.19}$$

- *For each $0 < j \leqslant m$, $0 < i \leqslant n$, and $s, s', s'' \in \mathcal{S}$, we have*

$$\left(s \circ_i^{(m)} s'\right) \circ_{i+j-1}^{(n)} s'' = s \circ_i^{(n+m-1)} \left(s' \circ_j^{(n)} s''\right). \tag{2.20}$$

Proof. We have

$$\left(s \circ_i^{(m)} s'\right) \circ_{j+m-1}^{(n)} s'' = \overset{j+m-1,n}{\circ\!\!\to}\overset{i,m}{\circ\!\!\to} s \oplus \overset{j+m-1,n}{\circ\!\!\to}\overset{0,i}{\circ\!\!\to} s' \oplus \overset{0,j+m-1}{\circ\!\!\to} s''.$$

But (2.12) implies $\overset{j+m-1,n}{\circ\!\!\to}\overset{i,m}{\circ\!\!\to} = \overset{i,m}{\circ\!\!\to}\overset{j,n}{\circ\!\!\to}$ and $\overset{j+m-1,n}{\circ\!\!\to}\overset{0,i}{\circ\!\!\to} = \overset{0,i}{\circ\!\!\to}\overset{j-i+m,n}{\circ\!\!\to}$. Since $i < j$ and $s' \in \mathcal{S}_m$, (2.18) implies $\overset{j-i+m,n}{\circ\!\!\to} s' = s'$. Furthemore by (2.13), one has $\overset{0,j+m-1}{\circ\!\!\to} s'' = \overset{i,m}{\circ\!\!\to}\overset{0,j}{\circ\!\!\to} s''$. Hence, we deduce

$$\left(s \circ_i^{(m)} s'\right) \circ_{j+m-1}^{(n)} s'' = \overset{i,m}{\circ\!\!\to}\overset{j,n}{\circ\!\!\to} s \oplus \overset{i,m}{\circ\!\!\to}\overset{0,j}{\circ\!\!\to} s'' \oplus \overset{0,i}{\circ\!\!\to} s' = \left(s \circ_j^{(n)} s''\right) \circ_i^{(m)} s'.$$

This proves (2.19).

Now let us prove (2.20). We have

$$\left(s \circ_i^{(m)} s'\right) \circ_{i+j-1}^{(n)} s'' = \overset{i+j-1,n}{\circ\!\!\to}\overset{i,m}{\circ\!\!\to} s \oplus \overset{i+j-1,n}{\circ\!\!\to}\overset{0,i}{\circ\!\!\to} s' \oplus \overset{0,i+j-1}{\circ\!\!\to} s''.$$

But from (2.13), one has $\overset{i+j-1,n}{\circ\!\!\to}\overset{i,m}{\circ\!\!\to} = \overset{i,m+n-1}{\circ\!\!\to}$, (2.12) implies $\overset{i+j-1,n}{\circ\!\!\to}\overset{0,i}{\circ\!\!\to} = \overset{0,i}{\circ\!\!\to}\overset{j,n}{\circ\!\!\to}$ and $\overset{0,i+j-1}{\circ\!\!\to} = \overset{0,i}{\circ\!\!\to}\overset{0,j}{\circ\!\!\to}$. Hence,

$$\left(s \circ_i^{(m)} s'\right) \circ_{i+j-1}^{(n)} s'' = \overset{i,m+n-1}{\circ\!\!\to} s \oplus \overset{0,i}{\circ\!\!\to}\overset{j,n}{\circ\!\!\to} s' \oplus \overset{0,i}{\circ\!\!\to}\overset{0,j}{\circ\!\!\to} s''.$$

This proves (2.20). □

Observe also that we have

$$\mathbf{0}_{\mathcal{S}} \circ_1^{(n)} s = s \circ_i^{(n)} \mathbf{0}_{\mathcal{S}} = s \tag{2.21}$$

for each $i \geqslant 0$, $n \geqslant 1$, and $s \in \mathcal{S}$.

2.3. **From precompositions to operads.** Let us consider a precomposition $\circ : \square \to \mathrm{Hom}(\mathcal{S}, \mathcal{S})$. From the commutative monoid $\mathcal{S}$, we define

$$\mathbb{S}_n := \left\{ a_s^{(n)} : s \in \mathcal{S}_n \right\}, \tag{2.22}$$

and $\mathbb{S} := \bigsqcup_{n \geqslant 1} \mathbb{S}_n$.

Claim 1. *Each $\mathbb{S}_n$ is naturally endowed with a structure of monoid. Furthermore, we have*

$$\mathbb{S}_1 \rightarrowtail \mathbb{S}_2 \rightarrowtail \cdots \rightarrowtail \mathbb{S}_n \rightarrowtail \cdots$$

where the arrows $\rightarrowtail$ denote injective morphisms of monoids.
Hence, $\mathcal{S}$ is nothing else but the inductive limit of the $\mathbb{S}_n$

$$\mathcal{S} = \varinjlim \mathbb{S}_n.$$

Now for any $1 \leqslant i \leqslant n$, we define the binary operator $\circ_i : \mathbb{S}_n \times \mathbb{S}_m \to \mathbb{S}_{k+m-1}$ by

$$a_s^{(n)} \circ_i a_{s'}^{(m)} := a_{s \circ_i^{(m)} s'}^{(n+m-1)}. \tag{2.23}$$

Proposition 4. *The set $\mathbb{S}$ endowed with the partial compositions $\circ_i$ is an operad.*

Proof. From (2.21), the unit of the structure is $\mathbf{1}_{\mathbb{S}} := a^{(1)}_{\mathbf{0}_{\mathcal{S}}}$. The associativity properties follows from (2.19) and (2.20). □

Example 2. Let us consider again the precomposition of Example 1. The binary operators $\circ_i^{(m)}$ are described in terms of infinite matrices by

$$v \circ_i^{(m)} v' = \mathfrak{n}_{i,n} v + \mathfrak{n}_{0,i} v' = \begin{bmatrix} v_1 \\ \vdots \\ v_{i-1} \\ v_i + v'_1 \\ v'_2 \\ \vdots \\ v'_m \\ v_{i+1} + v'_{m+1} \\ v_{i+2} + v'_{m+2} \\ \vdots \end{bmatrix}.$$

Remark that each symbol $a_v^{(m)}$ can be identified with a vector w of size m such that $v_i = w_i$ for each $i \leqslant m$. Hence, the compositions are given by

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \circ_i \begin{bmatrix} v'_1 \\ \vdots \\ v'_m \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_{i-1} \\ v_i + v'_1 \\ v'_2 \\ \vdots \\ v'_m \\ v_{i+1} \\ \vdots \\ v_n \end{bmatrix}.$$

For instance, we can illustrate (1.2) by remarking

$$(v \circ_i v') \circ_{j+m-1} v'' = \begin{bmatrix} v_1 \\ \vdots \\ v_{i-1} \\ v_i + v'_1 \\ v'_2 \\ \vdots \\ v'_m \\ v_{i+1} \\ \vdots \\ v_{j-1} \\ v_j + v''_1 \\ v''_2 \\ \vdots \\ v''_p \\ v_{j+1} \\ \vdots \\ v_n \end{bmatrix} = v \circ_i (v' \circ_j v'').$$

for $i < j$, $v \in \mathbb{B}^n$, $v' \in \mathbb{B}^m$, and $v'' \in \mathbb{B}^p$.

We denote by $\mathrm{OP}(\circ)$ the operad $(\mathbb{S}, \circ_i)$ as defined in the construction. For any $\phi \in \mathrm{Hom}(\circ, \triangleright)$, we define

$$\phi^{\mathrm{OP}} : \mathrm{OP}(\circ) \to \mathrm{OP}(\triangleright) \tag{2.24}$$

by

$$\phi^{\mathrm{OP}}(a_s^{(n)}) = a_{\phi(s)}^{(n)}. \tag{2.25}$$

We check that the following result holds:

Claim 2. *The arrow* $\mathrm{OP} : \mathrm{PreComp} \to \mathrm{Operad}$ *which associates with each precomposition* $\circ$ *the operad* $\mathrm{OP}(\circ)$ *and with each homomorphism* $\phi \in \mathrm{Hom}(\circ, \triangleright)$ *the operad morphism* ϕ^{OP} *is a functor.*

2.4. **Quotients of precompositions.** Let $\circ : \square \to \mathrm{Hom}(\mathcal{S}, \mathcal{S})$ be a precomposition and $\gamma : \mathcal{S} \to \mathcal{S}$ be an idempotent (that is, $\gamma^2 = \gamma$) monoid morphism sending $\mathcal{S}_n$ to $\mathcal{S}_n$ and satisfying: $\overset{i,n}{\circ\!\!\to}\gamma = \gamma\overset{i,n}{\circ\!\!\to}$. We define $\gamma : \mathbb{S} \to \mathbb{S}$ by $\gamma a_s^{(n)} := a_{\gamma s}^{(n)}$.

Lemma 3. *The two following assertions hold:*

(1) *for each* $s \in \mathcal{S}_n$, $s' \in \mathcal{S}_m$, *and* $1 \leqslant i \leqslant n$,

$$\gamma\left(\gamma(a_s^{(n)}) \circ_i \gamma(a_{s'}^{(m)})\right) = \gamma\left(a_s^{(n)} \circ_i a_{s'}^{(m)}\right), \tag{2.26}$$

(2) $\gamma(s_1) = \gamma(s'_1)$ *and* $\gamma(s_2) = \gamma(s'_2)$ *implies*

$$\gamma(a_{s_1}^{(n)} \circ_i a_{s_2}^{(m)}) = \gamma(a_{s'_1}^{(n)} \circ_i a_{s'_2}^{(m)}). \tag{2.27}$$

Proof. (1) We have $\gamma(a_s^{(n)}) \circ_i \gamma(a_{s'}^{(m)}) = a_{\gamma s}^{(n)} \circ_i a_{\gamma s'}^{(m)} = a_{s''}^{(n+m-1)}$ with

$$s'' = \gamma(s) \circ_i^{(m)} \gamma(s') = \overset{i,m}{\circ\!\!\rightarrow}(\gamma s) \oplus \overset{0,i}{\circ\!\!\rightarrow}(\gamma s') = \gamma\left(\overset{i,m}{\circ\!\!\rightarrow}(s) \oplus \overset{0,i}{\circ\!\!\rightarrow}(s')\right).$$

Hence $\gamma\left(\gamma(a_s^{(n)}) \circ_i \gamma(a_{s'}^{(m)})\right) = a_{s'''}^{(n+m-1)}$ with

$$\begin{aligned} s''' &= \gamma\left(\gamma\left(\overset{i,m}{\circ\!\!\rightarrow}(s) \oplus \overset{0,i}{\circ\!\!\rightarrow}(s')\right)\right) = \gamma\left(\overset{i,m}{\circ\!\!\rightarrow}(s) \oplus \overset{0,i}{\circ\!\!\rightarrow}(s')\right) \\ &= \gamma\left(\overset{i,m}{\circ\!\!\rightarrow}(a_s^{(k)}) \oplus \overset{0,i}{\circ\!\!\rightarrow}(a_{s'}^{(m)})\right) = s''. \end{aligned}$$

(2) Suppose $\gamma(s_1) = \gamma(s')$ and $\gamma(s_2) = \gamma(s_2')$ then we have

$$\gamma(a_{s_1}^{(n)} \circ_i a_{s_2}^{(m)}) = \gamma\gamma(a_{s_1}^{(n)} \circ_i a_{s_2}^{(m)}) = \gamma(a_{\gamma s_1}^{(n)} \circ_i a_{\gamma s_2}^{(m)}) = \gamma(a_{\gamma s_1'}^{(n)} \circ_i a_{\gamma s_2'}^{(m)}) = \gamma(a_{s_1'}^{(n)} \circ_i a_{s_2'}^{(m)}).$$

$\square$

Consider now the equivalence relation $\sim_\gamma$ on $\mathcal{S}$ defined for any $s, s' \in \mathcal{S}$ by $s \sim_\gamma s'$ if and only if $\gamma(s) = \gamma(s')$. By definition of γ, $\sim_\gamma$ is a monoid congruence of $\mathcal{S}$ and hence, $\mathcal{S}/_{\sim_\gamma}$ is a monoid. Consider also the equivalence relation $\equiv_\gamma$ on $\mathrm{OP}(\circ)$ defined for any $a_s^{(n)}, a_{s'}^{(n)} \in \mathrm{OP}(\circ)$ by $a_s^{(n)} \equiv_\gamma a_{s'}^{(n)}$ if and only if $s \sim_\gamma s'$. Lemma 3 shows that $\equiv_\gamma$ is actually an operadic congruence and hence, that $\mathrm{OP}(\circ)/_{\equiv_\gamma}$ is an operad.

Let the precomposition

$$\odot : \square \to \mathrm{Hom}\left(\mathcal{S}/_{\sim_\gamma}, \mathcal{S}/_{\sim_\gamma}\right) \tag{2.28}$$

defined for any $\sim_\gamma$-equivalence class $[s]_{\sim_\gamma}$ by $\overset{i,n}{\odot\!\!\rightarrow}\left([s]_{\sim_\gamma}\right) := [\overset{i,n}{\circ\!\!\rightarrow}(s)]_{\sim_\gamma}$. We then have

Theorem 2. *The operads* $\mathrm{OP}(\circ)/_{\equiv_\gamma}$ *and* $\mathrm{OP}(\odot)$ *are isomorphic.*

Proof. Let us denote by $\circ_i^\gamma$ the composition map of $\mathrm{OP}(\circ)/_{\equiv_\gamma}$. Let the map

$$\phi : \mathrm{OP}(\circ)/_{\equiv_\gamma} \to \mathrm{OP}(\odot) \tag{2.29}$$

defined for any $\equiv_\gamma$-equivalence class $[a_s^{(n)}]_{\equiv_\gamma}$ by

$$\phi([a_s^{(n)}]_{\equiv_\gamma}) := a_{[s]_{\sim_\gamma}}^{(n)}. \tag{2.30}$$

Let us show that ϕ is an operad morphism. For that, let $[a_s^{(n)}]_{\equiv_\gamma}$ and $[a_{s'}^{(m)}]_{\equiv_\gamma}$ be two $\equiv_\gamma$ equivalence classes. One has

$$\phi([a_s^{(n)}]_{\equiv_\gamma} \circ_i^\gamma [a_{s'}^{(m)}]_{\equiv_\gamma}) \;=\; \phi([a_s^{(n)} \circ_i a_{s'}^{(m)}]_{\equiv_\gamma}) \;=\; \phi([a_{s''}^{(n+m-1)}]_{\equiv_\gamma}) \;=\; a_{[s'']_{\sim_\gamma}}^{(n+m-1)}, \tag{2.31}$$

where $s'' := \overset{i,m}{\circ\!\!\rightarrow}(s) \oplus \overset{0,i}{\circ\!\!\rightarrow}(s')$. We moreover have

$$\phi([a_s^{(n)}]_{\equiv_\gamma}) \odot_i \phi([a_{s'}^{(m)}]_{\equiv_\gamma}) \;=\; a_{[s]_{\sim_\gamma}}^{(n)} \odot_i a_{[s']_{\sim_\gamma}}^{(m)} \;=\; a_{[s''']_{\sim_\gamma}}^{(n+m-1)}, \tag{2.32}$$

where $[s''']_{\sim_\gamma} := \overset{i,m}{\odot\!\!\to}([s]_{\sim_\gamma}) \oplus \overset{0,i}{\odot\!\!\to}([s']_{\sim_\gamma})$. Now, by using the fact that $\sim_\gamma$ is a monoid congruence, one has

$$\begin{aligned}[s''']_{\sim_\gamma} &= \overset{i,m}{\odot\!\!\to}([s]_{\sim_\gamma}) \oplus \overset{0,i}{\odot\!\!\to}([s']_{\sim_\gamma}) \\ &= [\overset{i,m}{\circ\!\!\to}(s)]_{\sim_\gamma} \oplus [\overset{0,i}{\circ\!\!\to}(s')]_{\sim_\gamma} \\ &= [\overset{i,m}{\circ\!\!\to}(s) \oplus \overset{0,i}{\circ\!\!\to}(s')]_{\sim_\gamma} \\ &= [s'']_{\sim_\gamma}.\end{aligned} \tag{2.33}$$

This shows that (2.31) and (2.32) are equal and hence, that ϕ is an operad morphism.

Furthermore, the definitions of $\sim_\gamma$ and $\equiv_\gamma$ imply that ϕ is a bijection. Therefore, ϕ is an operad isomorphism. □

3. Multi-tildes and precompositions

In [LMN13], we investigated several operads allowing to describe the behaviour of the multi-tilde operators. In this section, we show that some of them admit an alternative definition using the notion of precomposition.

3.1. The operad $\mathcal{T}$ revisited.

For any $m \geqslant 1$, let $\mathcal{S}^{\mathcal{T}}_m$ be the set of subsets of $\{(x,y) : 1 \leqslant x \leqslant y \leqslant m\} \subset \mathbb{N}^2$. Noting that $\mathcal{S}^{\mathcal{T}}_m \subset \mathcal{S}^{\mathcal{T}}_{m+1}$ we define $\mathcal{S}^{\mathcal{T}} := \bigcup_{m\geqslant 1} \mathcal{S}^{\mathcal{T}}_m$. Considering the binary operation $\cup$ as a product, the pair $(\mathcal{S}^{\mathcal{T}}, \cup)$ defines a commutative monoid whose unit is $\mathbf{1}_{\mathcal{S}^{\mathcal{T}}} = \emptyset \in \mathcal{S}^{\mathcal{T}}_1$. This monoid is generated by the set $\{\{(x,y)\} : 1 \leqslant x \leqslant y\}$.

Now define $\circ : \square \to \mathrm{Hom}(\mathcal{S}^{\mathcal{T}}, \mathcal{S}^{\mathcal{T}})$ by $\circ(\overset{i,n}{\square\!\!\Rightarrow}) := \overset{i,n}{\circ\!\!\to}$ where each homomorphism $\overset{i,n}{\circ\!\!\to}$ is defined by its values on the generators

$$\overset{i,n}{\circ\!\!\to}(\{(x,y)\}) := \begin{cases} \{(x,y)\} & \text{if } y < i, \\ \{(x, y+n-1)\} & \text{if } x \leqslant i \leqslant y, \\ \{(x+n-1, y+n-1)\} & \text{otherwise.}\end{cases} \tag{3.1}$$

Remark that $\circ$ is a monoid morphism. Indeed,

(1) The set of the homomorphisms $\overset{i,n}{\circ\!\!\to}$ generates a submonoid of $\mathrm{Hom}(\mathcal{S}^{\mathcal{T}}, \mathcal{S}^{\mathcal{T}})$ (with $\mathrm{Id}_{\mathcal{S}^{\mathcal{T}}}$ as unit)
(2) By construction, $\overset{i,n}{\circ\!\!\to} : \mathcal{S}^{\mathcal{T}}_m \to \mathcal{S}^{\mathcal{T}}_{m+n-1}$ and $\overset{i,n}{\circ\!\!\to}|_{\mathcal{S}^{\mathcal{T}}_m} = \mathrm{Id}_{\mathcal{S}^{\mathcal{T}}_m}$ if $m < i$.
(3) The operators $\overset{i,n}{\circ\!\!\to}$ satisfy (see [LMN13])
   - $\overset{i,n}{\circ\!\!\to} = \overset{0,n}{\circ\!\!\to}$ for each $i < 0$,
   - $\overset{i,1}{\circ\!\!\to} = \overset{0,1}{\circ\!\!\to} = \mathrm{Id}_{\mathcal{S}^{\mathcal{T}}}$ for each i,
   - $\overset{i,n}{\circ\!\!\to}\overset{j,n'}{\circ\!\!\to} = \overset{j+n-1,n'}{\circ\!\!\to}\ \overset{i,n}{\circ\!\!\to}$ if $i \leqslant j$ or $i, j \leqslant 0$,
   - $\overset{i+j,n}{\circ\!\!\to}\overset{i,n'}{\circ\!\!\to} = \overset{i,n+n'-1}{\circ\!\!\to}$ if $0 \leqslant j < n'$.

Hence $\circ$ is a precomposition. More precisely, the operad $\mathcal{T}$ can be seen as the operad constructed from the precomposition $\circ$:

Proposition 5. *The operads $\mathcal{T}$ and $\mathrm{OP}(\circ)$ are isomorphic.*

Proof. The isomorphism is given by the map from $\mathcal{T}_m$ to $\mathbb{S}_m$ sending any element T to $a_T^{(k)}$. $\square$

3.2. **The operad** RAS **revisited.** In [LMN13], we considered an operad RAS on reflexive and antisymmetric relations that are compatible with the natural order on integers (*i.e.*, $(x, y) \in \mathrm{RAS}$ implies $x \leqslant y$). Since the elements (x, x) do not play any role in the construction, we propose here an alternative construction based on antireflexive and antisymmetric relations.

For any $m \geqslant 1$, let $\mathcal{S}_m^{\diamond}$ be the set of subsets of $\{(x, y) : 1 \leqslant x < y \leqslant m + 1\} \subset \mathbb{N}^2$. By construction we have $\mathcal{S}_n^{\diamond} \subset \mathcal{S}_{m+1}^{\diamond}$. Endowed with the binary operation $\cup$, the set $\mathcal{S}^{\diamond} := \bigcup_{m \geqslant 1} \mathcal{S}_m^{\diamond}$ is a commutative monoid generated by $\{\{(x, y)\}_{1 \leqslant x < y}\}$.

Let us define $\diamond : \square \to \mathrm{Hom}(\mathcal{S}^{\diamond}, \mathcal{S}^{\diamond})$ by $\diamond(\overset{i,n}{\square\!\!\Rightarrow}) := \overset{i,n}{\diamond\!\!\to}$ with

$$\overset{i,k}{\diamond\!\!\to}(\{(x, y)\}) := \begin{cases} \{(x, y)\} & \text{if } y \leqslant i, \\ \{(x, y + n - 1)\} & \text{if } x \leqslant i < y, \\ \{(x + n - 1, y + n - 1)\} & \text{otherwise.} \end{cases} \tag{3.2}$$

Figure 1 illustrates the action of the operator $\overset{i,n}{\diamond\!\!\to}$ as an operation cutting one triangle representing the set of all the couples (x, y) with $1 \leqslant x < y \leqslant m + 1$ into two triangles and a rectangle and putting them back into a larger triangle.

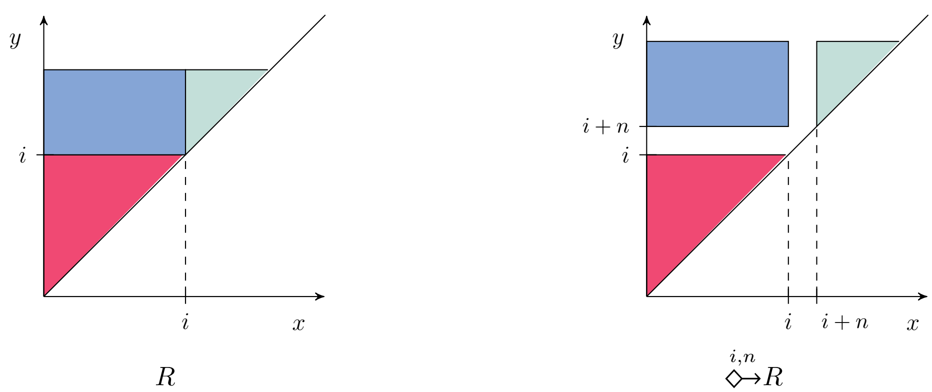


FIGURE 1. Action of RAS on pairs.

Similarly to Section 3.1, we consider the submonoid of $\mathrm{Hom}(\mathcal{S}^{\diamond}, \mathcal{S}^{\diamond})$ generated by the elements $\overset{i,n}{\diamond\!\!\to}$. We have $\overset{i,n}{\diamond\!\!\to} : \mathcal{S}_m^{\diamond} \to \mathcal{S}_{n+n-1}^{\diamond}$ and $\overset{i,n}{\diamond\!\!\to}|_{\mathcal{S}_m^{\diamond}} = \mathrm{Id}_{\mathcal{S}_m^{\diamond}}$ when $m < i$. Furthermore, the elements $\overset{i,n}{\diamond\!\!\to}$ satisfy the properties

- $\overset{i,n}{\diamond\!\!\to} = \overset{0,n}{\diamond\!\!\to}$ for each $i < 0$,
- $\overset{i,1}{\diamond\!\!\to} = \overset{0,1}{\diamond\!\!\to} = \mathrm{Id}_{\mathcal{S}^{\diamond}}$ for each i
- $\overset{i,n}{\diamond\!\!\to}\overset{j,n'}{\diamond\!\!\to} = \overset{j+n-1,n'}{\diamond\!\!\to}\ \overset{i,n}{\diamond\!\!\to}$ if $i \leqslant j$ or $i, j \leqslant 0$
- $\overset{i+j,n}{\diamond\!\!\to}\overset{i,n'}{\diamond\!\!\to} = \overset{i,n+n'-1}{\diamond\!\!\to}$ if $0 \leqslant j < n'$.

The map $\diamond$ is a monoid morphism and so a precomposition. We set $\mathrm{ARAS} := \mathrm{OP}(\diamond) = (\mathbb{S}^{\diamond}, \diamond)$. The operad ARAS is an alternative construction for the operad RAS as shown by:

Proposition 6. *The operads* RAS *and* ARAS *are isomorphic.*

Proof. The isomorphism is given by the map from RAS_n to $\mathbb{S}^{\diamond}_m$ sending any element R to $a^{(m)}_{R\setminus\Delta}$ where $\Delta := \{(x,x) : x \in \mathbb{N}\}$. ◻

3.3. **The operad** POSet **revisited.** The operad POSet is defined as a quotient of the operad RAS. In [LMN13], we showed that POSet is optimal in the sense that two of its operators have two different actions on languages.

Denote by $\gamma : S^{\diamond} \to S^{\diamond}$ the transitive closure. Remarking that $\gamma(R) : S^{\diamond}_n \to S^{\diamond}_n$ and $\overset{i,n}{\diamond\!\!\rightarrow}\gamma = \gamma\overset{i,n}{\diamond\!\!\rightarrow}$, we apply the result of Section 2.4 and define the precomposition $\diamond : \square \to \mathrm{Hom}(S^{\diamond}_{/\equiv_\gamma}, S^{\diamond}_{/\equiv_\gamma})$ by setting $\overset{i,n}{\diamond\!\!\rightarrow}([R]) := \left[\overset{i,n}{\diamond\!\!\rightarrow}(R)\right]$ where $[] : S^{\diamond} \to S^{\diamond}_{/\equiv_\gamma}$ denotes the natural morphism sending each element R of $S^{\diamond}$ to its equivalence class $[R]$.

The operad $\mathrm{OP}(\diamond)$ gives an alternative way to define the operad POSet using precompositions.

Proposition 7. *The operads* POSet*,* $\mathrm{OP}(\diamond)$ *and* $\mathrm{ARAS}_{/\equiv_\gamma}$ *are isomorphic.*

Proof. The isomorphism is given by the map from POSet_n to $\mathbb{S}^{\diamond}_n$ sending any element P to $a^{(n)}_{[(P\setminus\Delta)]}$ where $\Delta = \{(x,x) : x \in \mathbb{N}\}$. ◻

4. The operad of double multi-tildes

In [LMN13], we proved that the action of $\mathcal{T}$ on symbols allows us to denote all finite languages. In this section, we propose an extension of the operad $\mathcal{T}$ in order to represent infinite languages. New operators are required in order to describe the Kleene star operation *. In the last section of [LMN13], we introduced an operad $\mathcal{T}^*$ generated by $\mathcal{T}$ together with an additional operator $\star$ (denoting the Kleene star *). Albeit this operad allows the manipulation of regular languages, the equivalence of the operators, w.r.t. the action over languages, is difficult to model. In this section, we introduce a new operad $\mathcal{DT}$ which is composed of two kinds of multi-tildes: right and left multi-tildes. The * operation will be realized by a combination of right and left multi-tildes operations. Furthermore, we show that the expressiveness of these operators is higher than operators of $\mathcal{T}^\star$ for a given number of symbols. We start by considering that the two types of operators are independently composed. More precisely,

$$\mathcal{DT} := \mathbb{H}(\mathcal{T}, \mathcal{T}). \tag{4.1}$$

We mimic the construction of [LMN13] linking multi-tildes and reflexive antisymmetric relations in order to construct a new operad ARef, which elements are antireflexive relations, isomorphic to $\mathcal{DT}$.

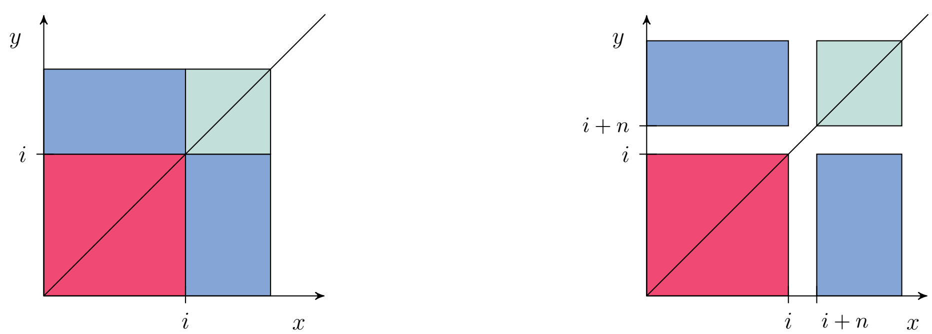
4.1. $\mathcal{DT}$ **and antireflexive relations.** We consider the graded set

$$\mathcal{S}^{\mathrm{ARef}} := \bigcup_{n\geqslant 1} \mathcal{S}_n^{\mathrm{ARef}}, \tag{4.2}$$

where $\mathcal{S}_n^{\mathrm{ARef}}$ is the set of subsets of $\{(x,y) : 1 \leqslant x, y \leqslant n+1,\, x \neq y\} \subset \mathbb{N}^2$. Endowed with the binary operation $\cup$, the set $\mathcal{S}^{\mathrm{ARef}}$ is a commutative monoid generated by $\{(x,y) : x \neq y\}$. We define the map $\diamond : \square \to \mathrm{Hom}(\mathcal{S}^{\mathrm{ARef}}, \mathcal{S}^{\mathrm{ARef}})$ by $\diamond(\overset{i,n}{\square\!\!\Rightarrow}) = \overset{i,n}{\diamond\!\!\rightarrow}$ where

$$\overset{i,n}{\diamond\!\!\rightarrow}(\{(x,y)\}) := \begin{cases} \{(x,y)\} & \text{if } x, y \leqslant i, \\ \{(x, y+n-1)\} & \text{if } x \leqslant i \text{ and } i < y, \\ \{(x+n-1, y)\} & \text{if } i < x \text{ and } y \leqslant i, \\ \{(x+n-1, y+n-1\} & \text{otherwise.} \end{cases} \tag{4.3}$$

We easily check that $\diamond$ is a precomposition and we set $\mathrm{ARef} := \mathrm{OP}(\diamond)$. Observe that a graphical representation of the action of ARef can be obtained from those of ARAS by replacing triangles by squares (see Figure 2).



R $\qquad\qquad$ $\overset{i,n}{\diamond\!\!\rightarrow} R$

FIGURE 2. Action of ARef on pairs.

Proposition 8. *The operad* ARef *is isomorphic to* $\mathbb{H}(\mathrm{ARAS}, \mathrm{ARAS})$.

Proof. Let us denote $\mathrm{rev}(x,y) = (y,x)$. If $R \in \mathrm{ARef}$ we will denote $\mathrm{rev}(R) = \{\mathrm{rev}(x,y) : (x,y) \in R\}$, $R^< = \{(x,y) \in R : x < y\}$, and $R^> = \{(x,y) \in R : x > y\} = \mathrm{rev}((\mathrm{rev}(R))^<)$ (note that $R = R^< \cup R^>$). Let $\Phi : \mathbb{H}(\mathrm{ARAS}, \mathrm{ARAS}) \to \mathrm{ARef}$ be the map defined by $\Phi(a_{R_1}^{(n)}, a_{R_2}^{(n)}) = a_{R_1 \cup \mathrm{rev}(R_2)}^{(n)}$. This map is a bijection which inverse is $\Phi^{-1}(a_R^{(n)}) = (a_{R^<}^{(n)}, a_{\mathrm{rev}(R^>)}^{(n)})$.
Let us prove that $a_R^{(n)} \diamond_i a_{R'}^{(m)} = \Phi(\Phi^{-1}(a_R^{(n)}) \diamond_i \Phi^{-1}(a_{R'}^{(m)}))$. We have

$$\Phi(\Phi^{-1}(a_R^{(n)}) \diamond_i \Phi^{-1}(a_{R'}^{(m)})) = \Phi(\Phi^{-1}(a_{R''}^{(n+m-1)})) = \Phi(a_{R''^<}^{(n+m-1)}, a_{\mathrm{rev}(R''^>)}^{(n+m-1)}),$$

where $R'' = \overset{i,m}{\diamond\!\!\rightarrow}(R) \cup \overset{0,i}{\diamond\!\!\rightarrow}(R')$. Let $* \in \{<, >\}$. Since, $\overset{i,k'}{\diamond\!\!\rightarrow}(R^*) = (\overset{i,m}{\diamond\!\!\rightarrow}(R))^*$ and $\overset{0,i}{\diamond\!\!\rightarrow}(R'^*) = (\overset{0,i}{\diamond\!\!\rightarrow}(R'))^*$, we have $(\overset{i,m}{\diamond\!\!\rightarrow}(R^*) \cup \overset{0,i}{\diamond\!\!\rightarrow}(R'^*)) = (R'')^*$. In other words, $R''^* = \overset{i,m}{\diamond\!\!\rightarrow}(R^*) \cup \overset{0,i}{\diamond\!\!\rightarrow}(R'^*)$. Hence,

$$\Phi(\Phi^{-1}(R) \diamond_i \Phi^{-1}(R')) = \Phi(a_{R''^<}^{(n+m-1)}, a_{\mathrm{rev}(R''^>)}^{(n+m-1}) = a_{R''^< \cup R''^>}^{(n+m-1} = a_{R''}^{(n+m-1)} = a_R^{(n)} \diamond_i a_{R'}^{(m)}.$$

This proves that ARef is an operad isomorphic to $\mathbb{H}(\mathrm{ARAS}, \mathrm{ARAS})$. $\square$

Corollary 1. *The operads* $\mathcal{DT}$, ARef, $\mathbb{H}(\mathrm{RAS},\mathrm{RAS})$, *and* $\mathbb{H}(\mathrm{ARAS},\mathrm{ARAS})$ *are isomorphic.*

In the aim to illustrate the isomorphism between ARef and $\mathcal{DT}$, we recall that the graded map $\zeta : \mathcal{T}_n \to \mathrm{RAS}_n$ defined by $\zeta(R) = \{(x, y+1) : (x,y) \in R\} \cup \{(1,1),\dots,(n+1,n+1)\}$ is an isomorphism of operads. According to the definition of ARAS, we obtain explicitly an isomorphism from $\mathcal{T}$ to ARAS by a slight modification of ζ: $\zeta^A(R) = a^{(n)}_{\zeta(R)\setminus\Delta}$. Since ARAS and $\mathcal{T}$ are isomorphic, this is also the case for $\mathcal{DT}$ and ARef (because ARef is isomorphic to $\mathbb{H}(\mathrm{ARAS},\mathrm{ARAS})$). From the construction described in Proposition 8, the map $\xi : \mathcal{DT} \to \mathrm{ARef}$ defined by $\xi(R_1,R_2) = a^{(n)}_{\zeta^A(R_1)\cup\mathrm{rev}(\zeta^A(R_2))}$, when $(R_1,R_2) \in \mathcal{DT}_n$, explicits the isomorphism.

Example 3. Consider $P_1 = (\{(1,3),(2,2),(3,4)\},\{(2,3)\}) \in \mathcal{DT}_5$ and $P_2 = (\{(2,3),(3,4)\},\{(1,2),(3,4)\}) \in \mathcal{DT}_4$. We have

$$\xi(P_1) = a^{(5)}_{\{(1,4),(2,3),(3,5),(4,2)\}} \text{ and } \xi(P_2) = a^{(4)}_{\{(2,4),(3,5),(3,1),(5,3)\}}.$$

Remark that

$$\begin{aligned} P_1 \circ_2 P_2 &= (\{(1,3),(2,2),(3,4)\} \circ_2 \{(2,3),(3,4)\}, \{(2,3)\} \circ_2 \{(1,2),(3,4)\}) \\ &= (\{(1,6),(2,5),(6,7),(3,4),(4,5)\}, \{(2,6),(2,3),(4,5)\}), \end{aligned}$$

and then

$$\xi(P_1 \circ_2 P_2) = a^{(8)}_{\{(1,7),(2,6),(6,8),(3,5),(4,6),(7,2),(4,2),(6,4)\}}.$$

Let us now compute $\xi(P_1)\diamond_2\xi(P_2)$:

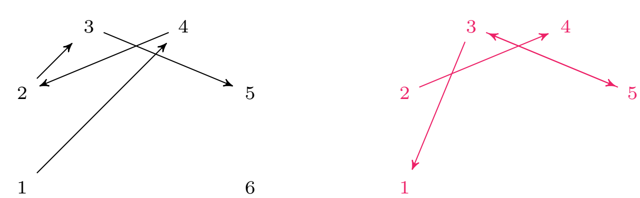
$$\xi(P_1)\diamond_2\xi(P_2) = a^{(5)}_{\{(1,4),(2,3),(3,5),(4,2)\}} \diamond_2 a^{(4)}_{\{(2,4),(3,5),(3,1),(5,3)\}} = a^{(8)}_R$$

with

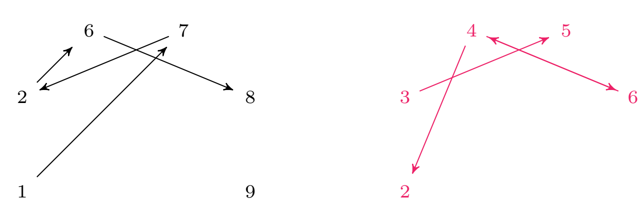
$$\begin{aligned} R &= \overset{2,4}{\diamond\!\!\rightarrow}(\{(1,4),(2,3),(3,5),(4,2)\}) \cup \overset{0,2}{\diamond\!\!\rightarrow}(\{(2,4),(3,5),(3,1),(5,3)\}) \\ &= \{(1,7),(2,6),(6,8),(7,2),(3,5),(4,6),(4,2),(6,4)\}. \end{aligned}$$

We observe that $\xi(P_1 \circ_2 P_2) = \xi(P_1)\diamond_2\xi(P_2)$.

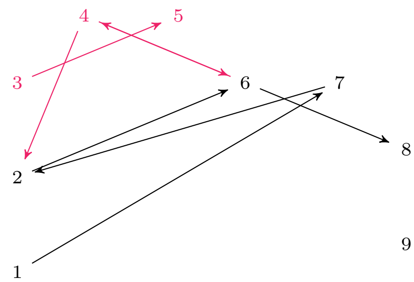
Graphically, the composition $\diamond_i$ can be illustrated in two steps corresponding to the operators $\overset{ik'}{\diamond\!\!\rightarrow}$ and $\overset{0,i}{\diamond\!\!\rightarrow}$ by drawing the graph of the relations. For instance, we start with the two graphs of the relations $\{(1,4),(2,3),(3,5),(4,2)\}$ and $\{(2,4),(3,5),(3,1),(5,3)\}$:



We rename the vertices $3 \to 6$, $4 \to 7$, $\dots$, $6 \to 9$ in the graphs of $\{(1,4),(2,3),(3,5),(4,2)\}$ and the vertices $1 \to 2, \dots, 5 \to 6$ in the graph of $\{(2,4),(3,5),(3,1),(5,3)\}$.



Then we identify the vertices which have the same label in the two graphs:



4.2. **An operad on quasiorders.** A *quasiorder* is a reflexive and transitive relation. If R is a relation we denote by $\gamma(R)$ its transitive closure. We also set $\gamma^{\mathrm{A}}(R) = \gamma(R) \setminus \{(n,n) : n \geqslant 1\}$ and $\gamma^{\mathrm{R}}(R) = \gamma(R) \cup \{(n,n) : n \geqslant 1\}$. Note that $\gamma^{\mathrm{R}}(R)$ is the smallest quasiorder which contains R. Since $\gamma^{\mathrm{A}} : \mathcal{S}^{\mathrm{ARef}} \to \mathcal{S}^{\mathrm{ARef}}$ is an idempotent monoid morphism sending S_n^{ARef} to S_n^{ARef} and satisfies $\overset{i,n}{\diamond\!\!\rightarrow}\gamma^A = \gamma^A \overset{i,n}{\diamond\!\!\rightarrow}$, following Section 2.4, we construct the precomposition

$$\diamond : \square \to \operatorname{Hom}\left(\mathcal{S}^{\mathrm{ARef}}/_{\equiv_{\gamma^A}}, \mathcal{S}^{\mathrm{ARef}}/_{\equiv_{\gamma^A}}\right) \tag{4.4}$$

defined by $\overset{i,n}{\diamond\!\!\rightarrow}([R]) := \left[\overset{i,n}{\diamond\!\!\rightarrow}(R)\right]$ where $[]$ denotes the natural morphism $\mathcal{S}^{\mathrm{ARef}} \to \mathcal{S}^{\mathrm{ARef}}/_{\equiv_{\gamma^A}}$ sending each relation to its class. Hence, we consider the operad $\mathrm{OP}(\diamond)$.

Alternatively, consider the set QOSet_n of quasiorder of $\{1,\dots,n+1\}$ and $\mathrm{QOSet} := \bigcup_n \mathrm{QOSet}_n$. Consider also the partial composition defined by $Q \diamond_i Q' = \gamma(\overset{i,m}{\diamond\!\!\rightarrow}(Q) \cup \overset{0,i}{\diamond\!\!\rightarrow}(Q'))$ if $Q \in \mathrm{QOSet}_n$, $Q' \in \mathrm{QOSet}_m$ and $i \leqslant n$.

Theorem 3. *The pair* $(\mathrm{QOSet}, \diamond)$ *is an operad isomorphic to* $\mathrm{OP}(\diamond)$.

Proof. Consider the map $\eta : \mathrm{QOSet} \to \mathrm{OP}(\diamond)$ given by $\eta(Q) = a^{(n)}_{[Q\setminus\Delta]}$. The map η is a graded bijection and its inverse is given by $\eta^{-1}(a^{(n)}_{[R]}) = \gamma^{\mathrm{R}}(R)$. Remarking that

$$\begin{aligned}\eta^{-1}(a^{(n)}_{[R']} \diamond_i a^{(n)}_{[R'']}) &= \gamma^{\mathrm{R}}(R' \diamond_i R'') \\ &= \gamma(\overset{i,m}{\diamond\!\!\rightarrow}(\gamma^{\mathrm{R}}(R'))) \cup \overset{0,i}{\diamond\!\!\rightarrow}(\gamma^{\mathrm{R}}(R'')) \\ &= \gamma^{\mathrm{R}}(R') \diamond_i \gamma^{\mathrm{R}}(R'') \\ &= \eta^{-1}(a^{(n)}_{[R']}) \diamond_i \eta^{-1}(a^{(n)}_{[R'']}),\end{aligned} \tag{4.5}$$

we prove that the set QOSet inherits from $\mathrm{OP}(\diamond)$ of a structure of operad. $\square$

Example 4. Let us give an example. Consider, as in Example 3, the antireflexive relations $R_1 = \{(1,4),(2,3),(3,5),(4,2)\}$ and $R_2 = \{(2,4),(3,5),(3,1),(5,3)\}$. We have

$\gamma^{\mathrm{R}}(R_1) = \{(1,4),(2,3),(3,5),(4,2),(1,2),(2,5),(4,3),(1,3),(4,5),(1,5),(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$

and

$$\gamma^{\mathrm{R}}(R_2) = \{(2,4),(3,5),(3,1),(5,3),(5,1),(1,1),(2,2),(3,3),(4,4),(5,5)\}.$$

We have

$$\begin{array}{rcl}\gamma^{\mathrm{R}}(R) & = & \gamma^{\mathrm{R}}(\overset{2,4}{\diamond\!\!\rightarrow}(R_1) \cup \overset{0,2}{\diamond\!\!\rightarrow}(R_2)) \\ & = & \gamma^{\mathrm{R}}(\{(1,7),(2,6),(6,8),(7,2),(3,5),(4,6),(4,2),(6,4)\}) \\ & = & \{(1,7),(2,6),(6,8),(7,2),(3,5),(4,6),(4,2),(6,4), \\ & & (1,2),(2,8),(2,4),(4,8),(7,8),(7,4),(6,2),(1,8),(1,4),(7,6),(1,6) \\ & & (1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9)\}.\end{array}$$

Also,

$$\begin{array}{rcl}\gamma(\overset{2,4}{\diamond\!\!\rightarrow}(\gamma^{R}(R_1)) \cup \overset{0,2}{\diamond\!\!\rightarrow}(\gamma^{R}(R_2))) & = & \gamma(\{(1,7),(2,6),(6,8),(7,2),(1,2),(2,8),(7,6),(1,6),(7,8),(1,8), \\ & & (1,1),(2,2),(6,6),(7,7),(8,8),(9,9)\} \\ & & \cup\{(3,5),(4,6),(4,2),(6,4),(6,3),(2,2),(3,3),(4,4),(5,5),(6,6)\}) \\ & = & \{(1,7),(2,6),(6,8),(7,2),(1,2),(2,8),(7,6),(1,6),(7,8),(1,8),(3,5),(4,6), \\ & & (4,2),(6,4),(6,2),(2,4),(4,8),(7,4),(1,4), \\ & & (1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9)\} \\ & = & \gamma^{\mathrm{R}}(R)\end{array}$$

4.3. **Infinite matrices again.** We consider the set $\mathcal{R}$ of infinite matrices indexed by $\mathbb{N}\setminus\{0\}$, whose entries are in the boolean semiring $\mathbb{B}$ and which have only a finite number of non zero entries. Consider the map which sends a set $s \in \mathcal{S}^{ARef}$ to the matrix M_s such that $M_s[i,j]=1$ is $(i,j)\in s$ and 0 otherwise. Through this map, $\mathcal{S}_n^{ARef}$ is in a one to one correspondence with a finite subset $\widetilde{\mathcal{S}_n} \subset \mathcal{R}$. More precisely,

$$\widetilde{\mathcal{S}_n} = \{(m_{i,j})_{i,j>0} : m_{i,j} = 0 \text{ if } i > n+1 \text{ or } j > n+1 \text{ or } i = j\}.$$

Furthermore each $\widetilde{\mathcal{S}_n}$ is stable for the sum and is isomorphic to the monoid $\mathcal{S}_n^{ARef}$ and we observe that the precomposition $\tilde{\diamond}$ defined by $\tilde{\diamond}(\overset{i,n}{\square\!\!\Rightarrow})(M) = \mathfrak{n}_{i,k} M\ {}^{t}\mathfrak{n}_{i,n}$ is isomorphic to $\diamond$.

Example 5. We have

$$M_{\{(1,4),(2,3),(3,5),(4,2)\}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

And we check that

$$\begin{array}{rcl}\mathfrak{n}_{24} M_{\{(1,4),(2,3),(3,5),(4,2)\}}\ {}^{t}\mathfrak{n}_{24} & = & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} = M_{\{(1,7),(2,6),(6,8),(7,2)\}} \\ & = & M_{\overset{2,4}{\diamond\!\!\rightarrow}\{(1,4),(2,3),(3,5),(4,2)\}}.\end{array}$$

Each symbol $a_M^{(m)}$ can be assimilated with a $(m+1)\times(m+1)$ matrix $M^{(m)}$ which is obtained by considering only the entries $M_{i,j}$ of M such that $i,j \leqslant m+1$. Hence, the composition $a_M^{(n)} \circ_i a_{M'}^{(m)}$ is obtained by summing two $(m+n)\times(m+n)$ matrices: $(n_{i,m}M\ {}^t\mathfrak{n}_{i,m})^{(n+m)}$ and $(n_{0,i}M'\ {}^t\mathfrak{n}_{0,i})^{(n+m)}$.

Example 6. One has

$$
\begin{aligned}
M^{(5)}_{\{(1,4),(2,3),(3,5),(4,2)\}} \circ_2 M^{(4)}_{\{(2,4),(3,5),(3,1),(5,3)\}} &= \begin{bmatrix} 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix} \circ_2 \begin{bmatrix} 0&0&0&0&0\\ 0&0&0&1&0\\ 1&0&0&0&1\\ 0&0&0&0&0\\ 0&0&1&0&0 \end{bmatrix} \\
&= \begin{bmatrix} 0&0&0&0&0&0&1&0&0\\ 0&0+0&0&0&0&1+0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&1&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0+0&0&1&0&0+0&0&1&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix} \\
&= M^{(9)}_{\{(1,7),(2,6),(3,5),(4,2),(4,6),(6,4),(6,8),(7,2)\}}.
\end{aligned}
$$

This is coherent with Example 3.

Set $M^+ = M + M^2 + \cdots$. Remarking that $M_s^+ = M_{\gamma(s)}$, we deduce that if $M \in \widetilde{\mathcal{S}_n}$ then $M^+ = M + M^2 + \cdots$ still belongs to $\widetilde{\mathcal{S}_n}$. Let $\widetilde{\mathcal{S}_n}^+ = \{M^+ : M \in \widetilde{\mathcal{S}_n}\}$ and $\widetilde{\mathcal{S}}^+ = \bigcup_n \widetilde{\mathcal{S}_n}^+$. The set $\widetilde{\mathcal{S}}^+$ endowed with the operation $M \oplus M' = (M+M')^+$ is a monoid isomorphic to QOSet. Hence, one easily checks that

Proposition 9. *The homomorphism* $\widetilde{\diamond} \in \mathrm{Hom}(\square, \mathrm{Hom}(\widetilde{\mathcal{S}}^+, \widetilde{\mathcal{S}}^+))$ *defined by* $\widetilde{\diamond}(\overset{i,k}{\square\!\Rightarrow})(M) = (\mathfrak{n}_{i,k}M\ {}^t\mathfrak{n}_{i,k})^+$ *is a precomposition isomorphic to* $\diamond$.

5. Action on languages

The aim of this section is to describe regular languages by using the operads defined above. More precisely, we show that the set of regular languages is a module on each of these operads. Furthermore, we prove that each regular language can be expressed by an operator acting on symbols or $\emptyset$. Finally, we show that the operad QOSet is optimal in the sense that its action is faithful.

5.1. **Action of** ARef**.** We recall that a grammar G is a 4-tuple (Σ, Γ, S, P) where Σ is a terminal alphabet, Γ a nonterminal alphabet, $S \in \Gamma$ an axiom and P a set of productions. The set of productions is a relation that contains couples of the form $X \to \alpha$, with X in Γ and α in $(\Sigma\cup\Gamma)^*$. We denote by $\Rightarrow$ the catenation stability closure of $\to$ *i.e.*, the transitive and reflexive closure of the relation $\to$. The language denoted by G is the set $\mathbb{L}(G) = \{w \in \Sigma^* \mid S_1 \Rightarrow w\}$.

An ε-automaton is a 5-tuple $(Q, \Sigma, \delta, i, F)$ where Q is a set of states, Σ is an alphabet, δ is a transition function from $Q \times (\Sigma \cup \{\varepsilon\})$ to 2^Q, $i \in Q$ is an initial state and $F \subset Q$ is a set of final states. The transition function δ can be extended for any integer n as the function $\delta_{\varepsilon,n}$ from Q to 2^Q as follows: for any $q \in Q$, $\delta_{\varepsilon,0}(q) = \delta(q, \varepsilon)$, $\delta_{\varepsilon,n+1}(q) = \bigcup_{q' \in \delta(q,\varepsilon)} \delta_{\varepsilon,n}(q')$. The transition function δ can also be extended as the function δ' from $2^Q \times \Sigma^*$ as follows: for any $Q \subset \Gamma_n$, for any word w in Σ^*,

$$\delta'(q, \varepsilon) = \{q\} \cup \{q' \mid q' \in \delta_{\varepsilon,n}(q) \text{ for some } n\},$$
$$\delta'(Q, w) = \bigcup_{q \in Q, q' \in \delta'(q,w)} \delta'(q', \varepsilon), \ \delta'(q, aw) = \delta'(\delta(q, a), w).$$

The language recognized by a ε-automaton is the set $\{w \mid \delta'(i, w) \cap F \neq \emptyset\}$.

We associate to each element $a_R^{(n)} \in \mathrm{ARef}_n$ a list of productions $\mathtt{P}(a_R^{(n)})$ defined by

(1) $\mathtt{S}_i \to \mathtt{a}_i \mathtt{S}_{i+1}$ for each $1 \leqslant i \leqslant k$,
(2) $\mathtt{S}_i \to \mathtt{S}_{i'}$ if $(i, i') \in R$,
(3) $\mathtt{S}_{k+1} \to \varepsilon$,

and we construct the grammar $\mathbf{G}_R^{(n)} := (\mathtt{A}_n, \Gamma_n, \mathtt{S}_1, \mathtt{P}(a_R^{(n)}))$ with $\mathtt{A}_n := \{\mathtt{a}_i : 1 \leqslant i \leqslant n\}$ and $\Gamma_n := \{\mathtt{S}_i : 1 \leqslant i \leqslant n+1\}$.

Example 7. Let $a^{(5)}_{\{(1,4),(2,3),(3,5),(4,2)\}}$, we have

$$\mathtt{P}(a^{(5)}_{\{(1,4),(2,3),(3,5),(4,2)\}}) = \begin{cases} \mathtt{S}_1 \to \mathtt{a}_1\mathtt{S}_2, \\ \mathtt{S}_1 \to \mathtt{S}_4, \\ \mathtt{S}_2 \to \mathtt{a}_2\mathtt{S}_3, \\ \mathtt{S}_2 \to \mathtt{S}_3, \\ \mathtt{S}_3 \to \mathtt{a}_3\mathtt{S}_4, \\ \mathtt{S}_3 \to \mathtt{S}_5, \\ \mathtt{S}_4 \to \mathtt{a}_4\mathtt{S}_5, \\ \mathtt{S}_4 \to \mathtt{S}_2, \\ \mathtt{S}_5 \to \mathtt{a}_5\mathtt{S}_6, \\ \mathtt{S}_6 \to \varepsilon. \end{cases}$$

Lemma 4. *The language $\mathbb{L}(\mathbf{G}_R^{(n)})$ denoted by the grammar $\mathbf{G}_R^{(n)}$ is regular.*

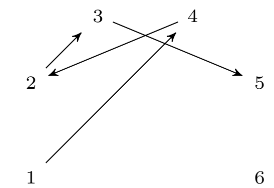
Proof. It is sufficient to remark that $\mathbb{L}(\mathbf{G}_R^{(n)})$ is recognized by the ε-automaton $\mathcal{A}(a_R^{(n)}) = (\Gamma_n, \mathtt{A}_n, \delta_R^{(n)}, \mathtt{S}_1, \{\mathtt{S}_{n+1}\})$ where the transitions $\delta_R^{(n)}$ are

$$\mathtt{S}_i \xrightarrow{\mathtt{a}_i} \mathtt{S}_{i+1} \quad \text{for each } 1 \leqslant i \leqslant k,$$
$$\mathtt{S}_i \xrightarrow{\varepsilon} \mathtt{S}_j \quad \text{for each } (i, j) \in R.$$

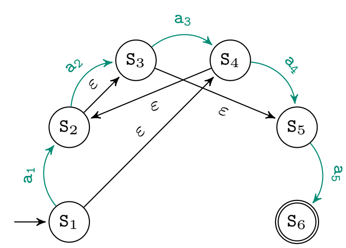
□

Note that the automaton $\mathcal{A}(a_R^{(n)})$ is just an interpretation of the relation R by adding transitions.

Example 8. We obtain the automaton $\mathcal{A}(a^{(5)}_{\{(1,4),(2,3),(3,5),(4,2)\}})$ from the graph of the relation $\{(1,4), (2,3), (3,5), (4,2)\}$



by adding transitions:



If $L_1, \ldots, L_n$ are languages, we define $\mathbf{G}_R^{(n)}(L_1, \ldots, L_n) = \mathbb{L}(\mathbf{G}_R^{(n)})|_{\mathtt{a}_i = L_i}$, that is the language $\mathbb{L}(G_R^{(n)})$ denoted by the grammar $\mathbf{G}_R^{(n)}$ where each letter $\mathtt{a}_i$ is replaced by the language L_i.

Example 9. Using the same relation than in Example 8 we find

$$\mathbb{L}(G_R^{(5)}) = (\mathtt{a}_1 + \varepsilon)(\mathtt{a}_3 + \mathtt{a}_2\mathtt{a}_3)^*(\mathtt{a}_5 + \mathtt{a}_2\mathtt{a}_5 + (\mathtt{a}_3 + \mathtt{a}_2\mathtt{a}_3)\mathtt{a}_4\mathtt{a}_5) + \mathtt{a}_4\mathtt{a}_5.$$

Therefore, if $L_1, \ldots, L_5$ are five languages,

$$G_R^{(5)}(L_1, \ldots, L_5) = (L_1 + \varepsilon)(L_3 + L_2L_3)^*(L_5 + L_2L_5 + (L_3 + L_2L_3)L_4L_5) + L_4L_5.$$

It is easy to see that this construction is compatible with the partial compositions in ARef. Indeed,

$$\mathbf{G}_{R\circ_i R'}^{(n+m-1)}(L_1, \ldots, L_{n+m-1}) = \mathbf{G}_R^{(n)}(L_1, \ldots, L_{i-1}, \mathbf{G}_{R'}^{(m)}(L_i, \ldots, L_{i+m-1}), L_{i+m}, \ldots, L_{n+m-1}), \tag{5.1}$$

for each $a_R^{(n)} \in \mathrm{ARef}_n$, $a_{R'}^{(m)} \in \mathrm{ARef}_m$, and $i \leqslant n$. Indeed,

$$\mathtt{P}(a_{R\circ_i R'}^{(n+m-1)}) = \begin{cases} \mathtt{S}_j \to \mathtt{a}_j\mathtt{S}_{j+1} \text{ for each } 0 \leqslant j \leqslant n+m-1, \\ \mathtt{S}_\ell \to \mathtt{S}_{\ell'} \text{ if } (\ell, \ell') = \overset{i,m}{\diamond\!\!\to}((j, j')) \text{ for } (j, j') \in R, \\ \mathtt{S}_\ell \to \mathtt{S}_{\ell'} \text{ if } (\ell, \ell') = \overset{0,i}{\diamond\!\!\to}((j, j')) \text{ for } (j, j') \in R', \\ \mathtt{S}_{n+m} \to \varepsilon. \end{cases}$$

Hence, we have

$$\begin{aligned} \mathtt{P}\left(a_{R\circ_i R'}^{(n+m-1)}\right) \quad = \quad & \{\mathtt{S}_j \to \mathtt{a}_j\mathtt{S}_{j+1} : 0 \leqslant j \leqslant n+m-1\} \\ & \cup\{\mathtt{S}_\ell \to \mathtt{S}_{\ell'} : \mathtt{S}_j \to \mathtt{S}_{j'} \in \mathtt{P}(a_R^{(n)}) \text{ and } (\ell, \ell') = \overset{i,m}{\diamond\!\!\to}((j, j))\} \\ & \cup\{\mathtt{S}_\ell \to \mathtt{S}_{\ell'} : \mathtt{S}_j \to \mathtt{S}_{j'} \in \mathtt{P}(a_{R'}^{(m)}) \text{ and } (\ell, \ell') = \overset{0,i}{\diamond\!\!\to}((j, j))\} \\ & \cup\{\mathtt{S}_{n+m-1} \to \varepsilon\} \end{aligned}$$

We deduce that

$$\mathbb{L}(\mathbf{G}_{R\circ_i R'}^{(n+m-1)}) = \mathbf{G}_R^{(n)}(\mathtt{a}_1, \ldots, \mathtt{a}_{i-1}, \mathbf{G}_{R'}^{(m)}(\mathtt{a}_i, \ldots, \mathtt{a}_{i+m-1}), \mathtt{a}_{i+m}, \ldots, \mathtt{a}_{n+m-1}).$$

This implies (5.1).

Remark 3. Alternatively, the construction on grammars can be described in terms of automata. The automaton $\mathcal{A}(a^{(n+m-1)}_{R\circ_i R'})$ is obtained by replacing the transition $\mathtt{S}_i \xrightarrow{\mathtt{a}_i} \mathtt{S}_{i+1}$ in $\mathcal{A}(a^{(n)}_R)$ by a copy of the automaton $\mathcal{A}(a^{(m)}_{R'})$ and hence relabeling the vertices and edges.

Setting $a^{(n)}_R.(L_1,\dots,L_n) = \mathbf{G}^{(n)}_R(L_1,\dots,L_n)$ we define an action of the operad ARef on languages.

Theorem 4. *The sets 2^{Σ^*} and $Reg(\Sigma)$ are* ARef*-modules.*

Proof. The fact that 2^{Σ^*} is a ARef-module is a direct consequence of (5.1).
Remarking that $\mathbb{L}(G^{(n)}_R) \in Reg(\mathtt{A}_n)$ (Lemma 4), we deduce that $\mathbf{G}^{(n)}_R(L_1,\dots,L_n) \in Reg(\Sigma)$ when $L_1,\dots,L_n \in Reg(\Sigma)$. Equivalently, $Reg(\Sigma)$ is ARef-module. □

Note that the action of ARef can be defined directly from $\mathcal{DT}$. For any $(a^{(n)}_{T_1}, a^{(n)}_{T_2}) \in \mathcal{DT}_n$, we construct the grammar $\mathbf{G}^{(n)}_{T_1,T_2} := (\mathtt{A}_n, \Gamma_n, \mathtt{S}_1, \mathtt{P}_{DT}(a^{(n)}_R))$ where the production rules $\mathtt{P}_{DT}\left(a^{(n)}_R\right)$ are

(1) $\mathtt{S}_i \to \mathtt{a}_i\mathtt{S}_{i+1}$ for each $1 \leqslant i \leqslant n$,
(2) $\mathtt{S}_i \to \mathtt{S}_{i'}$ if $(i', i-1) \in T_2$ or $(i, i'-1) \in T_1$,
(3) $\mathtt{S}_{n+1} \to \varepsilon$.

Example 10. Let $((13)(24)(34),(23)) \in \mathcal{DT}_5$. The grammar $\mathbf{G}^{(5)}_{(13)(22)(34),(23)}$ is

$$\left\{\begin{array}{lcl} \mathtt{S}_1 & \to & \mathtt{a}_1\mathtt{S}_2, \\ \mathtt{S}_1 & \to & \mathtt{S}_4, \\ \mathtt{S}_2 & \to & \mathtt{a}_2\mathtt{S}_3, \\ \mathtt{S}_2 & \to & \mathtt{S}_3, \\ \mathtt{S}_3 & \to & \mathtt{a}_3\mathtt{S}_4, \\ \mathtt{S}_3 & \to & \mathtt{S}_5, \\ \mathtt{S}_4 & \to & \mathtt{a}_4\mathtt{S}_5, \\ \mathtt{S}_4 & \to & \mathtt{S}_2, \\ \mathtt{S}_5 & \to & \mathtt{a}_5\mathtt{S}_6, \\ \mathtt{S}_6 & \to & \varepsilon. \end{array}\right. \tag{5.2}$$

Note that we recover the grammar $\mathbf{G}^{(5)}_{\{(1,4),(2,3),(3,5),(4,2)\}}$.

In general we have

Proposition 10. *For each $(a^{(n)}_{T_1}, a^{(n)}_{T_2}) \in \mathcal{DT}_n$, $\mathbf{G}^{(n)}_{T_1,T_2} = \mathbf{G}^{(n)}_{\xi(T_1,T_2)}$.*
Here ξ denotes the morphism from $\mathcal{DT}$ to ARef *as defined in Section 4.1.*

5.2. **Operadic expressions for regular languages.** The following proposition shows that any regular language admits an expression involving an operator of ARef and symbols of the alphabet or $\emptyset$.

Proposition 11. *Each regular language $L \in Reg(\Sigma)$ can be written as*

$$L = a^{(n)}_R(\alpha_1,\dots,\alpha_n)$$

for some $n \geqslant 1$, $a^{(n)}_R \in \mathrm{ARef}_n$, and $\alpha_1,\dots,\alpha_n \in \{\{\mathtt{a}\} : \mathtt{a} \in \Sigma\} \cup \{\emptyset\}$.

Proof. First note that $\{\mathtt{a}\} = a^{(1)}_{\emptyset}(\{\mathtt{a}\})$, $\{\varepsilon\} = a^{(1)}_{\{(1,2)\}}(\emptyset)$, and $\emptyset = a^{(1)}_{\emptyset}(\emptyset)$.
Suppose now that $L, L' \in \mathit{Reg}(\Sigma)$ are two regular languages satisfying

$$L = a^{(n)}_{R}(\alpha_1,\dots,\alpha_n) \text{ and } L' = a^{(m)}_{R'}(\alpha'_1,\dots,\alpha'_m)$$

for some $m \geqslant 1$, $a^{(n)}_{R} \in \mathrm{ARef}_n$, $a^{(m)}_{R'} \in \mathrm{ARef}_m$; and $\alpha_1,\dots,\alpha_n,\alpha'_1,\dots,\alpha'_m \in \{\{\mathtt{a}\} : \mathtt{a} \in \Sigma\} \cup \{\emptyset\}$. We have

$$L+L' = a^{(n+m+1)}_{R''}(\alpha_1,\dots,\alpha_n,\emptyset,\alpha'_1,\dots,\alpha'_m) \ \text{ with } \ R'' = R \cup \overset{0,n+1}{\diamond\!\!\to} R' \cup \{(1,n+2),(n+1,n+m+2)\}, \tag{5.3}$$

$$LL' = a^{(n+m+1)}_{R''}(\alpha_1,\dots,\alpha_n,\emptyset,\alpha'_1,\dots,\alpha'_m) \ \text{ with } \ R'' = R \cup \overset{0,n+1}{\diamond\!\!\to} R' \cup \{(n+1,n+2)\}, \tag{5.4}$$

$$L^* = a^{(n)}_{R\cup\{(n+1,1),(1,n+1)\}}(\alpha_1,\dots,\alpha_n). \tag{5.5}$$

The property is obtained by a straightforward induction. □

Remark 4. Note that in Formula (5.4), the symbol $\emptyset$ is important for the computation of the catenation. For instance, we have

$$\mathtt{a}^+\mathtt{b}^+ = a^{(1)}_{\{(2,1)\}}(\mathtt{a}) \cdot a^{(1)}_{\{(2,1)\}}(\mathtt{b}) = a^{(3)}_{\{(2,1),(2,3),(4,3)\}}(\mathtt{a},\emptyset,\mathtt{b}) \neq a^{(2)}_{\{(2,1),(3,2)\}}(\mathtt{a},\mathtt{b}) = (\mathtt{a}^+\mathtt{b}^+)^+.$$

But in some cases it may be omitted. For instance,

$$a^{(1)}_{\{(1,2)\}}(\mathtt{a}) \cdot a^{(1)}_{\{(1,2)\}}(\mathtt{b}) = a^{(2)}_{\{(1,2),(2,3)\}}(\mathtt{a},\mathtt{b}) = \varepsilon + \mathtt{a} + \mathtt{b} + \mathtt{ab}.$$

Let us give some examples. First we illustrate the construction described in the proof of Proposition 11.

Example 11. Consider the languages $L = \mathtt{b}(\mathtt{ab}^*) + \mathtt{a}^*$. We have $\{\mathtt{a}\} = a^{(1)}_{\emptyset}(\mathtt{a})$, $\{\mathtt{b}\} = a^{(1)}_{\emptyset}(\mathtt{b})$. So

$$\mathtt{b}^* = a^{(1)}_{(2,1),(1,2)}(\mathtt{b}),\ \mathtt{ab}^* = a^{(3)}_{(4,3),(3,4)}(\mathtt{a},\emptyset,\mathtt{b}) \text{ and } \mathtt{b}(\mathtt{ab}^*) = a^{(5)}_{(6,5),(5,6)}(\mathtt{b},\emptyset,\mathtt{a},\emptyset,\mathtt{b}).$$

On the other hand $\mathtt{a}^* = a^{(1)}_{(2,1),(1,2)}(\mathtt{a})$, hence

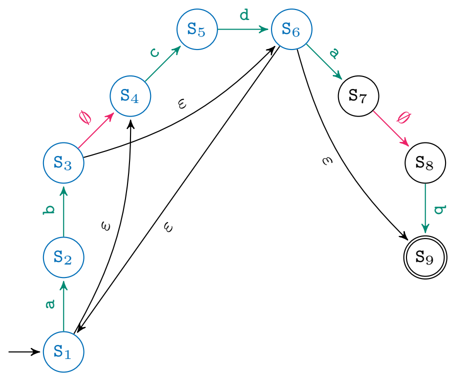
$$L = a^{(7)}_{(6,5),(5,6),(8,7),(7,8),(6,8),(1,7)}(\mathtt{b},\emptyset,\mathtt{a},\emptyset,\mathtt{b},\emptyset,\mathtt{a}).$$

Manipulating the relations allows to obtain some languages from others. We give here few constructions.

Example 12.

- Consider a language $L = a^{(n)}_{R}(\alpha_1,\dots,\alpha_n)$ with $R \in \mathrm{ARef}_n$ and $\alpha_i \in \{\{\mathtt{a}\} : \mathtt{a} \in \Sigma\}$. We define $R_P := R \cup \{(i,n+1) : 1 \leqslant i \leqslant n\}$. The language $a^{(n)}_{R_P}(\alpha_1,\dots,\alpha_n)$ is the set of the prefixes of L.
  For instance, consider $L = a^{(3)}_{(4,1),(1,4)}(\mathtt{a},\mathtt{b},\mathtt{c}) = (\mathtt{abc})^*$ we have $L = a^{(3)}_{(4,1),(1,4),(2,4),(3,4)}(\mathtt{a},\mathtt{b},\mathtt{c}) = (\mathtt{abc})^*\{\varepsilon,\mathtt{a},\mathtt{ab}\}$.
- For a more general regular language L, Proposition 11 implies that there exists $n > 0$, $R \in \mathrm{ARef}_n$, and $\alpha_i \in \{\{\mathtt{a}\} : \mathtt{a} \in \Sigma\} \cup \{\emptyset\}$ satisfying $L = a^{(n)}_{R}(\alpha_1,\dots,\alpha_n)$. An *admissible position* is an integer $1 \leqslant i \leqslant n+1$ such that there exists a path $i_1 = 1 \xrightarrow{\beta_1} i_2 \xrightarrow{\beta_2} i_3 \cdots i_{p-1} \xrightarrow{\beta_p} i_p = i_{n+1}$ in $\mathcal{A}(a^{(n)}_{R})$ with either $\beta_i = \varepsilon$ either $\beta_i = \mathtt{a}_i$ with $\alpha_i \neq \emptyset$ such that $i_\ell = i$ for some $1 \leqslant \ell \leqslant p-1$. The set of admissible positions is denoted by $\mathrm{Adm}(R;\alpha_1,\dots,\alpha_n)$. We define $R_P := R \cup \{(i,n+1) : i \in \mathrm{Adm}(R;\alpha_1,\dots,\alpha_n), i \neq n+1\}$. The language $a^{(n)}_{R_P}(\alpha_1,\dots,\alpha_n)$

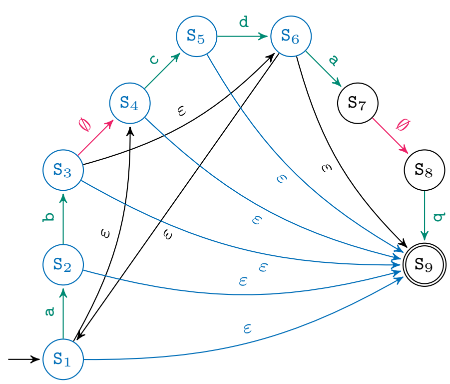
is the set of the prefixes of L.
For instance consider $L = a^{(8)}_{(1,4),(3,6),(6,1),(6,9)}(\mathtt{a}, \mathtt{b}, \emptyset, \mathtt{c}, \mathtt{d}, \mathtt{a}, \emptyset, \mathtt{b})$. We have $L = (\mathtt{ab} + \mathtt{cd})^+$,

$\mathcal{A}(a^{(8)}_{(1,4),(3,6),(6,1),(6,9)}) =$ 

and

$$\mathrm{Adm}(\{(1,4),(3,6),(6,1),(6,9)\}; \mathtt{a}, \mathtt{b}, \emptyset, \mathtt{c}, \mathtt{d}, \mathtt{a}, \emptyset, \mathtt{b}) = \{1,2,3,4,5,6\}.$$

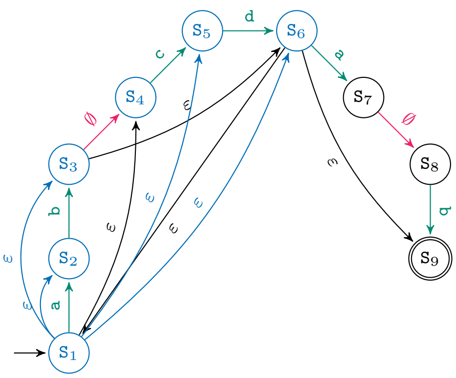
So $R_P = \{(1,4),(3,6),(6,1),(6,9),(1,9),(2,9),(3,9),(4,9),(5,9)\}$. We check that $a^{(8)}_{R_P}(\mathtt{a}, \mathtt{b}, \emptyset, \mathtt{c}) = (\mathtt{ab} + \mathtt{cd})^*(\varepsilon + \mathtt{a} + \mathtt{c}) = \mathrm{Pref}(L)$. Graphically,

$\mathcal{A}(a^{(8)}_{R_p}) =$ 

Indeed the language recognized by this automaton is $(\mathtt{a}_1\mathtt{a}_2\mathtt{a}_3\mathtt{a}_4\mathtt{a}_5 + \mathtt{a}_1\mathtt{a}_2 + \mathtt{a}_4\mathtt{a}_5)^*(\varepsilon + \mathtt{a}_1 + \mathtt{a}_1\mathtt{a}_2 + \mathtt{a}_1\mathtt{a}_2\mathtt{a}_3 + \mathtt{a}_1\mathtt{a}_2\mathtt{a}_3\mathtt{a}_4 + \mathtt{a}_4 + (\mathtt{a}_1\mathtt{a}_2\mathtt{a}_3\mathtt{a}_4\mathtt{a}_5 + \mathtt{a}_1\mathtt{a}_2 + \mathtt{a}_4\mathtt{a}_5)(\varepsilon + \mathtt{a}_6\mathtt{a}_7\mathtt{a}_8))$. Setting $\mathtt{a}_i = \alpha_i$ in this expression, we find $(\mathtt{ab} + \mathtt{cd})^*(\varepsilon + \mathtt{a} + \mathtt{ab} + \mathtt{c} + (\mathtt{ab} + \mathtt{cd})) = (\mathtt{ab} + \mathtt{cd})^*(\varepsilon + \mathtt{a} + \mathtt{c})$ as expected.

- Symmetrically, the language of the suffixes of L is obtained by considering the relation $R_S := R \cup \{(1,i) : i \in \mathrm{Adm}(R; \alpha_1, \ldots, \alpha_n),\ i \neq 1\}$. From the example above we obtain

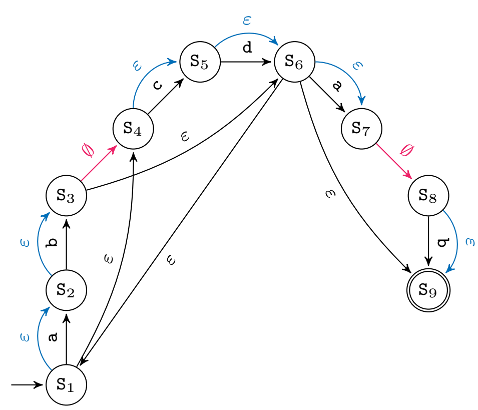
$R_S = \{(1,4),(3,6),(6,1),(6,9),(1,2),(1,3),(1,4),(1,5),(1,6)\}$. Graphically,

$\mathcal{A}(a^{(8)}_{R_S}) =$ 

- The language of the factors of L is obtained by first computing the prefixes and then the suffixes. Applying this construction to $L = a^{(8)}_{(1,4),(3,6),(6,1),(6,9)}(\mathtt{a},\mathtt{b},\emptyset,\mathtt{c},\mathtt{d},\mathtt{a},\emptyset,\mathtt{b})$, we find that the set of the factors of L is denoted by $a^{(8)}_{R_F}(\mathtt{a},\mathtt{b},\emptyset,\mathtt{c},\mathtt{d},\mathtt{a},\emptyset,\mathtt{b})$ with

$$R_F = \{(1,4),(3,6),(6,1),(6,9),(1,9),(2,9),(3,9),(4,9),(5,9),(1,2),(1,3),(1,4),(1,5),(1,6)\}.$$

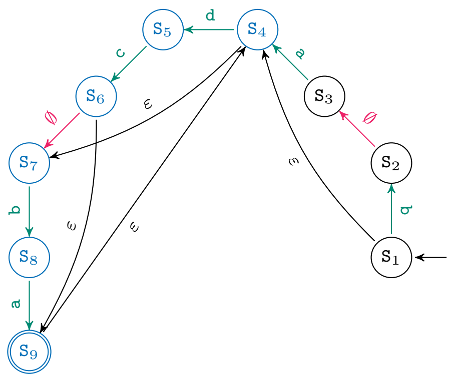
- The subwords of L are denoted by the expressions $a^{(n)}_S(\alpha_1,\dots,\alpha_n)$ where $S = R\cup\{(i,i+1) : \alpha_i \neq \emptyset\}$. Applying the construction to $L = a^{(8)}_{(1,4),(3,6),(6,1),(6,9)}(\mathtt{a},\mathtt{b},\emptyset,\mathtt{c},\mathtt{d},\mathtt{a},\emptyset,\mathtt{b})$, the language of the subwords of L is $a^{(8)}_{(1,4),(3,6),(6,1),(6,9),(1,2),(2,3),(4,5),(5,6),(6,7),(8,9)}(\mathtt{a},\mathtt{b},\emptyset,\mathtt{c},\mathtt{d},\mathtt{a},\emptyset,\mathtt{b})$. The associated automaton is

$\mathcal{A}(a^{(8)}_S) =$ 

- The mirror image of L is obtained by computing $a^{(n)}_M(\alpha_n,\dots,\alpha_1)$ where $M = \{(n+2-j, n+2-i) : (i,j) \in R\}$. Let us again illustrate the construction on $L = a^{(8)}_{(1,4),(3,6),(6,1),(6,9)}(\mathtt{a},\mathtt{b},\emptyset,\mathtt{c},\mathtt{d},\mathtt{a},\emptyset,\mathtt{b})$.

The mirror image of L is $a^{(8)}_{(1,4),(3,6),(6,1),(6,9)}(\mathtt{b}, \emptyset, \mathtt{a}, \mathtt{d}, \mathtt{c}, \emptyset, \mathtt{b}, \mathtt{a})$. Graphically,

$\mathcal{A}(a^{(8)}_M) =$



The language recognized by $\mathcal{A}(a^{(8)}_M)$ is $(\varepsilon + \mathtt{a}_1\mathtt{a}_2\mathtt{a}_3)(\mathtt{a}_4\mathtt{a}_5(\varepsilon + \mathtt{a}_6\mathtt{a}_7\mathtt{a}_8) + \mathtt{a}_7\mathtt{a}_8)^+$. Specializing to $\mathtt{a}_1 = \mathtt{b}$, $\mathtt{a}_2 = \emptyset$, $\mathtt{a}_3 = \mathtt{a}$, $\mathtt{a}_4 = \mathtt{d}$, $\mathtt{a}_5 = \mathtt{c}$, $\mathtt{a}_6 = \emptyset$, $\mathtt{a}_7 = \mathtt{b}$, and $\mathtt{a}_8 = \mathtt{a}$, we recover the language $(\mathtt{dc} + \mathtt{ba})^+$ that is the mirror image of L.

Some more examples:

Example 13. Let $\mathtt{a}_1, \dots, \mathtt{a}_n$ be n letters. We have

- $a^{(n)}_{\{(n+1,1),(1,n+1)\}}(\mathtt{a}_1, \dots, \mathtt{a}_n) = (\mathtt{a}_1 \cdots \mathtt{a}_n)^*$.
- $a^{(n)}_{\{(i,j):i \neq j\}}(\mathtt{a}_1, \dots, \mathtt{a}_n) = (\mathtt{a}_1 + \cdots + \mathtt{a}_n)^*$.
- $a^{(n)}_{\{(n+1,1)\} \cup \{(i,n+1):1 \leqslant i \leqslant n\}}(\mathtt{a}_1, \dots, \mathtt{a}_n) = (\mathtt{a}_1 + \mathtt{a}_1\mathtt{a}_2 + \cdots + \mathtt{a}_1 \cdots \mathtt{a}_n)^*$.
- $a^{(n)}_{\{(n+1,1)\} \cup \{(1,i+1):1 \leqslant i \leqslant n\}}(\mathtt{a}_1, \dots, \mathtt{a}_n) = (\mathtt{a}_n + \mathtt{a}_{n-1}\mathtt{a}_n + \cdots + \mathtt{a}_1 \cdots \mathtt{a}_n)^*$.
- $a^{(n)}_{\{(i+1,i):1 \leqslant i \leqslant n\}}(\mathtt{a}_1, \dots, \mathtt{a}_n) = \{w \in \{\mathtt{a}_1, \dots, \mathtt{a}_n\}^* : w = \mathtt{a}_1 w' \mathtt{a}_n$ and $w = u\mathtt{a}_i\mathtt{a}_j v$ implies $j \leqslant i + 1\}$.

5.3. **Action of** QOSet**.** Let $a^{(n)}_R \in \mathrm{ARef}_n$. If we compare the grammars $G^{(n)}_R$ and $G^{(n)}_{\gamma R}$ (γR being the transitive cloture of R), we observe that $S_i \to S_\ell \in P(a^{(n)}_{\tilde{\gamma} R})$ implies there exists $i_1 = i, i_2, \dots, i_p = \ell$ such that $S_{i_h} \to S_{h+1} \in P(a^{(n)}_R)$ for each $1 \leqslant h < \ell$. Hence, the languages $\mathbb{L}(G^{(n)}_R)$ and $\mathbb{L}(G^{(n)}_{\gamma R})$ are equal.

Example 14. Consider $R = \{(1,2), (2,3)\}$, we have $\tilde{\gamma}(R) = \{(1,2), (2,3), (1,3)\}$. We have

$$P(a^{(2)}_R) = \left\{ \begin{array}{l} \mathtt{S}_1 \to \mathtt{a}_1\mathtt{S}_2, \\ \mathtt{S}_1 \to \mathtt{S}_2, \\ \mathtt{S}_2 \to \mathtt{a}_2\mathtt{S}_3, \\ \mathtt{S}_2 \to \mathtt{S}_3, \\ \mathtt{S}_3 \to \varepsilon, \end{array} \right. \quad \text{and } P(a^{(2)}_{\tilde{\gamma} R}) = \left\{ \begin{array}{l} \mathtt{S}_1 \to \mathtt{a}_1\mathtt{S}_2, \\ \mathtt{S}_1 \to \mathtt{S}_2, \\ \mathtt{S}_1 \to \mathtt{S}_3, \\ \mathtt{S}_2 \to \mathtt{a}_2\mathtt{S}_3, \\ \mathtt{S}_2 \to \mathtt{S}_3, \\ \mathtt{S}_3 \to \varepsilon. \end{array} \right.$$

Hence, $\mathbb{L}(G^{(n)}_R) = \{\varepsilon, \mathtt{a}_1, \mathtt{a}_1\mathtt{a}_2, \mathtt{a}_2\} = \mathbb{L}(G^{(n)}_{\gamma R})$.

This allows to consider the action of $\mathrm{OP}(\diamondsuit)$ defined by $a^{(n)}_{[R]}(L_1, \dots, L_n) := a^{(n)}_R(L_1, \dots, L_n)$. Alternatively, the action of QOSet is defined by $Q(L_1, \dots, L_n) = a^{(n)}_{Q \setminus \Delta}(L_1, \dots, L_n)$. Observing that the operads QOSet and $\mathrm{OP}(\diamondsuit)$ are isomorphic and that the isomorphism η satisfies

$\eta(Q)(L_1,\dots,L_n) = a^{(n)}_{[Q\setminus\Delta]}(L_1,\dots,L_n) = a^{(n)}_{Q\setminus\Delta}(L_1,\dots,L_n) = Q(L_1,\dots,L_n)$, the action of QOSet is compatible with the partial compositions. Hence, Theorem 4 implies

Corollary 2. *The sets 2^{Σ^*} and $Reg(\Sigma)$ are* QOSet*-modules.*

Now, we prove that the operad QOSet is optimal in the sense that two different operators act in two different ways on regular languages. That is:

Theorem 5. *If Σ is an alphabet with at least two letters then $Reg(\Sigma)$ is a faithful* QOSet*-module.*

Proof. Let $Q_1 \neq Q_2 \in \mathrm{QOSet}_n$ be two quasiorders. Without loss of generality, we suppose that there exists $(i,j) \in Q_1$ such that $(i,j) \notin Q_2$. Let $\Sigma_n = \{\mathtt{a}_1,\dots,\mathtt{a}_n\}$ be an alphabet. The constructions above shows that the word $\mathtt{a}_1\mathtt{a}_2\dots\mathtt{a}_{i-1}\mathtt{a}_j\mathtt{a}_{j+1}\dots\mathtt{a}_n$ belongs to $Q_1(\{\mathtt{a}_1\},\{\mathtt{a}_2\},\dots,\{\mathtt{a}_n\})$ but not to $Q_2(\{\mathtt{a}_1\},\{\mathtt{a}_2\},\dots,\{\mathtt{a}_n\})$. Setting $\mathtt{a}_\ell = \mathtt{a}^{\ell-1}\mathtt{b}$ for each $\ell > 0$, this shows the result for an alphabet of size at least 2. □

Note that the number of elements of QOSet_n is known up to $n = 17$ (see [Slo11] `sequence A000798`):

$$4, 29, 355, 6942, 209527, 9535241, 642779354, 63260289423, \dots$$

Example 15.

- Let us examine the four operators of QOSet_1:

$Q_1 = \{(1,1),(2,2)\}, Q_2 = \{(1,1),(1,2),(2,2)\}, Q_3 = \{(1,1),(2,1),(2,2)\}, Q_4 = \{(1,1),(1,2),(2,1),(2,2)\}$,

The four languages are $Q_1(\mathtt{a}_1) = \mathtt{a}_1$, $Q_2(\mathtt{a}_1) = \varepsilon + \mathtt{a}_1$, $Q_3 = \mathtt{a}_1^+ (= \mathtt{a}_1\mathtt{a}_1^*)$, and $Q_4 = \mathtt{a}_1^*$.

- Let us examine the 29 operators of QOSet_2:

$Q\setminus\Delta$	$Q(\mathtt{a}_1,\mathtt{a}_2)$	$Q\setminus\Delta$	$Q(\mathtt{a}_1,\mathtt{a}_2)$	$Q\setminus\Delta$	$Q(\mathtt{a}_1,\mathtt{a}_2)$
$\emptyset$	$\mathtt{a}_1\mathtt{a}_2$	$\{(1,2)\}$	$\mathtt{a}_2+\mathtt{a}_1\mathtt{a}_2$	$\{(1,3)\}$	$\varepsilon+\mathtt{a}_1\mathtt{a}_2$
$\{(2,3)\}$	$\mathtt{a}_1+\mathtt{a}_1\mathtt{a}_2$	$\{(2,1)\}$	$\mathtt{a}_1^+\mathtt{a}_2$	$\{(3,1)\}$	$(\mathtt{a}_1\mathtt{a}_2)^+$
$\{(3,2)\}$	$\mathtt{a}_1\mathtt{a}_2^+$	$\{(1,2),(2,1)\}$	$\mathtt{a}_1^*\mathtt{a}_2$	$\{(1,3),(3,1)\}$	$(\mathtt{a}_1\mathtt{a}_2)^*$
$\{(2,3),(3,2)\}$	$\mathtt{a}_1\mathtt{a}_2^*$	$\{(1,2),(3,2)\}$	$(\varepsilon+\mathtt{a}_1)\mathtt{a}_2^+$	$\{(2,1),(2,3)\}$	$a_1^+(\varepsilon+\mathtt{a}_2)$
$\{(1,3),(2,3)\}$	$(\varepsilon+\mathtt{a}_1+\mathtt{a}_2)$	$\{(3,1),(3,2)\}$	$(\mathtt{a}_1\mathtt{a}_2^+)^+$	$\{(3,1),(2,1)\}$	$(\mathtt{a}_1^+\mathtt{a}_2)^+$
$\{(1,3),(1,2)\}$	$\varepsilon+\mathtt{a}_2+\mathtt{a}_1\mathtt{a}_2$	$\{(1,2),(2,3),(1,3)\}$	$\varepsilon+\mathtt{a}_1+\mathtt{a}_2+\mathtt{a}_1\mathtt{a}_2$	$\{(2,1),(3,2),(3,1)\}$	$(\mathtt{a}_1^+\mathtt{a}_2^+)^+$
$\{(1,3),(3,2),(1,2)\}$	$(\varepsilon+\mathtt{a}_1)\mathtt{a}_2^+ +\varepsilon$	$\{(3,1),(2,3),(2,1)\}$	$(\mathtt{a}_1^+(\varepsilon+\mathtt{a}_2))^+$	$\{(2,1),(1,3),(2,3)\}$	$\varepsilon+\mathtt{a}_1^+(\varepsilon+\mathtt{a}_2)$
$\{(1,2),(3,1),(3,2)\}$	$((\mathtt{a}_1+\varepsilon)\mathtt{a}_2^+)^+$				

$Q\setminus\Delta$	$Q(\{\mathtt{a}_1\},\{\mathtt{a}_2\})$	$Q\setminus\Delta$	$Q(\{\mathtt{a}_1\},\{\mathtt{a}_2\})$
$\{(1,2),(2,1),(2,3),(1,3)\}$	$\varepsilon+\mathtt{a}_1^*\mathtt{a}_2$	$\{(1,2),(2,1),(3,2),(3,1)\}$	$(\mathtt{a}_1^*\mathtt{a}_2^+)^+$
$\{(1,3),(3,1),(1,2),(3,2)\}$	$((\varepsilon+\mathtt{a}_1)\mathtt{a}_2^+)^*$	$\{(1,3),(3,1),(2,1),(2,3)\}$	$(\mathtt{a}_1^2(\varepsilon+\mathtt{a}_2))^*$
$\{(2,3),(3,2),(2,1),(3,1)\}$	$(\mathtt{a}_1^+\mathtt{a}_2^*)^+$	$\{(2,3),(3,2),(1,2),(1,3)\}$	$(\varepsilon+\mathtt{a}_1)\mathtt{a}_2^*$
$\{(1,2),(1,3),(2,3),(2,1),(2,3),(3,1)\}$	$(\mathtt{a}_1+\mathtt{a}_2)^*$		

We illustrate the proof of Theorem 5. Remarking that $(3,2) \in \{(2,3),(3,2),(2,1),(3,1)\}$, $(3,2) \notin \{(2,1),(1,3),(2,3)\}$, we have $\mathtt{a}_1\mathtt{a}_2\mathtt{a}_2 \in (\mathtt{a}_1^+\mathtt{a}_2^*)^+ = \{(2,3),(3,2),(2,1),(3,1)\}(\mathtt{a}_1,\mathtt{a}_2)$ and $\mathtt{a}_1\mathtt{a}_2\mathtt{a}_2 \notin \varepsilon+\mathtt{a}_1^+(\varepsilon+\mathtt{a}_2) = \{(2,1),(1,3),(2,3)\}(\mathtt{a}_1,\mathtt{a}_2)$.

5.4. **Back to (simple) multi-tildes.** The purpose of this section is to show that the restriction of the action to (simple) multi-tildes is compatible with the action described in [LMN13]. In this paper, the action of multi-tildes involve another operad: the operad of sets of boolean vectors $\mathcal{B} = \bigcup_n \mathcal{B}_n$ with $\mathcal{B}_n = 2^{\mathbb{B}^n}$ and $\mathbb{B} = \{0, 1\}$. The composition is defined by

$$E \circ_i F = \{[e_1, \ldots, e_{i-1}, e_i f_1, \ldots, e_i f_m, e_{i+1}, \ldots, e_n] : [e_1, \ldots, e_m] \in E, [f_1, \ldots, f_m] \in F\}$$

for $E \in \mathcal{B}_n$ and $F \in \mathcal{B}_m$. The action on the languages is defined by

$$E(L_1, \ldots, L_n) = \bigcup_{[e_1,\ldots,e_n] \in E} L_1^{e_1} \cdots L_n^{e_n}.$$

We denote $[x, z] = \{y : x \leqslant y \leqslant z\}$. For each $T \in \mathcal{T}_n$ we set $\mathcal{F}(T) = \{S \subset T : (x, y), (z, t) \in S$ implies $[x, y] \cup [z, t] = \emptyset\}$. Finally we define $V(T) = \{v(S) : S \in \mathcal{F}(T)\}$ with $v(S) = (v_1, \ldots, v_n)$ where $v_j = 0$ if $j \in \bigcup_{(x,y)\in S} [x, y]$ and 1 otherwise. In [LMN13] we proved that V is an operadic morphism and defined the action $T(L_1, \ldots, L_n) = V(T)(L_1, \ldots, L_n)$.
Remark that $\mathcal{T}$ is isomorphic to the suboperad of $\mathcal{DT}$ generated by $(a_T^{(n)}, a_\emptyset^{(n)})$ (the isomorphism sends each T to $(a_T^{(n)}, a_\emptyset^{(n)})$. So we have to prove that $T(L_1, \ldots, L_n) = (a_T^{(n)}, a_\emptyset^{(n)})(L_1, \ldots, L_n)$. Equivalently,

$$T(\mathtt{a}_1, \ldots, \mathtt{a}_n) = \mathbb{L}(\mathbf{G}_{T,\emptyset})(\mathtt{a}_1, \ldots, \mathtt{a}_n).$$

To this aim, we associate a set of boolean vectors to each grammar $\mathbf{G}_{T,\emptyset}$ in the following way: we consider the grammar $\mathbf{G}_{0,1}(T)$ which is obtained from $\mathbf{G}_{T,\emptyset}$ by substituting to each rule $\mathtt{S}_i \to \mathtt{a}_i\mathtt{S}_{i+1}$ the rule $\mathtt{S}_i \to 1\mathtt{S}_{i+1}$ and to each rule $\mathtt{S}_i \to \mathtt{S}_j$ the rule $\mathtt{S}_i \to 0^{j-i}\mathtt{S}_j$. Denote $\mathbb{L}_{0,1}(T) = \mathbb{L}(\mathbf{G}_{0,1}(T))$. Each word of $\mathbb{L}_{0,1}(T)$ has a length equal to n. Remark that

$$\mathbb{L}(\mathbf{G}_{T,\emptyset})(\mathtt{a}_1, \ldots, \mathtt{a}_n) = \{\mathtt{a}_1^{e_1} \cdots \mathtt{a}_n^{e_n} : e_1 \ldots e_n \in \mathbb{L}_{0,1}(T)\}.$$

Assimilating each word $e_1 \ldots e_n \in \mathbb{L}_{0,1}(T)$ to the boolean vector $(e_1, \ldots, e_n)$ we prove the following result:

Proposition 12. *For any $a_T^{(n)} \in \mathcal{T}_n$, we have $a_T^{(n)}(L_1, \ldots, L_n) = (a_T^{(n)}, a_\emptyset^{(n)})(L_1, \ldots, L_n)$.*

Proof. Let us first recall that a *closed* multi-tilde is a multi-tilde T satisfying

$$(i, j),\ (j + 1, \ell) \in T \Rightarrow (i, \ell) \in T.$$

The *normal form* $\widetilde{T}$ of a multi-tilde T is the smallest closed multi-tilde containing T as a subset (see *e.g.* [CCM11]). From the definition of the action of $\mathcal{T}$, we have $a_T^{(n)}(L_1, \ldots, L_n) = a_{\widetilde{T}}^{(n)}(L_1, \ldots, L_n)$. From the construction of $\mathbf{G}_{T,\emptyset}$ we observe that $\mathbb{L}_{0,1}(\tilde{T}) = \mathbb{L}_{0,1}(T)$. Indeed, it is sufficient to remark that one can add the rule $\mathtt{S}_i \to 0^{l-i}\mathtt{S}_i$ in $\mathbf{G}_{0,1}(T)$, when $\mathtt{S}_i \to 0^{j-i}\mathtt{S}_j$ and $\mathtt{S}_j \to 0^{l-j}$ are two rules of $\mathbf{G}_{0,1}(T)$, without modifying the language.

Thus, we have to prove $a_T^{(n)}(L_1, \ldots, L_n) = (a_T^{(n)}, a_\emptyset^{(n)})(L_1, \ldots, L_n)$ for any closed multi-tilde T. That is $v = 0^{i_1}10^{i_2} \cdots 10^{i_p} \in V(T)$ (considering the vector as a word) if and only if $v \in \mathbb{L}_{0,1}(T)$. The case when $p = 1$ means that $v = 0^{i_1} = 0^n$. For convenience, we set $i_0 = 1$. Obviously $(i_0, i_1), (i_0+i_1+1, i_0+i_1+i_2+1), \ldots, (i_0+i_1+\cdots+i_{\ell-1}+2(\ell-1)+1, i_0+i_1+\cdots+i_\ell+2(\ell-1)+1) \in T$ if and only if $\mathtt{S}_1 \xrightarrow{*} 0^{i_1}1 \ldots 10^{i_\ell}\mathtt{S}_{i_0+i_1+\cdots+i_\ell+2\ell}$ for any $0 \leqslant \ell \leqslant p$ (here $E \xrightarrow{*} w$ means that we can produce the word w from E by applying a finite sequence of rules). Equivalently $v \in V(T)$ if and only if $\mathtt{S}_1 \xrightarrow{*} v\mathtt{S}_{n+1} \to v$. This proves the result. $\square$

Conclusion and perspectives

We have described a faithful action of a combinatorial operad on regular languages. This means that we describe countable operations providing a new kind of expressions for denoting regular languages. One of the interest of the construction is that we propose expressions which are close to the representation by automata. The obtained expressions are more expressive in the sense that most of the complexity of the denoted language is concentrated at the operator. So this allows to define several measures of the complexity of a language. For instance, let us define $\mathrm{rank}_w(L) = \min\{k : \exists Q \in \mathrm{QOSet}_k,\ \alpha_1, \dots, \alpha_k \in \Sigma \cup \{\emptyset\}$ such that $L = Q(\alpha_1, \dots, \alpha_k)\}$ and $\mathrm{rank}_h(L) = \min\{h : \exists k \geqslant 1, O \in \mathcal{DT}_k,\ \alpha_1, \dots, \alpha_k \in \Sigma \cup \{\emptyset\}$ such that $L = O(\alpha_1, \dots, \alpha_k)$ and $\#O = h\}$. The two ranks rank_w and rank_h can be respectively interpreted as the width and the height of a language. The first one, rank_w, is the minimal number of occurrences of symbols or $\emptyset$ in the expression. The rank rank_h expresses the minimal complexity of an operator involved for denoting the languages. These measures will be investigated; in particular a parallel with the size of a minimal (in terms of states or transitions) automaton should be established.

The operads considered in this paper are `SET`-operads, that are operads that can be constructed from the category `SET`. We can also consider linear combinations of operators which consists to use `VECT`-operads based on the category of the vector spaces. By this way, we guess that the infinite matrices studied in our paper are good candidates to describe a weighted analogue of multi-tilde operators for rational series.

Another perspective is the extension of the conversion methods from automata to expressions using double multi-tildes. These conversions were studied in [CCM10] and in [CCM12]. By slightly modifying the action of our operads, we aim to extend these algorithms of conversions, and conversely from expressions to automata *e.g.*, the position functions [Glu61] or the expression derivatives [Ant96, Brz64].

A last perspective, suggested by the referee, is the following. By the Alexandroff correspondence [Ale37], quasiorders on finite sets are in bijection with finite topologies. The question consists in investigating if the action of the operad of quasiorders QOSet on languages (see Section 5.3) has a topological interpretation.

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Laboratoire d'Informatique Gaspard-Monge, Université Paris-Est Marne-la-Vallée, 5 boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France.
E-mail address: samuele.giraudo@univ-mlv.fr

Laboratoire LITIS - EA 4108 Université de Rouen. Avenue de l'Université - BP 8 76801 Saint-Étienne-du-Rouvray Cedex.
E-mail address: jean-gabriel.luque@univ-rouen.fr

Laboratoire LITIS - EA 4108 Université de Rouen. Avenue de l'Université - BP 8 76801 Saint-Étienne-du-Rouvray Cedex.
E-mail address: ludovic.mignot@univ-rouen.fr

Laboratoire LITIS - EA 4108 Université de Rouen. Avenue de l'Université - BP 8 76801 Saint-Étienne-du-Rouvray Cedex.
E-mail address: florent.nicart@univ-rouen.fr