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An improved method for estimating the domain of attraction of nonlinear systems containing rational functions

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Abstract. An optimization based method is proposed in this paper for the computation of Lyapunov functions and regions of attractions for nonlinear systems containing polynomial and rational terms. The Lyapunov function is given in a special quadratic form, and the negativity of its derivative is ensured using appropriate LMI conditions. The conservatism of the solution is reduced by utilizing Finsler's lemma. The number of monomial and rational terms in the computational problem is kept as low as possible using linear fractional transformation (LFT) and automatic model simplification steps. The operation of the method is illustrated on two examples taken from the literature.

1. Introduction

Finding or at least approximating the region of attraction of nonlinear dynamical systems is an important task in model analysis and controller design/evaluation, and numerous works have been devoted to this issue (see, e.g. [1]). An important early result in this field is the existence of so-called maximal Lyapunov functions for a wide class of nonlinear systems and the corresponding iterative procedure to approximate them [2]. In [3], maximal Lyapunov functions were defined and computed for hybrid (piecewise nonlinear) systems. At the same time, the use of linear matrix inequalities (LMI) and semidefinite programming (SDP) techniques for nonlinear systems has become very popular due to their advantageous properties and the availability of efficient numerical tools to solve LMI problems. These new techniques provide a powerful framework for stability analysis, robust control, and filtering problems. Ghaoui et.al. [4] used quadratic Lyapunov functions and linear fractional transformations (LFT) to represent a rational nonlinear system and defined convex conditions for stability analysis and state feedback design. The application of sum of squares (SOS) programming to maximize the estimate of the region of attraction can be found in [1, 5]. Stability conditions in both references are converted into LMIs using SOS relaxations and the generalization of the S-procedure. Topcu et.al. [6] utilized a further branch-and-bound type refinement in the parameter space to reduce the solution's conservatism. According to Trofino et.al. [7], LMI conditions can be obtained by using the Finsler's Lemma and the notion of annihilators. The newly introduced sufficient conditions

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for the stability are affine parameter dependent LMIs because they are characterized by affine functions of the state (x) and uncertain parameters (δ). Affine parameter dependent LMIs can be computationally handled by checking their feasibility at the corner points (vertices) of a polytopic region, on which the uncertain parameters are defined. In [7] it is shown that with some additional conservatism, the use of the vertices can be avoided by modifying the LMIs with the S-Procedure.

Based on the results of [7], in this work we present improved sufficient linear matrix inequality (LMI) conditions for local and regional asymptotic stability of polynomial and rational nonlinear systems. These LMI conditions are given through a Lyapunov function containing monomial and rational terms with the prescribed properties. We also present our computational results on illustrative examples taken from the literature.

2. Background

In this section we present the basic notions and known results on which our computational results are based.

2.1. System class, Lyapunov functions and domain of attraction

We consider nonlinear systems of the form

$$\dot{x} = f(x, \delta),\ x \in \mathbb{R}^n,\ x_0 \in \mathcal{X},\ \delta \in \mathcal{D},\ \dot{\delta} \in \mathcal{D}_d, \tag{1}$$

where x is the state vector, x_0 is the initial condition, and δ is the vector of (possibly time-dependent) uncertain parameters. Furthermore, it is assumed that $\mathcal{X}$, $\mathcal{D}$, and $\mathcal{D}_d$ are known polytopic regions. We assume that $x^* = 0 \in \mathbb{R}^n$ is a locally asymptotically stable equilibrium point of (1) for all $\delta \in \mathcal{D}$. The set of all initial conditions from which the solutions converge to x^* is called the *domain of attraction* (DOA). We search for Lyapunov functions in the form

$$V(x, \delta) = x^T P(x, \delta) x, \quad v_l(x) \le V(x, \delta) \le v_u(x) \tag{2}$$

with the property

$$\dot{V}(x) \le -v_d(x),\ \forall (x, \delta, \dot{\delta}) \in \mathcal{X} \times \mathcal{D} \times \mathcal{D}_d, \tag{3}$$

where P is a positive definite symmetric matrix function, and v_l, v_u and v_d are continuous positive functions on $\mathcal{X}$. Clearly, if (3) is fulfilled, then any closed level set of V completely inside $\mathcal{X}$ bounds an invariant region of the state space that is part of the domain of attraction.

The measure of *conservatism* is a property, which describes an invariant region. Let us consider two domains bounded by two different ε and γ level sets. We call ε less *conservative* than γ if ε is a better estimate of the actual region of attraction in the sense that the area/volume inside ε is larger than that of γ.

Roughly speaking, our main objective is to find a $V(x)$ function having a level set, which bounds the least conservative invariant region. In fact, introducing higher degree monomials into $V(x)$ generally results in better estimates, although a small increment in the number of monomials generates a huge increase in the dimension of the problem. Therefore, the rapidly growing computational burden must be taken into consideration through considering the possibilities of dimension reduction.

2.2. Two relevant approaches in the literature

The main differences between the methods proposed by Topcu et.al. [5] and by Trofino et.al. [7] are related to the definition of the sets, on which the conditions are stated, and the objective function, with which the size of the invariant region will be maximized.

Topcu prescribed the Lyapunov function to be positive definite on the whole $\mathbb{R}^n$ but the function's time derivative should be negative definite only in the inside of a level set, which would be the final invariant level set as it reached its maximal size[1]. In order to maximize the size of the level set, Topcu defined a variable sized region $\mathcal{P}_\beta = \{x \in \mathbb{R}^n \mid p(x) \le \beta\}$, which should lie inside the invariant level set, while the variable β is maximized. The polynomial $p(x)$ is a design factor, which determines the shape and the orientation of the inner region.

On the other hand, the authors in [7] do not prescribe any constraint outside a given $\mathcal{X} \subset \mathbb{R}^n$ polytopic set, but for every $x \in \mathcal{X}$ the Lyapunov function and its derivative are required to be positive and negative definite, respectively. Furthermore, an additional constraint was introduced, more specifically, that the level set $\varepsilon_1 = \{x \in \mathbb{R}^n \mid V(x) = 1\}$ should be located inside the $\mathcal{X}$ polytope. In this case, the objective function to be minimized constitutes the sum of values of the Lyapunov function in some $x \in \mathcal{X}$ points, which are strategically chosen. Such an objective function can enforce that the level set $V(x) = 1$ is as close to the boundary of $\mathcal{X}$ as possible.

2.3. Further notations

In the paper, we will use the following additional notations:

- $\mathrm{Co}(\mathcal{X})$ denotes the convex expansion of the set $\mathcal{X}$, more specifically, $\mathrm{Co}(v_i, i = 1 \dots n)$ is the convex hull of the set of vertices $\{v_i \mid i = \overline{1,n}\}$.
- $\vartheta(\mathcal{X})$ denotes the set of all vertices (corner points) of the polytope $\mathcal{X}$ (i.e. $\mathrm{Co}\big(\vartheta(\mathcal{X})\big) = \mathcal{X}$).
- $\nabla V(x)$ is the gradient of the scalar function $V(x)$.
- $O_{n\times m}$ and I_n denote the $n \times m$ zero matrix and $n \times n$ unit matrix, respectively.
- A *basis function* is an element of a particular basis for a function space, e.g. monomials of the form $x_1^{p_1} x_2^{p_2} \cdots x_k^{p_k}$ are a special class of basis functions. In this paper, we have considered every rational function with a monomial numerator as a basis function.

2.4. LMI stability conditions and Finsler's lemma

Let us suppose that we have the $\dot{x} = Ax$ linear time invariant (LTI) dynamical system, then the origin is globally asymptotically stable if there exists a Lyapunov function $V(x) = x^T P x$ such that the following conditions are satisfied:

$$\begin{aligned} &\forall x: \quad V(x) = x^T P x > 0 && \Longleftrightarrow P > 0 \quad \text{(P is positive definite)} \\ &\forall x: \quad \dot{V}(x) = \nabla V(x)\dot{x} = x^T(PA + A^T P)x < 0 && \Longleftrightarrow PA + A^T P < 0 \end{aligned} \tag{4}$$

This small example shows how can the Lyapunov conditions converted into a system of LMIs. As it is presented in [8] (Definition 1.37), every system of LMIs can be easily converted into a single LMI condition.

In the case of an uncertain linear system $\dot{x} = A(\delta)x$ with an affine $A(\delta)$ state transition matrix function, the second LMI of (4) will become parameter (δ) dependent LMI. According to Proposition 5.4 in [8], if we can specify a bounded polytopic domain in which the $\delta \in \mathcal{D}$ uncertain parameter operates, then it is enough to test the inequality at the corner points of the $\mathcal{D}$ polytope: $PA(\delta) + A(\delta)^T P < 0,\ \forall \delta \in \vartheta(\mathcal{D})$, being still *equivalent* to the original Lyapunov condition.

Consider the nonlinear system $\dot{x} = A(x)x$, where $A(x) \in \mathbb{R}^{n\times n}$. Let us choose again a simple quadratic Lyapunov function $V(x) = x^T P x$. Then the negative definiteness of $\dot{V}(x)$ constitutes $x^T K(x) x < 0\ \forall x$, where $K(x) = PA(x) + A^T(x)P$. In this case, prescribing that $K(x) < 0$ for

[1] for more details, see Lemma 1. in [5]

all x introduces conservatism, because of its being only a sufficient but not a necessary condition for $x^T K(x)\, x < 0$.

Alternatively, we can define a non-quadratic polynomial Lyapunov function of the form $V(x) = \pi^T P\, \pi$, where π is a column vector of monomials in x, eg. $\pi^T = \begin{bmatrix} x_1 & x_2 & x_1^2 & x_1 x_2 \end{bmatrix}$. Now we can see again that the condition $P > 0$ is only a sufficient condition for $\pi^T P \pi > 0$, because it is equivalent to $z^T P z > 0 \;\; \forall z \in \mathbb{R}^p$, which is definitely a more strict condition for P than the original one.

From the above examples it is clear that it would be advantageous to decrease the conservatism of the inequalities corresponding to the stability condition. It is shown in [7] that Finsler's lemma and the notion of annihilators will help us to bring more freedom into our LMIs.

Lemma 2.1 (Finsler's lemma) *Let $\mathcal{X} \subseteq \mathbb{R}^n$ be given a polytopic set, $P : \mathcal{X} \mapsto \mathbb{R}^{p\times p}$, $N : \mathcal{X} \mapsto \mathbb{R}^{r\times p}$ be given matrix functions, with P symmetric. Let $Q(x)$ be a basis for the null space of $N(x)$. Then, the following are equivalent:*

(i) $\forall x \in \mathcal{X} :\quad \pi^T P(x) \pi > 0$ *is satisfied* $\forall \pi \in \mathbb{R}^p, N(x)\pi = 0$

(ii) $\forall x \in \mathcal{X} :\quad \exists L : \mathcal{X} \mapsto \mathbb{R}^{q\times r}$ *matrix function such that* $P(x) + L(x)N(x) + N^T(x)L^T(x) > 0$

(iii) $\forall x \in \mathcal{X} :\quad Q^T(x)P(x)Q(x) > 0$ *is satisfied.*

Assuming that $P(x)$ and $N(x)$ are affine functions and L is constant matrix, we obtain a special case of the Finsler's lemma. Henceforth, the conditions (i) and (ii) are no longer equivalent, but (ii) shall continue to be a sufficient condition for (i), which in our case is a satisfactory result. Furthermore, (ii) will became a polytopic LMI (i.e. an affine parameter dependent LMI) of the following form

$$\forall x \in \mathcal{X} \;\; \exists L \in \mathbb{R}^{p\times r} : \quad P(x) + LN(x) + N^T(x)L^T > 0, \tag{5}$$

which again can be transformed into a single parameter independent LMI with a higher dimension. We have to design an $N(x)$ affine matrix function in such a way that $N(x)\pi = 0$ for all π. Therefore, $N(x)$ is called an annihilator of π. It is obvious that (5) is a less conservative sufficient condition for $\pi^T P\, \pi > 0$, than $P > 0$. In fact, increasing the size of the annihillator introduces more freedom into the computational problem.

2.5. Dynamical system representation

As a starting point, we use the same differential-algebraic representation of nonlinear models that was introduced in [7], namely:

$$\begin{aligned} \dot{x} &= f(x,\delta) = A\, x + B\, \pi && x_0 \in \mathcal{X} \\ 0 &= G(x,\delta)\, x + F(x,\delta)\, \pi && \delta \in \mathcal{D}, \dot{\delta} \in \mathcal{D}_d \end{aligned} \tag{6}$$

where $G : \mathcal{X} \times \mathcal{D} \mapsto \mathbb{R}^{p\times n}$ and $F : \mathcal{X} \times \mathcal{D} \mapsto \mathbb{R}^{p\times p}$ are affine matrix functions of (x, δ). This form separates the linear part of the system (Ax) from its nonlinear part $(B\pi)$. The second equation introduces a constraint representing the relation between x and π, where $\pi(x)$ is a vector of monomials in x. In [7], the authors propose that π contains all basic monomials of degree less than or equal to the maximal degree term in the system equation. This clearly causes a combinatorial explosion as the number of variables and their degrees increase. Therefore, in this work, we propose the application of LFT to decrease the number of monomials in π, hoping that the solutions conservatism will not increase significantly.

2.6. Linear fractional transformation (LFT)
The main objective of LFT is to separate the linear part of a system from the nonlinear and uncertain parts. The LFT requires the system model in the form $\dot{x} = \mathcal{A}(x)x$, therefore, in case of a system represented in a more general $\dot{x} = f(x)$ form, we have to find a matrix function $\mathcal{A} : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$, such that $\mathcal{A}(x)x = f(x)$. This computation step can be handled e.g. with the built-in functions of Matlab's Symbolic Math Toolbox. Let us consider the following uncertain (parameter dependent) nonlinear system:

$$\dot{x} = \mathcal{A}(x, \delta)\, x, \quad \delta \in \mathcal{D} \tag{7}$$

The linear fractional representation (LFR) of the system is the following:

$$\begin{aligned} \dot{x} &= M_{11}x + M_{12}w \\ z &= M_{21}x + M_{22}w \end{aligned} \Longrightarrow \begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \left(\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right) \cdot \begin{bmatrix} x \\ w \end{bmatrix} \tag{8}$$

$$w = \Delta(x, \delta)\, z \tag{9}$$

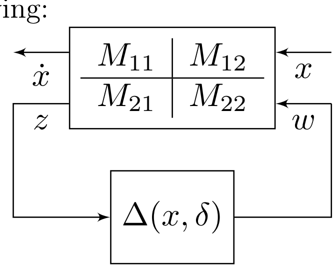

Figure 1: Block diagram of an LFR system

Equation (8) can be considered as a linear time invariant (LTI) system equipped with an input defined by the nonlinear uncertain relation (9). The block diagram of an LFR can be seen in Figure 1. Eliminating z and w from (8) and (9) we get the following:

$$\mathcal{A}(x, \delta) = \underbrace{M_{21}(I_n - \Delta M_{11})^{-1}\Delta M_{12}}_{\text{nonlinear part}} + \underbrace{M_{22}}_{\text{linear part}}\ , \tag{10}$$

where M_{ij} are constant matrices and Δ is a diagonal matrix containing monomials of the state variables and uncertain parameters.

3. Estimating the domain of attraction
In this section we describe the main steps of the improved method for estimating the domain of attraction.

3.1. Preliminary transformations of the model
Due to the fact that the LFR is a special case of (6), it can be easily converted into that form by introducing the following notations:

$$\begin{aligned} A &= M_{22} \\ B &= M_{21} \end{aligned}, \qquad \begin{aligned} G(x) &= -\Delta M_{12} \\ F(x) &= I_n + \Delta M_{11} \end{aligned}, \qquad \pi = (I_n - \Delta M_{11})^{-1}\Delta M_{12}\, x \tag{11}$$

In this form, π contains only nonlinear elements. The LFR may result in such M_{ij} and Δ matrices that π will contain polynomials and rational terms with polynomial numerator and denominator. In this case, these elements of π should be split into monomials and rational terms with monomial numerators, respectively. At the same time, B and $F(x)$ should be modified appropriately, to satisfy the model equations (6) with the modified π vector. The same steps should be performed on the matrix B.

Additionally, the LFT and the previous transformations may generate several linearly dependent and thus redundant entries in π (the same term can appear multiple times, optionally with different constant multipliers). This may bring a significant unnecessary increase in the dimensions of the representation. Therefore, we designed an algorithm which eliminates the repetitive terms from π. The basic principle is to merge two rows in π into a single row, consequently, we also have to merge two columns in B and $F(x)$. Let us denote with $p(x)$ the monomial appearing at least twice and with $a(x)$, $b(x)$ those two elements in an arbitrary row of B or $F(x)$, which will be multiplied by the two identical monomials in π. Using these notations, we can define the following transformation:

$$\underbrace{\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a(x) & \cdots & b(x) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}}_{B \text{ or } F(x)} \underbrace{\begin{bmatrix} \cdots \\ \alpha p(x) \\ \cdots \\ \beta p(x) \\ \cdots \end{bmatrix}}_{\pi} = \begin{bmatrix} \cdots & \cdots\cdots\cdots & \cdots \\ \cdots & \alpha a(x) + \beta b(x) & \cdots \\ \cdots & \cdots\cdots\cdots & \cdots \end{bmatrix} \begin{bmatrix} \cdots \\ p(x) \\ \cdots \end{bmatrix} \tag{12}$$

We have to note here that it is comfortable to keep $F(x)$ a square matrix, because in a few computation steps the left inverse of $F(x)$ is used. However, all multiplications by the left inverse can be avoided. We remark that the authors in [7] assumed $F(x)$ to be square matrix, i.e. $F(x) \in \mathbb{R}^{p\times p}$ and $\pi \in \mathbb{R}^p$. In order to ease this strict condition on $F(x)$, we will give a more general formula for the LMIs in Section 3.3. In that formula, we assume that $F(x) \in \mathbb{R}^{q\times p}$ is a general rectangular matrix. This relaxation does not affect the size of the LMIs, they still remain semidefinite problems (SDP). Although it is not necessary for $F(x)$ to be of maximum rank, it is advisable to remove the redundant rows from $C_b(x) = \begin{bmatrix} G(x) & F(x) \end{bmatrix}$ (i.e. to keep linearly independent rows only). The following pseudocode shows the entire simplification procedure:

Data:
  π, indices(π) = (first) $1 \dots p$ (last), having repetitive monomials
  B, $F(x)$, $G(x)$: system matrices
Result:
  π without repetitive monomials,
  modified B, $F(x)$, $G(x)$ model matrices corresponding to the new π
for $j = p$ *down to 2* **do**
  for i = *1 to j-1* **do**
    $\frac{\alpha\, p(x)}{r_i(x)} \leftarrow \pi[i]$, $\frac{\beta\, q(x)}{r_j(x)} \leftarrow \pi[j]$
    if $p(x) = q(x)$ *and* $r_i(x) = r_j(x)$ **then**
      Generate the new column in B and $F(x)$ (element-wise):
      B[column i] = α B[column i] + β B[column j]
      $F(x)$[column i] = α $F(x)$[column i] + β $F(x)$[column j]
      $\pi[i] = p(x)/r_i(x)$
      Remove column j from B and $F(x)$.
      Remove the jth element from π
    end
  end
end
$p \leftarrow$ size(π)
$C_b(x) \leftarrow \begin{bmatrix} G(x) & F(x) \end{bmatrix}$
Clear linearly dependent rows from $C_b(x)$

3.2. Annihillator generation
In the processed form, π contains only non-repetitive nonlinear basis functions (monomials and rational terms with monomial numerator). Let us introduce the notations $\pi_b = [x^T\ \pi^T]^T$ and $C_b(x) = \begin{bmatrix} G(x) & F(x) \end{bmatrix}$. Thus, the second equation in (6) can be modified to $0 = C_b(x)\pi_b$. It is clear that $C_b(x)$ is a linear annihilator of π_b, which represents the dependence of π upon x. However, the size of $C_b(x)$ as an annihilator is far from being maximal, therefore, finding a second annihilator with rows not appearing in $C_b(x)$ is essential. Using Matlab's symbolic toolbox, we have written an algorithm, which generates an affine annihilator $N_{\pi_b}(x)$ having a special form. Since π_b contains only basis functions (optionally multiplied by an arbitrary constant), it

is enough if in each row of the matrix there appear only two nonzero items[2], which will eliminate the two corresponding elements in π_b. In our algorithm, we search for an adequate $N_{\pi_b}(x)$ in the following form:

$$N_{\pi_b}(x) = \begin{bmatrix} \cdots & \alpha x_i & \cdots & \beta x_j & \cdots \\ \cdots & \alpha x_i & \cdots & \beta & \cdots \\ \cdots & \alpha & \cdots & \beta x_j & \cdots \end{bmatrix} \tag{13}$$

We have chosen two elements from π, let them be $a(x), c(x)$. If the numerator $\mathrm{num}(x)$ and the denominator $\mathrm{den}(x)$ of their simplified fraction $a(x)/c(x)$ are monomials of degree 0 or 1, then there exists $b(x) = \mathrm{den}(x)$, $d(x) = -\mathrm{num}(x)$ affine monomials such that $a(x)b(x)+c(x)d(x) = 0$. Finally, a new row can be appended to the annihilator matrix:

$$\begin{bmatrix} \cdots & b(x) & \cdots & d(x) & \cdots \end{bmatrix} \tag{14}$$

This procedure is evaluated on each pair of the elements of π. Possibly, $N_{\pi_b}(x)$ and $C_b(x)$ will have common rows, which can be skipped from $N_{\pi_b}(x)$. For this we used the built-in Matlab function $\left[\begin{smallmatrix} C_b(x) \\ N_{\pi_b}(x) \end{smallmatrix}\right]$=`unique(`$\left[\begin{smallmatrix} C_b(x) \\ N_{\pi_b}(x) \end{smallmatrix}\right]$`,'rows','static')`.

3.3. Finding an appropriate Lyapunov function

After the transformations presented in Section 3.1, we have the following model:

$$\begin{cases} \dot{x} = Ax + B\pi, & x \in \mathbb{R}^n, \quad \pi \in \mathbb{R}^p \\ 0 = G(x)x + F(x)\pi, & G(x) \in \mathbb{R}^{q\times n}, \quad F(x) \in \mathbb{R}^{q\times p} \end{cases} \tag{15}$$

Furthermore, $C_b(x) = \begin{bmatrix} G(x) & F(x) \end{bmatrix} \in \mathbb{R}^{q\times(n+p)}$ and $N_{\pi_b}(x) \in \mathbb{R}^{s_b\times(n+p)}$ are annihilators of π_b. A Lyapunov function candidate for this system will be given as $V(x) = x^T\mathbf{P}(x)x = \pi_b^T P\,\pi_b$, where $P \in \mathbb{R}^{(n+p)\times(n+p)}$ is a constant symmetric matrix. As it has been mentioned in Section 2.4, the positive definiteness of $V(x)$ can be ensured by a stricter inequality:

$$\forall x \in \mathcal{X} \ \ \exists L \in \mathbb{R}^{(n+p)\times(q+s_b)}: \quad P + L\begin{bmatrix} C_b(x) \\ N_{\pi_b}(x) \end{bmatrix} + \begin{bmatrix} C_b^T(x) & N_{\pi_b}^T(x) \end{bmatrix} L^T > 0 \tag{16}$$

According to Theorem 4.1 in [7], the negative definiteness of $\dot{V}(x) = \nabla V(x)\,\dot{x}$ in ensured by the following sufficient LMI condition:

$$\forall x \in \mathcal{X} \ \ \exists L_a \in \mathbb{R}^{(n^2+n+2p+np)\times s_b}: \quad P_a + P_a^T + L_a\begin{bmatrix} C_a(x) \\ N_{\pi_a}(x) \end{bmatrix} + \begin{bmatrix} C_a^T(x) & N_{\pi_a}^T(x) \end{bmatrix} L_a^T < 0 \tag{17}$$

In contrast to [7], we assume a general rectangular $F(x) \in \mathbb{R}^{q\times p}$ matrix (not necessarily square), therefore, the following variable is redefined as:

$$C_a(x) = \begin{bmatrix} G(x) & F(x) & O_{q\times p} & O_{q\times n^2} & O_{q\times np} \\ W_1(x) & W_2(x) & F(x) & O_{q\times n^2} & \bar{F}_a \\ W_3(x) & W_4(x) & O_{n^2\times p} & I_{n^2} & O_{n^2\times np} \\ O_{nq\times n} & O_{nq\times p} & O_{nq\times p} & -G_b(x) & F_b(x) \end{bmatrix} \tag{18}$$

where the matrices P_a, $N_{\pi_a}(x)$, $W_1(x)$, $W_2(x)$, $W_3(x)$, $W_4(x)$, $\bar{F}_a$, $G_b(x)$, $F_b(x)$ remain the same as they are defined in [7] in equations (39-42).

[2] In case of monomials, if an element $a(x)$ of π_b can be eliminated by two other elements $b(x), c(x)$, than $a(x)$ surely can be eliminated by using only one from $b(x)$ and $c(x)$. This statement is no longer valid when having monomials and basis rational terms, too. Consider the following simple example: $\begin{bmatrix} -1 & x & x \end{bmatrix}\begin{bmatrix} x & \frac{1}{x^2+1} & \frac{x^2}{x^2+1} \end{bmatrix}^T = 0$.

3.4. Finding the maximal level set, for a given $\mathcal{X}$

In order to find the maximal invariant level set, we adopted a combined method of the two approaches presented in Section 2.2. First of all, we defined a small $\mathcal{Y}$ polytope around the locally stable origin inside $\mathcal{X}$. With reference to [5], $\mathcal{Y}$ could be a variable size polytope, the size of which can be maximized during the optimization producing a maximal level set around $\mathcal{Y}$. The challenge is that the matrix inequality conditions are no longer linear using this approach, hence the need arises for further relaxations. In comparison, we set $\mathcal{Y}$ to have a small but constant size. In short, we have two polytopes around the origin: a larger one ($\mathcal{X}$) and a smaller one ($\mathcal{Y}$) in the inside of $\mathcal{X}$. We are looking for a level set $\varepsilon_\alpha = \{x \in \mathcal{X} \mid V(x) = \alpha,\ 1 \le \alpha\}$ that is outside of $\mathcal{Y}$ but inside of $\mathcal{X}$, subject to the 1-level set $\varepsilon_1 = \{x \in \mathcal{X} \mid V(x) = 1\}$ is in the same region. To rephrase, we have to maximize α, s.t.

(1) ε_α is located in the inside of $\mathcal{X}$,

(2) $\mathcal{Y}$ is inside ε_1 (without this condition, $V(x)$ can be scaled arbitrarily, resulting in an unbounded optimum).

Condition (C1) means that for each facet $\mathcal{F}_k^{(\mathcal{X})}$ of the $\mathcal{X}$ polytope the following condition must be satisfied:

$$V(x) = \pi_b^T P \pi_b \ge \alpha \Longleftrightarrow \begin{bmatrix} \pi_b^T & 1 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} \pi_b \\ 1 \end{bmatrix} \ge 0, \quad \forall x \in \vartheta\big(\mathcal{F}_k^{(\mathcal{X})}\big),\ \forall k = \overline{1, M_{\mathcal{X}}}, \tag{19}$$

where $M_{\mathcal{X}}$ is the number of facets of $\mathcal{X}$. According to [7] and using the equivalence between (i), (iii) of the Finsler's lemma, we can rewrite the condition (19) into

$$Q_k^T \big(P_{c_k}(x) + \Gamma_{c_k}(x)\big) Q_k > 0, \quad \forall x \in \vartheta\big(\mathcal{F}_k^{(\mathcal{X})}\big),\ \forall k = \overline{1, M_{\mathcal{X}}}, \tag{20}$$

where Q_k, $P_{c_k}(x)$, $\Gamma_{c_k}(x)$ are described in detail by equations (82-89) of [7]. With the same consideration, condition (C2) can be written as

$$V(x) = \pi_b^T P \pi_b \le 1 \Longleftrightarrow \begin{bmatrix} \pi_b^T & 1 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \pi_b \\ 1 \end{bmatrix} \le 0, \quad \forall x \in \vartheta\big(\mathcal{F}_k^{(\mathcal{Y})}\big),\ \forall k = \overline{1, M_{\mathcal{Y}}}, \tag{21}$$

which can be converted into the same form as (20). Finally, the optimization problem is traced back to a minimization of $-\alpha$ subject to eqs. (16), (17), (20) and (21).

3.5. Finding the most appropriate outer polytope

Finding the most suitable $\mathcal{X}$, in which the Lyapunov conditions could be fulfilled is an iterative problem. The basic concept is to choose an $\mathcal{X}^{(0)}$ inital polytope, which surely satisfies the LMI conditions, than in each step ($\mathcal{X}^{(k)}$) find out the $\varepsilon^{(k)}$ maximal level set and give a new, larger $\mathcal{X}^{(k+1)}$ polytope considering the shape of $\varepsilon^{(k)}$. One possible solution can be choosing some uniformly distributed discrete points placed on the $\varepsilon^{(k)}$ level set. These points will span a polytope, which should be enlarged with a given increment, such that its shape is not changed (practically, the coordinates of every corner are multiplied by an $1 < \gamma \ll 2$ scalar factor, since the stable equilibrium point is assumed to be at the origin).

4. Numerical examples and results

The results presented in this section have been computed in the Matlab environment. For symbolic computations we used Matlab's built-in Symbolic Math Toolbox based on Mupad. For LFT we used the Enhanced LFR-toolbox [9, 10]. To model and solve semidefinite optimization (SDP) problems we used the SeDuMi and Mosek solvers with YALMIP [11].

4.1. Van der Pol dynamics

The equation

$$\begin{aligned}\dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 - \varepsilon(1 - x_1^2)x_2\end{aligned} \quad \text{with} \quad \varepsilon = 1 \tag{22}$$

describes a time-inverted oscillator introduced by the Dutch electrical engineer and physicist Balthasar van der Pol. This system has an asymptotically stable equilibrium point at the origin, and it has a limit cycle, which defines the boundary of the region of attraction (ROA). This limit cycle and some trajectories of the system are illustrated in Figure 2 (dashed red line and black trajectories, respectively). In Figure 2, you can also see the maximal invariant level set (green line) generated by the method described in Section 3. The nonlinear basis functions of π used by Trofino et.al. were

$$\pi^T = \begin{bmatrix} {x_1}^2 & x_1 x_2 & x_2^2 & x_1^3 & x_1^2 x_2 & x_1 x_2^2 & x_2^3 \end{bmatrix} \tag{23}$$

In comparison, our method by applying the LFT generates a smaller number of monomials:

$$\pi^T = \begin{bmatrix} x_1^2 x_2 & x_1 x_2 \end{bmatrix}. \tag{24}$$

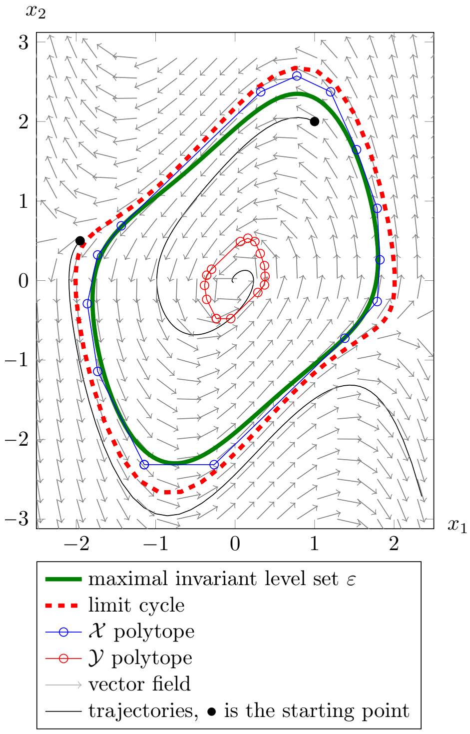

Figure 2: Van der Pol system, phase diagram. The ε level set (green line) approximates the limit cycle (dashed red line) in a quite acceptable manner. The blue and the red polygons constitutes the $\mathcal{X}$ and $\mathcal{Y}$, respectively

However, the area of the estimated invariant region is close enough to that of the true DOA. The matrices generated by the LFR Toolbox are the following:

$$M_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_{12} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_{22} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad \Delta = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix},$$

The final Lyapunov function and invariant level set we obtained is the following:

$$\begin{aligned} V(x) =& -0.05746\, x_1^4 x_2^2 + 0.03872\, x_1^3 x_2^2 + 2.235\, x_1^3 x_2 + 0.2525\, x_1^2 x_2^2 - 0.06664\, x_1^2 x_2 \\ &+ 5.961\, x_1^2 - 0.01476\, x_1 x_2^2 - 8.719\, x_1 x_2 + 5.05\, x_2^2 \\ \varepsilon =& \left\{ x \in \mathcal{X} \;\middle|\; V(x) = \alpha = 17.4086 \right\} \end{aligned} \tag{25}$$

4.2. Continuous fermentation process

Bioreactors often show strongly nonlinear dynamical behaviour, therefore, they can be interesting subjects for stability analysis. In our work, we have analysed the stability region of a widely used model presented in e.g. [12], which is a rational nonlinear system having a locally asymptotically stable equilibrium point. The equations of the normalized system are the following:

$$\dot{x} = \begin{bmatrix} \dot{\bar{X}} \\ \dot{\bar{S}} \end{bmatrix} = \begin{bmatrix} (\bar{X} + X_0)\cdot\mu(\bar{S} + S_0) - \dfrac{(\bar{X} + X_0)F_0}{V} \\ -\dfrac{(\bar{X} + X_0)\cdot\mu(\bar{S} + S_0)}{Y} + \dfrac{(S_F - (\bar{S} + S_0))F_0}{V} \end{bmatrix} \tag{26}$$

$$\mu(S) = \mu_{max} \frac{S}{K_2 S^2 + S + K_1}, \tag{27}$$

where the variables and parameters are explained in the following table:

Variables and parameters of the process				Steady-state operating point			
X	Biomass concentration		$[g/l]$	X_0	equilibrium point of X	4.8907	[g/l]
S	Substrate concentration		$[g/l]$	S_0	equilibrium point of S	0.2187	[g/l]
F	Feed flow rate		$[l/h]$	F_0	Inlet feed flow rate	3.2089	[l/h]
V	Volume	4	$[l]$				
S_F	Substrate feed concentration	10	$[g/l]$				
Y	Yield coefficient	0.5					
μ_{max}	maximal growth rate	1	$[l/h]$				
K_1	Saturation parameter	0.03	$[g/l]$				
K_2	Inhibition parameter	0.5	$[l/g]$				

We cannot apply the LFT in this form of the system, because there appear constant terms as well: if $\bar{X} = 0$ and $\bar{S} = 0$ then $\dot{x} = \begin{bmatrix} X_0 \cdot \mu(S_0) - X_0 F_0/V \\ -X_0 \cdot \mu(S_0)/Y - (S_F - S_0)F_0/V \end{bmatrix}$. Knowing that $F_0 = V\mu(S_0)$ and $X_0 = (S_0 - S_F)Y$, these constant terms can be eliminated. After the factorization ($f(x) = A(x)x$) we obtain the desired form:

$$\dot{x} = A(x)x, \quad \text{where} \quad A(x) = \begin{bmatrix} -\frac{c_2 F_0 \bar{S}^2 + c_1 F_0 \bar{S}}{q(\bar{S})V} & \frac{\mu_{max} V(\bar{X}+X_0) - X_0 F_0 (c_2\bar{S}+c_1)}{q(\bar{S})V} \\ -\frac{\mu_{max} S_0}{q(\bar{S})Y} & -\frac{F_0}{V} - \frac{\mu_{max} V(\bar{X}+X_0) + Y F_0 (S_0 - S_F)(c_2 \bar{S}+c_1)}{q(\bar{S})VY} \end{bmatrix} \tag{28}$$

$$c_2 = K_2$$
$$c_1 = 2\,K_2\,S_0 + 1$$
$$c_0 = K_2\,S_0^2 + S_0 + K_1 \qquad q(\bar{S}) = c_2\,\bar{S}^2 + c_1\,\bar{S} + c_0$$

Using the LFT we obtain a π with rational nonlinear elements having $q(\bar{S})$ as their denominator. The final $\mathcal{X}$ polytope with the corresponding invariant level set can be seen in Figure 3, as a gray and red line, respectively. We can see that the final level set adequately approximates the boundary of the DOA (dashed line). The final Lyapunov functions was the following:

$$V(x) = \frac{75232(-22.278\,x_1^2\,x_2^4 + 349.79\,x_1^2\,x_2^3 + 43.209\,x_1^2\,x_2^2 + 1.6758\,x_1^2\,x_2 + 0.093181\,x_1^2 + 123.37\,x_1\,x_2^5 + 577.76\,x_1\,x_2^4)}{1.3657\,x_2^4 + 6.6573\,x_2^3 + 9.6019\,x_2^2 + 3.6291\,x_2 + 0.40585}$$
$$+ \frac{75232(46.861\,x_1\,x_2^3 + 1.3259\,x_1\,x_2^2 + 0.038426\,x_1\,x_2 - 43.571\,x_2^6 - 120.6\,x_2^5 + 23.602\,x_2^4 + 1.3087\,x_2^3 + 0.031804\,x_2^2))}{1.3657\,x_2^4 + 6.6573\,x_2^3 + 9.6019\,x_2^2 + 3.6291\,x_2 + 0.40585}$$

4.3. Continuous fermentation process with a simple linear feedback

In the previous normalized process, we constrained the inlet feed flow rate to be constant (F_0). Here, we introduce a centered input u, and the actual feed flow rate will be $F = F_0 + u$. Consequently, the model should be modified with an input term as follows.

$$\dot{x} = A(x)x + g(x)u, \qquad g(x) = \begin{bmatrix} -\frac{X_0 + \bar{X}}{V} \\ -\frac{\bar{S} + S_0 - S_F}{V} \end{bmatrix} \tag{29}$$

In [12] it is shown that the zero dynamics of the model is globally stable if the output is the substrate concentration. Therefore, we chose a simple feedback of the form $u = k\bar{S}$ with $k \in \mathbb{R}$ ($k = -1$ was used for the computations). Then we can write:

$$g(x)u = \begin{bmatrix} -\frac{k\bar{S}}{V} & -\frac{kX_0}{V} \\ 0 & -\frac{k(\bar{S}+S_0-S_F)}{V} \end{bmatrix} \begin{bmatrix} \bar{X} \\ \bar{S} \end{bmatrix} \tag{30}$$

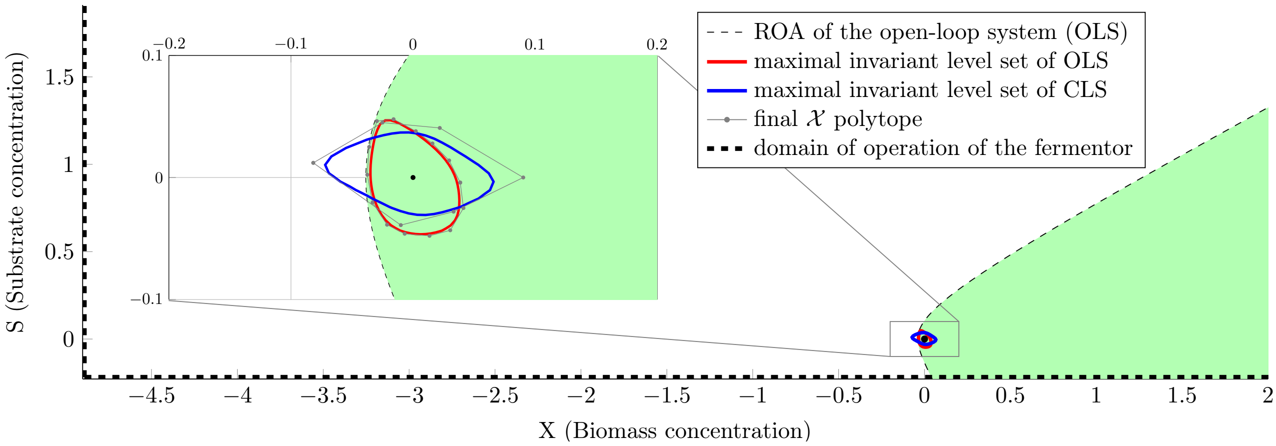

Figure 3: Region of attraction (ROA) of the continuous fermentation process without feedback (green area). The final invariant level set for the open-loop system (OLS) and for the closed-loop system (CLS) with $k = -1$ can be seen as a red and blue line, respectively. The most appropriate outer polytopes are illustrated by dotted gray lines.

The equation of the closed loop system can be written as $\dot{x} = \mathfrak{A}(x)x$, where

$$\mathfrak{A}(x) = \begin{bmatrix} -\frac{c_2 F_0 \bar{S}^2 + c_1 F_0 \bar{S}}{r(\bar{S})V} - \frac{k\bar{S}}{V} & \frac{\mu_{max} V(\bar{X}+X_0) - X_0 F_0 (c_2\bar{S}+c_1)}{r(\bar{S})V} - \frac{kX_0}{V} \\ -\frac{\mu_{max} S_0}{r(\bar{S})Y} & \frac{Y F_0 (S_F - S_0)(c_2\bar{S}+c_1) - \mu_{max} V(\bar{X}+X_0)}{r(\bar{S})VY} - \frac{k(\bar{S}+S_0-S_F)+F_0}{V} \end{bmatrix}$$

Due to numerical difficulties, we used an outer polytope ($\mathcal{X}$) with a reduced number of corner points with the purpose of reducing the SDP problem's dimension. Furthermore, the corners' position were chosen manually, according to the following procedure. First of all, we defined an initial polytope, which surely satisfies the LMI conditions. Then, in each step, we enlarged the polytope with small increments in a quasi-random manner ('mutation' of the polytope). In one step, only a few corners were modified, and they were strategically chosen by considering the distance of the facets to the maximal invariant level set of the previous iteration. The position of the chosen corner points were altered randomly by small amount with the intention of increasing the distance of the neighbouring facets to the level set. If the problem with the new outer polytope is feasible producing no numerical failures, than we continue with the next step, in the other case we try another mutation of the previous polytope. This procedure is based on quasi-random modifications of an initially given feasible polytope, therefore, it can be automated. The final Lyapunov function we achieved was $V(x) = p(x)/q(x)$, and its maximal invariant level set was $\varepsilon = \{x \in \mathcal{X} \mid V(x) = 12.886\}$, where:

$$\begin{aligned} p(x) = {} & 7893.1\, x_1^2 x_2^{12} + 47168\, x_1^2 x_2^{11} + 1.5464e6\, x_1^2 x_2^{10} + 1.2694e7\, x_1^2 x_2^9 + 4.3955e7\, x_1^2 x_2^8 + 7.9184e7\, x_1^2 x_2^7 + 8.0827e7\, x_1^2 x_2^6 \\ & + 4.8265e7\, x_1^2 x_2^5 + 1.5898e7\, x_1^2 x_2^4 + 2.6443e6\, x_1^2 x_2^3 + 1.7643e5\, x_1^2 x_2^2 + 1392\, x_1^2 x_2 + 147.6\, x_1^2 + 13824\, x_1 x_2^13 + 94230\, x_1 x_2^{12} \\ & - 6.131e5\, x_1 x_2^{11} - 8.0351e6\, x_1 x_2^{10} - 3.2513e7\, x_1 x_2^9 - 6.5635e7\, x_1 x_2^8 - 7.123e7\, x_1 x_2^7 - 4.1178e7\, x_1 x_2^6 - 1.261e7\, x_1 x_2^5 \\ & - 1.8956e6\, x_1 x_2^4 - 93978\, x_1 x_2^3 + 4561\, x_1 x_2^2 + 291.41\, x_1 x_2 + 46964\, x_2^14 + 5.953e5\, x_2^13 + 3.2591e6\, x_2^12 + 1.0158e7\, x_2^{11} \\ & + 1.9889e7\, x_2^{10} + 2.5173e7\, x_2^9 + 2.0184e7\, x_2^8 + 9.7704e6\, x_2^7 + 2.8587e6\, x_2^6 + 5.3869e5\, x_2^5 + 76424\, x_2^4 + 8727.2\, x_2^3 + 537.97\, x_2^2 \\ q(x) = {} & x_2^{10} + 12.187\, x_2^9 + 62.131\, x_2^8 + 171.36\, x_2^7 + 276.58\, x_2^6 + 265.61\, x_2^5 + 150.77\, x_2^4 + 50.925\, x_2^3 + 10.065\, x_2^2 + 1.0762\, x_2 \\ & + 0.048143 \end{aligned}$$

5. Conclusions

In this work, we presented an optimization-based computational method for determining Lyapunov functions and invariant regions for nonlinear dynamical systems. The starting point of the method is the approach presented in [7]. The improvements and new contributions can be summarized as follows: 1) The model transformation to the required form for optimization is done automatically using LFT with auxiliary algorithmic simplifications. This technique results in the dimension reduction of the problem compared to known solutions in the literature. 2) An algorithm was given for the generation of appropriate annihilators for the vector π. 3) An improved method was proposed for determining the possible largest invariant set for the dynamics using the computed Lyapunov function. 4) A generalized formula was given for the case when the system matrix $F(x)$ is not regular in the model (6). The operation of the approach was illustrated through examples taken from the literature. Although the developed method itself is capable of handling uncertain models, it will be the target of future work to test it on examples containing uncertainties.

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