Robust Internal Model Controller for Fractional-Order Systems

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Abstract – *The paper considers the control problem of a fractional order system with uncertain parameters. We describe the design of an Internal Model Control (IMC) scheme and apply it to a fractional order system. Due to plant uncertainty, we propose a robust IMC design scheme. The resulting IMC controller is also of fractional order. To tune the controller we minimize the infinity norm of the robust performance index.*

Keywords: fractional order systems, IMC controllers, robust fractional controllers.

1. Introduction

It is interesting to observe that fractional order systems (FOS) appear in a wide range of physical processes, as of heat equation, diffusion equation, acoustic waves, economic prolems, robot manipulators [7,8,15]. From the 80's there have been growing interest regarding fractional systems. One of the earliest work was the development of CRONE by Oustaloup [2,12]. In the early 90's Vajta (1991) and Charef (1992) proposed - independently of each other - an approximation technique based on frequency domain characteristics. Vajta's method was specifically tuned to approximate transfer functions with $\sinh(s^{1/2})$ and $\cosh(s^{1/2})$ terms and proved in experiments remarkable accurate [16]. Podlubny described the pheasibility of a $PI^{\mu} D^{\gamma}$ controller [1314]. Fractional order controllers may provide better performance for they are more "flexible". This is because FOS are infinite dimensional systems for they can be described by infinite number of states, like distributed parameter systems. In general to design a controller we may have two basic approaches:

i.) approximate the process by a finite order model and design the controller for this reduced order model, or

ii.) design the controller using the fractional order (infinite dimensional) process model and approximate the resulting controller.

We consider now the second case (for case i.) see [17) and shall design the controller using the fractional order process model. An well-suited controller design technique is the Internal Model Controller (IMC) [7,11] In a previous paper we described the IMC design for a fractional heat process [18] which naturally led to a non-rational controller. Now we consider systems described by fractional order linear differential euqations in the time domain, or by fractional order transfer functions in the frequency domain. We describe the basic design principles and present the IMC controller. To tune the controller one can choose different design specifications, we choose to minimize the infinity norm of robust performance.

2. Problem statement

Consider now a physical process described by the fractional order differential equation:

$$\begin{aligned} &b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + \ldots + b_1 D^{\beta_1} u(t) + b_0 u(t) = \\ &= a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \ldots + a_1 D^{\alpha_1} y(t) + a_0 y(t) \end{aligned} \tag{1}$$

where $D^{\gamma} \equiv D_{0,t}^{\gamma}$ denotes fractional derivative of order γ with respect to variable t, $\{a_k, b_j\} \in \mathbb{R}$ and $\{\alpha_k, \beta_j\} \in \mathbb{R}$ are real numbers, $0 < \alpha_1 < \ldots < \alpha_n$; and $0 < \beta_1 < \ldots < \beta_m$. There are different definitions of fractional derivatives (Riemann-Liouville, Grünwald-Letnikov or Caputo's derivative) which are equivalent under some conditions. We apply Caputo's definition due to its advantages concerning the initial conditions. The Caputo derivative with order $\alpha > 0$ of a given function x(t), $t \in [0,T]$ is defined [13]:

$$D_{0,t}^{\alpha} x(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau)\, d\tau \tag{2}$$

where $\Gamma(x)$ is the gamma function, m is a positive integer satisfying $m-1 < \alpha < m \in \mathbb{Z}_+$ and t>0. The Laplace transform of x(t) is then given:

$$L\left\{D_{0,t}^{\alpha} x(t)\right\} = s^{\alpha} X(s) - \sum_{k=0}^{m} s^{\alpha-k-1} x^{(k)}(0) \tag{3}$$

Assuming zero initial conditions we can directly express the process transfer function as:

$$P(s^{\alpha}) = \frac{B(s^{\beta_j})}{A(s^{\alpha_k})} = \frac{b_0 + b_1 s^{\beta_1} + b_2 s^{\beta_2} + \ldots + b_m s^{\beta_m}}{a_0 + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_n s^{\alpha_n}} \tag{4}$$

where $b_0 \ldots b_m$, $a_0 \ldots a_n$ are the coefficients of the numerator and denumerator polynomal, respectively. We assume the process to be stable. Some of the process parameters may not be known exactly but we assume that they all lie in a given interval. Our goal is to design an Internal Mode Controller which satisfies some design specifications and is robust to parameter uncertainties.

3. Internal Model Control (IMC) Scheme

Internal Model Control (IMC) has been widely studied and applied after the pioneering work of Morari [11]. It has recently been extended to distributed parameter systems [18]. The advantage of the IMC design is that - for stable plants - it is based on the open-loop characteristics. Note, that the IMC design is equivalent with the Youla parametrization [7]. Figure 1 shows the IMC control-loop and the equivalent feedback control loop. The control loop consists of the process model $P(s,\mathbf{p})$ with uncertain parameters and a nominal process model $M(s,\mathbf{p}_N)$. The closed-loop transfer function is:

$$G(s^{\alpha},\mu) = \frac{Q(s^{\alpha},\mu)\,P(s^{\alpha},\mathbf{p})}{1 + Q(s^{\alpha},\mu)\left[P(s^{\alpha},\mathbf{p}) - M(s^{\alpha},\mathbf{p}_N)\right]} \tag{5}$$

where $Q(s^{\alpha},\mu)$ is the IMC controller, $\mathbf{p}$ denotes the uncertain process parameter vector and $\mathbf{p}_N$ denotes its nominal value. The controller explicitly contains the nominal process model and therefore it is easy to denote the set of all stabilizing controllers. The model get the same input as the process and only the difference of the measured and simulated output is fed back. We can also express the equivalent conventional feedback controller by the IMC controller $Q(s^{\alpha},\mu)$ and the nominal model $M(s^{\alpha},\mathbf{p}_N)$:

$$C(s^{\alpha},\mu) = \frac{Q(s^{\alpha},\mu)}{1 - M(s^{\alpha},\mathbf{p}_N)\,Q(s^{\alpha},\mu)} \tag{6}$$

First we give some useful definitions.

Process uncertainties: In almost all practical problems the process is not exactly known. It may contain either unknown parameters or unmodelled dynamics. There are different ways to describe the model uncertainty. We consider the following family of plants with multiplicative uncertainty:

$$\Pi = \left\{P : P = \left(1 + \Delta W_2\right)M\right\} \tag{7}$$

where $\|\Delta\|_{\infty} \le 1$ and $P(s^{\alpha},\mathbf{p})$ and $M(s^{\alpha},\mathbf{p}_N)$ have the same number of unstable poles. $M(s^{\alpha},\mathbf{p}_N)$ is stricly proper and the uncertainty weight W_2 and perturbation Δ are stable (possibly irrational) transfer functions [11,18].

Nominal performance: The closed-loop process attains *nominal performance* if it is stable and

$$\|W_1 S\|_{\infty} = \sup_{\omega\in\mathbb{R}} \left|W_1(j\omega)S(j\omega)\right| < 1 \tag{8}$$

where $W_1(s)$ is the performance weighting function and $S(s)$ is the sensitivity function. At low frequencies the performance weighting function $W_1(s)$ is usually large and small at high frequencies.

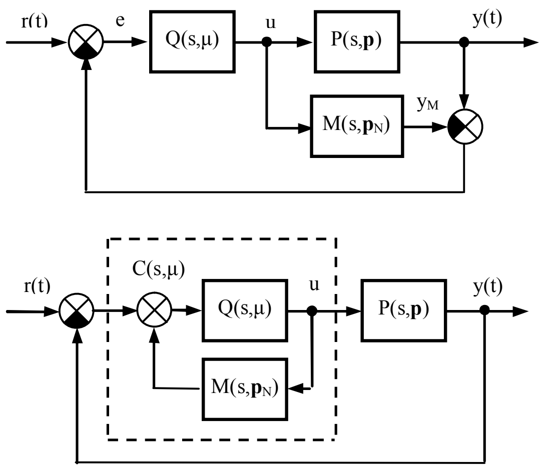

Figure 1: Block diagram of an IMC control-loop (top) and the equivalent feedback control-loop (bottom).

Robust stability and performance: the closed-loop system is stable iff the nominal system is stable and

$$\|W_2 T\| < 1 \tag{9}$$

where $T(s)$ is the complementary sensitivity function. A necessary and sufficient condition for *robust performance* is

$$\left\| |W_1 S| + |W_2 T| \right\|_{\infty} < 1 \tag{10}$$

Figure 2 shows the graphical interpretation of W_1 and W_2 where L is the open-loop transfer function. For more theoretical background see [6,7].

Design of an IMC controller: We generally choose the controller $Q(s)$ such that it minimizes the sensitivity function with nominal plant:

$$S(s) = 1 - M(s^{\alpha},\mathbf{p}_N)\tilde{Q}(s^{\alpha}) \tag{11}$$

subject to the constraint that $M(s^{\alpha},\mathbf{p}_N)$ is causal and stable.

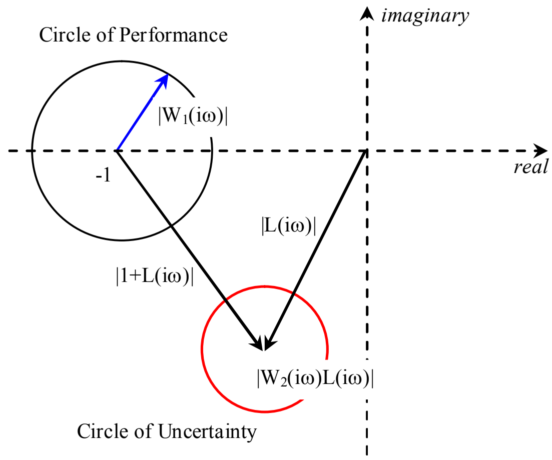

Figure 2: Graphical interpretation of W_1 and W_2
(L is the open-loop transfer function).

The sensitivity function S(s) reaches its absolute minimum for:

$$\tilde{Q}(s^{\alpha},\mathbf{p}_N) = M^{-1}(s^{\alpha},\mathbf{p}_N) \qquad (12)$$

This ideal controller is usually improper, therefore we augment it with a IMC filter $F(s^{\alpha},\mu)$:

$$Q(s^{\alpha},\mu) = \tilde{Q}(s^{\alpha},\mathbf{p}_N)\,F(s^{\alpha},\mu) \qquad (14)$$

For fractional order systems the IMC filter is also of *fractional order*. Typical choices for $F(s^{\alpha},\mu)$ are:

$$F_1(s^{\alpha},\mu) = \frac{1}{1+\mu s^{\gamma}}; \quad \text{or}$$
$$F_2(s^{\alpha},\mu) = \frac{1}{(1+\mu s)^{\gamma}}; \qquad (15)$$

where μ is a design parameter and γ is usually $\gamma = \alpha_n - \beta_m$. The IMC filter must satisfy the following conditions:

$$\lim_{s\to 0} F(s^{\alpha},\mu) = 1;$$
$$\lim_{\mu\to 0} F(s^{\alpha},\mu) = 1; \qquad (16)$$

To design an IMC controller we need:

- nominal process Model $M(s^{\alpha},\mathbf{p}_N)$,
- design specification (performance weighting function $W_1(s)$),
- uncertainty weight $W_2(s)$.

By tuning the controller parameter μ one can satisfy the design specifications. We may trade off speed with robustness to model uncertainty or insensitivity to measurement noise. There are many ways to make compromise. One way is to minimize the infinity norm or robust performance:

$$\mu_{\text{opt}} = \min_{\mu} \left\| \left|W_1\, S\right| + \left|W_2\, T\right| \right\|_{\infty} \qquad (17)$$

As $Q(s^{\alpha},\mu)$ is *infinite dimensional*, so is $C(s^{\alpha},\mu)$. To garantee zero steady-state error the condition $Q(0,\mu) = M(0,p_N)^{-1}$ is required. Note, that for stable processes, increasing μ slows the closed-loop dynamics and increases robustness to model uncertainties.

4. Fractional order controller

We demonstrate the design procedure with an example. Consider the following family of plants defined by a fractional order transfer function:

$$P = \left\{ \frac{1}{1+p\,s^{\alpha}} \;:\; p_{\min} \le p \le p_{\max} \right\} \qquad (18)$$

where the system parameter p lies in an interval:

(13)

$$p_{\min} \le p_N \le p_{\max} \qquad (19)$$

We have choosen the following parameter values:

$$\alpha = 0{,}6$$
$$0{,}8 \le p_N \le 3{,}0$$
$$p_N = 1{,}5.$$

The plant uncertainty can best be demonstrated by the step-response which is given ($p > 0$) by:

$$y_{step}(t) = L^{-1}\left(\frac{1}{s} * \frac{1}{1+p\,s^{\alpha}}\right) = 1(t) - E_{\alpha,1}\left(-a\,t^{\alpha}\right) \qquad (20)$$

where $a = 1/p$ and $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function defined by [1,13]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \qquad (21)$$

The Mittag-Leffler function is an entire function for $(\alpha,\beta) \in \mathbb{R}_+$. Figure 3 shows the effect of the plant uncertainty. The step-response is sentitive to the parameter p (shaded area), as we can see. The elongated step-response indicates that the fractional plan is infinite dimensional. The process model with nominal parameter value is:

$$M(s^{\alpha},p_N) = \frac{1}{1+p_N s^{\alpha}} \qquad (22)$$

The IMC controller with filter $F(s^{\alpha},\mu)$ and the feedback controller $C(s^{\alpha},\mu)$ is given by:

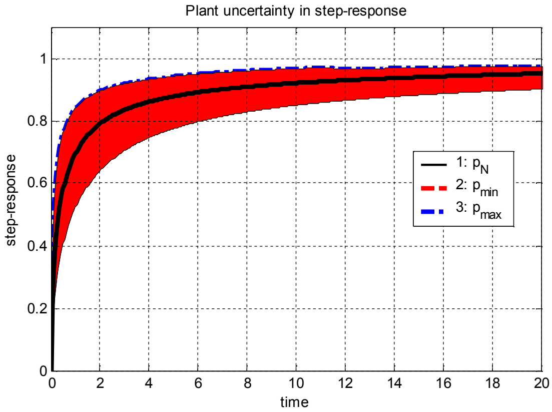

Figure 3. Step response of plant P(s^{α},p) with uncertain parameter p, (α = 0,6 and 0.8 < p_N < 3,0).

$$Q(s^{\alpha},\mu) = F(s^{\alpha},\mu)\tilde{Q}(s^{\alpha}) = \frac{1+p_N s^{\alpha}}{1+\mu\, p_N s^{\alpha}} \tag{23}$$

$$C(s^{\alpha},\mu) = \frac{Q(s^{\alpha},\mu)}{1-M(s^{\alpha},p_N)Q(s^{\alpha},\mu)} = \frac{1+p_N s^{\alpha}}{\mu\, p_N s^{\alpha}} \tag{24}$$

The sensitivity function S(s^{α},μ) and complementary sensitivity function T(s^{α},μ) is also of *fractional order* and is given by:

$$S(s^{\alpha},\mu) = \frac{e(s)}{r(s)} = 1-M\,Q = \frac{\mu\, p_N s^{\alpha}}{1+\mu\, p_N s^{\alpha}} \tag{25}$$

$$T(s^{\alpha},\mu) = \frac{y(s)}{r(s)} = 1-S(s^{\alpha},\mu) = \frac{1}{1+\mu\, p_N s^{\alpha}} \tag{26}$$

We can determine the uncertainty weight $W_2(s)$ from

$$\left|1-\frac{P(s^{\alpha},p)}{M(s^{\alpha},p_N)}\right| = \left|\frac{(p_N-p)\,s^{\alpha}}{1+p_N s^{\alpha}}\right| \le |W_2(s)| \tag{27}$$

With the given parameters it is easy to find an upper bound to the multiplicative uncertainty error in the form:

$$W_2(s) = \frac{T_{w1}\, s^{\alpha}}{1+T_{w2}\, s^{\alpha}} \tag{28}$$

with T_{w1} = 1,35 and T_{w2} = 1,6. As for the performance weight we choose:

$$W_1(s) = \frac{1}{K_i}\left(1+\frac{1}{T_i\, s^{\alpha}}\right) \tag{29}$$

with K_i = 2,5 (i.e. peak sensitivity less then 2,5) and T_i = 0,80 ("closed-loop time constant"). The nominal performance and robust stability can also be expressed in closed-form:

$$|W_1\, S| = \left|\frac{1+p_N s^{\alpha}}{K_i\, T_i\, s^{\alpha}}\right| \tag{30}$$

$$|W_2\, T| = \left|\frac{T_{w1}\, s^{\alpha}}{1+T_{w2}\, s^{\alpha}} * \frac{1}{1+\mu\, p_N\, s^{\alpha}}\right| \tag{31}$$

Figure 4 shows the nominal performance $|W_1\,S|$, the robust stability $|W_2\,T|$ and robust performance index $|W_1\,S|+|W_2\,T|$. As we can see in figure 4c. the infinity norm varies as a function of the controller parameter μ. Figure 5 shows the infinity norm of the robust performance index as a function of μ. It has an absolute minimum around $\mu_{opt} \approx 0{,}25$. So the optimal robust fractional controller (satisfying the design specifications) can now be expressed:

$$Q_{opt}(s^{\alpha},\mu) = \frac{1+s^{0.6}}{1+0{,}25\, s^{0{,}6}} \tag{32}$$

Clearly, this is a PD^{α} type controller. Consequently, the classical feedback controller is of type PI^{α}. To implement the IMC controller one needs to approximate its fractional order transfer function. There are several ways to approximate Q(s^{α},μ) [2,5,10]. We applied continued fraction expansion (CFE) for it converges faster then power series expansion (PSE) based approximations. In the Appendix we provide the CFE of s^{α} and of the second type of IMC filter F_2(s^{α},μ). Figure 6 shows the frequency diagram of the fractional controller Q(s^{α},μ) and its approximation by CFE. A 5th order approximation is sufficiently accurate. Note the extra wide frequency range of 6 decades which is due to the infinite dimension of the process!

<u>Closed-loop performance.</u> Figure 7 demonstrates the closed-loop performance with optimal fractional controller. As we can see the controller drives the process output $y_P(t)$ to its required value, in spite of the large parameter uncertainty.

<u>Lumped, finite order IMC filter.</u> It is a legitimate and interesting question whether a lumped, finite order IMC filter can performe as well as the optimal fractional controller. The answer is simply <u>no</u>. Take for example

$$Q_L(s^{\alpha},\mu) = F(s,\mu)\tilde{Q}(s^{\alpha}) = \frac{1+p_N s^{\alpha}}{1+\mu\, p_N s} \tag{33}$$

Note, that the controller remains of fractional order, but the IMC filter is a first-order filter. Nevertheless, the minimal infinity norm with Q_L(s^{α},μ) is far larger then the one with Q(s^{α},μ). Table 1 contains the numerical values for the two controllers.

For a second order example consider [19].

	$\left\| \|W_1 S\| + \|W_2 T\| \right\|_\infty$	μ_{opt}
Fractional $F(s^\alpha,\mu)$	0,76	0,25
Lumped $F(s,\mu)$	0,95	0,13

Table 1. Minimum value of the infinity norm with fractional- and with lumped IMC filter.

5. Conclusions

We considered and extended the design of an IMC controller for fractional order systems. The basic steps of the design procedure are given. Naturally, the resulting controller is also of fractional order. To tune the IMC controller parameter μ, we minimized the infinity norm of the robust performance index. To increase performance one may apply a two-degree-of-freedom configuration as well.

6. Appendix

There are several ways to realize the fractional power of s^α. We used a continued fraction expansion (CFE) given by:

$$s^\alpha = 1 + \cfrac{\alpha(s-1)}{1 - \cfrac{(\alpha-1)(s-1)}{2 + \cfrac{(\alpha+1)(s-1)}{3 - \cfrac{(\alpha-2)(s-1)}{2 + \cfrac{(\alpha+2)(s-1)}{5 - \cfrac{(\alpha-3)(s-1)}{2 + \cfrac{(\alpha+3)(s-1)}{7 - \cfrac{(\alpha-4)(s-1)}{2 + \ldots}}}}}}}}$$

To approximate the fractional IMC filter $F_2(s^\alpha,\mu)$ we may use the following CFE expression:

$$\frac{1}{(1+\mu s)^\alpha} = 1 - \cfrac{\mu\alpha s}{1 + \cfrac{\mu(\alpha+1)s}{2 - \cfrac{\mu(\alpha-1)s}{3 + \cfrac{\mu(\alpha+2)s}{2 - \cfrac{\mu(\alpha-2)s}{5 + \cfrac{\mu(\alpha+3)s}{2 - \cfrac{\mu(\alpha-3)s}{7 + \ldots}}}}}}}$$

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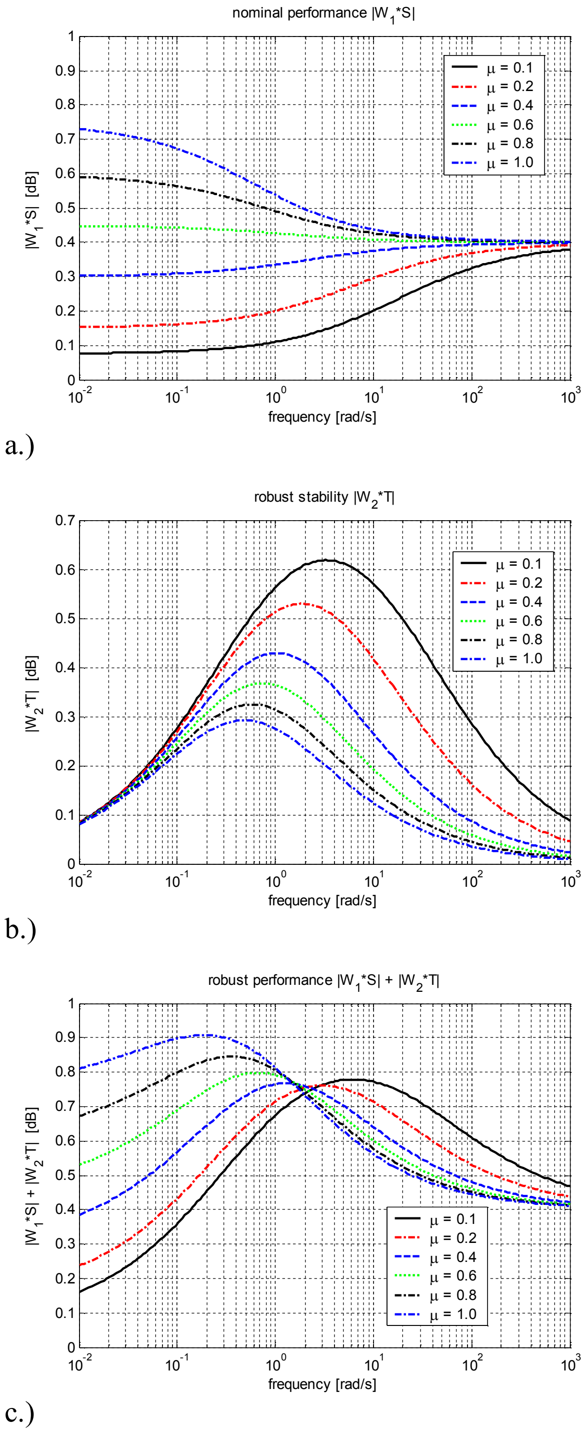

Figure 4. Frequency diagram of nominal performance (a), robust stability (b) and robust performance (c) with fractional controller $Q(s^{\alpha},\mu)$ ($\alpha = 0{,}6$ and $0{,}8 < p_N < 3{,}0$).

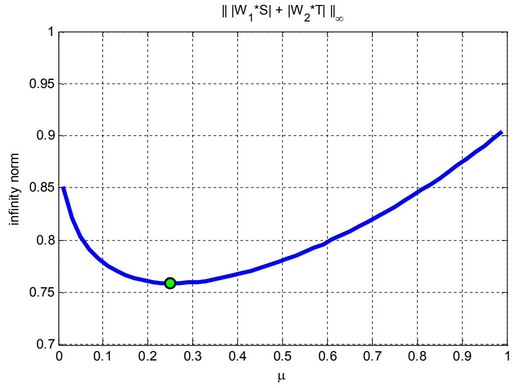

Figure 5: Infinity norm of robust performance index as a function of controller parmeter μ.

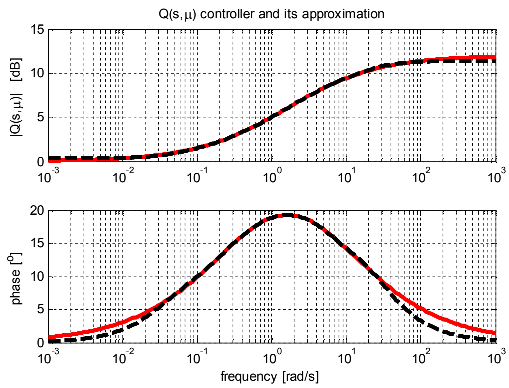

Figure 6: Realization of fractional controller $Q(s^{\alpha},\mu_{opt})$ ($\mu_{opt} = 0{,}25$).

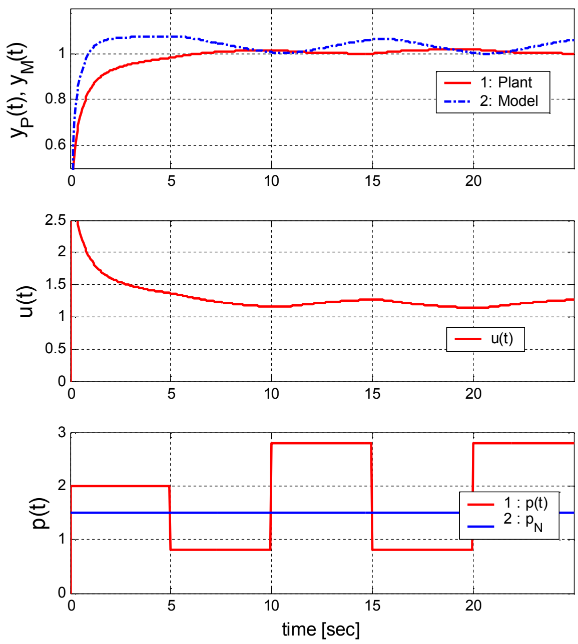

Figure 7. Closed-loop performance with fractional controller ($\alpha = 0{,}6$ and $0.8 < p_N < 3{,}0$).