# A converse to Fortin's Lemma in Banach spaces

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**Abstract**

We establish the converse of Fortin's Lemma in Banach spaces. This result is useful to assert the existence of a Fortin operator once a discrete inf-sup condition has been proved. The proof uses a specific construction of a right-inverse of a surjective operator in Banach spaces. The key issue is the sharp determination of the stability constants.

## 1 Introduction

Let $V$ and $W$ be two complex Banach spaces equipped with the norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. We adopt the convention that dual spaces are denoted with primes and are composed of antilinear forms; complex conjugates are denoted by an overline. Let $a$ be a sesquilinear form on $V \times W$ (linear w.r.t. its first argument and antilinear w.r.t. its second argument). We assume that $a$ is bounded, i.e.,

$$\|a\| := \sup_{v\in V} \sup_{w\in W} \frac{|a(v,w)|}{\|v\|_V \|w\|_W} < \infty, \tag{1}$$

and that the following inf-sup condition holds:

$$\alpha := \inf_{v\in V} \sup_{w\in W} \frac{|a(v,w)|}{\|v\|_V \|w\|_W} > 0. \tag{2}$$

Here and in what follows, arguments in infima and suprema are implicitly assumed to be nonzero.

Assume that we have at hand two sequences of finite-dimensional subspaces $\{V_h\}_{h\in\mathcal{H}}$ and $\{W_h\}_{h\in\mathcal{H}}$ with $V_h \subset V$ and $W_h \subset W$ for all $h \in \mathcal{H}$, where the parameter $h$ typically refers to a family of underlying meshes. The spaces $V_h$

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and $W_h$ are equipped with the norms of $V$ and $W$, respectively. A question of fundamental importance is to assert the following discrete inf-sup condition:

$$\hat{\alpha}_h := \inf_{v_h \in V_h} \sup_{w_h \in W_h} \frac{|a(v_h, w_h)|}{\|v_h\|_V \|w_h\|_W} > 0. \tag{3}$$

The aim of this Note is to prove the following result.

**Theorem 1 (Fortin's Lemma with converse)** *Under the above assumptions, consider the following two statements:*

(i) *There exists a map $\Pi_h : W \to W_h$ and a real number $\gamma_{\Pi_h} > 0$ such that $a(v_h, \Pi_h w - w) = 0$, for all $(v_h, w) \in V_h \times W$, and $\gamma_{\Pi_h} \|\Pi_h w\|_W \leq \|w\|_W$ for all $w \in W$.*

(ii) *The discrete inf-sup condition* (3) *holds.*

*Then,* (i) $\Rightarrow$ (ii) *with $\hat{\alpha}_h \geq \gamma_{\Pi_h} \alpha$. Conversely,* (ii) $\Rightarrow$ (i) *with $\gamma_{\Pi_h} = \frac{\hat{\alpha}_h}{\|a\|}$, and $\Pi_h$ can be constructed to be idempotent. Moreover, $\Pi_h$ can be made linear if $W$ is a Hilbert space.*

The statement (i) $\Rightarrow$ (ii) in Theorem 1 is classical and is known in the literature as Fortin's Lemma, see [5] and [1, Prop. 5.4.3]. It provides an effective tool to prove the discrete inf-sup condition (3) by constructing explicitly a Fortin operator $\Pi_h$. We briefly outline a proof that (i) $\Rightarrow$ (ii) for completeness. Assuming (i), we have

$$\begin{aligned}\sup_{w_h \in W_h} \frac{|a(v_h, w_h)|}{\|w_h\|_W} &\geq \sup_{w \in W} \frac{|a(v_h, \Pi_h w)|}{\|\Pi_h w\|_W} = \sup_{w \in W} \frac{|a(v_h, w)|}{\|\Pi_h w\|_W} \\ &\geq \gamma_{\Pi_h} \sup_{w \in W} \frac{|a(v_h, w)|}{\|w\|_W} \geq \gamma_{\Pi_h} \alpha \|v_h\|_V,\end{aligned}$$

since $a$ satisfies (2) and $V_h \subset V$. This proves (ii) with $\hat{\alpha}_h \geq \gamma_{\Pi_h} \alpha$.

The proof of the converse (ii) $\Rightarrow$ (i) is the main object of this Note. This property is useful when it is easier to prove the discrete inf-sup condition directly rather than constructing a Fortin operator. Another application of current interest is the analysis framework for discontinuous Petrov–Galerkin methods (dPG) recently proposed in [3] which includes the existence of a Fortin operator among its key assumptions. The proof of the converse is not so straightforward if one wishes to establish a sharp stability bound for $\Pi_h$, i.e., that indeed one can take $\gamma_{\Pi_h} = \frac{\hat{\alpha}_h}{\|a\|}$. Incidentally, we observe that there is a gap in the stability constant $\gamma_{\Pi_h}$ between the direct and converse statements, since the ratio of the two is equal to $\frac{\|a\|}{\alpha}$ (which is independent of the discrete setting).

This Note is organized as follows. In Section 2, we establish a sharp bound on the stability of the right-inverse of surjective operators in Banach spaces. Since this result can be of independent theoretical interest, we present it in an infinite-dimnesional setting. Then in Section 3, we prove the converse of Fortin's Lemma. The proof is relatively simple once the sharp stability estimate from Section 2 is available.

## 2 Right-inverse of surjective Banach operators

Let $Y$ and $Z$ be two complex Banach spaces equipped with the norms $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. Let $B : Y \to Z$ be a bounded linear map. The following result is a well-known consequence of Banach's Open Mapping and Closed Range Theorems, see, e.g., [2, Thm. 2.20] or [4, Lem. A.36 & A.40].

**Lemma 2 (Surjectivity)** *The following three statements are equivalent:*

(i) $B : Y \to Z$ *is surjective.*

(ii) $B^* : Z' \to Y'$ *is injective and* $\mathrm{im}(B^*)$ *is closed in* $Y'$.

(iii) *The following holds:*

$$\inf_{z'\in Z'} \frac{\|B^*z'\|_{Y'}}{\|z'\|_{Z'}} = \inf_{z'\in Z'} \sup_{y\in Y} \frac{|\langle B^*z', y\rangle_{Y',Y}|}{\|z'\|_{Z'}\|y\|_Y} =: \beta > 0. \tag{4}$$

Let us now turn to the main result of this section. To motivate the result, assume that (4) holds; then $B$ is surjective and thus admits a bounded right-inverse. The crucial question is whether the stability of this right-inverse can be formulated using precisely the constant $\beta > 0$ from (4).

**Lemma 3 (Right inverse)** *Assume that* (4) *holds and that* $Y$ *is reflexive. Then there is a right-inverse map* $B^\dagger : Z \to Y$ *such that*

$$\forall z \in Z, \quad (B \circ B^\dagger)(z) = z \quad \text{and} \quad \beta\|B^\dagger z\|_Y \le \|z\|_Z. \tag{5}$$

*Moreover, this right-inverse map* $B^\dagger$ *is linear if* $Y$ *is a Hilbert space.*

**Proof** Parts of this result can be found in [4, Lem. A.42]; for completeness, we present a proof. Owing to Lemma 2, $B^*$ is injective and $R := \mathrm{im}(B^*)$ is closed in $Y'$. Since the operator $B^*$ is injective, it admits a left-inverse linear map $B^{*\ddagger} : R \to Z'$ such that $(B^{*\ddagger} \circ B^*)(z') = z'$ for all $z' \in Z'$. Moreover, the inf-sup condition (4) implies that $\|B^{*\ddagger}y'\|_{Z'} \le \beta^{-1}\|y'\|_{Y'}$ for all $y' \in R$. Consider now the adjoint $B^{*\ddagger*} : Z'' \to R'$. Let $E^{\mathrm{HB}}_{R'Y''}$ be the Hahn–Banach extension operator that extends antilinear forms over $R \subset Y'$ into antilinear forms over $Y'$ (see [2, Prop. 11.23]); $E^{\mathrm{HB}}_{R'Y''}$ maps from $R'$ to $Y''$. Let $J_Y$ (resp., $J_Z$) be the canonical isometry from $Y$ to $Y''$ (resp., $Z$ to $Z''$), and observe that $J_Y$ is an isomorphism since $Y$ is assumed to be reflexive. Let us set

$$B^\dagger := J_Y^{-1} \circ E^{\mathrm{HB}}_{R'Y''} \circ B^{*\ddagger*} \circ J_Z : Z \to Y, \tag{6}$$

and let us verify that $B^\dagger$ satisfies the expected properties. We have, for all $(z', z) \in Z' \times Z$,

$$\begin{aligned}
\langle z', B(B^\dagger(z))\rangle_{Z',Z} &= \langle B^*z', B^\dagger(z)\rangle_{Y',Y} = \overline{\langle J_Y(B^\dagger(z)), B^*z'\rangle_{Y'',Y'}} \\
&= \overline{\langle E^{\mathrm{HB}}_{R'Y''}(B^{*\ddagger*}(J_Z z)), B^*z'\rangle_{Y'',Y'}} = \overline{\langle B^{*\ddagger*}(J_Z z), B^*z'\rangle_{R',R}} \\
&= \overline{\langle J_Z z, B^{*\ddagger}B^*z'\rangle_{Z'',Z'}} = \overline{\langle J_Z z, z'\rangle_{Z'',Z'}} = \langle z', z\rangle_{Z',Z},
\end{aligned}$$

where we have used that $B^* z' \in R$ to pass from the first to the second line. This shows that $(B \circ B^\dagger)(z) = z$. Moreover, since $J_Y$ is an isometry and the extension operator $E^{\mathrm{HB}}_{R'Y''}$ preserves the norm, we observe that, for all $z \in Z$,

$$\begin{aligned}
\|B^\dagger z\|_Y = \|B^{*\ddagger *}(J_Z z)\|_{R'} &= \sup_{z' \in Z'} \frac{|\langle B^{*\ddagger *}(J_Z z), B^* z'\rangle_{R',R}|}{\|B^* z'\|_{Y'}} \\
&= \sup_{z' \in Z'} \frac{|\langle J_Z z, z'\rangle_{Z'',Z'}|}{\|B^* z'\|_{Y'}} \le \sup_{z' \in Z'} \frac{\|z'\|_{Z'}}{\|B^* z'\|_{Y'}} \|z\|_Z.
\end{aligned}$$

We conclude from (4) that $\beta \|B^\dagger z\|_Y \le \|z\|_Z$. Finally, if $Y$ is a Hilbert space, we can consider the orthogonal complement of $R$ in $Y'$ (recall that $R$ is a closed subspace of $Y'$) and write $Y' = R \oplus R^\perp$. Then, the Hahn–Banach extension operator $E^{\mathrm{HB}}_{R'Y''}$ in (6) can be replaced by the linear map $E^{\perp}_{R'Y''}$ such that, for all $\phi \in R'$, $\langle E^{\perp}_{R'Y''}\phi, y'\rangle_{Y'',Y'} = \langle \phi, r\rangle_{R',R}$ for all $y' \in Y'$ with $y' = r + r^\perp$, $r \in R$, $r^\perp \in R^\perp$. $\square$

# 3 Proof of the converse in Theorem 1

Let $A_h : V_h \to W_h'$ be the operator defined by $\langle A_h v_h, w_h\rangle_{W_h',W_h} := a(v_h, w_h)$ for all $(v_h, w_h) \in V_h \times W_h$. We identify $V_h''$ with $V_h$ and $W_h''$ with $W_h$ (since these spaces are finite-dimensional). We consider the adjoint operator $A_h^* : W_h \to V_h'$, and identify $A_h^{**}$ with $A_h$. We apply Lemma 3 to $Y := W_h$, $Z := V_h'$, and $B := A_h^*$. Owing to the discrete inf-sup condition (3), we infer that (4) holds with $\beta = \hat{\alpha}_h$. Therefore, there exists a right-inverse map $A_h^{*\dagger} : V_h' \to W_h$ such that, for all $\theta_h \in V_h'$, $(A_h^* \circ A_h^{*\dagger})(\theta_h) = \theta_h$ and $\hat{\alpha}_h \|A_h^{*\dagger}\theta_h\|_W \le \|\theta_h\|_{V_h'}$. Let us now set

$$\Pi_h := A_h^{*\dagger} \circ \Theta : W \to W_h, \tag{7}$$

with the linear map $\Theta : W \to V_h'$ such that, for all $w \in W$, $\langle \Theta(w), v_h\rangle_{V_h',V_h} := \overline{a(v_h, w)}$ for all $v_h \in V_h$. We then infer that

$$\begin{aligned}
a(v_h, \Pi_h(w)) &= \langle A_h v_h, A_h^{*\dagger}(\Theta(w))\rangle_{W_h',W_h} = \overline{\langle A_h^*(A_h^{*\dagger}(\Theta(w))), v_h\rangle_{V_h',V_h}} \\
&= \overline{\langle \Theta(w), v_h\rangle_{V_h',V_h}} = a(v_h, w),
\end{aligned}$$

which establishes that $a(v_h, \Pi_h(w) - w) = 0$ for all $w \in W$. Moreover,

$$\hat{\alpha}_h \|\Pi_h(w)\|_W = \hat{\alpha}_h \|A_h^{*\dagger}(\Theta(w))\|_W \le \|\Theta(w)\|_{V_h'} \le \|a\| \|w\|_W,$$

which proves that $\frac{\hat{\alpha}_h}{\|a\|}\|\Pi_h(w)\|_W \le \|w\|_W$. In addition, we observe that

$$\begin{aligned}
\langle \Theta(A_h^{*\dagger}(\theta_h)), v_h\rangle_{V_h',V_h} &= \overline{\langle A_h v_h, A_h^{*\dagger}(\theta_h)\rangle_{W_h',W_h}} \\
&= \langle A_h^*(A_h^{*\dagger}(\theta_h)), v_h\rangle_{V_h',V_h} = \langle \theta_h, v_h\rangle_{V_h',V_h},
\end{aligned}$$

for all $v_h \in V_h$, which proves that $\Theta(A_h^{*\dagger}(\theta_h)) = \theta_h$ for all $\theta_h \in V_h'$. As a result, $\Pi_h(\Pi_h(w)) = A_h^{*\dagger}(\Theta \circ A_h^{*\dagger}(\Theta(w))) = A_h^{*\dagger}(\Theta(w)) = \Pi_h(w)$, i.e., $\Pi_h$ is idempotent. Finally, if $W$ is a Hilbert space, the right-inverse map $A_h^{*\dagger}$ is linear by Lemma 3, and so is the operator $\Pi_h$ defined from (7).

*Remark 1 (Value of $\gamma_{\Pi_h}$) Without the use of Lemma 3, one only knows that $A_h^*$ has a stable right-inverse, but a stability bound for this right-inverse is not available. In particular, if the discrete inf-sup condition* (3) *holds uniformly with respect to $h$, i.e., if there is $\hat{\alpha}_0 > 0$ such that $\hat{\alpha}_h \geq \hat{\alpha}_0$ for all $h \in \mathcal{H}$, then a uniform stability bound for $\Pi_h$ is $\gamma_{\Pi_h} \geq \gamma_{\Pi_0} = \frac{\hat{\alpha}_0}{\|a\|}$ for all $h \in \mathcal{H}$.*

*Remark 2 (Linearity) Even in the case of Banach spaces, the linearity of the map $\Pi_h$ can be asserted if one has at hand a stable decomposition $W_h = \ker(A_h^*) \oplus K_h$ such that there is $\kappa_h > 0$ such that the induced projector $\pi_{K_h} : W_h \to K_h$ satisfies $\kappa_h \|\pi_{K_h} w_h\|_W \leq \|w_h\|_W$ for all $w_h \in W_h$ (this property holds in the Hilbertian setting with $\kappa_h = 1$). Then, one can adapt the reasoning at the end of the proof of Lemma 3 to build a stable, linear right-inverse map $A_h^{*\dagger}$. The mild price to be paid is that the stability constant of $\Pi_h$ now becomes $\gamma_{\Pi_h} = \frac{\kappa_h \hat{\alpha}_h}{\|a\|}$.*

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