Existence result for degenerate cross-diffusion system with constraint: application to seawater intrusion in confined aquifer

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Abstract: We consider a strongly-coupled nonlinear parabolic system which arises from seawater intrusion in confined aquifers. The global existence of a nonnegative solution is obtained after establishing a suitable entropy estimate.

1 Introduction

Seawater intrusion is one of the major concerns commonly found in coastal aquifers. It is the movement of seawater into the freshwater aquifers. In the modelling of such phenomenon, Jazar and Monneau proposed in [5] two reduced models in confined and unconfined aquifers, where the freshwater and the saltwater are assumed to be immiscible and one of the dimension is negligible with respect to the tow others. In this paper, we are concerned with the confined case (see Figure 1): we consider $\{z = 0\}$ the interface between the saltwater and the bedrock, $\{z = g(t,x)\}$ the interface between the saltwater and the freshwater and $\{z = h(t,x) + g(t,x) = 1\}$ the interface between the freshwater and the impermeable layer. Then the confined model reads

$$\left\{ \begin{array}{rcll} \partial_t h &=& \operatorname{div}\{h\nabla(p+\nu(h+g))\} & \text{in } [0,\infty)\times\mathbb{R}^N, \\ \partial_t g &=& \operatorname{div}\{g\nabla(p+\nu h+g)\} & \text{in } [0,\infty)\times\mathbb{R}^N, \\ h+g &=& 1 & \text{in } [0,\infty)\times\mathbb{R}^N, \end{array} \right. \tag{1.1}$$

where $N = 2,3$, p is the pressure on the top confining rock and $\nu = 1-\varepsilon_0 \in (0,1)$ with

$$\varepsilon_0 = \frac{\gamma_s - \gamma_f}{\gamma_s}$$

and γ_s and γ_f are the specific weight of the saltwater and freshwater respectively.

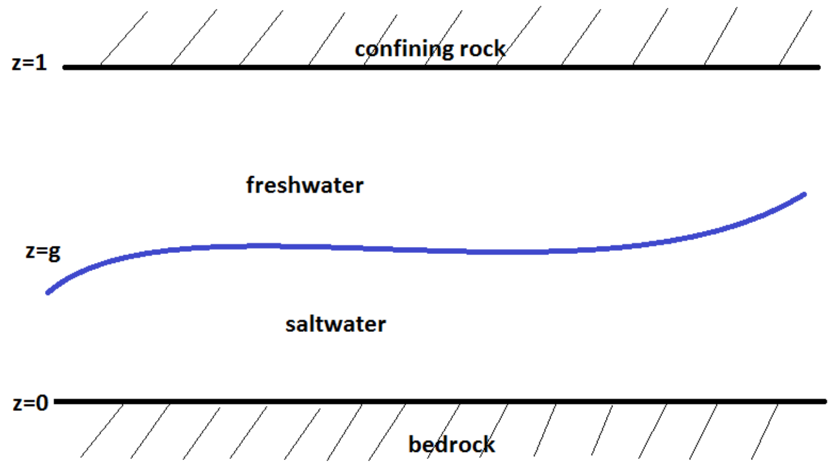

Figure 1: Confined aquifer

In this paper, we show existence result for a more generalized model of the form

$$\begin{cases} \partial_t u^i = \operatorname{div}\left\{ u^i \nabla \left(p + \sum_{j=1}^{m} A_{ij} u^j \right) \right\} & \text{in } \Omega_T, \quad \text{for } i = 1, \dots, m, \\ \sum_{i=1}^{m} u^i(t,x) = 1 & \text{in } \Omega_T, \end{cases} \tag{1.2}$$

where $\Omega_T := (0,T) \times \Omega$ with $T > 0$ and $\Omega := \mathbb{T}^N = (\mathbb{R}/\mathbb{Z})^N$, with $N \geq 1$. Here, p appears as a Lagrange multiplier of the constraint on $u = (u^i)_{1 \leq i \leq m}$, given by the second line of (1.2). This system has been studied in [3] without the constraint and with $p = 0$. We will follow the strategy proposed in [3] to which we will often refer. Note that, a different kind of seawater intrusion model in confined aquifers, which consists in a coupled system of an elliptic and a degenerate parabolic equation, has been studied in [6].

In a more general setting, we should replace the second equation of (1.2) by $\sum_{i=1}^{m} u^i(t,x) = f(x)$ on Ω_T. Here, we considered $f \equiv 1$ for the sake of simplicity, though the same result holds true with some additional conditions of f. Namely, we think that a nondecreasing condition on f in the direction of the freshwater flux is sufficient.

The strategy of the proof and the organization of the paper are as follow: after discretizing in time and regularizing we will consider a modified linear elliptic model to which we can apply Lax-Milgram theorem in Subsection 2.1. Then, in Subsection 2.2 we apply a fixed point theorem to get the existence of a solution of the nonlinear problem and we establish an entropy estimate. Finally, in Subsection 2.3, we pass to the limit in all added parameters in order to recover the expected result.

1.1 Main result

To introduce our main result, some definitions and assumptions are given.
The space $\mathbf{H}^1(\Omega)/\mathbb{R}$:
We define $H^1(\Omega)/\mathbb{R}$ as the space of functions of $H^1(\Omega)$, up to addition of constants. A natural norm is

$$\|p\|_{(H^1(\Omega)/\mathbb{R})} = \inf_{c\in\mathbb{R}} \|p-c\|_{H^1(\Omega)} = \left\| p - \frac{1}{|\Omega|}\int_\Omega p \right\|_{H^1(\Omega)}. \tag{1.3}$$

The function Ψ:
We define the nonnegative function Ψ as

$$\Psi(a) - \frac{1}{e} = \begin{cases} a\ln a & \text{for} \quad a>0, \\ 0 & \text{for} \quad a=0, \\ +\infty & \text{for} \quad a<0, \end{cases} \tag{1.4}$$

which is minimal for $a = \frac{1}{e}$.
The positivity condition:
The real $m\times m$ matrix $A = (A_{ij})_{1\le i,j\le m}$ is not necessarily symmetric and satisfies the following positivity condition: there exists $\delta_0 > 0$, such that we have

$$\xi^T A\,\xi \ge \delta_0|\xi|^2, \qquad \text{for all } \xi\in\mathbb{R}^m. \tag{1.5}$$

This condition, as in [3], can be weaken: there exist two positive definite diagonal $m\times m$ matrices L and R and $\delta_0>0$, such that we have

$$\zeta^T LAR\,\zeta \ge \delta_0|\zeta|^2, \quad \text{for all} \quad \zeta\in\mathbb{R}^m. \tag{1.6}$$

In the core of this paper we will assume (1.5) for the sake of simplicity.

Now we state our main result.

Theorem 1.1 ***(Existence for (1.2))***
Assume that A satisfies (1.5). For $i=1,\dots,m$, let $u_0^i\ge 0$ in Ω satisfying

$$\sum_{i=1}^m \int_\Omega \Psi(u_0^i) < +\infty, \tag{1.7}$$

where Ψ is given in (1.4). Then there exists a function $u=(u^i)_{1\le i\le m}\in (L^2(0,T;H^1(\Omega))\cap C([0,T);(W^{1,\infty}(\Omega))'))^m$, and a function $p\in L^2(0,T;H^1(\Omega)/\mathbb{R})$ such that (u,p) is a solution in the sense of distributions of (1.2), with $u^i\ge 0$ a.e. in Ω_T, for $i=1,\dots,m$. Moreover, u satisfies the following entropy estimate for a.e. $t_1,t_2\in(0,T)$, with $u^i(t_2)=u^i(t_2,\cdot)$:

$$\sum_{i=1}^m \int_\Omega \Psi(u^i(t_2)) + \delta_0 \sum_{i=1}^m \int_{t_1}^{t_2}\int_\Omega |\nabla u^i|^2 \le \sum_{i=1}^m \int_\Omega \Psi(u_0^i), \tag{1.8}$$

and p satisfies:

$$\int_{\Omega_T} |\nabla p|^2 \leq m \|A\|^2 \sum_{i=1}^m \int_{\Omega_T} \left|\nabla u^i\right|^2. \tag{1.9}$$

Here $\|A\|$ is the matrix norm defined as

$$\|A\| = \sup_{|\xi|=1} |A\xi|. \tag{1.10}$$

Notice that (1.8) also allows us to define the products $u^i \sum_{i=1}^m A_{ij} \nabla u^j$ and $u^i \nabla p$ in (1.2).

Remark 1.2 *(Decreasing energy)*
If A is a symmetric matrix then a solution (u, p) of system (1.2) satisfies

$$\frac{d}{dt}\left(\sum_{i=1}^m \sum_{j=1}^m \int_\Omega \frac{1}{2} A_{ij} u^i u^j\right) = -\int_\Omega \left(\sum_{i=1}^m u^i \left|q^i\right|^2 - \left|\sum_{i=1}^m u^i q^i\right|^2\right),$$

where $q^i = \sum_{j=1}^m A_{ij} \nabla u^j$.

2 Existence result

In order to get the existence of (1.1) and an entropy estimate we introduce the following modified linear discretized in time system (see [3] for details)

$$\left\{\begin{array}{rcl} \dfrac{u^{i,n+1} - u^{i,n}}{\Delta t} & = & \operatorname{div}\left\{T^{\epsilon,\ell}(v^{i,n+1})\left(\nabla p^{n+1} + \sum\limits_{j=1}^m A_{ij} \nabla \rho_\eta \star \rho_\eta \star u^{j,n+1} + \delta \nabla u^{i,n+1}\right)\right\}, \\ \sum\limits_{j=1}^m u^{i,n+1} & = & 1, \end{array}\right. \tag{2.1}$$

where a is a given function in $H^1(\Omega)$, $\Delta t = T/K$ is the time step with $K \in \mathbb{N}^*$, η, $\delta > 0$, $0 < \varepsilon < 1 < \ell < +\infty$, $T^{\varepsilon,\ell}$ is the truncation operator defined by

$$T^{\varepsilon,\ell}(a) := \left\{\begin{array}{ll} \varepsilon & \text{if } a \leq \varepsilon, \\ a & \text{if } \varepsilon \leq a \leq \ell, \\ \ell & \text{if } a \geq \ell, \end{array}\right. \tag{2.2}$$

and the mollifier $\rho_\eta(x) = \eta^{-N} \rho(x/\eta)$ with $\rho \in C_c^\infty(\mathbb{R}^N)$, $\rho \geq 0$, $\int_{\mathbb{R}^N} \rho = 1$, $\int_{\mathbb{R}^N} \nabla\rho = 1$ and $\rho(-x) = \rho(x)$. Note that we consider the $\mathbb{Z}^N$- periodic extension on $\mathbb{R}^N$ of $u^{j,n+1}$. We will look for fixed points solutions $v^{i,n+1} = u^{i,n+1}$ of (2.1). Finally, we will recover the existence result by passing to the limit in all parameters.

2.1 Existence for the linear elliptic problem (2.1)

In this subsection we prove the existence, via Lax-Milgram theorem, of the unique solution for the linear elliptic confined system (2.1).
Let us recall our linear elliptic confined system: Assume that A is any $m\times m$ real matrix. Let $v^{n+1}=(v^{i,n+1})_{1\le i\le m}\in (L^2(\Omega))^m$ and $u^n=(u^{i,n})_{1\le i\le m}\in (H^1(\Omega))^m$. Set

$$F^i_{\epsilon,\ell,\eta,\delta}(v^{n+1},u^{n+1},p^{n+1})=T^{\epsilon,\ell}(v^{i,n+1})\left(\nabla p^{n+1}+\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star u^{j,n+1}+\delta\nabla u^{i,n+1}\right), \tag{2.3}$$

where $T^{\varepsilon,\ell}$ is given in (2.2). Then for all Δt, ε, ℓ, η, $\delta>0$, with $\varepsilon<1<\ell$ and $\Delta t<\tau$ where τ is given in (2.5), we look for the solution $(u^{n+1},p^{n+1})=((u^{i,n+1})_{1\le i\le m},p^{n+1})$ of the following system:

$$\begin{cases} \dfrac{u^{i,n+1}-u^{i,n}}{\Delta t} = \mathrm{div}\left\{F^i_{\epsilon,\ell,\eta,\delta}(v^{n+1},u^{n+1},p^{n+1})\right\} & \text{in} \quad \mathcal{D}'(\Omega),\\[2ex] \displaystyle\sum_{i=1}^m u^{i,n+1}(x) = 1 & \text{in} \quad \Omega, \end{cases} \tag{2.4}$$

where $F^i_{\epsilon,\ell,\eta,\delta}$ is given in (2.3).

Proposition 2.1 ***(Existence for system (2.4))***
Assume that A is any $m\times m$ real matrix. Let Δt, ε, ℓ, η, $\delta>0$, with $\varepsilon<1<\ell$ such that

$$\Delta t<\frac{\delta\varepsilon\eta^2}{\ell^2\left\|A\right\|^2}:=\tau. \tag{2.5}$$

Then for $n\in\mathbb{N}$, for a given $v^{n+1}=(v^{i,n+1})_{1\le i\le m}\in(L^2(\Omega))^m$ and $u^n=(u^{i,n})_{1\le i\le m}\in (H^1(\Omega))^m$, there exists a unique function $(u^{n+1},p^{n+1})=((u^{i,n+1})_{1\le i\le m},p^{n+1})\in(H^1(\Omega))^m\times H^1(\Omega)/\mathbb{R}$ solution of system (2.4). Moreover, this solution (u^{n+1},p^{n+1}) satisfies the following estimate

$$\left(\delta-\frac{\delta\Delta t}{\tau}\right)\left\|u^{n+1}\right\|^2_{(L^2(\Omega))^m}+\varepsilon\delta^2\Delta t\left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m}+m\varepsilon\Delta t\left\|\nabla p\right\|^2_{L^2(\Omega)}\le\delta\left\|u^n\right\|^2_{(L^2(\Omega))^m}. \tag{2.6}$$

Proof of Proposition 2.1.
We use Lax-Milgram theorem to prove the existence result.
First of all, let us define for all $(u^{n+1},p^{n+1})=((u^{i,n+1})_{1\le i\le m},p^{n+1})$ and $(\varphi,q)=$

$((\varphi^i)_{1\le i\le m}, q) \in (H^1(\Omega))^m \times H^1(\Omega)/\mathbb{R}$, the following bilinear continuous form

$$\begin{aligned} b\left((u^{n+1},p^{n+1}),(\varphi,q)\right) &= \delta\sum_{i=1}^m\int_\Omega u^{i,n+1}\varphi^i + \Delta t\sum_{i=1}^m\int_\Omega T^{\varepsilon,\ell}(v^{i,n+1})\nabla p^{n+1}(\delta\nabla\varphi^i+\nabla q) \\ &\quad +\Delta t\sum_{i=1}^m\int_\Omega T^{\varepsilon,\ell}(v^{i,n+1})\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star u^{j,n+1}(\delta\nabla\varphi^i+\nabla q) \\ &\quad +\delta\Delta t\sum_{i=1}^m\int_\Omega T^{\varepsilon,\ell}(v^{i,n+1})\nabla u^{i,n+1}(\delta\nabla\varphi^i+\nabla q), \end{aligned}$$

and the following linear continuous form:

$$K(\varphi,q) = \delta\sum_{i=1}^m\int_\Omega u^{i,n}\varphi^i.$$

Note that we have $\displaystyle\sum_{i=1}^m\int_\Omega\left(u^{i,n+1}-u^{i,n}\right)q = 0.$

Step 1: Existence by Lax-Milgram

It remains to prove the coercivity of b to get the existence, by Lax-Milgram theorem, of a unique solution for system (2.4).

For all $(\varphi,q) = ((\varphi^i)_{1\le i\le m}, q) \in (H^1(\Omega))^m \times H^1(\Omega)/\mathbb{R}$, we have that

$$\begin{aligned} b((\varphi,q),(\varphi,q)) &= \delta\sum_{i=1}^m\int_\Omega|\varphi^i|^2 + \Delta t\sum_{i,j=1}^m\int_\Omega T^{\varepsilon,\ell}(v^{i,n+1})(\delta\nabla\varphi^i+\nabla q)^2 \\ &\quad +\Delta t\sum_{i=1}^m\int_\Omega T^{\varepsilon,\ell}(v^{i,n+1})\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star\varphi^j\nabla q \\ &\quad +\delta\Delta t\sum_{i=1}^m\int_\Omega T^{\varepsilon,\ell}(v^{i,n+1})\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star\varphi^j\nabla\varphi^i \\ &= b_0((\varphi,q),(\varphi,q)) + b_1((\varphi,q),(\varphi,q)), \end{aligned}$$

where

$$b_0((\varphi,q),(\varphi,q)) = \delta\sum_{i=1}^m\int_\Omega|\varphi^i|^2 + \Delta t\sum_{i=1}^m\int_\Omega T^{\varepsilon,\ell}(\varphi^{i,n+1})(\delta\nabla\varphi^i+\nabla q)^2,$$

and

$$\begin{aligned} b_1((\varphi,q),(\varphi,q)) &= \Delta t\sum_{i=1}^m\int_\Omega T^{\varepsilon,\ell}(\varphi^{i,n+1})\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star\varphi^j\nabla q \\ &\quad + \delta\Delta t\sum_{i=1}^m\int_\Omega T^{\varepsilon,\ell}(\varphi^{i,n+1})\sum_{j=1}^m A_{ij}\nabla\rho_\eta\star\rho_\eta\star\varphi^j\nabla\varphi^i. \end{aligned}$$

On the one hand, since $\sum_{i=1}^m \int_\Omega \nabla q \cdot \nabla \varphi^i = 0$, we already have the coercivity of b_0:

$$b_0((\varphi,q),(\varphi,q)) \;\geq\; \delta \|\varphi\|^2_{(L^2(\Omega))^m} + m\varepsilon\Delta t \|\nabla q\|^2_{L^2(\Omega)} + \varepsilon\delta^2\Delta t \|\nabla\varphi\|^2_{(L^2(\Omega))^m}.$$

On the other hand, we have

$$\begin{aligned}
|b_1(\varphi,\varphi)| \;\leq\;& \Delta t\ell \|A\| \sqrt{m} \|\nabla\rho_\eta \star \rho_\eta \star \varphi\|_{(L^2(\Omega))^m} \|\nabla q\|_{L^2(\Omega)} \\
&+\delta\Delta t\ell \|A\| \|\nabla\rho_\eta \star \rho_\eta \star \varphi\|_{(L^2(\Omega))^m} \|\nabla\varphi\|_{(L^2(\Omega))^m} \\
\leq\;& \Delta t\ell \|A\| \sqrt{m} \left(\frac{1}{2d} \|\nabla\rho_\eta \star \rho_\eta \star \varphi\|^2_{(L^2(\Omega))^m} + \frac{d}{2} \|\nabla q\|^2_{L^2(\Omega)}\right) \\
&+\delta\Delta t\ell \|A\| \left(\frac{1}{2c} \|\nabla\rho_\eta \star \rho_\eta \star \varphi\|^2_{(L^2(\Omega))^m} + \frac{c}{2} \|\nabla\varphi\|^2_{(L^2(\Omega))^m}\right) \\
\leq\;& \frac{\varepsilon\delta^2\Delta t}{2} \|\nabla\varphi\|^2_{(L^2(\Omega))^m} + \frac{m\varepsilon\Delta t}{2} \|\nabla q\|^2_{L^2(\Omega)} + \frac{\Delta t\ell^2 \|A\|^2}{\varepsilon\eta^2} \|\varphi\|^2_{(L^2(\Omega))^m},
\end{aligned}$$

where in the second line we have used Young's inequality, and chosen $c = \dfrac{\varepsilon\delta}{\ell\|A\|}$ and $d = \dfrac{\varepsilon\sqrt{m}}{\ell\|A\|}$ in the third line, with $\|A\|$ is given in (1.10).
So we get that

$$b((\varphi,q),(\varphi,q)) \geq \left(\delta - \frac{\delta\Delta t}{\tau}\right) \|\varphi\|^2_{(L^2(\Omega))^m} + \frac{\varepsilon\delta^2\Delta t}{2} \|\nabla\varphi\|^2_{(L^2(\Omega))^m} + \frac{m\varepsilon\Delta t}{2} \|\nabla q\|^2_{L^2(\Omega)}, \quad (2.7)$$

is coercive, since $\Delta t < \tau$.

Step 2: Proof of estimate (2.6)
Using (2.7) and the fact that $b((u^{n+1},p),(u^{n+1},p)) = K((u^{n+1},p))$ we get

$$\begin{aligned}
\left(\delta - \frac{\delta\Delta t}{\tau}\right) \left\|u^{n+1}\right\|^2_{(L^2(\Omega))^m} + \frac{\varepsilon\delta^2\Delta t}{2} \left\|\nabla u^{n+1}\right\|^2_{(L^2(\Omega))^m} \;&+\; \frac{m\varepsilon\Delta t}{2} \|\nabla p\|^2_{L^2(\Omega)} \\
&\leq\; \frac{\delta}{2} \|u^n\|^2_{(L^2(\Omega))^m} + \frac{\delta}{2} \left\|u^{n+1}\right\|^2_{(L^2(\Omega))^m},
\end{aligned}$$

which gives us the estimate (2.6). □

2.2 Existence for the nonlinear time-discrete problem

In this subsection we prove the existence, using Schaefer's fixed point theorem, of a solution for the nonlinear time discrete confined system (2.10) given below.

For this purpose, let the function

$$\Psi_{\varepsilon,\ell}(b) - \frac{1}{e} = \begin{cases} \frac{b^2}{2\varepsilon} + b\ln\varepsilon - \frac{\varepsilon}{2} & \text{if } b \le \varepsilon, \\ b\ln b & \text{if } \varepsilon < b \le \ell, \\ \frac{b^2}{2\ell} + b\ln\ell - \frac{\ell}{2} & \text{if } b > \ell, \end{cases} \tag{2.8}$$

which is continuous, convex and satisfies that $\Psi''_{\varepsilon,\ell}(x) = \dfrac{1}{T^{\varepsilon,\ell}(x)}$, where $T^{\varepsilon,\ell}$ is given in (2.2). Let us introduce our nonlinear time discrete system: Assume that A satisfies (1.5). Let $u^0 = (u^{i,0})_{1\le i\le m} := u_0 = (u_0^i)_{1\le i\le m}$ that satisfies

$$C_1 := \sum_{i=1}^m \int_\Omega \Psi_{\varepsilon,\ell}(u_0^i) < +\infty, \tag{2.9}$$

such that $u_0^i \ge 0$ in Ω for $i = 1,\dots,m$. Then for all Δt, ε, ℓ, η, $\delta > 0$, with $\varepsilon < 1 < \ell$ and $\Delta t < \tau$, for $n \in \mathbb{N}$, we look for a solution $(u^{n+1}, p^{n+1}) = ((u^{i,n+1})_{1\le i\le m}, p^{n+1})$ of the following nonlinear system:

$$\begin{cases} \dfrac{u^{i,n+1} - u^{i,n}}{\Delta t} & = & \operatorname{div}\left\{F^i_{\varepsilon,\ell,\eta,\delta}((u^{n+1}, u^{n+1}), p^{n+1})\right\} & \text{in} \quad \mathcal{D}'(\Omega), \\ \displaystyle\sum_{i=1}^m u^{i,n+1} & = & 1 & \text{in} \quad \Omega, \end{cases} \tag{2.10}$$

where $F^i_{\epsilon,\ell,\eta,\delta}$ is given in (2.3).

Proposition 2.2 ***(Existence for system (2.10))***
There exists a sequence of functions $u^{n+1} = (u^{i,n+1})_{1\le i\le m} \in (H^1(\Omega))^m$ and a function $p^{n+1} \in H^1(\Omega)/\mathbb{R}$ such that (u^{n+1}, p^{n+1}) is a solution of system (2.10), that satisfies the entropy estimate

$$\sum_{i=1}^m \int_\Omega \Psi_{\varepsilon,\ell}(u^{i,n+1}) + \delta\Delta t \sum_{k=0}^n \left\|\nabla u^{k+1}\right\|^2_{(L^2(\Omega))^m} + \delta_0 \Delta t \sum_{k=0}^n \left\|\nabla \rho_\eta \star u^{k+1}\right\|^2_{(L^2(\Omega))^m} \le \sum_{i=1}^m \int_\Omega \Psi_{\varepsilon,\ell}(u_0^i), \tag{2.11}$$

and

$$\Delta t \left\|\nabla p^{n+1}\right\|^2_{L^2(\Omega)} \le C_2, \tag{2.12}$$

where

$$C_2 = \frac{\ell^2 m(\|A\| + \delta)^2 C_1}{\delta}, \tag{2.13}$$

where C_1 is given in (2.9).

Proof of Proposition 2.2

Step 1: Existence of a solution for (2.10)

We define, for a given $w := u^n = (u^{i,n})_{1\le i\le m} \in (L^2(\Omega))^m$ and $v := v^{n+1} = (v^{i,n+1})_{1\le i\le m} \in (L^2(\Omega))^m$, the map θ as:

$$\begin{array}{rcccc} \theta & : & (L^2(\Omega))^m & \to & (L^2(\Omega))^m \\ & & v & \mapsto & u \end{array}$$

where $u := u^{n+1} = (u^{i,n+1})_{1\le i\le m} = \theta(v^{n+1}) \in (H^1(\Omega))^m$ is such that (u, p^{n+1}) is the unique solution of system (2.4), given by Proposition 2.1. Moreover, we can prove that θ is continuous, compact mapping and the set $\{u \in X, \quad u = \lambda\Phi(u) \quad \text{for some} \quad \lambda \in [0,1]\}$ is bounded. Then θ has a fixed point u^{n+1} on $(L^2(\Omega))^m$ by the Schaefer's fixed point theorem [2, Theorem 4 page 504]. This implies the existence of a solution (u^{n+1}, p^{n+1}) for system (2.10).

Step 2: Proof of estimate (2.11)

Since $\Psi_{\varepsilon,\ell}$ is convex we have

$$\begin{aligned}
&\sum_{i=1}^m \int_\Omega \frac{\Psi_{\varepsilon,\ell}(u^{i,n+1}) - \Psi_{\varepsilon,\ell}(u^{i,n})}{\Delta t} \le \sum_{i=1}^m \int_\Omega \left(\frac{u^{i,n+1} - u^{i,n}}{\Delta t}\right) \Psi'_{\varepsilon,\ell}(u^{i,n+1}) \\
= \; & -\sum_{i=1}^m \int_\Omega \Bigg(T^{\varepsilon,\ell}(u^{i,n+1})\nabla p^{n+1} + \delta T^{\varepsilon,\ell}(u^{i,n+1})\nabla u^{i,n+1} \\
& + T^{\varepsilon,\ell}(u^{i,n+1}) \sum_{j=1}^m A_{ij} \nabla\rho_\eta \star \rho_\eta \star u^{j,n+1} \Bigg) \Psi''_{\varepsilon,\ell}(u^{i,n+1})\nabla u^{i,n+1} \\
= \; & -\sum_{i=1}^m \left\{ -\int_\Omega \nabla p^{n+1}\cdot\nabla u^{i,n+1} + \delta \int_\Omega |\nabla u^{i,n+1}|^2 + \int_\Omega \sum_{j=1}^m \nabla\rho_\eta \star u^{i,n+1} A_{ij} \nabla\rho_\eta \star u^{j,n+1} \right\} \\
\le \; & -\sum_{i=1}^m \delta \int_\Omega |\nabla u^{i,n+1}|^2 - \delta_0 \sum_{i=1}^m \int_\Omega |\nabla\rho_\eta \star u^{i,n+1}|^2,
\end{aligned}$$

which, after a reccurence, gives us (2.11).

Now, taking the sum on i of system (2.10) then by multiplying by p^{n+1} we get

$$\begin{aligned}
0 = \sum_{i=1}^m \int_\Omega T^{\varepsilon,\ell}(u^{i,n+1}) \left|\nabla p^{n+1}\right|^2 \; &+ \; \sum_{i=1}^m \int_\Omega T^{\varepsilon,\ell}(u^{i,n+1}) \sum_{j=1}^m A_{ij} \left(\nabla\rho_\eta \star \rho_\eta \star u^{j,n+1}\right)\cdot\nabla p^{n+1} \\
&+ \; \delta \sum_{i=1}^m \int_\Omega T^{\varepsilon,\ell}(u^{i,n+1})\nabla u^{i,n+1}\cdot\nabla p^{n+1}.
\end{aligned}$$

Thus we have that

$$
\begin{aligned}
\left\|\nabla p^{n+1}\right\|_{L^2(\Omega)}^2 &\leq \ell \|A\| \sqrt{m} \left\|\nabla\rho_\eta \star \rho_\eta \star u^{n+1}\right\|_{(L^2(\Omega))^m} \left\|\nabla p^{n+1}\right\|_{L^2(\Omega)} \\
&+ \ell\delta\sqrt{m} \left\|\nabla u^{n+1}\right\|_{(L^2(\Omega))^m} \left\|\nabla p^{n+1}\right\|_{L^2(\Omega)} \\
&\leq \ell\sqrt{m}\,(\delta + \|A\|) \left\|\nabla u^{n+1}\right\|_{(L^2(\Omega))^m} \left\|\nabla p^{n+1}\right\|_{L^2(\Omega)} \\
&\leq \ell\sqrt{m}\,(\delta + \|A\|) \left(\frac{1}{2d} \left\|\nabla u^{n+1}\right\|_{(L^2(\Omega))^m}^2 + \frac{d}{2} \left\|\nabla p^{n+1}\right\|_{L^2(\Omega)}^2 \right) \\
&\leq \frac{1}{2} \left\|\nabla p^{n+1}\right\|_{L^2(\Omega)}^2 + \frac{\ell^2 m(\|A\| + \delta)^2}{2} \left\|\nabla u^{n+1}\right\|_{(L^2(\Omega))^m}^2 ,
\end{aligned}
$$

where we have used in the fourth line the Young's inequality, chosed in the fifth line $d = \dfrac{1}{\ell\sqrt{m}(\|A\| + \delta)}$. Moreover using, from the entropy estimate (2.11), the fact that $\Delta t \left\|\nabla u^{n+1}\right\|_{(L^2(\Omega))^m}^2 \leq \dfrac{C_1}{\delta}$ we get estimate (2.12). □

2.3 Proof of Theorem1.1

Passage to the limit in all parameters
The techniques used here are very similar to [3].
Step 1: Passage to the limit when $(\Delta t, \varepsilon) \to (0,0)$ ([3, Proposition 3.3])
For all $n \in \{0, \ldots, K-1\}$ set $t_n = n\Delta t$ and let the piecewise continuous in time functions:

$$
u^{\Delta t}(t,x) = (u^{i,\Delta t}(t,x))_{1\leq i\leq m} := (u^{i,n+1}(x))_{1\leq i\leq m}, \quad \text{for } t \in (t_n, t_{n+1}]. \tag{2.14}
$$

$$
p^{\Delta t}(t,x) := p^{n+1}(x), \quad \text{for } t \in (t_n, t_{n+1}]. \tag{2.15}
$$

The first term on the left-hand side of (2.11) implies that $u^{i,\Delta t} \in L^\infty(0,T;L^1(\Omega))$ for $i = 1, \cdots, m$. Now, using (2.11) with a suitable interpolation we can find $q > 2$ and a constant C independent of ε and Δt such that

$$
\left\|u^{i,\Delta t}\right\|_{L^q(0,T;L^2(\Omega))} + \left\|u^{i,\Delta t}\right\|_{L^1(0,T;H^1(\Omega))} + \left\|u^{i,\Delta t}\right\|_{\mathrm{Var}([0,T);H^{-1}(\Omega))} \leq C,
$$

where $\left\|u^{i,\Delta t}\right\|_{\mathrm{Var}([0,T);H^{-1}(\Omega))} = \displaystyle\sum_{n=0}^{K-1} \left\|u^{i,\Delta t}(t_{n+1}) - u^{i,\Delta t}(t_n)\right\|_{H^{-1}(\Omega)}$. Therefore, by a variant of Simon's Lemma [3, Theorem 2.4] there exists a function $u = (u^i)_{1\leq i\leq m} \in \left(L^2(0,T;H^1(\Omega))\right)^m$ such that $u^{\Delta t} \to u$ strongly in $\left(L^2(0,T;L^2(\Omega))\right)^m$. Moreover, (2.12) and Poincaré-Wirtinger's inequality imply that there exists a function $p \in L^2(0,T;H^1(\Omega)/\mathbb{R})$ such that $p^{\Delta t} \to p$ weakly in $(L^2(0,T;H^1(\Omega)/\mathbb{R}))^m$. Passing to the limit as $(\Delta t, \varepsilon) \to (0,0)$ in (2.10), by using in particular the weak L^2 - strong L^2 convergence in the products, we obtain that

(u,p) is a solution of the following system

$$\begin{cases} \partial_t u^i & = & \operatorname{div}\left\{F^i_{0,\ell,\eta,\delta}(u,p)\right\} & \text{in } \mathcal{D}'(\Omega_T), \\ \displaystyle\sum_{i=1}^m u^i(t,x) & = & 1 & \text{in } \Omega_T, \\ u^i(x,0) & = & u^i_0(x) & \text{in } \Omega, \end{cases} \tag{2.16}$$

where

$$F^i_{0,\ell,\eta,\delta}(u,p) = T^{0,\ell}(u^i)\left(\nabla p + \sum_{j=1}^m A_{ij}\nabla\rho_\eta \star \rho_\eta \star u^j + \delta\nabla u^i\right),$$

the initial condition is recovered by Aubin's Lemma [4, Proposition 2.1 and Theorem 3.1, Chapter 1] and [3, Proposition 5.1] since (2.11) implies that $\partial_t(u^{\Delta t} \star \rho_{\Delta t})$ is uniformaly bounded in $L^2(0,T;H^{-1}(\Omega))$ for some mollifier $\rho_{\Delta t}$ (see [3, Proposition 3.3]).
Moreover, the boundedness on the entropy function due to (2.11) implies the nonnegativity of the limit u and then together with $\sum_{i=1}^m u^i = 1$ imply that $0 \leq u^i \leq 1$ and $T^{0,\ell}(u^i) = u^i$. So we do not need to pass to the limit as $\ell \to \infty$.
Therefore, taking the lim inf as $(\varepsilon, \Delta t) \to (0,0)$ in (2.11) we obtain that (u,p) satisfies the following entropy estimate for a.e. $t_1, t_2 \in (0,T)$

$$\int_\Omega \sum_{i=1}^m \Psi(u^i(t_2)) + \delta \sum_{i=1}^m \left\|\nabla u^i\right\|^2_{L^2(t_1,t_2;L^2(\Omega))} + \delta_0 \sum_{i=1}^m \left\|\nabla\rho_\eta \star u^i\right\|^2_{L^2(t_1,t_2;L^2(\Omega))} \leq \int_\Omega \sum_{i=1}^m \Psi(u^i_0). \tag{2.17}$$

The same calculation leading to (2.12), but with integration in time and space, yields us to

$$\|\nabla p\|^2_{L^2(0,T;L^2(\Omega))} \leq C_4, \tag{2.18}$$

where $C_4 := \dfrac{m(\|A\| + \delta)^2 C_0}{\delta}$ and $C_0 := \sum_{i=1}^m \int_\Omega \Psi(u^i_0) < +\infty$.
Step 2: Passage to the limit when $\eta \to 0$ ([3, Proposition 3.5])
Let u^η a solution of (2.16). By (2.17) and a suitable interpolation we can find $q > 2$ such that

$$\left\|u^{i,\eta}\right\|_{L^q(0,T;L^2(\Omega))} + \left\|u^{i,\eta}\right\|_{L^1(0,T;H^1(\Omega))} + \left\|\partial_t u^{i,\eta}\right\|_{L^1(0,T;(W^{1,\infty}(\Omega))')} \leq C.$$

Then, by Simon's Lemma [7] we can find a function $u = (u^i)_{1\leq i\leq m} \in \left(L^2(0,T;H^1(\Omega))\right)^m$ and a function $p \in L^2(0,T;H^1(\Omega)/\mathbb{R})$ such that, as $(\eta) \to (\infty, 0)$, $u^\eta \to u$ in $\left(L^2(0,T;L^2(\Omega))\right)^m$, $\nabla u^\eta \to \nabla u$ weakly in $\left(L^2(0,T;L^2(\Omega))\right)^m$ and $p^\varepsilon \to p$ weakly in $L^2(0,T;H^1(\Omega)/\mathbb{R})$. Note that $u^i \geq 0$ a.e. in Ω_T. Therefore, passing to the limit as $\eta \to 0$ in (2.16) we get that (u,p)

is a solution of the following system

$$\begin{cases} \partial_t u^i &= \operatorname{div}\left\{u^i\left(\nabla p + \sum_{j=1}^m A_{ij}\nabla u^j + \delta\nabla u^i\right)\right\} & \text{in } \mathcal{D}'(\Omega_T), \\ \sum_{i=1}^m u^i(t,x) &= 1 & \text{in } \Omega_T \\ u^i(x,0) &= u_0^i(x) & \text{in } \Omega, \end{cases} \tag{2.19}$$

where the initial condition is recovered by [3, Proposition 5.1] since we have that $\partial_t u^\eta$ is uniformly bounded in $L^s(0,T;(W^{1,\infty}(\Omega))')$ with $s = \frac{2q}{q+2} > 1$. Moreover, u satisfies for a.e. $t_1, t_2 \in (0,T)$

$$\int_\Omega \sum_{i=1}^m \Psi(u^i(t_2)) + \delta_0 \sum_{i=1}^m \left\|\nabla u^i\right\|^2_{L^2(t_1,t_2;L^2(\Omega))} \leq \int_\Omega \sum_{i=1}^m \Psi(u_0^i), \tag{2.20}$$

and p satisfies

$$\|\nabla p\|^2_{L^2(0,T;L^2(\Omega))} \leq C_5, \tag{2.21}$$

where $C_5 := \dfrac{m\,\|A\|\,C_0}{\delta_0}$.

Step 3: Passage to the limit when $\delta \to 0$ ([3, Theorem 1.1])

Similarly, we pass to the limit when $\delta \to 0$ in system (2.19) to get the existence result, announced in Theorem 1.1, of a solution for system (1.2) that satisfies (1.8) and (1.9). □

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